An inverse problem in photon transport theory: identification of the boundary surface of an interstellar cloud

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Abstract

We study photon transport in an interstellar cloud, contained in a region $V \subset \mathbb{R}^3$ bounded by an unknown surface Σ. Assuming that the cross section and the sources are given, we identify the shape of Σ, provided that the value of the photon number density is known at some location far from V. The mathematical techniques employed to solve this inverse problem are rather simple and are based on some monotony properties of the photon density.

1 Introduction

Interstellar clouds are astronomical objects that occupy large regions of the galactic space: the diameter of an average cloud may range from 10^{-1} to 10 parsec, i.e. from 10^3 to 10^5 times the diameter of the Solar System. Clouds are composed of a low density mixture of gases and dust grains (mainly hydrogen molecules with some (1-2%) silicon grains). Typical particle densities may be of the order of 10^4 particles/cm^3, i.e. 10^{-5} times the density of earth atmosphere at sea level. Such a mixture of particles is exposed to UV-radiation produced by stars within the cloud or external to it [DW97].

Since interactions between UV-photons and particles play a crucial role in the chemistry and in the evolution of interstellar clouds, it is of great interest to study

a) the spatial distribution of UV-photons within a given cloud;

b) the spatial behaviour of some physical quantities (such as the cross sections and the photon sources) or the form of the ingoing photon density;

c) the shape of the surface Σ that bounds the cloud.

As far as a) is concerned, the evaluation of the photon number density, as a function of space, energy and time variables, is a classic problem in transport theory. It requires the knowledge of the cross section, of the sources (including the ingoing photon density) and of the shape of Σ [Pom73].

On the other hand, in problem b), the cross section and/or the sources, or the ingoing density are unknown and must be evaluated starting from the knowledge of the (outgoing) photon density, measured at a location "far" from the cloud. This is a typical *inverse* problem in photon transport theory, see for instance [Ago91, Bal00, Cho92, CS99, Dre89, Gon86, GN92, Hs88, Lar88, McC86, MS97, MK94, Sha96, Sie01, YY89, Zwe99].

Finally, in problem c), a *geometric* quantity (the equation of the surface Σ) is the unknown, see for instance [BMR02]. This is a less common inverse problem in photon transport theory and will be studied in the following sections. More precisely, starting from the knowledge of the cross sections, of the sources and of the ingoing density that characterize the interstellar cloud under consideration and of some *a priori* information on the shape of Σ, we shall determine the surface Σ which bounds the cloud, provided that the value of the UV-photon number density is known at a location far from the cloud.

2 The integral form of the transport equation

Let our interstellar cloud be contained in a closed bounded region $V \subset \mathbb{R}^3$, bounded by the closed "regular" surface Σ. Assume that V is strictly convex (i.e. $\forall \mathbf{x}', \mathbf{x}'' \in \Sigma$, then $\lambda \mathbf{x}' + (1-\lambda)\mathbf{x}'' \in V_i$ $\forall \lambda \in (0,1)$, where V_i is the interior of $V = V_i \cap \Sigma$, see Figure 1).

Under the assumptions that the total cross section σ, thr scattering cross section σ_s and the source q are given positive constants within V and are zero outside, the integral version of the transport equation for UV-photons has the form, [Pom73]:

$$n(\mathbf{x},\mathbf{u}) = Q(\mathbf{x},\mathbf{u}) + (Bn)(\mathbf{x},\mathbf{u}), \qquad (\mathbf{x},\mathbf{u}) \in V \times S \tag{1}$$

where

$$Q(\mathbf{x},\mathbf{u}) = \frac{q}{\sigma}\left\{1 - \exp\left[-\sigma R(\mathbf{x},\mathbf{u})\right]\right\} \tag{2}$$

$$(Bn)(\mathbf{x},\mathbf{u}) = \frac{c\sigma}{4\pi}\int_0^{R(\mathbf{x},\mathbf{u})} dr\ \exp(-\sigma r)\int_S n(\mathbf{x}-r\mathbf{u},\mathbf{u}')\,d\mathbf{u}'. \tag{3}$$

Moreover, if $\mathbf{x} \notin V$ we have

$$n(\mathbf{x}, \mathbf{u}) = n(\mathbf{z}, \mathbf{u}) \qquad \text{if } \gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq \emptyset \tag{4}$$

$$n(\mathbf{x}, \mathbf{u}) = 0 \qquad \text{if } \gamma_{(\mathbf{x},\mathbf{u})} \cap V_i = \emptyset. \tag{5}$$

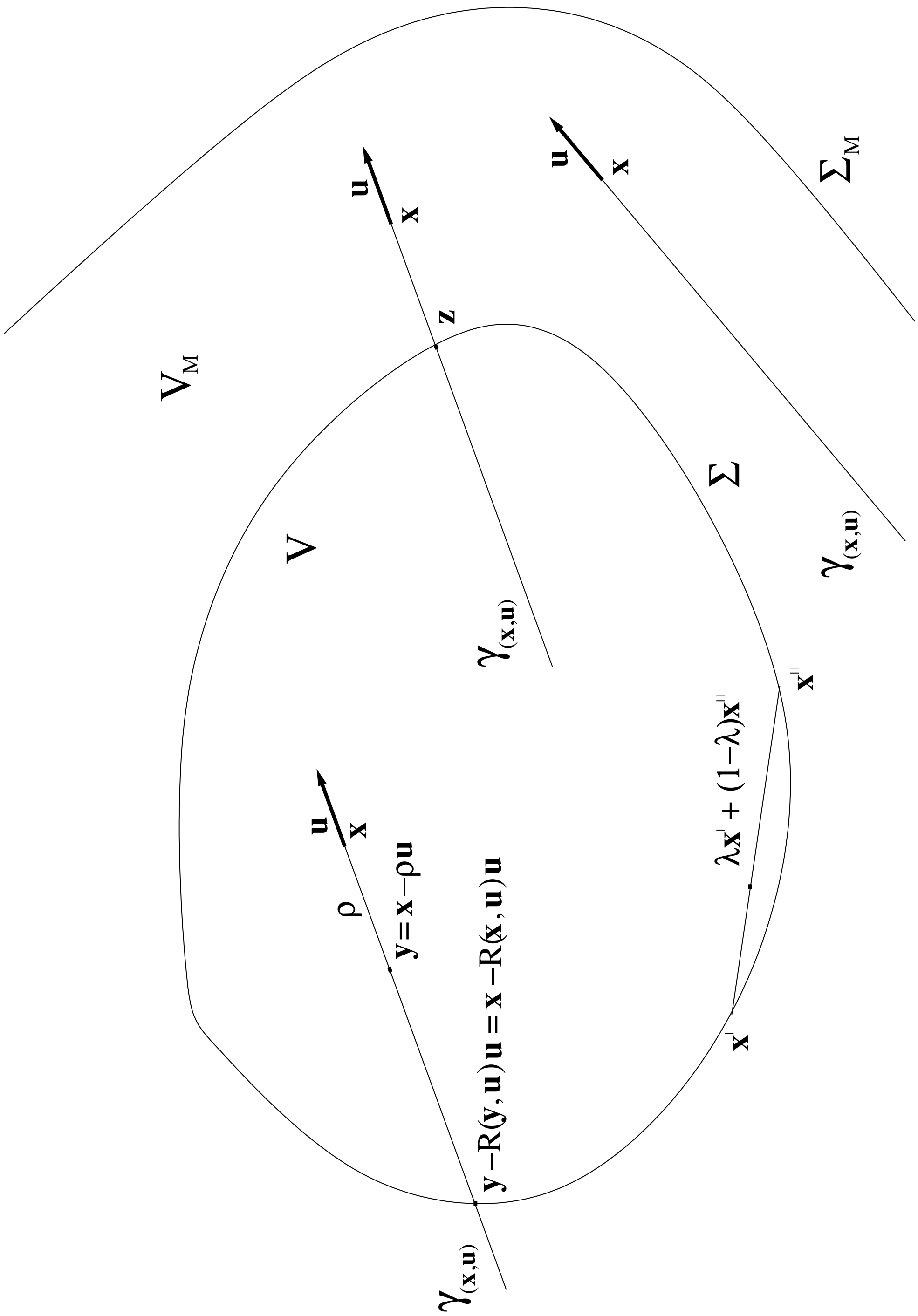

Figure 1: The region $V = V_i \cap \Sigma \subset \mathbb{R}^3$, occupied by the interstellar cloud, and the "large" region V_M.

In (1 - 5), $n(\mathbf{x}, \mathbf{u})$ is the number density of photons that are at $\mathbf{x}$ and move with velocity parallel to the unit vector $\mathbf{u} \in S$, where S is the surface of the unit sphere. Moreover, $R(\mathbf{x}, \mathbf{u})$ is such that $\mathbf{x} - R(\mathbf{x}, \mathbf{u})\mathbf{u} \in \Sigma \; \forall (\mathbf{x}, \mathbf{u}) \in V \times S$, see Figure 1, with $R(\mathbf{x}, \mathbf{u}) = 0$ if $\mathbf{x} \in \Sigma$ and $\mathbf{u}$ is directed towards V_i; the surface Σ is assumed to be regular enough to ensure that $R(\mathbf{x}, \mathbf{u})$ is a continuous funcion of $(\mathbf{x}, \mathbf{u}) \in V \times S$, see also property b) in Section 3 and (ii) of Remark 3.1. Finally $c = \sigma_s/\sigma$ ($c < 1$ because $\sigma = \sigma_s + \sigma_c$ where $\sigma_c > 0$ is the capture cross section), $\gamma_{(\mathbf{x},\mathbf{u})} = \{\mathbf{y} \colon \mathbf{y} = \mathbf{x} - r\mathbf{u}, r \geq 0\}$ is the half straight line passing through $\mathbf{x}$ and parallel to $\mathbf{u}$, and $\mathbf{z} \in \Sigma$ is the "first" intersection of $\gamma_{(\mathbf{x},\mathbf{u})}$ with Σ if $\mathbf{x} \notin V$ and $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq \emptyset$, see Figure 1.

For a reason that will become clear later on, equations (1),(4) and (5) will be studied in the real Banach space $X = C(V_M \times S)$ with a norm $\|f\| = \{\max |f(\mathbf{x}, \mathbf{u})|, (\mathbf{x}, \mathbf{u}) \in V_M \times S\}$, where V_M is a "large" closed convex region of $\mathbb{R}^3$, bounded by a "regular" surface Σ_M and such that $V_{M,i}$, the interior of V_M, contains V.

Correspondingly, it is convenient to modify (1)-(5) as follows:

$$
\begin{aligned}
n(\mathbf{x}, \mathbf{u}) &= Q(\mathbf{x}, \mathbf{u}) + (Bn)(\mathbf{x}, \mathbf{u}), \quad (\mathbf{x}, \mathbf{u}) \in V_M \times S, && (6)\\
Q(\mathbf{x}, \mathbf{u}) &= \frac{q}{\sigma}\left\{1 - \exp\left[-\sigma R(\mathbf{x}, \mathbf{u})\right]\right\}, \quad (\mathbf{x}, \mathbf{u}) \in V \times S, && (7)\\
Q(\mathbf{x}, \mathbf{u}) &= \frac{q}{\sigma}\left\{1 - \exp\left[-\sigma R(\mathbf{z}, \mathbf{u})\right]\right\}, \quad \mathbf{x} \in V_M \setminus V, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq \emptyset && (8)\\
Q(\mathbf{x}, \mathbf{u}) &= 0, \qquad \mathbf{x} \in V_M \setminus V, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_i = \emptyset, && (9)\\
(Bn)(\mathbf{x}, \mathbf{u}) &= \frac{c\sigma}{4\pi} \int_0^{R(\mathbf{x},\mathbf{u})} dr \, \exp(-\sigma r) \int_S n(\mathbf{x} - r\mathbf{u}, \mathbf{u}') \, d\mathbf{u}', \\
&\qquad (\mathbf{x}, \mathbf{u}) \in V \times S, && (10)\\
(Bn)(\mathbf{x}, \mathbf{u}) &= \frac{c\sigma}{4\pi} \int_0^{R(\mathbf{z},\mathbf{u})} dr \, \exp(-\sigma r) \int_S n(\mathbf{z} - r\mathbf{u}, \mathbf{u}') \, d\mathbf{u}', \\
&\qquad \mathbf{x} \in V_M \setminus V, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq \emptyset, && (11)\\
(Bn)(\mathbf{x}, \mathbf{u}) &= 0, \qquad \mathbf{x} \in V_M \setminus V, \qquad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_i = \emptyset. && (12)
\end{aligned}
$$

We remark that $Q \in X$ and $Bn \in X \; \forall n \in X$ because we assumed that the region V is strictly convex.

From (7-9) we have

$$0 \leq Q(\mathbf{x}, \mathbf{u}) \leq \frac{q}{\sigma}\left\{1 - \exp(-\sigma\delta)\right\} \qquad \forall (\mathbf{x}, \mathbf{u}) \in V_M \times S$$

where $\delta = \max\{R(\mathbf{x},\mathbf{u}), (\mathbf{x},\mathbf{u}) \in V \times S\}$ is the diameter of V. Hence,

$$\|Q\| \leq \frac{q}{\sigma}\{1 - \exp(-\sigma\delta)\}. \tag{13}$$

Moreover, (10-12) give

$$|(Bn)(\mathbf{x},\mathbf{u})| \leq c\sigma\|n\| \int_0^\delta \exp(-\sigma r)\,dr = c\{1 - \exp(-\sigma\delta)\}\|n\|,$$
$$\forall n \in X, \quad \forall(\mathbf{x},\mathbf{u}) \in V_M \times S$$

and so

$$\|B\| \leq c\{1 - \exp(-\sigma\delta)\} < c < 1, \tag{14}$$

i.e. the linear operator B is strictly contractive [BMM98]

Inequality (14) implies that the unique solution $n \in X$ of equation (6) has the form

$$n(\mathbf{x},\mathbf{u}) = ((I - B)^{-1}Q)(\mathbf{x},\mathbf{u}), \qquad (\mathbf{x},\mathbf{u}) \in V_M \times S \tag{15}$$

with

$$\|n\| \leq \frac{\|Q\|}{1 - \|B\|} \leq \frac{q}{\sigma}\frac{1 - \exp(-\sigma\delta)}{1 - c\{1 - \exp(-\sigma\delta)\}} \leq \frac{q}{\sigma}\frac{1}{1 - c}. \tag{16}$$

The results, listed in Lemmas 2.1-2.4, will be used in the sequel.

Lemma 2.1

(i) $Q \in X_+ = \{f : f \in X, f(\mathbf{x},\mathbf{u}) \geq 0 \;\; \forall(\mathbf{x},\mathbf{u}) \in V_M \times S\}$ *(X_+ is the closed positive cone of X);*

(ii) $Q(\mathbf{x},\mathbf{u}) > 0$ *if* $\mathbf{x} \in V_M$ *and* $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq 0$*;*

(iii) $\displaystyle\int_S Q(\mathbf{x},\mathbf{u})\,du > 0 \quad \forall \mathbf{x} \in V_M$.

Proof. (i) immediatly follows from definitions (7-9).

(ii) holds because $R(\mathbf{x},\mathbf{u}) > 0$ $\forall \mathbf{u} \in S$ if $\mathbf{x} \in V_i$, and $R(\mathbf{z},\mathbf{u}) > 0$ if $\mathbf{x} \in V_M \setminus V_i$ and $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq \emptyset$.

(iii) follows from (ii). □

Lemma 2.2

(i) $Bf \in X_+$ $\forall f \in X_+$*;*

(ii) if $\displaystyle\int_S f(\mathbf{y},\mathbf{u}')\,d\mathbf{u}' > 0$ $\forall \mathbf{y} \in V$*, then* $(Bf)(\mathbf{x},\mathbf{u}) > 0$ $\forall(\mathbf{x},\mathbf{u})$ *with* $\mathbf{x} \in V_M$ *and* $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq \emptyset$.

Proof. (i) and (ii) directly follow from definitions (10-12). □

Lemma 2.3

(i) $(I - B)^{-1} f \in X_+ \ \forall f \in X_+$;

(ii) *if* $f \in X_+$ *and* $\int_S f(\mathbf{y}, \mathbf{u}')\, d\mathbf{u}' > 0 \ \forall \mathbf{y} \in V$, *then* $((I - B)^{-1} f)(\mathbf{x}, \mathbf{u}) > 0$ $\forall (\mathbf{x}, \mathbf{u})$ *with* $\mathbf{x} \in V_M$ *and* $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq \emptyset$.

Proof. Since $((I - B)^{-1} f)(\mathbf{x}, \mathbf{u}) = \sum_{j=0}^{\infty} (B^j f)(\mathbf{x}, \mathbf{u}) \geq f(\mathbf{x}, \mathbf{u}) + (Bf)(\mathbf{x}, \mathbf{u})$, $\forall f \in X_+$, (i) and (ii) follow from Lemma 2.2. □

Remark 2.1 *Lemmas 2.1 and 2.3 imply that the photon density* $n(\mathbf{x}, \mathbf{u})$, *given by (15), is such that* $n(\mathbf{x}, \mathbf{u}) > 0 \ \forall (\mathbf{x}, \mathbf{u})$ *with* $\mathbf{x} \in V_M$ *and* $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq \emptyset$. *In any case,* n *belongs to* X_+, *as it must be from a physical viewpoint.*

We shall now prove that $n(\mathbf{x}, \mathbf{u})$ increases as $\mathbf{x}$ "approaches" the boundary surface Σ, in the sense specified in the following lemma. The crucial role of this result will become clear in Sections 3 and 4

Lemma 2.4 *Let* $(\mathbf{x}, \mathbf{u}) \in V \times S$, *with* $\mathbf{u}$ *such that* $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \neq \emptyset$. *If* $\mathbf{y} = \mathbf{x} - \rho \mathbf{u}$ *with* $0 < \rho \leq R(\mathbf{x}, \mathbf{u})$, *see Figure 1, then* $n(\mathbf{x}, \mathbf{u}) > n(\mathbf{y}, \mathbf{u})$.

Proof. Since both $\mathbf{x}$ and $\mathbf{y} = \mathbf{x} - \rho \mathbf{u}$ belong to V and $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_i \supset \gamma_{(\mathbf{y},\mathbf{u})} \cap V_i \neq \emptyset$, (6)+(7)+(10) give

$$
\begin{aligned}
n(\mathbf{x}, \mathbf{u}) - n(\mathbf{y}, \mathbf{u}) \ &= \ \frac{q}{\sigma} \{1 - \exp[-\sigma R(\mathbf{x}, \mathbf{u})]\} - \frac{q}{\sigma} \{1 - \exp[-\sigma R(\mathbf{y}, \mathbf{u})]\} + \\
&+ \ \frac{c\sigma}{4\pi} \int_0^{R(\mathbf{x},\mathbf{u})} dr \, \exp(-\sigma r) \int_S n(\mathbf{x} - r\mathbf{u}, \mathbf{u}')\, d\mathbf{u}' - \\
&- \ \frac{c\sigma}{4\pi} \int_0^{R(\mathbf{y},\mathbf{u})} dr \, \exp(-\sigma r) \int_S n(\mathbf{y} - r\mathbf{u}, \mathbf{u}')\, d\mathbf{u}'
\end{aligned}
$$

where $R(\mathbf{y}, \mathbf{u}) = R(\mathbf{x}, \mathbf{u}) - \rho$. Hence, if we put

$$
\Delta(\mathbf{x}, \mathbf{u}) = n(\mathbf{x}, \mathbf{u}) - n(\mathbf{y}, \mathbf{u}) = n(\mathbf{x}, \mathbf{u}) - n(\mathbf{x} - \rho \mathbf{u}, \mathbf{u})
$$

we obtain

$$
\begin{aligned}
\Delta(\mathbf{x},\mathbf{u}) &= \left\{ \frac{q}{\sigma}\left[1-\exp(-\sigma\rho)\right]\exp\left[-\sigma R(\mathbf{y},\mathbf{u})\right] + \right. \\
&+ \left. \frac{c\sigma}{4\pi}\int_{R(\mathbf{y},\mathbf{u})}^{R(\mathbf{x},\mathbf{u})} dr\,\exp(-\sigma r)\int_S n(\mathbf{x}-r\mathbf{u},\mathbf{u}')\,d\mathbf{u}' \right\} + \\
&+ \frac{c\sigma}{4\pi}\int_0^{R(\mathbf{y},\mathbf{u})} dr\,\exp(-\sigma r)\int_S \Delta(\mathbf{x}-r\mathbf{u},\mathbf{u}')\,d\mathbf{u}' \qquad (17)
\end{aligned}
$$

because $\Delta(\mathbf{x}-r\mathbf{u},\mathbf{u}') = n(\mathbf{x}-r\mathbf{u},\mathbf{u}') - n(\mathbf{x}-r\mathbf{u}-\rho\mathbf{u},\mathbf{u}') = n(\mathbf{x}-r\mathbf{u},\mathbf{u}') - n(\mathbf{y}-r\mathbf{u},\mathbf{u}')$.

It is not difficult to check that the unique solution $\Delta(\mathbf{x},\mathbf{u})$ of the integral equation (17) belongs to the closed positive cone of the Banach space $C(V\times S)$. Moreover $\Delta(\mathbf{x},\mathbf{u}) > 0\ \forall(\mathbf{x},\mathbf{u})\in V\times S$ with $\gamma_{(\mathbf{x},\mathbf{u})}\cap V_i \neq \emptyset$ because the known term in (17) is positive. □

3 The family of the boundary surfaces Σ_h

Assume that the value $\hat{n} = n(\hat{\mathbf{x}},\hat{\mathbf{u}})$ of the photon density at $(\hat{\mathbf{x}},\hat{\mathbf{u}})$ is known, where $\hat{\mathbf{x}}$ is "far" from V (i.e., $\hat{\mathbf{x}} \in V_M \setminus V$) and $\hat{\mathbf{u}}$ is such that $\gamma_{\hat{\mathbf{x}},\hat{\mathbf{y}}}\cap V_i \neq \emptyset$: $\hat{n}$ may be the result of experimental measurements made by terrestrial astronomers. The question is whether it is possible to determine the shape of the boundary surface Σ from the knowledge of the measured value $\hat{n}$.

To answer this question, we introduce the family of closed surfaces

$$
\mathcal{F} = \{\Sigma_h\colon \varphi(x,y,z;h) = 0, h\in[h_m,h_M]\}, \qquad (18)
$$

with the following properties:

a) Σ_h is a closed surface, i.e. if (ρ,α,β) are spherical coordinates and $\Phi(\rho,\alpha,\beta;h) = 0$ is the corresponding equation of Σ_h (thus $\Phi(\rho,\alpha,\beta;h) = \varphi(x,y,z;h)$ with $x = \rho\sin\alpha\cos\beta$, $y = \rho\sin\alpha\sin\beta$, $z = \rho\cos\alpha$), then we have that $\Phi(\rho,\alpha+2\pi,\beta+2\pi;h) = \varphi(x,y,z;h)\ \forall(\rho,\alpha,\beta)\in V_M$;

b) $\varphi(x,y,z;h)$ is a continuous function of $(x,y,z;h)\in V_M\times[h_m,h_M]$, and the point (x,y,z) is external to Σ_h if and only if $\varphi(x,y,z;h) > 0$;

c) the region V_h, bounded by Σ_h, is bounded, closed, strictly convex and contained within V_{Mi}, the interior of V_M;

d) if $h < h'$, $V_h \subset V_{h'i}$, where $V_{h'i}$ is the interior of $V_{h'}$ (i.e., if (x,y,z) is such that $\varphi(x,y,z;h') = 0$, then $\varphi(x,y,z;h) > 0$). (See also the Appendix)

Remark 3.1 *(i) assumption c) is necessary because we shall be working in the Banach space* $X = C(V_M \times S)$*, where* V_M *is a "large" closed and convex region such that* $V_{h_m} \subseteq V_h \subseteq V_{h_M} \subset V_{Mi}$*.*
(ii) Let $\mathbf{x} = (x_1, x_2, x_3) \in V_h$ *and assume that* $\mathbf{u} = (u_1, u_2, u_3)$ *is such that* $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h,i} \neq \emptyset$ *(with* $u_1^2+u_2^2+u_3^2 = 1$*). The intersection point of* $\gamma_{(\mathbf{x},\mathbf{u})}$ *with* Σ *is* $(x_1 - Ru_1, x_2 - Ru_2, x_3 - Ru_3)$*, where* $R = R(\mathbf{x}, \mathbf{u}, h)$ *satisfies the equation* $\varphi((x_1 - Ru_1, x_2 - Ru_2, x_3 - Ru_3; h) = 0$ *(such a point is unique because* V_h *is strictly convex). In an analogous way, let* $\mathbf{x}' = (x_1', x_2', x_3') \in V_h$ *and take* $\mathbf{u}' = (u_1', u_2', u_3')$ *such that* $\gamma_{(\mathbf{x}',\mathbf{u}')} \cap V_{h,i} \neq \emptyset$*. The unique intersection point of* $\gamma_{(\mathbf{x}',\mathbf{u}')}$ *with* $\Sigma_{h'}$ *is* $(x_1' - R'u_1', x_2' - R'u_2', x_3' - R'u_3')$*, where* $R' = R(\mathbf{x}', \mathbf{u}'; h')$ *satisfies the equation* $\varphi(x_1' - R'u_1', x_2' - R'u_2', x_3' - R'u_3'; h') = 0$*. Since* φ *is continuous due to assumption b), we have*

$$\begin{aligned} 0 &= \lim_{(\mathbf{x}',\mathbf{u}';h')\to(\mathbf{x},\mathbf{u};h)} \varphi(x_1' - R'u_1' \, , \, x_2' - R'u_2' \, , \, x_3' - R'u_3'; h') = \\ &= \varphi(x_1 - (\lim R')u_1 \, , \, x_2 - (\lim R')u_2 \, , \, x_3 - (\lim R')u_3; h'). \end{aligned}$$

Hence, we have at the same time

$$\varphi(x_1 - Ru_1, x_2 - Ru_2, x_3 - Ru_3) = 0$$

and

$$\varphi(x_1 - \lim(R')u_1 \, , \, x_2 - (\lim R')u_2 \, , \, x_3 - (\lim R')u_3; h') = 0.$$

On the other hand, the intersection point of $\gamma_{(\mathbf{x},\mathbf{u})}$ *with* Σ_h *is unique and so we obtain that*

$$x_i - \lim R'u_i = x_i - Ru_i, \qquad i = 1, 2, 3, \qquad (u_1^2 + u_2^2 + u_3^2 = 1).$$

As a consequence, $\lim R' = R$ *i.e.* $R(\mathbf{x}', \mathbf{u}'; h') \to R(\mathbf{x}, \mathbf{u}; h)$ *as* $(\mathbf{x}', \mathbf{u}'; h') \to (\mathbf{x}, \mathbf{u}; h)$*.*

We conclude that $R(\mathbf{x}, \mathbf{u}; h)$ *is continuous (and hence uniformly continuous) on some closed and bounded set* $\Omega_h = V_h \times S \times [h, \overline{h}]$*.*

(iii) If $\chi(h, h') = \max\{[R(\mathbf{x}, \mathbf{u}; h') - R(\mathbf{x}, \mathbf{u}; h)], \ (\mathbf{x}, \mathbf{u}) \in V_h \times S\}$ *is the "maximum crossing" of the region* $V_{h'} \backslash V_h$*, see Figure 2, then (ii) implies that* $\chi(h, h') \to 0$*. In this sense, we can say that the family* $\mathcal{F}$ *is "continuous" with respect to the parameter* h*.*

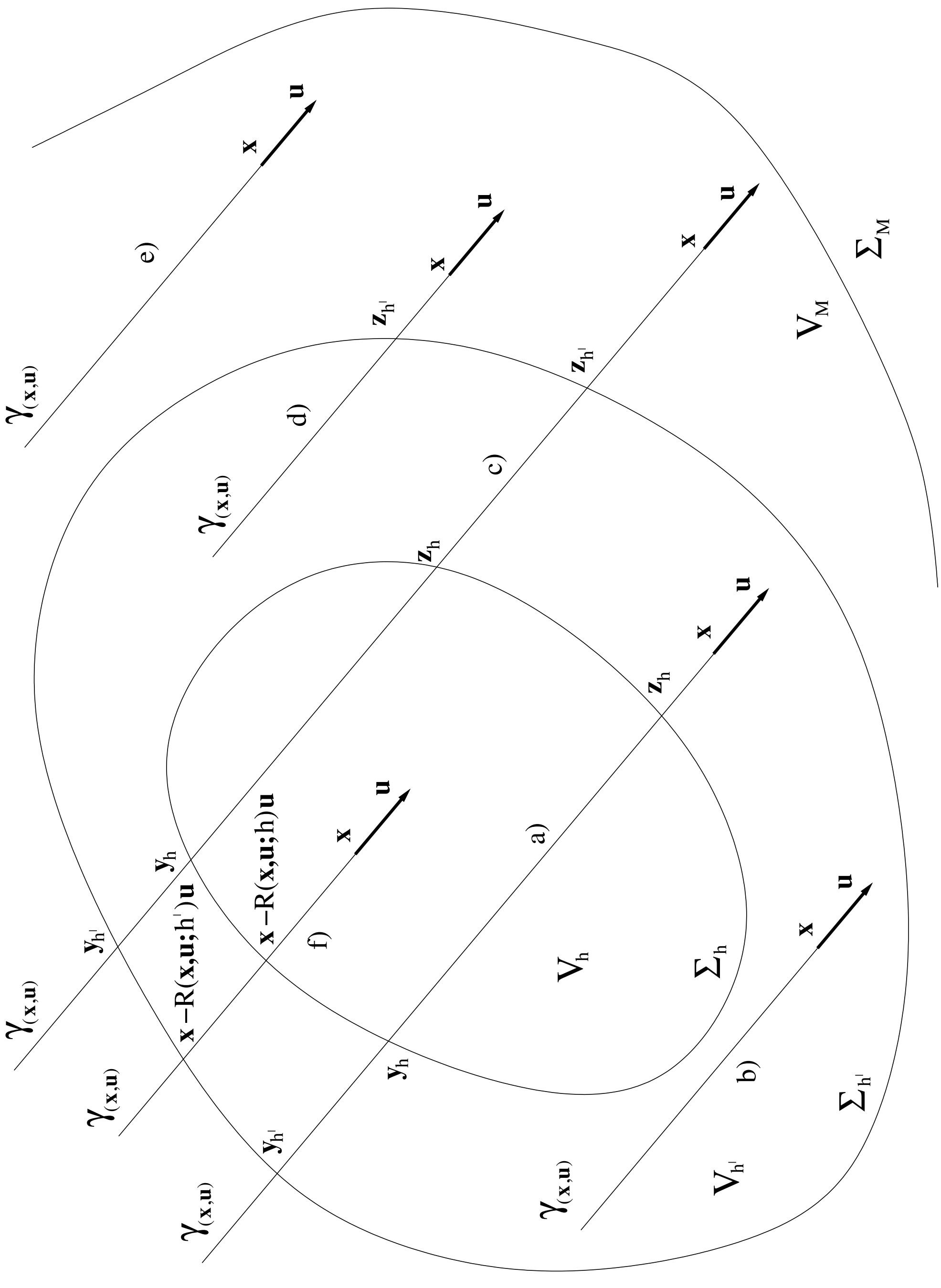

Figure 2: The regions V_h and $V_{h'}$ with $h < h'$, and the "large" region V_M ($V_{Mi} \supset V_{h_M}$). $[R(\mathbf{x}, \mathbf{u}; h') - [R(\mathbf{x}, \mathbf{u}; h)]$ is the length of the crossing of the region $V_h - V_{h'}$, corresponding to $(\mathbf{x}, \mathbf{u}) \in V_h \times S$. In case a), $\mathbf{y}_h = \mathbf{z}_h - R(\mathbf{z}_h, \mathbf{u}; h)\mathbf{u}$ and $\mathbf{y}_{h'} = \mathbf{z}_{h'} - R(\mathbf{z}_{h'}, \mathbf{u}; h)\mathbf{u}$.

For clarity, we now rewrite (6)-(12) observing that, if $V = V_h$, the quantities n, R, Q and B depend on $(\mathbf{x}, \mathbf{u})$ and on the parameter $h \in [h_m, h_M]$

$$
\begin{aligned}
n(\mathbf{x},\mathbf{u};h) &= Q_h(\mathbf{x},\mathbf{u};h) + (B_h n)(\mathbf{x},\mathbf{u};h), \quad (\mathbf{x},\mathbf{u}) \in V_M \times S, && (19)\\
Q_h(\mathbf{x},\mathbf{u};h) &= \frac{q}{\sigma}\left\{1-\exp\left[-\sigma R(\mathbf{x},\mathbf{u};h)\right]\right\}, \quad (\mathbf{x},\mathbf{u}) \in V_h \times S, && (20)\\
Q_h(\mathbf{x},\mathbf{u};h) &= \frac{q}{\sigma}\left\{1-\exp\left[-\sigma R(\mathbf{z}_h,\mathbf{u};h)\right]\right\}, \\
&\qquad \mathbf{x} \in V_M \setminus V_h, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{hi} \neq \emptyset && (21)\\
Q_h(\mathbf{x},\mathbf{u};h) &= 0, \qquad \mathbf{x} \in V_M \setminus V_h, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{hi} = \emptyset, && (22)\\
(B_h n)(\mathbf{x},\mathbf{u};h) &= \frac{c\sigma}{4\pi}\int_0^{R(\mathbf{x},\mathbf{u};h)} dr\ \exp(-\sigma r)\int_S n(\mathbf{x}-r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}', \\
&\qquad (\mathbf{x},\mathbf{u}) \in V_h \times S, && (23)\\
(B_h n)(\mathbf{x},\mathbf{u};h) &= \frac{c\sigma}{4\pi}\int_0^{R(\mathbf{z}_h,\mathbf{u};h)} dr\ \exp(-\sigma r)\int_S n(\mathbf{z}_h-r\mathbf{u},\mathbf{u}')\,d\mathbf{u}', \\
&\qquad \mathbf{x} \in V_M \setminus V_h, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{hi} \neq \emptyset, && (24)\\
(B_h n)(\mathbf{x},\mathbf{u};h) &= 0, \qquad \mathbf{x} \in V_M \setminus V_h, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{hi} = \emptyset && (25)
\end{aligned}
$$

Given h, h' (with $h_m \le h < h' \le h_M$) and $(\mathbf{x},\mathbf{u}) \in V_h \times S$, let $\Lambda(\mathbf{x},\mathbf{u}) = n(\mathbf{x},\mathbf{u};h') - n(\mathbf{x},\mathbf{u},h)$.

Then, (19-25) give

$$
\Lambda(\mathbf{x},\mathbf{u}) = F(\mathbf{x},\mathbf{u}) + G(\mathbf{x},\mathbf{u}) + (B_h\Lambda)(\mathbf{x},\mathbf{u}), \tag{26}
$$

where B_h is defined by (23) and

$$
\begin{aligned}
F(\mathbf{x},\mathbf{u}) &= \frac{q}{\sigma}\left\{\exp\left[-\sigma R(\mathbf{x},\mathbf{u};h)\right] - \exp\left[-\sigma R(\mathbf{x},\mathbf{u};h')\right]\right\} > 0\\
G(\mathbf{x},\mathbf{u}) &= \frac{c\sigma}{4\pi}\int_{R(\mathbf{x},\mathbf{u};h)}^{R(\mathbf{x},\mathbf{u};h')} dr\ \exp(-\sigma r)\int_S n(\mathbf{x}-r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' > 0
\end{aligned}
$$

because $R(\mathbf{x},\mathbf{u};h) > R(\mathbf{x},\mathbf{u};h')$.

It follows that the unique solution $\Lambda(\mathbf{x},\mathbf{u})$ of equation (26) in the Banach space $X_h = C(V_h \times S)$ (with norm $\|f\|_h = \max\{|f(\mathbf{x},\mathbf{u})|, (\mathbf{x},\mathbf{u}) \in V_h \times S\}$) is positive, i.e. $\Lambda(\mathbf{x},\mathbf{u}) = n(\mathbf{x},\mathbf{u};h') - n(\mathbf{x},\mathbf{u};h) > 0\ \forall(\mathbf{x},\mathbf{u}) \in V_h \times S$. (Note that B_h maps the closed positive cone of X_h into itself and that $\|B_h\|_h < c < 1$.)

Further, we have from (26) that $\|\Lambda\|_h \le \|F\|_h + \|G\|_h + c\|\Lambda\|_h$ and so

$$
\|\Lambda\|_h \le \frac{1}{1-c}\left[\,\|F\|_h + \|G\|_h\,\right].
$$

On the other hand, the definitions of F and G (with $h < h'$; the case $h > h'$ is analogous) give

$$
\begin{aligned}
0 < F(\mathbf{x},\mathbf{u}) &= \frac{q}{\sigma}\exp\left[-\sigma R(\mathbf{x},\mathbf{u};h)\right]\left\{1-\exp\left[-\sigma(R(\mathbf{x},\mathbf{u};h')-R(\mathbf{x},\mathbf{u};h))\right]\right\} \leq \\
&\leq q\left[R(\mathbf{x},\mathbf{u};h')-R(\mathbf{x},\mathbf{u};h)\right] \leq q\chi(h,h'), \\
0 < G(\mathbf{x},\mathbf{u}) &\leq c\sigma\|n\| \int_{R(\mathbf{x},\mathbf{u};h)}^{R(\mathbf{x},\mathbf{u};h')} \exp(-\sigma r)\,dr = \\
&= c\|n\|\left\{\exp\left[-\sigma R(\mathbf{x},\mathbf{u};h)\right]\right\} - \exp\left[-\sigma R(\mathbf{x},\mathbf{u};h')\right] \leq \\
&\leq c\|n\|\,\sigma\left[R(\mathbf{x},\mathbf{u};h')-R(\mathbf{x},\mathbf{u};h)\right] \leq c\|n\|\,\sigma\,\chi(h,h') \leq q\frac{c}{1-c}\chi(h,h')
\end{aligned}
$$

where $\chi(h,h')$ was defined in (iii) of Remark 3.1 and we used (16).

We conclude that

$$
\|\Lambda\|_h \leq \frac{q}{(1-c)^2}\,\chi(h,h') \to 0 \quad \text{as} \quad h' \to h.
$$

Thus, we can state the following Theorem.

Theorem 3.1

i) If $h_m \leq h < h' \leq h_M$, then $n(\mathbf{x},\mathbf{u};h') > n(\mathbf{x},\mathbf{u};h)$ at any $(\mathbf{x},\mathbf{u}) \in V_h \times S$;

ii) $\|n' - n\|_h \to 0$ as $h' \to 0$, where $n = n(\mathbf{x},\mathbf{u};h)$, $n' = n(\mathbf{x},\mathbf{u};h')$ and $\|\cdot\|_h$ is the norm in X_h. □

To complete the comparison between $n(\mathbf{x},\mathbf{u};h')$ and $n(\mathbf{x},\mathbf{u};h)$ with $h < h'$, we still have to examine the five cases, see Figure 2:

a) $\mathbf{x} \in V_{h'} \setminus V_h, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h,i} \neq \emptyset,$

b) $\mathbf{x} \in V_{h'} \setminus V_h, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h,i} = \emptyset, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h'i} \neq \emptyset,$

c) $\mathbf{x} \notin V_{h'}, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h,i} \neq \emptyset,$

d) $\mathbf{x} \notin V_{h'}, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h,i} = \emptyset, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h'i} \neq \emptyset,$

e) $\mathbf{x} \notin V_{h'}, \quad \gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h'i} = \emptyset,$

Case a): $n(\mathbf{x},\mathbf{u};h') > n(\mathbf{z}_h,\mathbf{u};h')$ because of lemma 2.4 with $V = V_{h'}$; $n(\mathbf{z}_h,\mathbf{u};h') > n(\mathbf{z}_h,\mathbf{u};h)$ because of Theorem 3.1; $n(\mathbf{z}_h,\mathbf{u};h) = n(\mathbf{x},\mathbf{u};h)$ because of (19)+(21)+(24). Hence $n(\mathbf{x},\mathbf{u};h') > n(\mathbf{x},\mathbf{u};h)$.

Case b): $n(\mathbf{x},\mathbf{u};h') > n(\mathbf{x},\mathbf{u};h) = 0$, see (19)+(22)+(25).

Case c): $n(\mathbf{x},\mathbf{u};h') > n(\mathbf{z}_{h'},\mathbf{u};h')$ because of (19)+(21)+(24) with h' instead of h; $n(\mathbf{z}_{h'},\mathbf{u};h') > n(\mathbf{z}_h,\mathbf{u};h')$ because of Lemma 2.4 with $V = V_{h'}$; $n(\mathbf{z}_h,\mathbf{u};h') > n(\mathbf{z}_h,\mathbf{u};h))$ because of Theorem 3.1; $n(\mathbf{z}_h,\mathbf{u};h) = n(\mathbf{x},\mathbf{u};h)$ because of (19)+(21)+(24). Hence $n(\mathbf{x},\mathbf{u};h') > n(\mathbf{x},\mathbf{u};h)$.

Case d): $n(\mathbf{x},\mathbf{u};h') > n(\mathbf{x},\mathbf{u};h) = 0$ see (19)+(22)+(25).

Case e): $n(\mathbf{x},\mathbf{u};h') = n(\mathbf{x},\mathbf{u};h) = 0$ see (19)+(22)+(25).

The above discussion and i) of Theorem 3.1 lead to the following Theorem.

Theorem 3.2 *If $h_m \le h < h' \le h_M$, then $n(\mathbf{x},\mathbf{u};h') > n(\mathbf{x},\mathbf{u};h)$ $\forall(\mathbf{x},\mathbf{u}) \in V_M \times S$, except when $\mathbf{x} \notin V_{h'}$ and $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h'i} = \emptyset$ (in this case $n(\mathbf{x},\mathbf{u};h') = n(\mathbf{x},\mathbf{u};h) = 0$).*

Moreover, ii) of Theorem 3.1 leads to the following property of the photon density:

Theorem 3.3 *$n(\mathbf{x},\mathbf{u};h') \to n(\mathbf{x},\mathbf{u};h)$ as $h' \to h$ if $x \in V_M \setminus V_h$ and $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h,i} = \emptyset$*

Proof: Let $(\mathbf{x},\mathbf{u})$ be such that $\mathbf{x} \in V_M \setminus V_h$ and $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h,i} \neq \emptyset$ (hence, $\mathbf{x} \in V_M \setminus V_{h'}$ if h' is close enough to h, see case c) of Figure 2). From (19)+(21)+(24) with h substituted by h', we have

$$\begin{aligned} n(\mathbf{x},\mathbf{u};h') &= n(\mathbf{z}_{h'},\mathbf{u};h') = \frac{q}{\sigma}\{1-\exp[-\sigma R(\mathbf{z}_{h'},\mathbf{u};h')]\} + \\ &+ \frac{c\sigma}{4\pi}\int_0^{R(\mathbf{z}_{h'},\mathbf{u};h')} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_{h'}-r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' \end{aligned}$$

and also

$$\begin{aligned} n(\mathbf{x},\mathbf{u};h') &= \frac{q}{\sigma}\{1-\exp[-\sigma R(\mathbf{z}_{h'},\mathbf{u};h')]\} + \\ &+ \frac{c\sigma}{4\pi}\int_0^{\eta} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_{h'}-r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' \\ &+ \frac{c\sigma}{4\pi}\int_\eta^{R(\mathbf{z}_h,\mathbf{u};h)} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_{h'}-r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' \\ &+ \frac{c\sigma}{4\pi}\int_{R(\mathbf{z}_h,\mathbf{u};h)}^{R(\mathbf{z}_{h'},\mathbf{u};h')} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_{h'}-r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' \end{aligned}$$

where $\eta = |\mathbf{z}_{h'} - \mathbf{z}_h|$, see case c) of Figure 2, and where

$$\begin{aligned} &\int_\eta^{R(\mathbf{z}_h,\mathbf{u};h)} dr\,\exp(-\sigma r')\int_S n(\mathbf{z}_{h'}-r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' = \\ &\qquad = \exp(-\sigma\eta)\int_0^{R(\mathbf{z}_h,\mathbf{u};h)-\eta} dr'\,\exp(-\sigma r')\int_S n(\mathbf{z}_h - r'\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' \end{aligned}$$

because, if $r' = r - \eta$, then $\mathbf{z}_{h'} - r\mathbf{u} = \mathbf{z}_{h'} - \eta\mathbf{u} - r'\mathbf{u} = \mathbf{z}_h - r'\mathbf{u}$. On the other hand, from (19)+(21)+(24) we also obtain

$$n(\mathbf{x},\mathbf{u};h) = n(\mathbf{z}_h,\mathbf{u};h) = \frac{q}{\sigma}\{1-\exp[-\sigma R(\mathbf{z}_h,\mathbf{u},h)]\} + \\ + \frac{c\sigma}{4\pi}\int_0^{R(\mathbf{z}_h,\mathbf{u};h)} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}'$$

and so

$$\begin{aligned} n(\mathbf{x},\mathbf{u};h) =& \frac{q}{\sigma}\{1-\exp[-\sigma R(\mathbf{z}_h,\mathbf{u},h)]\} + \\ &+ \frac{c\sigma}{4\pi}\exp(-\sigma\eta)\int_0^{R(\mathbf{z}_h,\mathbf{u};h)-\eta} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}' + \\ &+ \frac{c\sigma}{4\pi}[1-\exp(-\sigma\eta)]\int_0^{R(\mathbf{z}_h,\mathbf{u};h)-\eta} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}' + \\ &+ \frac{c\sigma}{4\pi}\int_{R(\mathbf{z}_h,\mathbf{u};h)-\eta}^{R(\mathbf{z}_h,\mathbf{u};h)} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}'. \end{aligned}$$

It follows that

$$\begin{aligned} n(\mathbf{x},\mathbf{u};h') - n(\mathbf{x},\mathbf{u};h) =& \frac{q}{\sigma}\{\exp[-\sigma R(\mathbf{z}_h,\mathbf{u};h)] - \exp[-\sigma R(\mathbf{z}_{h'},\mathbf{u};h')]\} + \\ &+ \frac{c\sigma}{4\pi}\int_0^{\eta} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_{h'} - r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' + \\ &+ \frac{c\sigma}{4\pi}\int_{R(\mathbf{z}_h,\mathbf{u};h)}^{R(\mathbf{z}_{h'},\mathbf{u};h')} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_{h'} - r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' - \\ &- \frac{c\sigma}{4\pi}[1-\exp(-\sigma\eta)]\int_0^{R(\mathbf{z}_h,\mathbf{u};h)-\eta} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)\,d\mathbf{u}' - \\ &- \frac{c\sigma}{4\pi}\int_{R(\mathbf{z}_h,\mathbf{u};h)-\eta}^{R(\mathbf{z}_h,\mathbf{u};h)} dr\,\exp(-\sigma r)\int_S n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h')\,d\mathbf{u}' + \\ &+ \frac{c\sigma}{4\pi}\exp(-\sigma\eta)\int_0^{R(\mathbf{z}_h,\mathbf{u};h)-\eta} dr\,\exp(-\sigma r) \\ &\qquad \int_S [n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h') - n(\mathbf{z}_h - r\mathbf{u},\mathbf{u}';h)]\,d\mathbf{u}'. \end{aligned}$$

Thus, we have

$$\begin{aligned} |n(\mathbf{x},\mathbf{u};h') - n(\mathbf{x},\mathbf{u};h)| \le\ & q\{R(\mathbf{z}_{h'},\mathbf{u};h') - R(\mathbf{z}_h,\mathbf{u};h)\} + c\sigma\|n\|\eta + \\ &+ c\sigma\|n\|\{R(\mathbf{z}_{h'},\mathbf{u};h') - R(\mathbf{z}_h,\mathbf{u};h)\} + c\sigma\|n\|[1-\exp(-\sigma\eta)]\{R(\mathbf{z}_h,\mathbf{u};h) - \eta\} + \\ &+ c\sigma\|n\|\eta + c\sigma\|n' - n\|_h\{R(\mathbf{z}_h,\mathbf{u};h) - \eta\}. \end{aligned}$$

Since

$$
\begin{aligned}
0 &\le R(z_{h'}, \mathbf{u}; h') - R(\mathbf{z}_h, \mathbf{u}; h) = |\mathbf{z}_{h'} - \mathbf{z}_h| + |\mathbf{y}_{h'} - \mathbf{y}_h| \le 2\chi(h, h') \\
0 &\le \eta = |\mathbf{z}_{h'} - \mathbf{z}_h| \le \chi(h, h'), \\
0 &\le R(\mathbf{z}_h, \mathbf{u}; h) - \eta \le R(\mathbf{z}_h, \mathbf{u}; h) \le \delta_M,
\end{aligned}
$$

where δ_M is the diameter of V_M, we obtain from the preceding inequality

$$
\begin{aligned}
|n(\mathbf{x}, \mathbf{u}; h') - n(\mathbf{x}, \mathbf{u}; h)| \le& \\
&\le 2q\chi(h, h') + c\sigma\|n\|\chi(h, h') + 2c\sigma\|n\|\chi(h, h') + \\
&\quad + c\sigma^2\|n\|\chi(h, h')\delta_M + c\sigma\|n\|\chi(h, h') + c\sigma\|n - n'\|_h\delta_M = \\
&= [2q + 4c\sigma\|n\| + c\sigma^2\delta_M\|n\|]\chi(h, h') + c\sigma\delta_M\|n - n'\|_h.
\end{aligned}
$$

We conclude that $n(\mathbf{x}, \mathbf{u}; h') \to n(\mathbf{x}, \mathbf{u}; h)$ as $h' \to h$ because $\chi(h, h') \to 0$ (see *iii)* of Remark 3.1) and $\|n - n'\|_h \to 0$ (see *ii)* of Theorem 3.1). □

4 Identification of the boundary surface

As in Sections 2 and 3, we assume that the positive quantities σ_s σ and q (that characterize the physical behaviour of the interstellar cloud) are given.

Then, at *each given* $(\mathbf{x}, \mathbf{u}) \in V_M \times S$, the photon density is a function of the parameter $h \in [h_m, h_M]$, i.e. it depends on the region V_h occupied by the cloud. As a consequence, we can write

$$
n(\mathbf{x}, \mathbf{u}; h) = K(h)(\mathbf{x}, \mathbf{u}), \quad (\mathbf{x}, \mathbf{u}) \in V_M \times S, \tag{27}
$$

where

$$
K(h)(\mathbf{x}, \mathbf{u}) = ((I - B_h)^{-1}Q_h)(\mathbf{x}, \mathbf{u}), \quad (\mathbf{x}, \mathbf{u}) \in V_M \times S, \tag{28}
$$

see (19-25).

Note that K is a *nonlinear* operator acting on h through $R(\mathbf{x}, \mathbf{u}; h)$, with domain $D(K) = [h_m, h_M]$ and range $R(K) \subset X_+$. The main properties of K are summarized in the following lemma.

Lemma 4.1

i) If $h < h'$, $K(h)(\mathbf{x}, \mathbf{u}) < K(h')(\mathbf{x}, \mathbf{u})$, $\forall(\mathbf{x}, \mathbf{u}) \in V_M \times S$, except when $\mathbf{x} \notin V_{h'}$ and $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h'i} = \emptyset$ (in this case $K(h)(\mathbf{x}, \mathbf{u}) = K(h')(\mathbf{x}, \mathbf{u}) = 0$);

ii) $K(h')(\mathbf{x}, \mathbf{u}) \to K(h)(\mathbf{x}, \mathbf{u})$ as $h' \to h$, $\forall \mathbf{x} \in V_M$ and $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h'i} \neq \emptyset$.

Proof. (i) follows from (27) and Theorem 3.2. (ii) follows from ii) of Theorem 3.1 and from Theorem 3.3. □

Remark 4.1 *Lemma 4.1 implies that*

i) $K(h)(\mathbf{x},\mathbf{u}) < K(h')(\mathbf{x},\mathbf{u})$ *if* $h_m \leq h < h' \leq h_M$, $\mathbf{x} \in V_M$ *and* $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h_m i} \neq \emptyset$ *(hence,* $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h'i} \neq \emptyset$*);*

ii) $K(h')(\mathbf{x},\mathbf{u}) - K(h)(\mathbf{x},\mathbf{u}) \to 0$ *as* $h' \to h$, $\forall \mathbf{x} \in V_M$ *and* $\gamma_{(\mathbf{x},\mathbf{u})} \cap V_{h_m i} \neq \emptyset$.

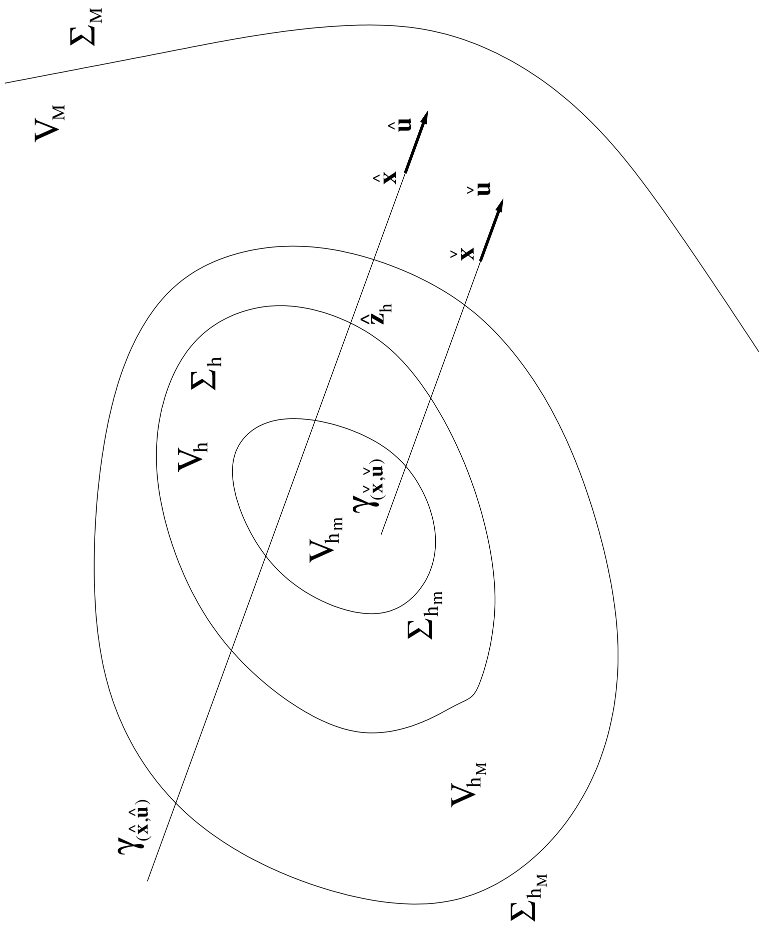

Figure 3: The locations $\hat{\mathbf{x}}$ and $\check{\mathbf{x}}$ "far" from V_{h_M}: $\hat{\mathbf{x}}, \check{\mathbf{x}} \in V_M - V_{h_M}$; $\hat{\mathbf{u}}$ and $\check{\mathbf{u}}$ are such that $\gamma_{\hat{\mathbf{x}},\hat{\mathbf{u}}} \cap V_{h_m i} \neq \emptyset$ and $\gamma_{\check{\mathbf{x}},\check{\mathbf{u}}} \cap V_{h_m i} \neq \emptyset$

As the beginning of Section 3, assume now that the value $\hat{n} = n(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ of the photon density is known, where $\hat{\mathbf{x}} \in V_M \setminus V_{h_M}$ and $\gamma_{(\hat{\mathbf{x}},\hat{\mathbf{u}})} \cap V_{h_m i} \neq \emptyset$ (with $V_{Mi} \supset V_{h_M}$, see Figure 3). If we put

$$K(h)(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \hat{K}(h), \qquad D(\hat{K}) = [h_m, h_M], \quad R(\hat{K}) \subset \mathbb{R}_+ \tag{29}$$

then Lemma 4.1 and Remark 4.1 imply that $\hat{K}(h)$ is a *continuous* and *strictly increasing* function of $h \in [h_m, h_M]$.

As a consequence, we have

Theorem 4.1 *If the family $\mathcal{F}$, defined by (18) and assumptions a) - d), is "suitably chosen" (i.e., if it is such that $\hat{K}(h_m) \leq \hat{n} \leq \hat{K}(h_M)$), then a unique $\hat{h} \in [h_m, h_M]$ exists, for which $\hat{K}(\hat{h}) = \hat{n}$.* □

Note that the value $\hat{h}$ can be found by using some standard successive approximation method, see also Remark 5.1 in Section 5.

Correspondingly, the surface $\Sigma_{\hat{h}}$ is identified, which bounds the region $V_{\hat{h}}$ that produces a photon density distribution $n(\mathbf{x}, \mathbf{u}; \hat{h})$ such that $n(\hat{\mathbf{x}}, \hat{\mathbf{u}}; \hat{h}) = \hat{n}$.

5 Concluding remarks

Remark 5.1 *Since $n(\hat{\mathbf{x}}, \hat{\mathbf{u}}; h) = n(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}; h)$, see Figure 3, we have from (27) and (29) that $\hat{K}(h) = K(h)(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = K(h)(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}; h)$. Hence, (28) gives*

$$\begin{aligned} \hat{K}(h) =& ((I - B_h)^{-1} Q_h)(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}) = Q_h(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}; h) + \sum_{j=1}^{\infty} (B_h^j Q_h))(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}) = \\ &= Q_h(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}; h) + \sum_{j=1}^{m} (B_h^j Q_h))(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}) + O(c^{m+1}). \end{aligned} \tag{30}$$

Relation (30) implies that

$$\hat{K}(h) \simeq Q_h(\hat{\mathbf{z}}_h, \mathbf{u}; h) + \sum_{j=1}^{m} (B_h^j Q_h))(\hat{\mathbf{z}}_h, \hat{\mathbf{u}}) \tag{31}$$

may be considered as the explicit expression of $\hat{K}(h)$ as a function of h (with an error of the order of c^{m+1}). Correspondingly, (31) can be used in any successive approximation procedure to determine the value of $\hat{h}$, such that $\hat{K}(\hat{h}) = \hat{n}$.

Remark 5.2 *If the family $\mathcal{F}$ is not "suitably chosen" (from a physical viewpoint), it may happen that $\hat{n} < \hat{K}(h_m)$ or $\hat{K}(h_m) < \hat{n}$. Obviously, in this case, the family $\mathcal{F}$ must be changed.*

Remark 5.3 *Assume that another family* $\mathcal{F}_1$ *(with* $\mathcal{F} \cap \mathcal{F}_1 = \emptyset$*) is considered, e.g.* $\mathcal{F}_1 = \left\{ \sum_h^{(1)} : \varphi_1(x, y, z; h) = 0, h \in [h_m, h_M]) \right\}$. *This will lead to a value* $\hat{h}_1$ *possibly different from* $\hat{h}$ *(and so to a surface* $\Sigma_{\hat{h}_1}^{(1)}$ *possibly different from* $\Sigma_{\hat{h}}$*). Such a "non -uniqueness" result is not surprising because* $\Sigma_{\hat{h}}$ *must be considered as the "best approximation within* $\mathcal{F}$*" of the true physical surface* Σ_{ph}*, derived from the unique measured photon density* $\hat{n}$.

Remark 5.4 *Further, assume that another measured value of the photon density* $\check{n} = n(\check{\mathbf{x}}, \check{\mathbf{u}})$ *is known, with* $(\check{\mathbf{x}}, \check{\mathbf{u}}) \neq (\hat{\mathbf{x}}, \hat{\mathbf{u}})$ *and with* $\check{\mathbf{x}} \in V_M \setminus V_{h_M}$ *and* $\gamma_{(\check{\mathbf{x}},\check{\mathbf{u}})} \cap V_{h_m i} \neq \emptyset$. *Then, the corresponding* $\check{h}$ *may be different from* $\hat{h}$ *because we have that* $\hat{K}(\hat{h}) = K(\hat{h})(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \hat{n}$ *but not necessarily that* $K(\hat{h})(\check{\mathbf{x}}, \check{\mathbf{u}}) = \check{n}$ *(however,* $K(\check{h})(\check{\mathbf{x}}, \check{\mathbf{u}}) = \check{n}$*). In other words, if the terrestrial astronomers are partiiculary lucky, it may happen that they choose a family* $\mathcal{F}$*, such that* $K(\hat{h})(\mathbf{x}, \mathbf{u})$ *is equal to the measured photon density* $n(\mathbf{x}, \mathbf{u})$ *at any* $(\mathbf{x}, \mathbf{u}) \in V_M \times S$. *In this case,* $K(\hat{h})(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \hat{n}$ *and* $K(\hat{h})(\check{\mathbf{x}}, \check{\mathbf{u}}) = \check{n}$*; correspondingly,* $V_{\hat{h}}$ *is really the region occupied by the cloud (i.e.* $\Sigma_{\hat{h}} = \Sigma_{ph}$*). However, in general,* $\Sigma_{\hat{h}}$ *is only an approximate representation of* Σ_{ph}.

6 Appendix: Examples of families $\mathcal{F}$

6.1 Homothetic families

Let $\eta(x_1, x_2, x_3) = 0$ be the equation of a closed surface Σ_0, that bounds the closed, bounded and strictly convex region $V_0 \subset \mathbb{R}^3$, and assume that η is a continuous function of

$$(x_1, x_2, x_3) \in \{(x_1, x_2, x_3) : (x_1, x_2, x_3) = (x/h, y/h, z/h), (x, y, z) \in V_M,\ 0 < h_m \leq h < h_M\} \tag{32}$$

If the family $\mathcal{F}$ is defined by

$$\mathcal{F} = \{\Sigma_h : \varphi(x, y, z; h) = 0,\ \varphi(x, y, z; h) = \eta(x/h, y/h, z/h),\ h \in [h_m, h_M]\},$$

then it is easy to see that each Σ_h is homothetic to Σ_0. In fact, let $P_h = (x_h, y_h, z_h) \in \Sigma_h$, i.e. $\eta(x_h/h, y_h/h, z_h/h) = 0$;
correspondingly $P_{h'} = (x_{h'}, y_{h'}, z_{h'})$ with $x_{h'} = (h/h')x_h$, $y_{h'} = (h/h')y_h$, $z_{h'} = (h/h')z_h$ belongs to $\Sigma_{h'}$ and we have that $\overline{P_{h'}O} = (h'/h)\overline{P_hO}$.
Note that $\mathcal{F}$ has the following properties:

a) Il $\mathbf{u}$ is a unit vector, let $\gamma_{0,\mathbf{u}} = \{\mathbf{y}\colon \mathbf{y} = t\mathbf{u},\ t \geq 0\}$ be the half straight line passing through the origin (the center of the homothety). Then, for each $\mathbf{u} \in S$, the intersection points of $\gamma_{0,\mathbf{u}}$ with Σ_0, Σ_h and $\Sigma_{h'}$ (with $h' > h$) are $\mathbf{x}_0 = t_0\mathbf{u}$, $\mathbf{x}_h = t_h\mathbf{u}$, $\mathbf{x}_{h'} = t_{h'}\mathbf{u}$, respectively, with $\eta(t_0u_1, t_0u_2, t_0u_3) = 0$, $t_h = ht_0$, $t_{h'} = h't_0 > t_h$. We conclude that $V_h \subset V_{h'i}$.

b) That the region V_h, bounded by Σ_h, is still strictly convex immediatly follows from the fact that strict convexity is preserved by homothetic transformations.

6.2 "Linear" families

Let the "minimal" closed surface Σ_0 and the "maximal" closed surface Σ_1 have equations $\varphi_0(x, y, z) = 0$ and $\varphi_1(x, y, z) = 0$, respectively. Assume that φ_0 and φ_1 are continuous $\forall\ (x, y, z) \in V_M$ and that the regions V_0 and V_1, bounded by Σ_0 and by Σ_1, are closed, bounded and strictly convex. Further, if $V_{0i} = \{(x, y, z) : \varphi_0(x, y, z) < 0\}$ and $V_{1i} = \{(x, y, z) : \varphi_1(x, y, z) < 0\}$ are the interior of V_0 and V_1, we shall also assume that $V_0 \subset V_{1i} \subset V_1 \subset V_{Mi}$. Under the above assumption the "linear" family $\mathcal{F}$, defined by

$$\begin{aligned} \mathcal{F} = \{&\Sigma_h : \varphi(x, y, z; h) = 0, \\ &\varphi(x, y, z; h) = (1 - h)\varphi_0(x, y, z) + h\varphi_1(x, y, z), \quad h \in [0, 1]\} \end{aligned} \tag{33}$$

has the following properties

a) Σ_h is a closed surface, see property b) in Section 3, and $\varphi(x, y, z; h)$ is continuous $\forall\ (x, y, z, h) \in V_M \times [0, 1]$.

b) If $\tilde{P} = (\tilde{x}, \tilde{y}, \tilde{z}) \in \Sigma_h$ with $0 < h < 1$, then $(1 - h)\varphi_0(\tilde{x}, \tilde{y}, \tilde{z}) + h\varphi_1(\tilde{x}, \tilde{y}, \tilde{z}) = 0$. As a consequence, it must be $\varphi_0(\tilde{P}) > 0$ and $\varphi_1(\tilde{P}) < 0$ because the other possible eight cases ($\varphi_0(\tilde{P}) > 0$ and $\varphi_1(\tilde{P}) = 0$, $\varphi_0(\tilde{P}) > 0$ and $\varphi_1(\tilde{P}) > 0$, etc) can be easily excluded. Hence $V_0 \subset V_{hi}$ and $V_h \subset V_{1i}$.

c) Assume that $P \in \Sigma_h \cap \Sigma_k$, with $h \neq k$. Then, we have that $(1 - h)\varphi_0(P) + h\varphi_1(P) = 0$, $(1 - k)\varphi_0(P) + k\varphi_1(P) = 0$. It follows that $\varphi_0(P) = 0$ *and* $\varphi_1(P) = 0$, i.e. $P \in \Sigma_0 \cap \Sigma_1$. We conclude that the Σ_h containing P is unique because $\Sigma_0 \cap \Sigma_1 = \emptyset$. Note that, given P, the surface Σ_h is defined by the value of h, such that $(1 - h)\varphi_0(P) + h\varphi_1(P) = 0$, i.e. $h = \varphi_0(P)/[\varphi_0(P) + |\varphi_1(P)|]$.

d) The region V_h, bounded by Σ_k, is strictly convex. To prove this, we introduce the regions V_{0a} and V_{1b}, bounded by the surfaces Σ_{0a} and

Σ_{1b} (where Σ_{0a} is defined by the equation $\varphi_0(x,y,z) = a$ and Σ_{1b} by $\varphi_1(x,y,z) = b$). Under the assumption that $V_0 = V_{00}$ (bounded by $\Sigma_0 = \Sigma_{00}$ defined by the equation $\varphi_0(x,y,z) = 0$) and $V_1 = V_{1,0}$ (bounded by $\Sigma_1 = \Sigma_{10}$ defined by the equation $\varphi_1(x,y,z) = 0$) are *chosen* so that V_{0a} and V_{0b} are still strictly convex, it is not difficult to show that V_{0a} is contained in the interior of $V_{0a'}$ and V_{0b} in the interior of $V_{0b'}$ if $a < a'$ and $b < b'$.

Assume now that V_h is *not* convex. Correspondengly, Σ_h may have the shape represented in Figure 4, and take $A, B \in \Sigma_h$. Since $A \in \Sigma_h$, we have that $(1-h)\varphi_0(A) + h\varphi_1(A) = 0$, i.e. $(1-h)a + hb = 0$. On the other hand, $(1-h)\varphi_0(B) + h\varphi_1(B) = (1-h)a' + hb' > (1-h)a + hb$, i.e. $(1-h)\varphi_0(B) + h\varphi_1(B) > 0$ and B can not belong to Σ_h. Since the case in which V_h is convex but *not* strictly convex may be dealt with in a similar way, we conclude that V_h is strictly convex.

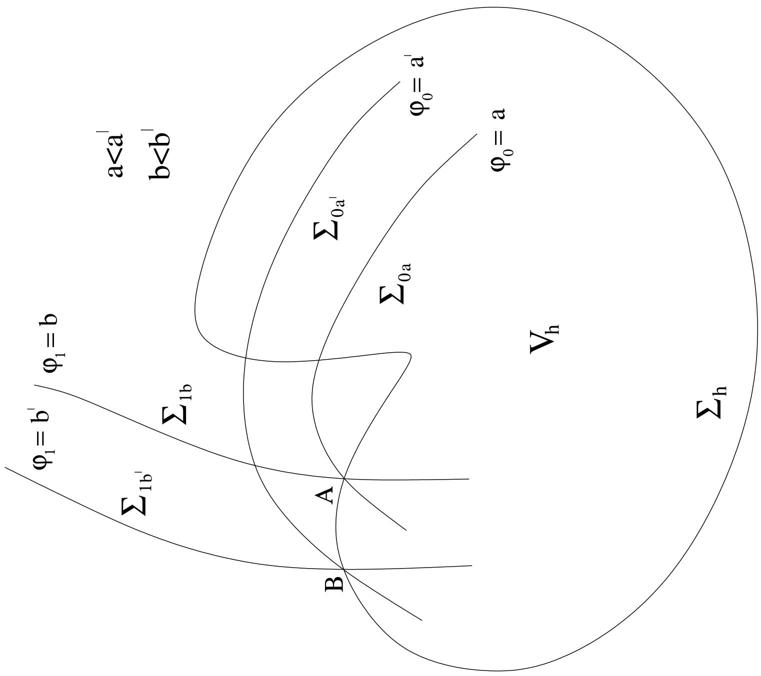

Figure 4: If V_h is not convex, Σ_h may have the shape represented above.

e) $V_{h,i} = \{(x,y,z) \colon \varphi(x,y,z;h) < 0\}$. That $P = (x_1, x_2, x_3)$ is an interior point of V_h if and only if $(1-h)\varphi_0(P) + h\varphi_1(P) < 0$ can also be proved by using the surfaces Σ_{0a} and Σ_{0b}. In fact, take $\gamma_{\mathbf{x},\mathbf{u}} = \{\mathbf{y} \colon \mathbf{y} = (y_1, y_2, y_3),\ y_i = x_i - tu_i\ i = 1,2,3,\ t \geq 0\}$ and let $P_h = \gamma_{\mathbf{x},\mathbf{u}} \cap$

Σ_h. If P_h also belongs to $\Sigma_{0a'}$ and to $\Sigma_{1b'}$, we have that $(1-h)\varphi_0(P_h) + h\varphi_1(P_h) = (1-h)a' + hb' = 0$. On the other hand, P belongs to Σ_{0a} and to Σ_{1b} with $a < a'$ and $b < b'$; hence, $(1-h)\varphi_0(P) + h\varphi_1(P) = (1-h)a + hb < (1-h)a' + hb' = 0$.

f) If $0 \leq h < h' \leq 1$, $V_h \subset V_{h'i}$. In fact, if $P \in \Sigma_{h'}$ we have that $(1-h')\varphi_0(P) + h'\varphi_1(P) = 0$.

It follows that

$$(1-h)\varphi_0(P) + h\varphi_1(P) =$$
$$= (1-h')\varphi_0(P) + h'\varphi_1(P) + (h'-h)[\varphi_0(P) - \varphi_1(P)] =$$
$$= (h'-h)\,[\varphi_0(P) + |\varphi_1(P)|] > 0$$

because $\varphi_0(P) > 0$ and $\varphi_1(P) < 0$. Thus, P is external to Σ_h.

A sufficient condition for the convexity of a linear family $\mathcal{F}$ will be stated in the last Section.

6.3 "Nonlinear" families

We shall give a simple example of a nonlinear family of surfaces Σ_h (which can not be written as a linear combination of the form considered in 6.2, and which is not a homotethic family).

If $g = g(t)$ is defined by $g(t) = t + 1/t$, $t > 0$, let $t_1(h)$ and $t_2(h)$ be the solutions of the equation $g(t) = h$, where h is given so that $2 < h_1 \leq h \leq h_2 < \infty$: $t_1(h) = (h - \sqrt{h^2-4})/2$ and $t_2(h) = (h + \sqrt{h^2-4})/2$. Then, $\mathcal{F} = \{\Sigma_h \colon h \in [h_1, h_2]\}$ is the family of the spherical surfaces Σ_h of radius $(t_2 - t_1)/2 = \sqrt{h^2-4}/2$ and centered at $((t_1+t_2)/2, 0, 0) = (h/2, 0, 0)$, i.e. $\Sigma_h : \varphi(x, y, z; h) = (x - h/2)^2 + y^2 + z^2 - (h^2-4)/4 = 0$.

Note that $V_h = \{(x, y, z) : \varphi(x, y, z; h) \leq 0\}$ is obviously strictly convex; moreover, it is easy to check that $V_{h_1} \subseteq V_{hi}$, $V_h \subseteq V_{h_2 i}$, and $V_h \subseteq V_{h'i}$ if $h < h'$.

The function $g(t)$ can be replaced by any convex function that admits a bounded level set (i.e. by a function $g(t)$ which has a minimum).

6.4 "Level sets" families

Let $G : \mathbb{R}^3 \to \mathbb{R}$ be a positive, continous, strictly convex function. Consider the family

$$\mathcal{F} = \{\Sigma_h = \{(x, y, z) \colon G(x, y, z) = h\},\ 0 \leq m \leq h \leq M\}$$

of the surfaces given by the level sets of G. The family $\mathcal{F}$ has the following properties:

a) For $h \geq m$, let $V_h = \{(x, y, z)\colon G(x, y, z) < h\}$. For the strict convexity of function G, V_h is an open convex set bounded by Σ_h;

b) Let $h_1 < h_2$. Consider V_{h_1} and V_{h_2} (interior of Σ_{h_1} and Σ_{h_2} respectively). The sets V_{h_i} are level sets of function G, hence

$$V_{h_1} = \{(x, y, z)\colon G(x, y, z) < h_1\} \subset \{(x, y, z)\colon G(x, y, z) < h_2\} = V_{h_2}$$

i.e. the family $\mathcal{F}$ is *monotone*.

From the properties of the level sets families, we obatin a sufficient condition for the convexity of a linear family. Let φ_0 and φ_1 as in 6.2. Define $G = \varphi_0/(\varphi_0 - \varphi_1)$ and consider the level set $T_h = \{(x, y, z)\colon G(x, y, z) = h\}$. Note that $(x, y, z) \in T_h$ if and only if $(1 - h)\varphi_0(x, y, z) + h\varphi_1(x, y, z) = 0$ i.e. if and only if $(x, y, z) \in \Sigma_h$ (see (33))

Hence, the linear family defined starting from φ_0 and φ_1 concides with the level set family of G if $h \in [0, 1]$. For the consideration above, a sufficient condition of the convexity of surfaces Σ_h is the strict convexity of function G.

Acknoledgements

This work was partially supported by MURST research funds as well as by GNFM (Gruppo Nazionale Fisica Matematica). The authors are also grateful to Professor Mario Landucci for his useful suggestions on the properties of the various families of surfaces $\mathcal{F}$.

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