COMBINATORIA - ESERCIZÎ

Titolo nota 17/10/2010

UNA PERMUTAZIONE DI n ELEMENTI SENZA PUNTI FISSI È UNA funzione $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ BIGETTIVA, TALE CHE $f(i) \neq i \quad \forall\, i \in \{1, \ldots, n\}$.

DETTO ALTRIMENTI VOGLIO TROVARE IL NUMERO DI MODI CHE HANNO n PERSONE DI SCAMBIARSI DI POSTO SENZA CHE NESSUNO RITORNI NELLO STESSO POSTO.

1. n persone $1, \dots, n$

$1, \dots, k$

$$\underbrace{\quad a \quad}\ \boxed{\quad k \quad}\ \underbrace{n-k-a}$$

$$\underbrace{\quad a \quad}\ \bullet\ \underbrace{n-k-a}$$

$$n-k-\cancel{a}+\cancel{a}+1 = n-k+1$$

$(n-k+1)!$ modi per scegliere l'ordine in cui parlano le n persone con le mie k persone speciali "accorpate" in una sola persona "collettiva"

$$N = (n-k+1)! \cdot k!$$

2

1\.   . 2
3\.   . 4
5\.   . 6
7\.   . 8

$C_1 \quad C_2 \quad \ldots \quad C_8$

$S_1 \quad S_2 \quad \ldots \quad S_8$

$C_1 C_2 S_1 S_2 C_3 S_3 \ldots C_8 S_8$

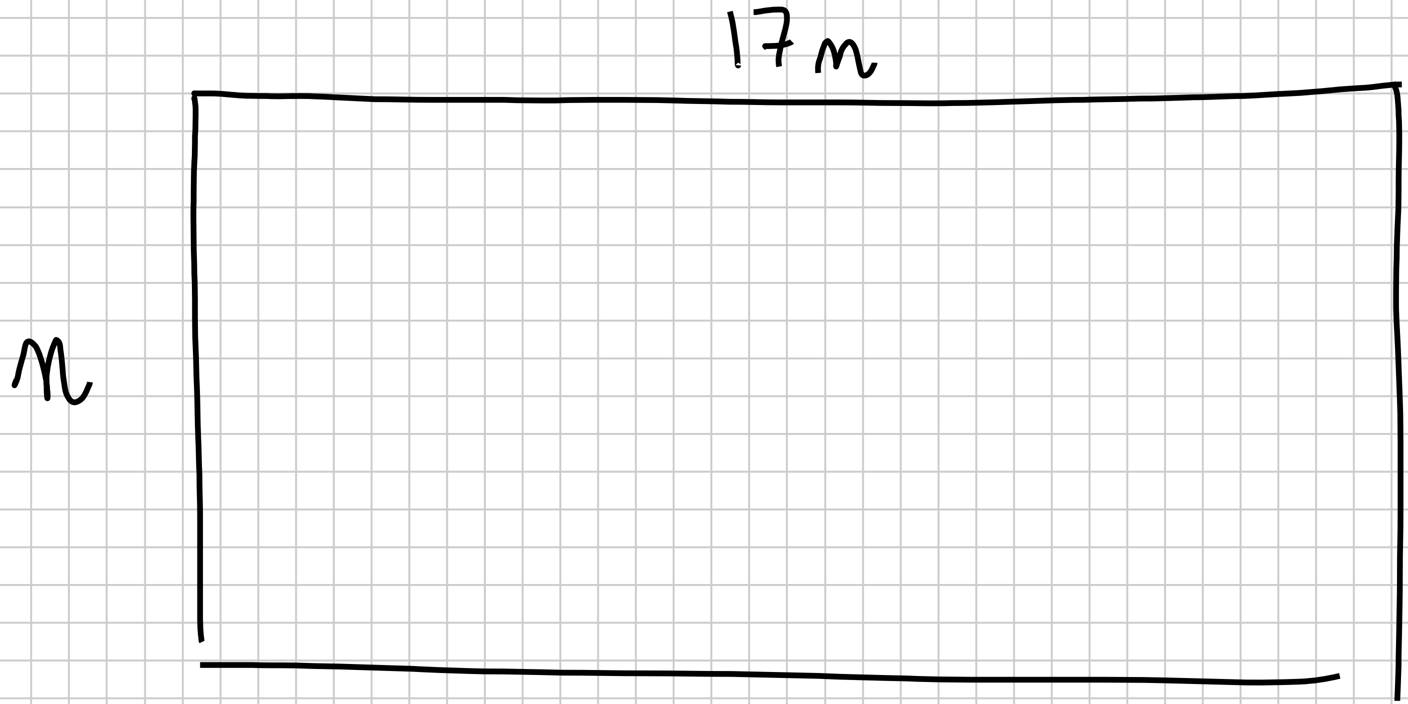
$C,$ 1 2 1 2 3 3 . . . 8 8

2 1 3 1 2 3 4 - - - - 7 8

$c_2\, c_1\, c_3\, s_1\, s_2\, s_3\, c_4 \qquad s_7\, s_8$

$$\frac{16!}{2^8}$$

③ DOUBLE - COUNTING



- in ogni colonna ci sono 3 pedoni
- in ogni riga c'è un numero di

pedoni compreso tra 1 ed n e
ogni riga ha un numero diverso
di pedoni

n° totale di pedoni $= 17n \cdot 3 = 51n$

$$\parallel$$

$$1+2+3+ \cdots + n = \frac{n(n+1)}{2}$$

$$\frac{\cancel{n}(n+1)}{2} = 51\cancel{n} \qquad n \neq 0$$

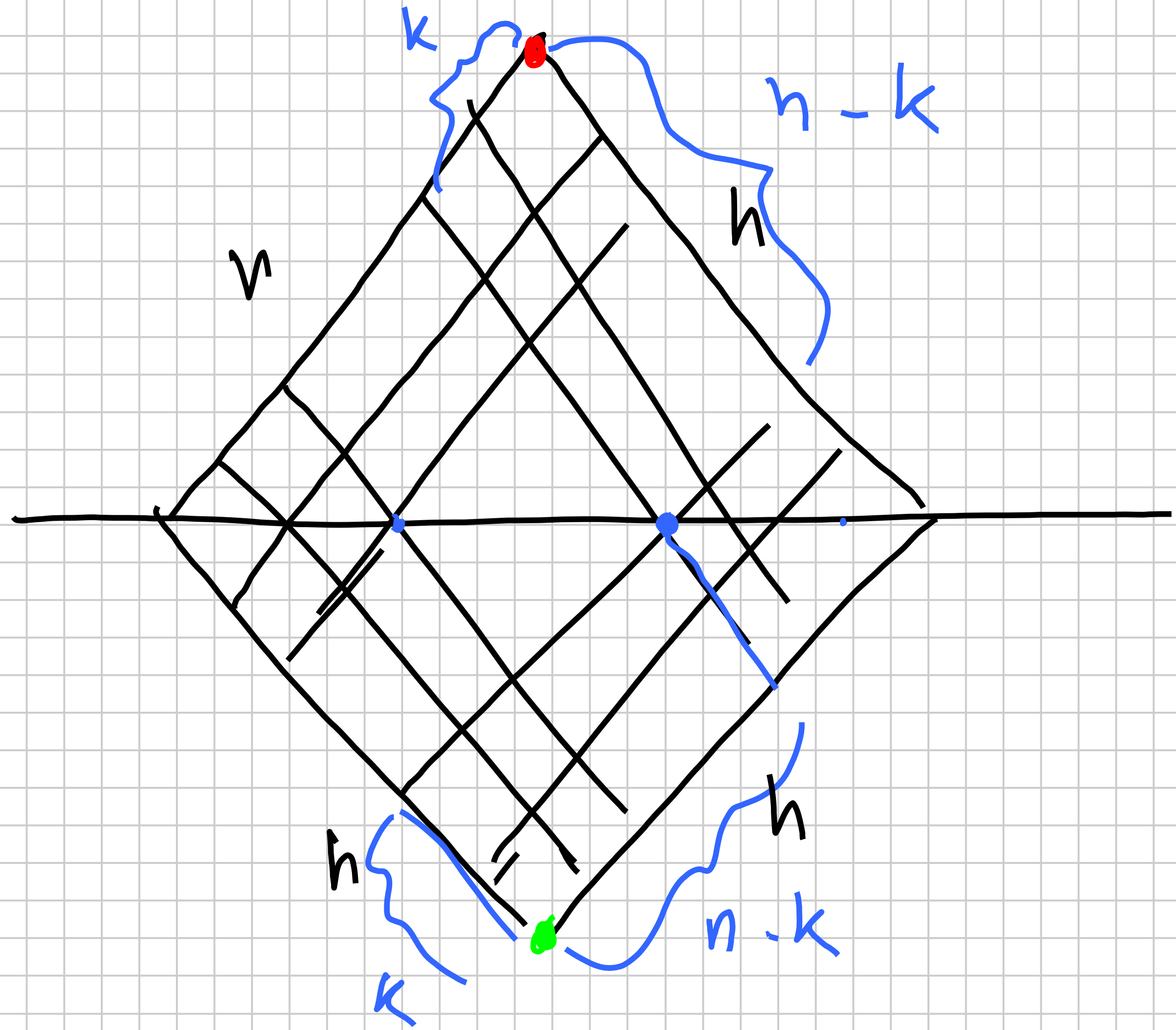
$$n = 101$$

5

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

1	2	3	4	.	n	n+1	.	2n

$$\binom{2n}{n} = \sum_{k=0}^{n} \underbrace{\binom{n}{k}\binom{n}{n-k}}_{\left\{\substack{\text{n° di sott di n el.} \\ \text{di } \{1,..,2n\} \text{ che hanno k elementi tra i primi n}}\right\}} = \sum_{k=0}^{n} \binom{n}{k}^2$$



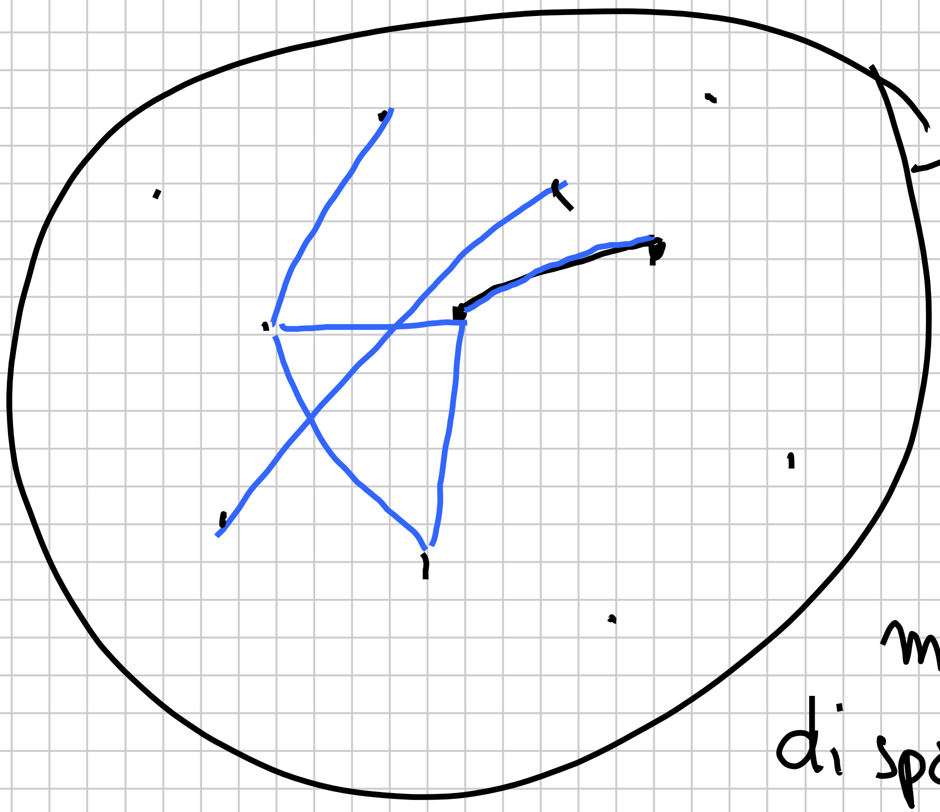
n° di modi di andare da qui a lì.

$(k, n-k)$

percorsi che passano da $(k, n-k) = \binom{n}{k} \cdot \binom{n}{k}$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

6



→ 54 persone

n° di persone che hanno stretto la mano ad un numero dispari di persone e' pari

n_i = # strette di mano che ha fatto la persona i-esima

$$\underbrace{\sum_{i=1}^{54} n_i} = 2 \cdot \#\text{strette di mano}$$

$D + D = P$

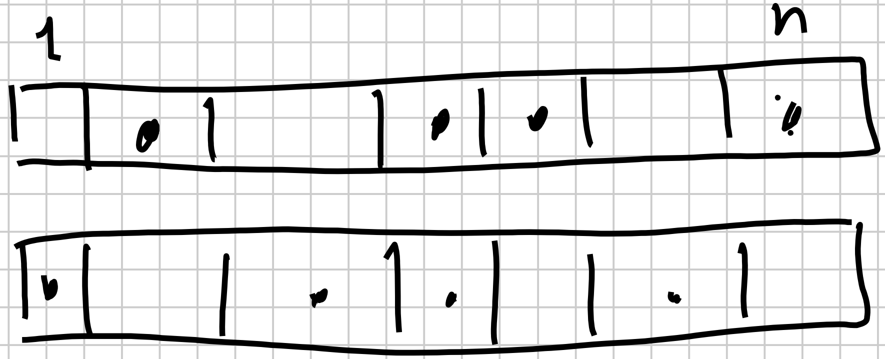
$P + D = D$

$$\underbrace{D + D \cdots D}_{k} = \begin{cases} D & \text{se } k = D \\ P & \text{se } k \cdot P \end{cases}$$

8 $B = \{1, \ldots, n\}$ $B_k = \{A \subseteq B, |A| = k\}$

$$x_k = \frac{1}{n+1} \sum_{A \in B_k} [\min(A) + \max(A)]$$

A $n+1 - A = \{n+1-j, \ j \in A\}$



A

$n+1 - A$

$$\{1, 2\} \qquad n+1-\{1,2\} = \{n, n-1\}$$

$$\cap \qquad\qquad\qquad\qquad \|$$

$$\{1, \ldots, n\} \qquad\qquad \{n+1-1, n+1-2\}$$

$$\{1, n\} \longrightarrow \{n, 1\}$$

$$\min(A) + \max(n+1-A) = n+1$$

$$\max(A) + \min(n+1-A) = n+1$$

$$\max_{i \in A} (n+1-i) = n+1 - \min_{i \in A} i$$

$$\frac{1}{n+1} \sum_{A \in B_k} \min A + \max A =$$

$$= \frac{1}{2} \frac{1}{n+1} \left(\sum_{A \in B_k} \min A + \max A + \sum_{A \in B_k} \min A + \max A \right) =$$

$$= \frac{1}{2} \frac{1}{n+1} \left(\sum_{A \in B_k} \min A + \max A + \sum_{A \in B_k} \min (n+1-A) + \max (n+1-A) \right.$$

$$= \cancel{\frac{1}{2}\,\frac{1}{n+1}}\left(\sum_{A\in B_k} \cancel{n+1} \;\overset{1}{+}\; \cancel{n+1}\right) =$$

$$= |B_k| = \binom{n}{k} = x_k$$

$$\Rightarrow \quad x_k \text{ e' intero}$$

$$x_1 + x_2 + \cdots\cdots + x_{n-1} = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} =$$

$$= 2^n - 2 = (2^{n-1} - 1)\cdot 2$$

9 bis

$$\sum_{A \in B_k} \min A = 1 \cdot \binom{n-1}{k-1} + 2\binom{n-2}{k-1} + 3\binom{n-3}{k-1} + \dots + (n-k+1)\binom{k-1}{k-1}$$

$$A = \{1, \underbrace{\dots}_{k-1 \text{ scelti tra } \{2,\dots,n\}}\}$$

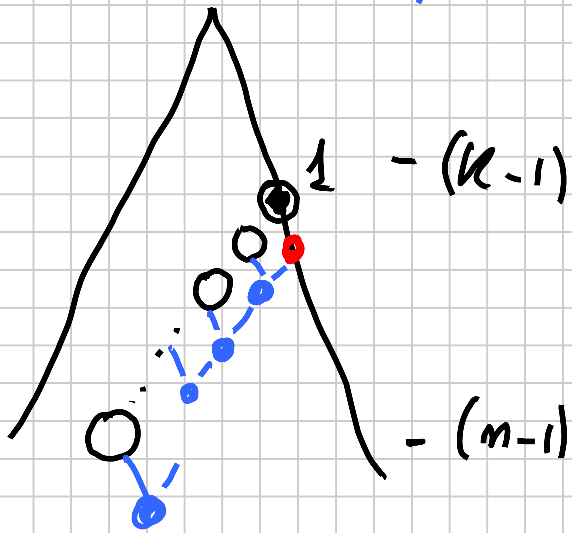
$$\sum_{j=1}^{n-k+1} j \cdot \binom{n-j}{k-1}$$

$$\sum \max A = n \cdot \binom{n-1}{k-1} + (n-1)\binom{n-2}{k-1} + (n-2)\binom{n-3}{k-1} + \dots + k\binom{k-1}{k-1}$$

$$\sum_{A \in B_k} \min A = 1 \cdot \binom{n-1}{k-1} + 2\binom{n-2}{k-1} + 3\binom{n-3}{k-1} + \dots + (n-k+1)\binom{k-1}{k-1}$$

$$(n+1)\binom{n-1}{k-1} + (n+1)\binom{n-2}{k-1} + (n+1)\binom{n-3}{k-1} + \dots$$

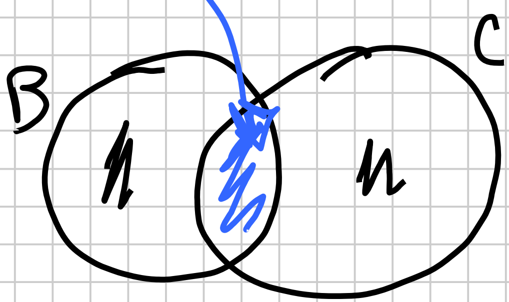
$$X_k = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \dots + \binom{k-1}{k-1} = \binom{n}{k}$$



9 $A = \{1, \dots, n\}$, (B, C) $B \subseteq A$, $C \subseteq A$ $B \cap C \neq \emptyset$

$0 < |B \cap C| \leq n$ $K = \# B \cap C$

$$\sum_{k=1}^{n} \binom{n}{k} 3^{n-k} =$$

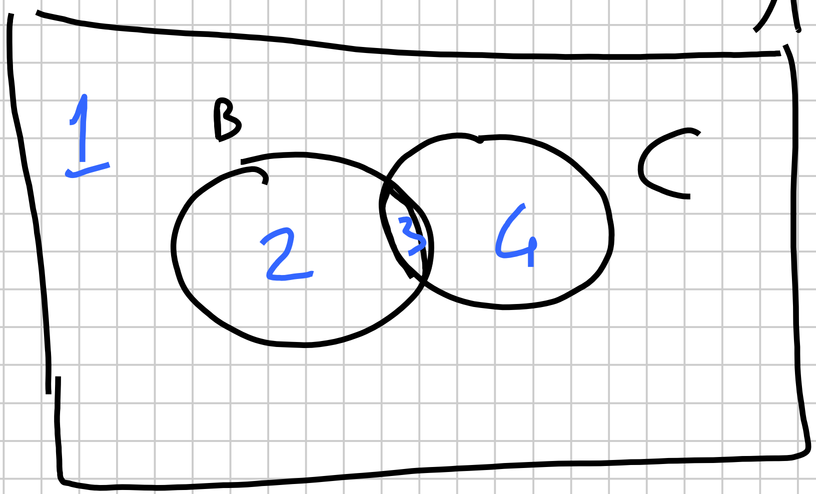


$n - k$ $\to B$
$\to C$
$\to$ né in B
né in C

$$\sum_{k=0}^{n} \binom{n}{k} 1^{k}\, 3^{n-k} = (1+3)^{n} = 4^{n}$$

$$\sum_{k=1}^{n} \ell = 4^{n} - 1 \cdot 3^{n}$$

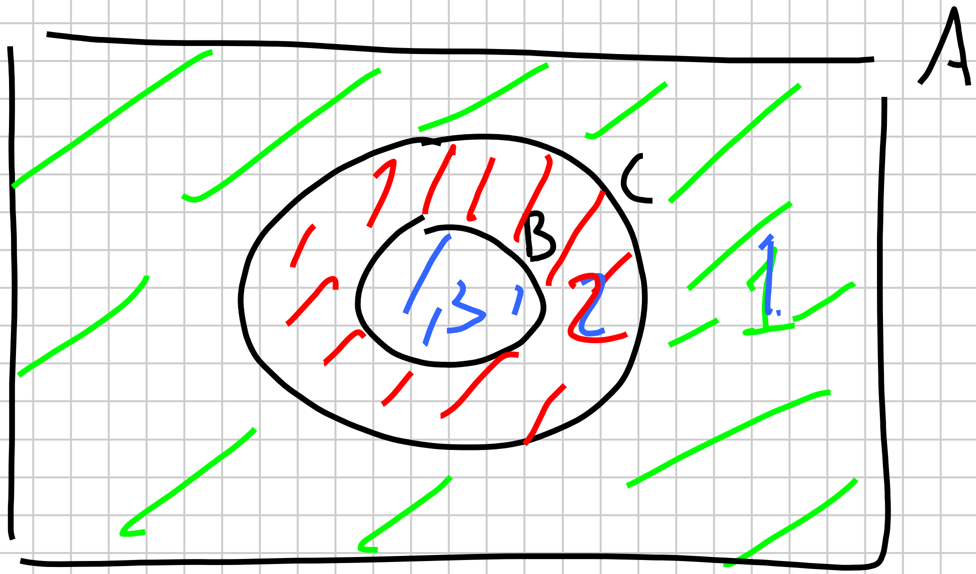
$A = \{1, \dots, n\}$



$ZONA_3 \neq \emptyset$

$B = 2 \cup 3$ $\qquad$ $C = 3 \cup 4$

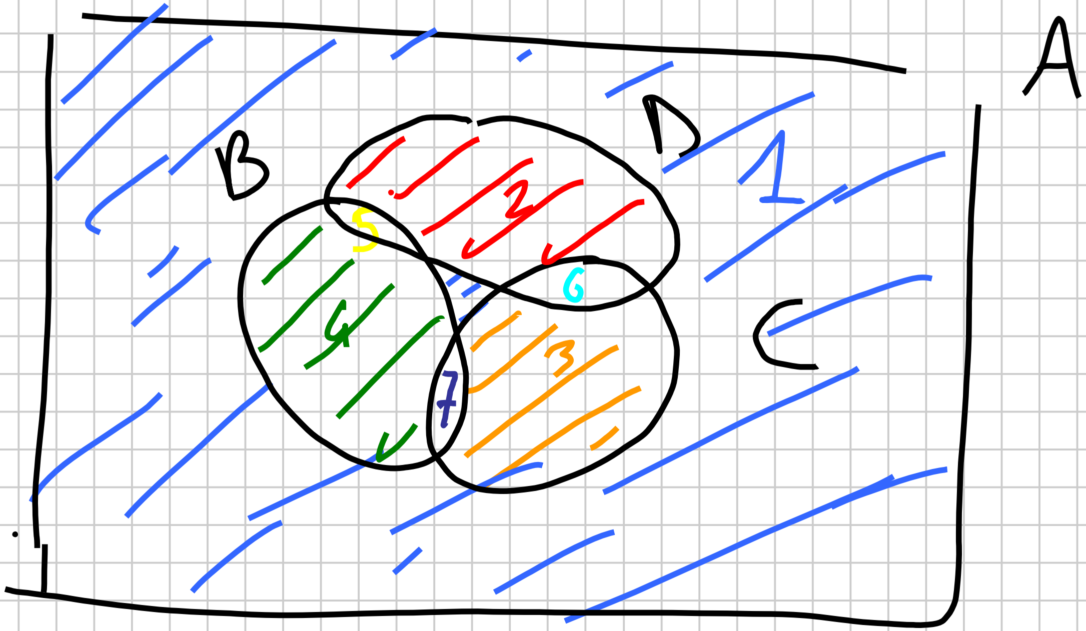
$$4^n - \{ \text{zona}_3 = \phi \} = 4^n - 3^n$$



$2 \neq \phi$

$3 \neq \phi$

$$3^n - 2^n - 2^n + 1 = 3^n - 2^{n+1} + 1$$

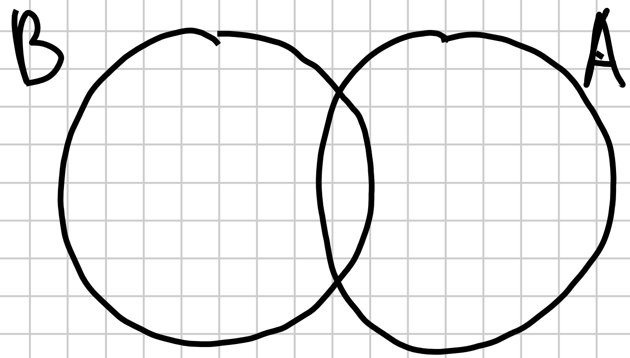


$7 \neq \phi$

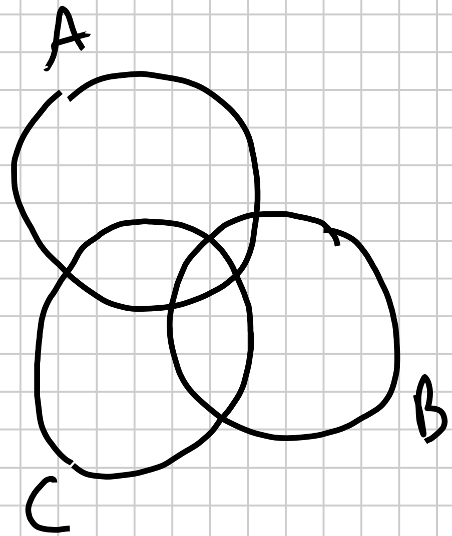
$5 \neq \phi$

$$7^n - 6^n - 6^n + 5^n = 7^n - 2 \cdot 6^n + 5^n$$

PRINCIPIO DI INCLUSIONE ESCLUSIONE



$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$|A \cup B \cup C| = |A| + |B| + |C| +$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$

Caso generale

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

$$- \left(|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| \; \cdot \right)$$

$$+ \left(|A_1 \cap A_2 \cap A_3| + \cdot \quad \cdot \quad \cdot \right)$$

$$- \left(|A_1 \cap A_2 \cap A_3 \cap A_4| \, | \cdot \quad \cdot \; \cdot \; \cdot \right)$$

$$+ (5\text{-uple}) - (6\text{-uple}) \cdot \; \cdot \; (-1)^n |A_1 \cap A_2 \ldots \cap A_n|$$

4) $A_i = \{ f$ biiettiva tale che $f(i) = i \}$

$\bigcup_{i=1}^{n} A_i = \{ f$ biiettiva con almeno 1 punto fisso$\}$

tale $N = n! - \left| \bigcup_{i=1}^{n} A_i \right|$

$\left| \bigcup_{i=1}^{n} A_i \right| = |A_1| + |A_2| + \dots + |A_n| \rightarrow = n \cdot (n-1)!$

$\binom{n}{2}(n-2)! \leftarrow - |A_{1,2}| - |A_{2,3}| - \dots - |A_{n,n-1}|$

$\binom{n}{3}(n-3)! \leftarrow + |\text{terne}| - \dots -$

$\binom{n}{k}(n-k)! = ($ — $|\overset{\text{[illegible]}}{\_\ \_\ \_}|$ . . .

$$|A_1| = (n-1)! = |A_2| = |A_3| \cdots = |A_n|$$

$$|A_1 \cap A_2| = (n-2)! =$$

$$|A_1 \cap A_2 \cap A_3| = (n-3)!$$

$$|A_1 \cap \cdots \cap A_k| = (n-k)!$$

$$\left|\bigcup_i A_i\right| = \sum_{i=1}^{n} \binom{n}{i} (n-i)!\,(-1)^{i-1}$$

$$N = n! - \sum_{i=1}^{n} \frac{n!}{i!\,\cancel{(n-i)!}}\,\cancel{(n-i)!}\,(-1)^{i-1}$$

$$N = n! \underbrace{\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \frac{(-1)^n}{n!}\right)}_{\frac{1}{e}}$$

$$N \approx \frac{n!}{e}$$

(10) coppie di studenti

$$\#((s_1, s_2), c) = \binom{10001}{2}$$

$$\| $$

$$\sum_{c \in C} \binom{2m_c+1}{2} = \sum_{c \in C} m_c(2m_c+1)$$

$$\#(s, c, e) = \#S \cdot \#e = k \cdot 10001$$

$$\|$$

$$\sum (2m_c+1)\, m_c$$

$c \in C$

$$\binom{10001}{2} = K \cdot 10001 \implies K = 5000$$

\# edifici di uno studente $= \displaystyle\sum_{\substack{c \text{ corsi} \\ \text{che segue}}} m_c$

$2m_c + 1 = \#$ partecipanti al corso $c \implies$

$m_c = \#$ n° edifici del corso c

$$\sum_{c \text{ corsi}} 2m_c = \sum_{c \text{ corsi}} (\text{compagni di corso in } c) =$$

$= 10001 - 1 = 10000$

$\sum m_c = \frac{1}{2} \sum 2 m_c = 5000 = \#$ edifici tot