Identification of the boundary surface of an interstellar cloud from a measurement of the photon far-field

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Abstract

We study an inverse problem for photon transport in a host medium (e.g. an interstellar cloud), that occupies a bounded and strictly convex region $\Omega \subset \mathbb{R}^3$. Under the assumption that the cross sections and the sources are known, we identify the boundary surface $\Sigma = \partial\Omega$ (within a suitable family $\mathcal{F}$ of surfaces), provided that one value of the photon number density is measured at some given location far from Ω.

1 Introduction

In photon transport theory, three types of problems are usually considered in the literature:

α) evaluation of the photon number density (as a function of the position $\mathbf{x}$ and of the direction of propagation $\mathbf{u}$), starting from the knowledge of the various cross sections, of the sources, of the ingoing flux and of the shape of the surface that bounds the host medium (e.g. an interstellar cloud);

β) identification of the spatial behaviour of the cross sections and/or of the sources (or of some other *physical* quantity, such as the ingoing flux), starting from the knowledge of the exiting photon flux;

γ) identification of the surface that bounds the host medium, starting from the knowledge of the photon far-field (i.e. of the photon number density measured in locations far from the medium), and assuming that the relevant physical quantities of the medium (cross sections, etc.) are known.

We remark that (α) are standard *direct* problems, where the unknown is the photon number density $n(\mathbf{x}, \mathbf{u})$, see for instance [GvdMP87, Pom73].

Note that, in astrophysics, the knowledge of $n(\mathbf{x}, \mathbf{u})$ is particularly important because interactions between UV-photons and the particles of an interstellar cloud play a crucial role in the chemistry and in the evolution of the cloud [DW97]. On the other hand, (β) and (γ) are typical *inverse* problems in radiative transfer theory and may be of interest in astrophysics as well as in metereology, in glass manifacturing, in modelling the radioactive properties of porous insulating materials, in tomography and in semiconductor theory.

In problems (β) and (γ), the number density $n(\mathbf{x}, \mathbf{u})$ is not the crucial unknown: the main interest is directed to evaluate some physical or geometrical quantities of the host medium (e.g. an interstellar cloud, the earth atmosphere, a porous insulator, a semiconductor, etc.). We also remark that (β) and (γ) are two *different* kinds of inverse problems. In fact, in (β) some *physical* quantities (e.g., the cross sections, the sources, the ingoing photon flux, etc) are the main unknowns, whereas in (γ) a *geometrical* quantity (e.g the shape of the boundary surface, the shape of the conduction and of the valence bands, etc.) is what one is looking for.

The literature on problems of type (β) is rather abundant: see the references, where we list most of the recent papers [AB88, Ago91, Bal00a, Bal00b, Cho92, CS99, Dre89, Gao92, Gon86, Gri00, GN92, Hs88, JN99, Lar88, McC86, MS97, MKZ00, MK94, Rom97, Sha96, SG00, Sie02b, Sie02a, Tam02, YY89, Zwe99] (for the less recent see [McC86]). On the other hand, there are only a few examples of photons transport problems of type (γ), see for instance [BMR02].

Note that in most of problems (β), the *whole* exiting flux is assumed to be known (or measured), whereas in this paper and in [BMR02] (a problem of type (γ)) only the value of the exiting flux, corresponding to a given direction, is taken to be known. For instance, if an inverse problem of type (β) is studied in the slab $\{x : 0 \leq x \leq a\}$, the ingoing particle densities $n(0, \mu)$ with $\mu \in (0, 1]$ and $n(a, \mu)$ with $\mu \in (-1, 0]$ are given, whereas, for instance the exiting density $n(a, \mu)$ is thought to be measured at *any* $\mu \in (0, 1]$ and not only for a given $\mu \in (0, 1]$. Finally, it is worth noticing that a rather unusual inverse problem is studied in [Zwe99]: given the exiting flux, identify the ingoing one. This may be of great interest in astrophysics to evaluate UV-protons sources "behind" a given interstellar cloud.

In this paper, we study the following inverse problem of type (γ): assuming that the cross sections of an interstellar cloud, the UV-photon sources and the ingoing flux are known, is it possible to identify the boundary surface of the cloud (within a suitable family of surfaces), provided that a single value of the UV-proton number density $n(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is, measured at some location $\hat{\mathbf{x}}$ far from the cloud? We shall prove that such an identification is possible because the density $n(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is, in some sense, a strictly monotonic function of the dimensions of the surface that bounds the cloud.

2 The Boltzmann-like model

Let the host medium (e.g., the interstellar cloud) occupy the bounded region $\Omega = \Omega_i \cup \Sigma \subset \mathbb{R}^3$, where Ω_i is the interior of Ω and Σ is the closed "regular" surface that bounds Ω, see Figure 1. In what follows, we shall also assume that Ω is strictly convex i.e., if $\mathbf{x}'$ and $\mathbf{x}''$ belong to Σ, then $\mathbf{x} = \lambda\mathbf{x}' + (1 - \lambda)\mathbf{x}'' \in \Omega_i \ \forall \lambda \in (0, 1)$.

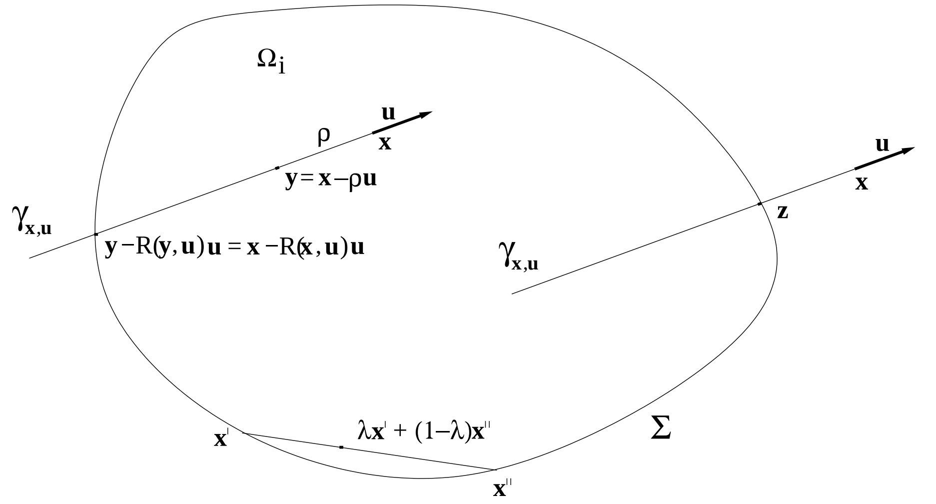

Figure 1: The strictly convex region $\Omega = \Omega_i \cup \Sigma$.

Furthermore, let $n(\mathbf{x},\mathbf{u})$ be the particle number density (e.g. the UV-photon number density), so that $n(\mathbf{x},\mathbf{u})\, d\mathbf{x}d\mathbf{u}$ is the expected number of particles in the volume element $d\mathbf{x}$ centered at $\mathbf{x}$ and having velocities within the solid angle $d\mathbf{u} = \sin\vartheta\, d\vartheta d\varphi$ around the unit vector $\mathbf{u}$. Then, $n(\mathbf{x},\mathbf{u})$ satisfies the following stationary Boltzmann-like equation [Pom73]

$$-\mathbf{u}\cdot\nabla n(\mathbf{x},\mathbf{u}) - \sigma(\mathbf{x})n(\mathbf{x},\mathbf{u}) + \frac{1}{4\pi}\sigma_s(\mathbf{x})\int_S n(\mathbf{x},\mathbf{u}')\, d\mathbf{u}' + q(\mathbf{x}) = 0, \qquad (\mathbf{x},\mathbf{u}) \in \mathbb{R}^3 \times S. \tag{1}$$

In equation (1), $\sigma(\mathbf{x})$ is the total cross section (i.e., $\sigma(\mathbf{x}) = \sigma_c(\mathbf{x}) + \sigma_s(\mathbf{x})$, where σ_c and σ_s are respectively the capture and the scattering cross section), $q(\mathbf{x})$ is the particle source at $\mathbf{x}$ (e.g., the UV-photon source at $\mathbf{x}$), and scattering is assumed to be isotropic for simplicity. Moreover, S is the surface of the unit sphere and $d\mathbf{u}' = \sin\vartheta'\, d\vartheta'\, d\varphi'$ is the infinitesimal solid angle around the unit vector $\mathbf{u}'$.

In what follows, we shall also assume that

$$\sigma_s(\mathbf{x}) = \sigma_s = \text{ a positive constant if } \mathbf{x} \in \Omega, \qquad \sigma_s(\mathbf{x}) = 0 \text{ if } \mathbf{x} \notin \Omega, \tag{2}$$

$$\sigma(\mathbf{x}) = \sigma = \text{ a positive constant if } \mathbf{x} \in \Omega, \qquad \sigma(\mathbf{x}) = 0 \text{ if } \mathbf{x} \notin \Omega, \tag{3}$$

$$q(\mathbf{x}) = q = \text{ a positive constant if } \mathbf{x} \in \Omega, \qquad q(\mathbf{x}) = 0 \text{ if } \mathbf{x} \notin \Omega \tag{4}$$

Remark 2.1

i) Equation (1) is in fact a quasi-static version of the time dependent Boltzmann-like equation and the number density n also depends on the "parameter" t. However, since the "speed of variation" of the surface Σ is much smaller than the speed of light, the quasi-static equation (1) is a very good approximation [BMR02],[RS02]

ii) Since we shall study the integral version of (1) in a Banach space of continuous functions, the assumption that Ω in a strictly convex region takes care of the fact that the cross section and the source term are not continuous when $\mathbf{x}$ *crosses Σ, see* [Pom73].

The integral version of equation (1) has the form

$$n(\mathbf{x},\mathbf{u}) = Q(\mathbf{x},\mathbf{u}) + (Bn)(\mathbf{x},\mathbf{u}), \qquad (\mathbf{x},\mathbf{u}) \in \Omega \times S \tag{5}$$

where

$$Q(\mathbf{x},\mathbf{u}) = \frac{q}{\sigma}\{1 - \exp[-\sigma R(\mathbf{x},\mathbf{u})]\}, \tag{6}$$

$$(Bn)(\mathbf{x},\mathbf{u}) = \frac{\nu\sigma}{4\pi}\int_0^{R(\mathbf{x},\mathbf{u})} dr\, \exp(-\sigma r)\int_S n(\mathbf{x}-r\mathbf{u},\mathbf{u}')\,d\mathbf{u}'. \tag{7}$$

$$\tag{8}$$

Moreover, if $x \notin \Omega$, we have

$$n(\mathbf{x},\mathbf{u}) = n(\mathbf{z},\mathbf{u}) \qquad \text{if } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset, \tag{9}$$

$$n(\mathbf{x},\mathbf{u}) = 0 \qquad \text{if } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i = \emptyset, \tag{10}$$

In (6) and (7), $\nu = \sigma_s/\sigma < 1$ and $R(\mathbf{x},\mathbf{u})$ is such that $\mathbf{x} - R(\mathbf{x},\mathbf{u})\mathbf{u} \in \Sigma$ for each $(\mathbf{x},\mathbf{u}) \in \Omega \times S$, whereas $R(\mathbf{x},\mathbf{u}) = 0$ if $\mathbf{x} \in \Sigma$ and $\mathbf{u}$ is directed towards Ω_i, see Figure 1 (the surface Σ is, by assumption, "regular" enough to ensure that $R(\mathbf{x},\mathbf{u})$ is a continuous function of $(\mathbf{x},\mathbf{u}) \in \Omega \times S$). Furthermore, in (9) and (10) $\gamma_{\mathbf{x},\mathbf{u}} = \{\mathbf{y} \colon \mathbf{y} = \mathbf{x} - r\mathbf{u}, r \geq 0\}$ is the half straight line passing through $\mathbf{x}$ and parallel to $\mathbf{u}$, and $\mathbf{z} \in \Sigma$ is the "first" intersection of $\gamma_{\mathbf{x},\mathbf{u}}$ with Σ if $\mathbf{x} \notin \Omega$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset$.

Since we shall study equations (5), (9) and (10) in a Banach space of continuous functions, we need to introduce a "large" (but bounded) convex region $\Omega_M = \Omega_{Mi} \cup \Sigma_M \subset \mathbb{R}^3$, bounded by the "regular" surface Σ_M. The region Ω_M is chosen so that its interior Ω_{Mi} contains Ω. Then, equations (5), (9) and (10) will be studied in the real Banach space $X = C(\Omega_M \times S)$, with norm $\|f\|_X = \{\max|f(\mathbf{x},\mathbf{u})|, (\mathbf{x},\mathbf{u}) \in \Omega_M \times S\}$. Correspondingly we rewrite (5)-(10) as follows:

$$n(\mathbf{x},\mathbf{u}) = Q(\mathbf{x},\mathbf{u}) + (Bn)(\mathbf{x},\mathbf{u}), \qquad (\mathbf{x},\mathbf{u}) \in \Omega_M \times S, \tag{11}$$

$$Q(\mathbf{x},\mathbf{u}) = \frac{q}{\sigma}\{1 - \exp[-\sigma R(\mathbf{x},\mathbf{u})]\} \qquad \text{if } (\mathbf{x},\mathbf{u}) \in \Omega \times S, \tag{12}$$

$$Q(\mathbf{x},\mathbf{u}) = \frac{q}{\sigma}\{1 - \exp[-\sigma R(\mathbf{z},\mathbf{u})]\}$$
$$\text{if } \mathbf{x} \in \Omega_M \setminus \Omega_i \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset, \tag{13}$$

$$Q(\mathbf{x},\mathbf{u}) = 0 \qquad \text{if } \mathbf{x} \in \Omega_M \setminus \Omega_i \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i = \emptyset, \tag{14}$$

$$(Bn)(\mathbf{x},\mathbf{u}) = \frac{\nu\sigma}{4\pi}\int_0^{R(\mathbf{x},\mathbf{u})} dr\, \exp(-\sigma r)\int_S n(\mathbf{x}-r\mathbf{u},\mathbf{u}')\,d\mathbf{u}'$$
$$\text{if } (\mathbf{x},\mathbf{u}) \in \Omega \times S, \tag{15}$$

$$(Bn)(\mathbf{x},\mathbf{u}) = \frac{\nu\sigma}{4\pi}\int_0^{R(\mathbf{z},\mathbf{u})} dr\, \exp(-\sigma r)\int_S n(\mathbf{z}-r\mathbf{u},\mathbf{u}')\,d\mathbf{u}'$$
$$\text{if } \mathbf{x} \in \Omega_M \setminus \Omega \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset, \tag{16}$$

$$(Bn)(\mathbf{x},\mathbf{u}) = 0 \qquad \text{if } \mathbf{x} \in \Omega_M \setminus \Omega \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i = \emptyset. \tag{17}$$

Remark 2.2 *$Q \in X = C(\Omega_M \times S)$ and $Bn \in X$ $\forall n \in X$, because the region Ω is assumed to be strictly convex.*

Note that, if $\delta = \max\{R(\mathbf{x}, \mathbf{u}), (\mathbf{x}, \mathbf{u}) \in \Omega \times S\}$ is the diameter of Ω, relations (12)–(14) give

$$0 \leq Q(\mathbf{x}, \mathbf{u}) \leq \frac{q}{\sigma} \{1 - \exp[-\sigma\delta]\} \qquad \forall (\mathbf{x}, \mathbf{u}) \in \Omega_M \times S$$

and so

$$\|Q\| \leq \frac{q}{\sigma} \{1 - \exp[-\sigma\delta]\} \tag{18}$$

In an analogous way we have from (15)-(16)

$$\begin{aligned} |(Bn)(\mathbf{x}, \mathbf{u})| \quad \leq \quad & \nu\sigma \|n\| \int_0^\delta \exp(-\sigma r)\, dr = \nu \{1 - \exp[-\sigma\delta]\} \|n\|, \\ & \qquad \forall n \in X, \ (\mathbf{x}, \mathbf{u}) \in \Omega_M \times S; \end{aligned}$$

thus,

$$\|B\| \leq \nu\{1 - \exp(\sigma\delta)\} < \nu < 1, \tag{19}$$

i.e. the operator B is strictly contractive, [BMM98]. It follows from inequality (19) that the unique solution $n \in X$ of the integral equation 18) has the form

$$n(\mathbf{x}, \mathbf{u}) = ((I - B)^{-1} Q)(\mathbf{x}, \mathbf{u}), \quad (\mathbf{x}, \mathbf{u}) \in \Omega_M \times S, \tag{20}$$

with

$$\|n\| \leq \frac{\|Q\|}{1 - \|B\|} \leq \frac{q}{\sigma} \frac{1 - \exp(-\delta\sigma)}{1 - \nu\{1 - \exp(-\delta\sigma)\}} \leq \frac{q}{\sigma} \frac{1}{1 - \nu}. \tag{21}$$

3 Some Technical Lemmas

The results listed in the following lemmas will be used in the sequel, in connection with equation (11).

Lemma 3.1

i) $Q \in X_+$, *where* $X_+ = \{f\colon f \in X, f(\mathbf{x}, \mathbf{u}) \geq 0, \forall (\mathbf{x}, \mathbf{u}) \in \Omega_M \times S\}$ *is the closed positive cone of* X*;*

ii) $Q(\mathbf{x}, \mathbf{u}) > 0$ *if* $\mathbf{x} \in \Omega_M$ *and* $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset$*;*

iii) $\displaystyle\int_S Q(\mathbf{x}, \mathbf{u})\, d\mathbf{u} > 0 \ \forall \mathbf{x} \in \Omega_M$.

Proof. i) That $Q \in X_+$ follows from definitions (12)-(14).

ii) Since $R(\mathbf{x}, \mathbf{u}) > 0 \ \forall \mathbf{u} \in S$ provided that $\mathbf{x} \in \Omega_i$, and $R(\mathbf{x}, \mathbf{u}) > 0$ if $\mathbf{x} \in \Omega_M \backslash \Omega_i$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset$, definitions (12)-(14) lead to the strict positivity of $Q(\mathbf{x}, \mathbf{u})$.

iii) immediately follows from *ii)*. □

Lemma 3.2

i) $Bf \in X_+ \ \forall f \in X$;

ii) *if* $\int_S f(\mathbf{y}, \mathbf{x}')\, d\mathbf{u}' > 0 \ \forall \mathbf{y} \in \Omega$, *then* $(Bf)(\mathbf{x}, \mathbf{u}) > 0 \ \forall (\mathbf{x}, \mathbf{u})$, *with* $\mathbf{x} \in \Omega_M$ *and* $\mathbf{u}$ *such that* $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset$.

Proof. Definitions (15)-16) immediately lead to *i)* and *ii)*. □

Lemma 3.3

i) $(I - B)^{-1} f \in X_+ \ \forall f \in X_+$;

ii) *if* $\int_S f(\mathbf{y}, \mathbf{u}')\, d\mathbf{u}' > 0 \ \forall \mathbf{y} \in \Omega$, *then* $((I - B)^{-1} f)(\mathbf{x}, \mathbf{u}) > 0 \ \forall (\mathbf{x}, \mathbf{u})$, *with* $\mathbf{x} \in \Omega_M$ *and* $\mathbf{u}$ *such that* $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset$.

Proof. Since $((I - B)^{-1} f)(\mathbf{x}, \mathbf{u}) = \sum_{j=0}^{\infty} (B^j f)(\mathbf{x}, \mathbf{u}) \geq f(\mathbf{x}, \mathbf{u}) + (Bf)(\mathbf{x}, \mathbf{u})$ $\forall f \in X_+$, (i) and (ii) follow from Lemma 3.2. □

Lemmas 3.1 and 3.3 show that the particle density $n(\mathbf{x}, \mathbf{u})$, as given by (20), is strictly positive if $\mathbf{x} \in \Omega_M$ and $\mathbf{u}$ is such that $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset$; in any case, $n \in X_+$ as it must be from a physical viewpoint.

Another rather important property of $n(\mathbf{x}, \mathbf{u})$ is specified in Lemma 3.4, where $n(\mathbf{x}, \mathbf{u})$ is shown to be an "increasing" function of $\mathbf{x}$ as $\mathbf{x}$ "approaches" the boundary surface Σ.

Lemma 3.4 *Let* $(\mathbf{x}, \mathbf{u}) \in \Omega \times S$, *with* $\mathbf{u}$ *such that* $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset$; *if* $\mathbf{y} = \mathbf{x} - \rho \mathbf{u}$ *with* $0 < \rho \leq R(\mathbf{x}, \mathbf{u})$, *then* $n(\mathbf{x}, \mathbf{u}) > n(\mathbf{y}, \mathbf{u})$, *see Figure 1.*

Proof. Since both $\mathbf{x}$ and $\mathbf{y} = \mathbf{x} - \rho \mathbf{u}$ belong to Ω, see Figure 1, and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega \supset \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset$, (11), (12) and (15) give

$$
\begin{aligned}
\Delta(\mathbf{x}, \mathbf{u}) &= \Bigg\{ \frac{q}{\sigma} [1 - \exp(-\sigma\rho)] \exp[-\sigma R(\mathbf{y}, \mathbf{u})] + \\
&+ \frac{\nu\sigma}{4\pi} \int_{R(\mathbf{y},\mathbf{u})}^{R(\mathbf{x},\mathbf{u})} dr \exp(-\sigma r) \int_S n(\mathbf{x} - r\mathbf{u}, \mathbf{u}')\, d\mathbf{u}' \Bigg\} + \\
&+ \frac{\nu\sigma}{4\pi} \int_0^{R(\mathbf{y},\mathbf{u})} dr \exp(-\sigma r) \int_S \Delta(\mathbf{x} - r\mathbf{u}, \mathbf{u}')\, d\mathbf{u}'. \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta(\mathbf{x}, \mathbf{u}) &= n(\mathbf{x}, \mathbf{u}) - n(\mathbf{y}, \mathbf{u}) = n(\mathbf{x}, \mathbf{u}) - n(\mathbf{x} - \rho\mathbf{u}, \mathbf{u}), \\
\Delta(\mathbf{x} - r\mathbf{u}, \mathbf{u}') &= n(\mathbf{x} - r\mathbf{u}, \mathbf{u}') - n(\mathbf{x} - r\mathbf{u} - \rho\mathbf{u}, \mathbf{u}') = \\
&= n(\mathbf{x} - r\mathbf{u}, \mathbf{u}') - n(\mathbf{y} - r\mathbf{u}, \mathbf{u}')
\end{aligned}
$$

and

$$R(\mathbf{y}, \mathbf{u}) = R(\mathbf{x}, \mathbf{u}) - \rho.$$

Equation (22) can be studied in the Banach space $X_0 = C(\Omega \times S)$ and it is not difficult to prove that the unique solution Δ belongs to the closed positive cone of X_0. Moreover, $\Delta(\mathbf{x}, \mathbf{u}) > 0 \ \forall (\mathbf{x}, \mathbf{u}) \in \Omega \times S$, provided that $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_i \neq \emptyset$, because the "known term" on the right hand side of (22) is then positive. □

4 The Inverse Problem

Assume that the value $\hat{n} = n(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ of the particle density at $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is known, where $\hat{\mathbf{x}}$ is a location "far" from Ω (i.e., $\hat{\mathbf{x}} \in \Omega_M \setminus \Omega$) and $\hat{\mathbf{u}}$ is such that $\gamma_{\hat{\mathbf{x}},\hat{\mathbf{u}}} \cap \Omega_i \neq \emptyset$. If Ω is occupied by an interstellar cloud, $\hat{n}$ might be the result of some experimental measurements of the emitted UV-photons radiation, made by terrestrial astronomer. In this case, $\hat{\mathbf{x}}$ might be the location of a radiotelescope on the earth and $\gamma_{\hat{\mathbf{x}},\hat{\mathbf{u}}}$ joins $\hat{\mathbf{x}}$ with points within the cloud.

Our inverse problem can be stated as follows: is it possible to identify the shape of the boundary surface Σ from the knowledge of the measured value $\hat{n}$? We shall prove that, given the value $\hat{n}$, an *unique* surface $\hat{\Sigma}$ can be determined *within* a *suitable* family $\mathcal{F}$ of surfaces. Such a surface is such that, if $n = n(\mathbf{x}, \mathbf{u})$ is the particle density "produced" by the host medium contained within $\hat{\Sigma}$, then the relation $n(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \hat{n}$ is satisfied.

However, if another family $\mathcal{F}_1$ is chosen (with $\mathcal{F} \cap \mathcal{F}_1 = \emptyset$), then another surface $\hat{\Sigma_1}$ is determined. This kind of "non-uniqueness" will be discussed later on.

We then introduce the one-parameter family of surfaces

$$\mathcal{F} = \{\Sigma_h \colon \varphi(x, y, z; h) = 0, \ h \in [h_m, h_M]\} \tag{23}$$

that satisfies the following assumptions

a1) $\varphi(x, y, z; h) = 0$ is the equation of a closed surface for each $h \in [h_m, h_M]$;

a2) $\varphi(x, y, z; h)$ is a continuous function of $(x, y, z; h) \in \Omega_M \times [h_m, h_M]$ and the point (x, y, z) is external to Σ_h if and only if $\varphi(x, y, z; h) > 0$;

a3) the region $\Omega_h = \Omega_{hi} \cup \Sigma_h$, which is bounded by Σ_h and whose interior is Ω_{hi}, is bounded, closed, strictly convex and contained in Ω_{Mi} (the interior of Ω_M);

a4) if $h < h'$, then $\Omega_h \subset \Omega_{h'i}$, i.e. if (x, y, z) is such that $\varphi(x, y, z; h') = 0$, then $\varphi(x, y, z; h) > 0$.

Remark 4.1

i) Perhaps, the simplest way to construct the family $\mathcal{F}$ is to chose two suitable (i.e. physically reasonable) surfaces Σ_0 and Σ_1 (with Σ_0 "small" and contained within the "large" Σ_1), and then assume that Σ_h is defined by the equation $0 = \varphi(x, y, z; h) = (1-h)\varphi(x, y, z; 0) + h\varphi(x, y, z; 1)$, $h \in [0, 1]$, where $\varphi(x, y, z; 0) = 0$ and $\varphi(x, y, z; 1) = 0$ are the equations of Σ_0 and Σ_1 respectively.

ii) Another rather simple way to construct a family $\mathcal{F}$ is to assume that $\eta(x,y,z) = 0$ is the equation of a closed surface Σ_0 that bounds the closed, strictly convex and bounded region Ω_0. Under suitable assumptions on the continuity of the function $\eta = \eta(x,y,z)$, a family $\mathcal{F}$ of homothetic surfaces may be defined as follows:

$$\mathcal{F} = \left\{\Sigma_h \colon \varphi(x,y,z;h) = 0, \varphi(x,y,z;h) = \eta\left(\frac{x}{h},\frac{y}{h},\frac{z}{h}\right), \quad h \in [h_m, h_M]\right\}$$

with $0 < h_m < h_M$.

Under the assumption that a suitable family $\mathcal{F}$ defined by (23) and satisfying a1)-a4) has been chosen, consider a given $\Sigma_h \in \mathcal{F}$.

Then, in (11)–(17), $\Omega = \Omega_h$, $\Omega_i = \Omega_{hi}$ and the quantities n, R, Q and B depend on $(\mathbf{x},\mathbf{u})$ *and* on the parameter $h \in [h_m, h_M]$. Correspondingly, we rewrite (11)–(17) (for each $h \in [h_m, h_M]$) as follows

$$n(\mathbf{x},\mathbf{u};h) = Q_h(\mathbf{x},\mathbf{u};h) + (B_h n)(\mathbf{x},\mathbf{u};h), \qquad (\mathbf{x},\mathbf{u}) \in \Omega_M \times S, \tag{24}$$

$$Q_h(\mathbf{x},\mathbf{u};h) = \frac{q}{\sigma}\left\{1 - \exp[-\sigma R(\mathbf{x},\mathbf{u};h)]\right\}, \qquad \text{if } (\mathbf{x},\mathbf{u}) \in \Omega_h \times S, \tag{25}$$

$$Q_h(\mathbf{x},\mathbf{u};h) = \frac{q}{\sigma}\left\{1 - \exp[-\sigma R(\mathbf{z}_h,\mathbf{u};h))]\right\}, \qquad \text{if } \mathbf{x} \in \Omega_M \setminus \Omega_h \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{hi} \neq 0 \tag{26}$$

$$Q_h(\mathbf{x},\mathbf{u};h) = 0, \qquad \text{if } \mathbf{x} \in \Omega_M \setminus \Omega_h \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{hi} = \emptyset \tag{27}$$

$$(B_h n)(\mathbf{x},\mathbf{u};h) = \frac{\nu\sigma}{4\pi}\int_0^{R(\mathbf{x},\mathbf{u};h)} dr\, \exp(-\sigma r)\int_S n(\mathbf{x} - r\mathbf{u},\mathbf{u}';h)\, d\mathbf{u}' \qquad \text{if } (\mathbf{x},\mathbf{u}) \in \Omega_h \times S, \tag{28}$$

$$(B_h n)(\mathbf{x},\mathbf{u};h) = \frac{\nu\sigma}{4\pi}\int_0^{R(\mathbf{z}_h,\mathbf{u};h)} dr\, \exp(-\sigma r)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\, d\mathbf{u}' \qquad \text{if } \mathbf{x} \in \Omega_M \setminus \Omega_h \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{hi} \neq \emptyset \tag{29}$$

$$(Bn)(\mathbf{x},\mathbf{u};h) = 0 \qquad \text{if } \mathbf{x} \in \Omega_M \setminus \Omega_h \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{hi} = \emptyset. \tag{30}$$

In (25)-(30), $R(\mathbf{x},\mathbf{u};h)$ is of course such that $\mathbf{x} - R(\mathbf{x},\mathbf{u};h)\mathbf{u} \in \Sigma_h$ with $(\mathbf{x},\mathbf{u}) \in \Omega_h \times S$, and $\mathbf{z}_h \in \Sigma_h$ is the first intersection point of $\gamma_{\mathbf{x},\mathbf{u}}$ with Σ_h if $\mathbf{x} \in \Omega_M \setminus \Omega_h$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{hi} \neq \emptyset$, see Figure 2.

Assume now that h and h' are given (with $h_m \leq h < h' \leq h_M$) and that $(\mathbf{x},\mathbf{u}) \in \Omega_h \times S$ (we recall that $\Omega_h \subset \Omega_{h'i}$ because of assumption a4) and, consequently, $(\mathbf{x},\mathbf{u}) \in \Omega_{h'} \times S$). In the following theorem, we compare $n(\mathbf{x},\mathbf{u};h')$ (the particle density when the host medium occupies $\Omega_{h'}$) with $n(\mathbf{x},\mathbf{u};h)$ (the particle density when the host medium occupies $\Omega_h \subset \Omega_{h'}$).

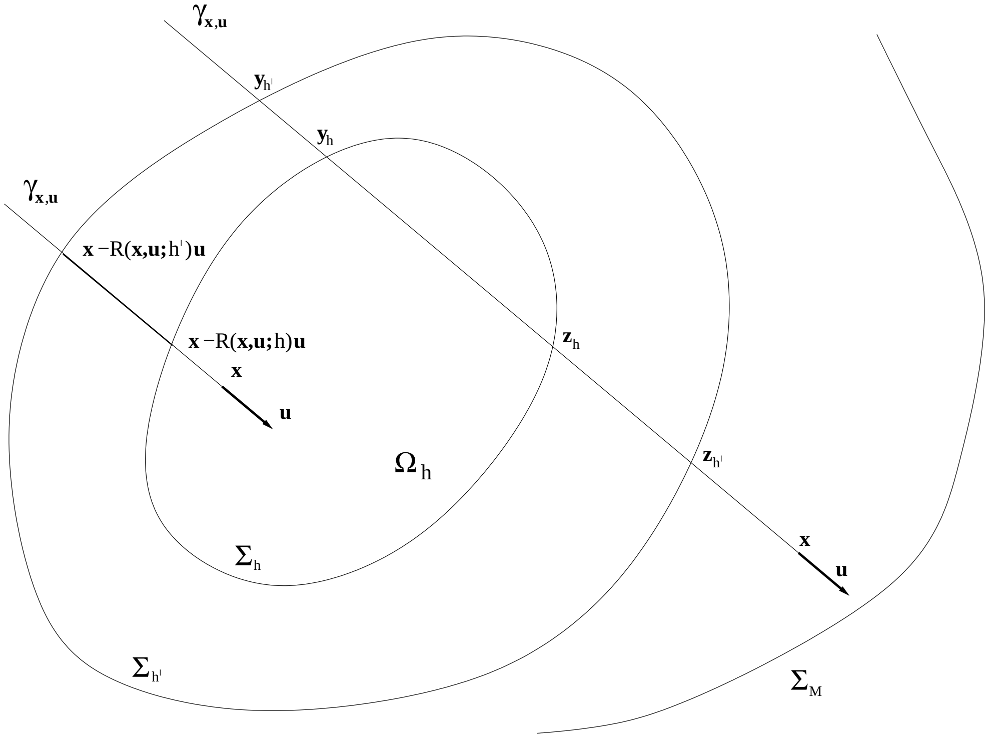

Figure 2: The surfaces Σ_h and $\Sigma_{h'}$ that bound the regions Ω_h and $\Omega_{h'}$, with $h < h'$.

Theorem 4.1

i) If $h_m \leq h < h' \leq h_M$, then $n(\mathbf{x}, \mathbf{u}; h') > n(\mathbf{x}, \mathbf{u}; h)$ at any $(\mathbf{x}, \mathbf{u}) \in \Omega_h \times S$;

ii) $\|n - n'\|_h \to 0$ as $h' \to h$, where $n' = n(\mathbf{x}, \mathbf{u}; h')$, $n = n(\mathbf{x}, \mathbf{u}; h)$ and $\| \cdot \|_h$ is the norm in $X_h = C(\Omega_h \times S)$

Proof. If we put (for h and h' *given* as specified in *i)*) $\Lambda(\mathbf{x}, \mathbf{u}) = n(\mathbf{x}, \mathbf{u}; h') - n(\mathbf{x}, \mathbf{u}; h)$, then (24)-(30) lead to the equation

$$\Lambda(\mathbf{x}, \mathbf{u}) = [F(\mathbf{x}, \mathbf{u}) + G(\mathbf{x}, \mathbf{u})] + (B_h \Lambda)(\mathbf{x}, \mathbf{u}), \qquad (\mathbf{x}, \mathbf{u}) \in \Omega_h \times S. \tag{31}$$

In (31), B_h is defined by (28) and

$$F(\mathbf{x}, \mathbf{u}) = \frac{q}{\sigma} \{\exp\,[-\sigma R(\mathbf{x}, \mathbf{u}; h)] - \exp\,[-\sigma R(\mathbf{x}, \mathbf{u}; h')]\}$$

$$G(\mathbf{x}, \mathbf{u}) = \frac{\nu\sigma}{4\pi} \int_{R(\mathbf{x},\mathbf{u};h)}^{R(\mathbf{x},\mathbf{u};h')} dr \exp(-\sigma r) \int_S n(\mathbf{x} - r\mathbf{u}, \mathbf{u}'; h')\, d\mathbf{u}'$$

where $F(\mathbf{x}, \mathbf{u}) > 0$ and $G(\mathbf{x}, \mathbf{u}) > 0$ because $R(\mathbf{x}, \mathbf{u}; h') > R(\mathbf{x}, \mathbf{u}; h)$ and $n \in X_{h+}$, see Lemma 3.3.

It follows that the unique solution $\Lambda(\mathbf{x}, \mathbf{u})$ of equation (31) in the Banach space $X_h = C(\Omega_h \times S)$ (with norm $\|f\|_h = \max\{|f(\mathbf{x}, \mathbf{u}), (\mathbf{x}, \mathbf{u}) \in \Omega_h \times S\}$) is positive, i.e. $\Lambda(\mathbf{x}, \mathbf{u}) > 0\ \forall (\mathbf{x}, \mathbf{u}) \in \Omega_h \times S$. Furthermore, we have from (31) that

$\|\Lambda_h\| \leq \|F\|_h + \|G\|_h + \nu\|\Lambda\|_h$ because $\|B_h\Lambda\|_h \leq \nu\|\Lambda\|_h$; hence we obtain $\|\Lambda\|_h \leq \frac{1}{1-\nu}\left[\|F\|_h + \|G\|_h\right]$. Since it is not difficult to show that $\|F\|_h \to 0$ and $\|G\|_h \to 0$ as $h' \to h$, we conclude that $\|\Lambda\|_h = \|n' - n\|_h \to 0$ as $h' \to h$. □

Theorem 4.1 takes care of the case in which $\mathbf{x} \in \Omega_h$; on the other hand, if $\mathbf{x} \in \Omega_M \setminus \Omega_h$, Lemma 3.4 and Theorem 4.1 immediately lead to the following results.

Theorem 4.2 *If $h_m \leq h < h' \leq h_M$, then $n(\mathbf{x}, \mathbf{u}; h') > n(\mathbf{x}, \mathbf{u}; h)$ $\forall(\mathbf{x}, \mathbf{u}) \in \Omega_M \times S$, except when $\mathbf{x} \notin \Omega_{h'}$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{h'i} = \emptyset$ (in this case, $n(\mathbf{x}, \mathbf{u}; h') = n(\mathbf{x}, \mathbf{u}; h) = 0$).*

Finally, *ii*) of Theorem 4.1 implies that the following property of the particle density $n(\mathbf{x}, \mathbf{u}; h)$ (still with $\mathbf{x} \in \Omega_M \setminus \Omega_h$) holds.

Theorem 4.3 *If $\mathbf{x} \in \Omega_M \setminus \Omega_h$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{hi} \neq \emptyset$, then $n(\mathbf{x}, \mathbf{u}; h') \to n(\mathbf{x}, \mathbf{u}; h)$ as $h' \to h$.*

Proof. Let $\mathbf{x}$ and $\mathbf{u}$ be such that $\mathbf{x} \in \Omega_M \setminus \Omega_h$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{hi} \neq \emptyset$ (hence, $\mathbf{x} \in \Omega_M \setminus \Omega_{h'}$, if h' is close enough to h, see Figure 2). Equations (9), (24), (26) and (29), with h' instead of h, give:

$$\begin{aligned} n(\mathbf{x}, \mathbf{u}; h') &= n(\mathbf{z}_{h'}, \mathbf{u}; h') = \frac{q}{\sigma}\left\{1 - \exp\left[-\sigma R(\mathbf{z}_{h'}, \mathbf{u}; h')\right]\right\} + \\ &+ \frac{\nu\sigma}{4\pi}\int_0^{R(\mathbf{z}_{h'},\mathbf{u};h')} dr \exp(-\sigma r) \int_S n(\mathbf{z}_{h'} - r\mathbf{u}, \mathbf{u}'; h')\, d\mathbf{u}'. \end{aligned}$$

Thus, we have

$$\begin{aligned} n(\mathbf{x}, \mathbf{u}; h') &= \frac{q}{\sigma}\left\{1 - \exp\left[-\sigma R(\mathbf{z}_{h'}, \mathbf{u}; h')\right]\right\} + \\ &+ \frac{\nu\sigma}{4\pi}\int_0^{\eta} dr \exp(-\sigma r) \left\{\int_S n(\mathbf{z}_{h'} - r\mathbf{u}, \mathbf{u}'; h')\, d\mathbf{u}'\right\} + \\ &+ \frac{\nu\sigma}{4\pi}\int_\eta^{R(\mathbf{z}_h,\mathbf{u};h)} dr \exp(-\sigma r) \left\{\int_S n(\mathbf{z}_{h'} - r\mathbf{u}, \mathbf{u}'; h')\, d\mathbf{u}'\right\} + \\ &+ \frac{\nu\sigma}{4\pi}\int_{R(\mathbf{z}_h,\mathbf{u};h)}^{R(\mathbf{z}_{h'},\mathbf{u};h')} dr \exp(-\sigma r) \left\{\int_S n(\mathbf{z}_{h'} - r\mathbf{u}, \mathbf{u}'; h')\, d\mathbf{u}'\right\}. \end{aligned}$$

where $\eta = |\mathbf{z}_{h'} - \mathbf{z}_h|$, see Figure 2, and where

$$\begin{aligned} &\frac{\nu\sigma}{4\pi}\int_\eta^{R(\mathbf{z}_h,\mathbf{u};h)} dr \exp(-\sigma r) \left\{\int_S n(\mathbf{z}_{h'} - r\mathbf{u}, \mathbf{u}'; h')\, d\mathbf{u}'\right\} = \\ &= \exp(-\sigma\eta) \int_0^{R(\mathbf{z}_h,\mathbf{u};h)-\eta} dr' \exp(-\sigma r') \left\{\int_S n(\mathbf{z}_{h'} - r'\mathbf{u}, \mathbf{u}'; h')\, d\mathbf{u}'\right\} \end{aligned}$$

because, if $r' = r - \eta$, then $\mathbf{z}_{h'} - r\mathbf{u} = \mathbf{z}_{h'} - \eta\mathbf{u} - r'\mathbf{u} = \mathbf{z}_h - r'\mathbf{u}$.

On the other hand, (9),(24),(26),(29) also give:

$$n(\mathbf{x},\mathbf{u};h) = n(\mathbf{z}_h,\mathbf{u};h) = \frac{q}{\sigma}\left\{1-\exp\left[-\sigma R(\mathbf{z}_h,\mathbf{u};h)\right]\right\} +$$
$$+\frac{\nu\sigma}{4\pi}\int_0^{R(\mathbf{z}_h,\mathbf{u};h)} dr\,\exp\left(-\sigma r\right)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}'$$

and so

$$\begin{aligned} n(\mathbf{x},\mathbf{u};h) &= \frac{q}{\sigma}\left\{1-\exp\left[-\sigma R(\mathbf{z}_h,\mathbf{u};h)\right]\right\} + \\ &+ \frac{\nu\sigma}{4\pi}\exp\left(-\sigma\eta\right)\int_0^{R(\mathbf{z}_h,\mathbf{u};h)-\eta} dr\,\exp\left(-\sigma r\right)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}' + \\ &+ \frac{\nu\sigma}{4\pi}\left[1-\exp\left(-\sigma\eta\right)\right]\int_0^{R(\mathbf{z}_h,\mathbf{u};h)-\eta} dr\,\exp\left(-\sigma r\right)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}' + \\ &+ \frac{\nu\sigma}{4\pi}\int_{R(\mathbf{z}_h,\mathbf{u};h)-\eta}^{R(\mathbf{z}_h,\mathbf{u};h)} dr\;\exp\left(-\sigma r\right)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}'. \end{aligned}$$

Correspondingly, we obtain

$$\begin{aligned} n(\mathbf{x},\mathbf{u};h') &- n(\mathbf{x},\mathbf{u};h) = \frac{q}{\sigma}\left\{\exp\left[-\sigma R(\mathbf{z}_h,\mathbf{u};h)\right] - \exp\left[-\sigma R(\mathbf{z}_{h'},\mathbf{u};h')\right]\right\} + \\ &+ \frac{\nu\sigma}{4\pi}\int_0^{\eta} dr \exp(-\sigma r)\int_S n(\mathbf{z}_{h'} - r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' + \\ &+ \frac{\nu\sigma}{4\pi}\int_{R(\mathbf{z}_h,\mathbf{u};h)}^{R(\mathbf{z}_{h'},\mathbf{u};h')} dr\exp(-\sigma r)\int_S n(\mathbf{z}_{h'} - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}' - \\ &- \frac{\nu\sigma}{4\pi}\left[1-\exp\left(-\sigma\eta\right)\right]\int_0^{R(\mathbf{z}_h,\mathbf{u};h)} \exp(-\sigma r)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}' - \\ &- \frac{\nu\sigma}{4\pi}\int_{R(\mathbf{z}_{h'},\mathbf{u};h)-\eta}^{R(\mathbf{z}_h,\mathbf{u};h)} dr\exp(-\sigma r)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}' - \\ &+ \frac{\nu\sigma}{4\pi}\exp(-\sigma\eta)\int_0^{R(\mathbf{z}_h,\mathbf{u};h)-\eta} dr\exp(-\sigma r) \\ &\int_S\left[n(\mathbf{z}_{h'} - r\mathbf{u},\mathbf{u}';h') - n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\right]\;,d\mathbf{u}'. \end{aligned}$$

It follows that

$$\begin{aligned} |n(\mathbf{x},\mathbf{u};h') - n(\mathbf{x},\mathbf{u};h)| &\leq q\{R(\mathbf{z}_{h'},\mathbf{u};h') - R(\mathbf{z}_h,\mathbf{u};h)\} + \nu\sigma\|n\|\eta + \\ &+ \nu\sigma\|n\|\{R(\mathbf{z}_{h'},\mathbf{u};h') - R(\mathbf{z}_h,\mathbf{u};h)\} + \\ &+ \nu\sigma\|n\|\left[1-\exp\left(-\sigma\eta\right)\right]\{R(\mathbf{z}_h,\mathbf{u};h) - \eta\} + \\ &+ \nu\sigma\|\eta\|\eta + \nu\sigma\|n - n'\|_h\{R(\mathbf{z}_h,\mathbf{u};h) - \eta\}, \end{aligned}$$

where we recall that $\|\cdot\|$ is the norm in $X = C(\Omega_M \times S)$ and $\|\cdot\|_h$ is the norm in $X_h = C(\Omega_h \times S)$. Further, if δ_M is the diameter of Ω_M and

$$\chi(h, h') = \max\{\, [R(\mathbf{x}, \mathbf{u}; h) - R(\mathbf{x}, \mathbf{u}; h')], \quad (\mathbf{x}, \mathbf{u}) \in \Omega_h \times S\,\}$$

is the "maximum crossing" of the region $\Omega_{h'} \setminus \Omega_h$ (where h and h' are given, with $h < h'$), see Figure 2, then we have

$$\begin{aligned}
0 &\leq R(\mathbf{z}_{h'}, \mathbf{u}; h') - R(\mathbf{z}_h, \mathbf{u}; h) = |\mathbf{z}_{h'} - \mathbf{z}_h| + |\mathbf{y}_{h'} - \mathbf{y}_h| \leq 2\chi(h, h') \\
0 &\leq \eta = |\mathbf{z}_{h'} - \mathbf{z}_h| \leq \chi(h, h'), \\
0 &\leq R(\mathbf{z}_h, \mathbf{u}; h) - \eta \leq R(\mathbf{z}_h, \mathbf{u}; h) \leq \delta_M.
\end{aligned}$$

Hence,

$$|n(\mathbf{x}, \mathbf{u}; h') - n(\mathbf{x}, \mathbf{u}; h)| \leq [2q + 4\nu\sigma\|n\| + \nu\sigma^2\delta_M\|n\|]\chi(h, h') + \nu\sigma\delta_M\|n - n'\|_h.$$

We conclude that $n(\mathbf{x}, \mathbf{u}; h') \to n(\mathbf{x}, \mathbf{u}; h)$ as $h' \to h$, because $\chi(h, h') \to 0$ and $\|n' - n\|_h \to 0$, see *ii)* of Theorem 1. □

5 Identification of the boundary surface $\widehat{\Sigma} \in \mathcal{F}$

As in the preceding sections, we assume that the positive quantities σ_s, σ and q, that characterize the physical behaviour of the host medium (e.g., the interstellar cloud) are given; on the other hand, the boundary surface $\Sigma_h \in \mathcal{F}$ is a priori unknown. Then, the particle density n is, at *each given* $(\mathbf{x}, \mathbf{u}) \in \Omega_M \times S$, a function of the parameter $h \in [h_m, h_M]$, i.e. it depends on the shape of the region Ω_h occupied by the host medium and bounded by $\Sigma_h \in \mathcal{F}$. As a consequence, we can write that

$$n(\mathbf{x}, \mathbf{u}; h) = K(h)(\mathbf{x}, \mathbf{u}), \quad (\mathbf{x}, \mathbf{u}; h) \in \Omega_M \times S \times [h_m, h_M] \tag{32}$$

where

$$K(h)(\mathbf{x}, \mathbf{u}) = ((I - B_h)^{-1}Q_h)(\mathbf{x}, \mathbf{u}), \quad D(K) = [h_m, h_M], \quad R(K) \subset X_+ \tag{33}$$

see [JN99] and [Rom97, Sha96, MK94].

The operator K, defined by (33), is *nonlinear* and acts on $h \in [h_m, h_M] = D(K)$ through $R(\mathbf{x}, \mathbf{u}; h)$. The main properties of K directly follow from Theorems 4.1, 4.2, 4.3 and are listed in Lemma 5.1.

Lemma 5.1

i) *If $h' < h$, then $K(h)(\mathbf{x}, \mathbf{u}) < K(h')(\mathbf{x}, \mathbf{u})$ $\forall(\mathbf{x}, \mathbf{u}) \in \Omega_M \times S$, except when $\mathbf{x} \notin \Omega_{h'}$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{h'i} = \emptyset$ (in this case, $K(h)(\mathbf{x}, \mathbf{u}) = K(h')(\mathbf{x}, \mathbf{u}) = 0$);*

ii) *$K(h')(\mathbf{x}, \mathbf{u}) \to K(h)(\mathbf{x}, \mathbf{u})$ as $h' \to h$, $\forall \mathbf{x} \in \Omega_M$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{h'i} \neq \emptyset$;*

iii) *$K(h)(\mathbf{x}, \mathbf{u}) < K(h')(\mathbf{x}, \mathbf{u})$ if $h_m \leq h < h' \leq h_M$, $\mathbf{x} \in \Omega_M$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{h_m i} \neq \emptyset$;*

iv) *$K(h')(\mathbf{x}, \mathbf{u}) \to K(h)(\mathbf{x}, \mathbf{u})$ as $h' \to h$, $\forall \mathbf{x} \in \Omega_M$ and $\gamma_{\mathbf{x},\mathbf{u}} \cap \Omega_{h_m i} \neq \emptyset$.*

As at the beginning of Section 4, assume that the value $\hat{n} = n(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ of the particle density is known, where now $\hat{\mathbf{x}} \in \Omega_M \setminus \Omega_{h_M}$ and $\hat{\mathbf{u}}$ is such that $\gamma_{\hat{\mathbf{x}},\hat{\mathbf{u}}} \cap \Omega_{h_m i} \neq \emptyset$ (with $\Omega_{h_M} \subset \Omega_{Mi}$, see Figure 3): $\hat{n}$ is usually called a "far field" measurement of the particle density.

If we put

$$K(h)(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \hat{K}(h), \quad D(\hat{K}) = [h_m, h_M], \quad R(\hat{K}) = [\hat{K}(h_m), \hat{K}(h_M)] \subset R^+ \tag{34}$$

then Lemma 5.1 implies that $\hat{K}(h)$ is a *continuous* and *strictly increasing* function of $h \in D(\hat{K})$. As a consequence, we may state our main theorem.

Theorem 5.1 *In the family $\mathcal{F}$, defined by (23) and satisfying assumptions a1)–a4), is "suitably" chosen (i.e., if it is such that $\hat{n} \in R(\hat{K})$), then a unique $\hat{h} \in D(\hat{K}) = [h_m, h_M]$ exists such that $\hat{K}(\hat{h}) = \hat{n}$.*

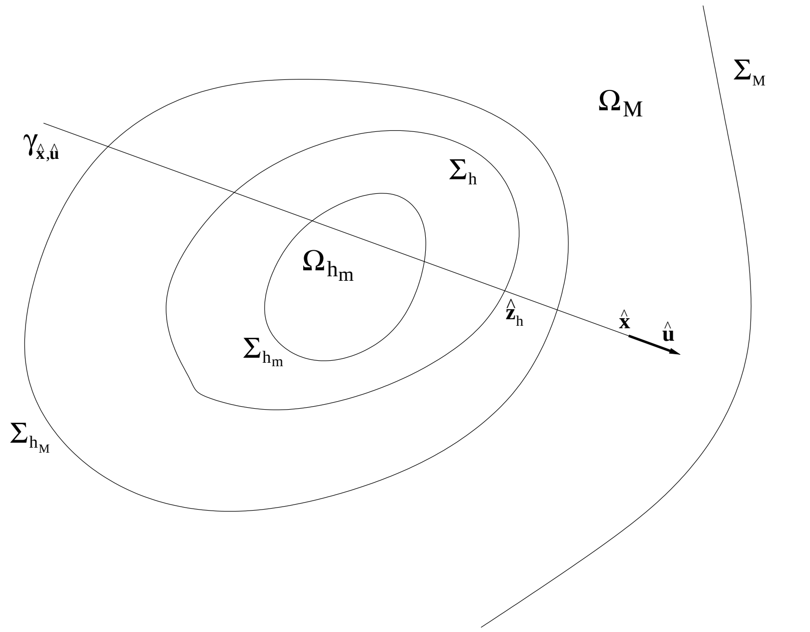

Figure 3: $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$, with $\hat{\mathbf{x}} \in \Omega_M \setminus \Omega_{h_m}$ and $\hat{\mathbf{u}}$ such that $\gamma_{\hat{\mathbf{x}},\hat{\mathbf{u}}} \cap \Omega_{h_m i} \neq \emptyset$.

The value $\hat{h}$, satisfying the equation $\hat{K}(\hat{h}) = \hat{n}$, can be found by using some standard successive approximation method, see also Section 6. Correspondingly, the surface $\Sigma_{\hat{h}} \in \mathcal{F}$ is identified, in such a way that the region $\Omega_{\hat{h}}$ "produces" a particle density $n(\mathbf{x}, \mathbf{u}; \hat{h})$, characterized by the fact that $n(\hat{\mathbf{x}}, \hat{\mathbf{u}}; \hat{h})$ is equal to the measured $\hat{n}$.

6 Concluding remarks

Definition (34) and relation (32) imply that

$$\hat{K}(h) = K(h)(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = n(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = n(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}) = K(h)(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}),$$

see Figure 3; hence (33) gives

$$\hat{K}(h) = ((I - B_h)^{-1} Q_h)(\hat{\mathbf{z}_h}, \hat{\mathbf{u}}) = Q(\hat{\mathbf{z}_h}, \hat{\mathbf{u}}) + \sum_{j=1}^{\infty} (B_h^j Q_h)(\hat{\mathbf{z}_h}, \hat{\mathbf{u}}) =$$

$$= Q_h(\hat{\mathbf{z}_h}, \hat{\mathbf{u}}) + \sum_{j=1}^{m} (B_h^j Q_h)(\hat{\mathbf{z}_h}, \hat{\mathbf{u}}) + O(\nu^{m+1}).$$

We conclude that

$$\hat{K}(h) = Q_h(\hat{\mathbf{z}_h}, \hat{\mathbf{u}}) + \sum_{j=1}^{m} (B_h^j Q_h)(\hat{\mathbf{z}_h}, \hat{\mathbf{u}}) \tag{35}$$

may be considered as the *explicit* expression of $\hat{K}(h)$ as a function of $h \in [h_m, h_M]$ (with an error of the order of ν^{m+1}). Correspondingly, (35) may be used in any successive approximantion procedure to determine the unique value $\hat{h}$ such that $\hat{K}(\hat{h}) = \hat{n}$. Of course, if the family $\mathcal{F} = \{\Sigma_h \colon \varphi(x, y, z; h) = 0, [h_m, h_M]\}$ is not "suitably" chosen (i.e., if $\hat{n} < \hat{K}(h_m)$ or $\hat{n} > \hat{K}(h_M)$), then the equation has no solution belonging to $[h_m, h_M]$. In this case, $\mathcal{F}$ must be obviously changed.

Assume now that another family $\mathcal{F}_1 = \{\Sigma_h^{(1)} \colon \varphi_1(x, y, z; h) = 0, [h_m, h_M]\}$ is considered, with $\mathcal{F} \cap \mathcal{F}_1 = \emptyset$. This will lead to a value $\hat{h}_1$ and so to a surface $\Sigma_{\hat{h}_1}^{(1)}$ different from $\Sigma_{\hat{h}}$. Such a "non-uniqueness" result is not surprising because $\Sigma_{\hat{h}}$ is the "best-approximation within $\mathcal{F}$ to the true physical surface Σ_{ph}, obtained from the unique measured particle density $\hat{n}$.

Finally, suppose that another value of the particle density $\check{n} = n(\check{\mathbf{x}}, \check{\mathbf{u}})$ is known, with $(\check{\mathbf{x}}, \check{\mathbf{u}}) \neq (\hat{\mathbf{x}}, \hat{\mathbf{u}})$ and with $\check{\mathbf{x}} \in \Omega_M \setminus \Omega_{h_M}$, $\gamma_{\check{\mathbf{x}}, \check{\mathbf{u}}} \cap \Omega_{h_m i} \neq \emptyset$. Then, the corresponding $\check{h}$ may be different from $\hat{h}$ because we have that $\hat{K}(\hat{h}) = K(\hat{h})(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \hat{n}$, but not necessarily that $K(\hat{h})(\check{\mathbf{x}}, \check{\mathbf{u}}) = \check{n}$. Only if one is particularly lucky (or wise), it may happen that he choses a family $\mathcal{F}$ such that $K(\hat{h})(\mathbf{x}, \mathbf{u})$ is equal to particle density $n(\mathbf{x}, \mathbf{u})$ at any $(\mathbf{x}, \mathbf{u}) \in \Omega_M \times S$. In this case, $K(\hat{h})(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \hat{n}$ and $K(\hat{h})(\check{\mathbf{x}}, \check{\mathbf{u}}) = \check{n}$; correspondingly, $\Omega_{\hat{h}}$ is the true physical region occupied by the host medium (i.e. $\Sigma_{\hat{h}} = \Sigma_{ph}$). However, in general, $\Sigma_{\hat{h}}$ is only an approximate representation of Σ_{ph}.

Note that, if the host medium is an interstellar cloud, $n(\mathbf{x}, \mathbf{u})$ is the UV-photon density at $(\mathbf{x}, \mathbf{u})$ and $\Sigma_{\hat{h}}$ is an approximate representation of the surface that bounds the cloud.

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