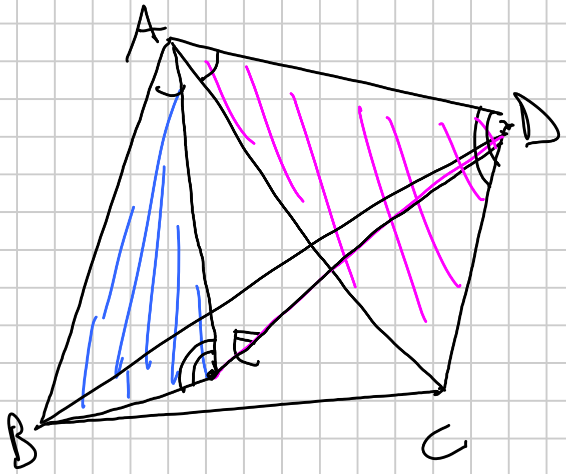
TEOREMI di TOLOMEO, CEVA, MENELAO

Titolo nota 06/12/2009

DISUGUAGLIANZA DI TOLOMEO



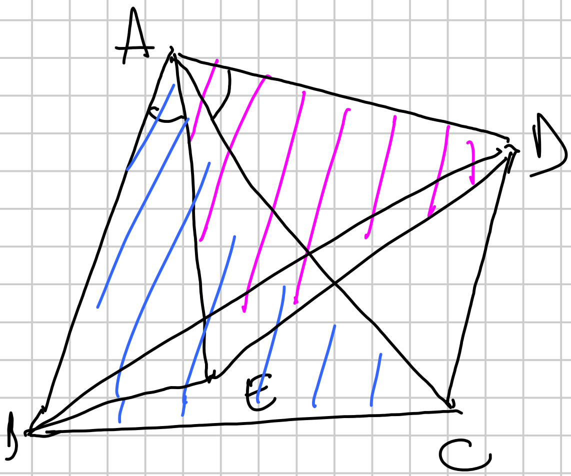
$$AB\cdot CD + BC\cdot AD \geq AC\cdot BD$$

vale l'uguaglianza ⟺ ABCD è ciclico

$B\hat{A}E = C\hat{A}D$ $\quad B\hat{E}A = C\hat{D}A$

Considero $A\overset{\triangle}{B}E$ e $A\overset{\triangle}{C}D$ sono simili!

$$\frac{BE}{CD} = \frac{AB}{AC} \qquad \boxed{BE = \frac{AB\cdot CD}{AC}}$$



Considero $A\overset{\triangle}{B}E$ e $A\overset{\triangle}{E}D$

$$\frac{AB}{AE} = \frac{AC}{AD} \qquad B\hat{A}C = E\hat{A}D$$

→ sono simili

$$\frac{ED}{BC} = \frac{AD}{AC} \rightarrow \boxed{ED = \frac{AD\cdot BC}{AC}}$$

Disug. triangolare su BED

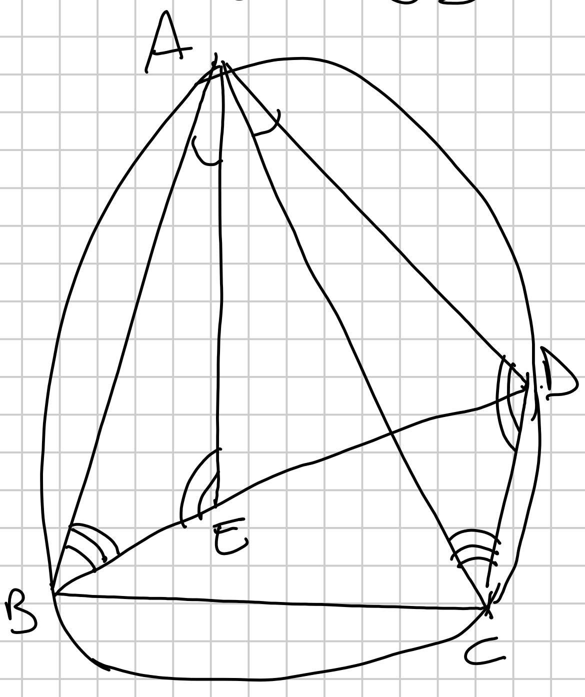
$$BD \leq BE + ED$$

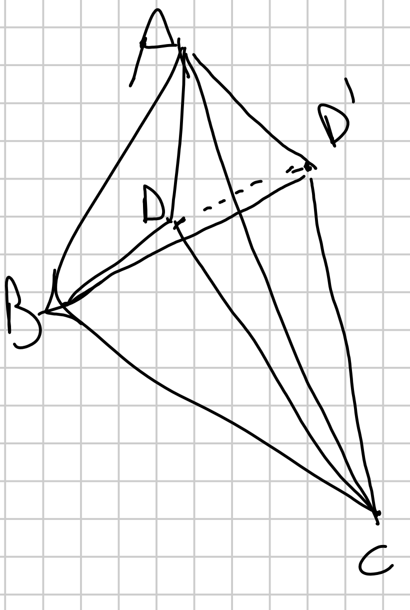
$$BD \leq \frac{AB \cdot CD}{AC} + \frac{AD \cdot BC}{AC}$$

$$AB \cdot CD + AD \cdot BC \geq BD \cdot AC$$

Vale l'uguaglianza $\iff E \in BD$

$E \in BD \iff ABCD$ è ciclico





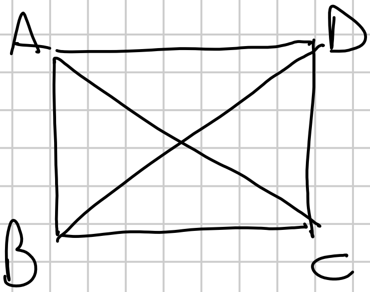
$$AB \cdot DC + BC \cdot AD =$$

$$= AB \cdot D'C + BC \cdot AD' \geq AC \cdot BD' >$$

$$AC \cdot BD$$

APPLICAZIONI

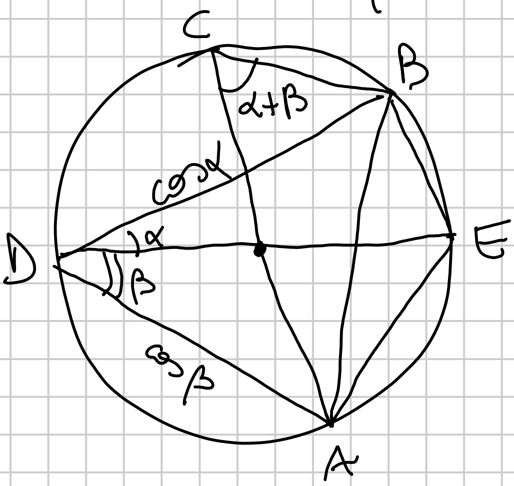
1)



$$AB \cdot DC + AD \cdot BC = AC \cdot BD$$

$$AB^2 + BC^2 = AC^2$$

2) $\operatorname{sen}(\alpha+\beta) = \operatorname{sen}\alpha \cos\beta + \operatorname{sen}\beta \cos\alpha$



DIAMETRO UNITARIO

$AE = \operatorname{sen}\beta$

$BE = \operatorname{sen}\alpha$

$AB = \operatorname{sen}(\alpha+\beta)$

perché $D\hat{B}E$ e $D\hat{A}C$ rett.

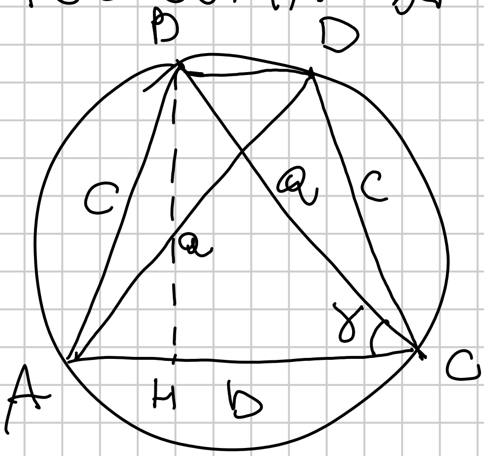
CBA tr. rett.

DBEA è quad ciclico

$$DB \cdot EA + DA \cdot BE = DE \cdot BA$$

$$\cos\alpha \operatorname{sen}\beta + \cos\beta \operatorname{sen}\alpha = 1 \cdot \operatorname{sen}(\alpha+\beta)$$

TEOREMA di CARNOT



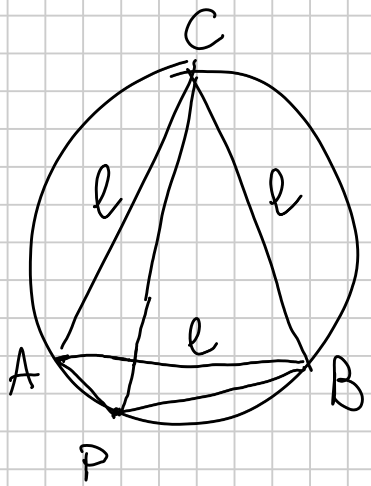
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$AH = AC - HC =$$
$$= b - a\cos\gamma$$

$$BD = AC - 2AH = b - 2(b - a\cos\gamma) =$$
$$= 2a\cos\gamma - b$$

$$AC \cdot BD + AB \cdot DC = AD \cdot BC$$
$$b(2a\cos\gamma - b) + c \cdot c = a \cdot a$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

es)



$\overset{\triangle}{ABC}$ equilatero

$$PA + PB = PC$$

$$AP \cdot CB + AC \cdot PB = AB \cdot PC$$
$$AP \cdot \ell + \ell \cdot PB = \ell \cdot PC$$ da cui la tesi

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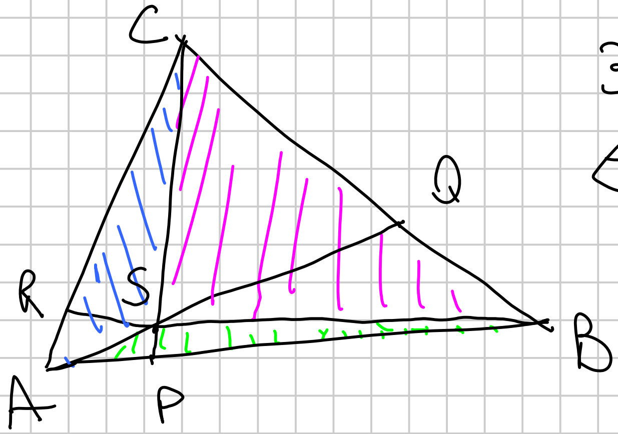
TEOREMA di CEVA



3 ceviane sono concorrenti

$$\Leftrightarrow \quad \frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RA} = 1$$

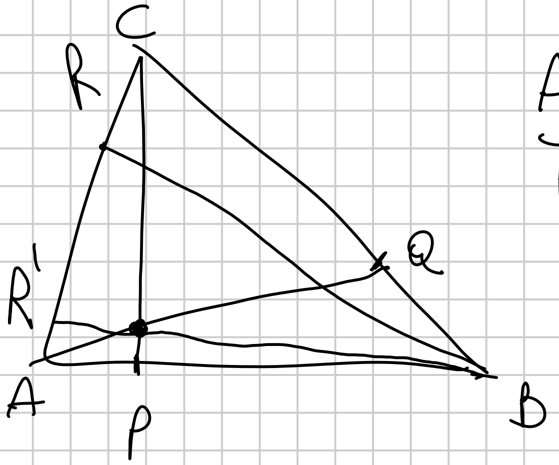
$\boxed{\Rightarrow}$ $$\frac{AP}{PB} = \frac{\text{area}(ASP)}{\text{area}(PSB)} = \frac{\text{area}(ACP)}{\text{area}(PCB)} = \frac{\text{area}(ASC)}{\text{area}(CSB)}$$

$$\frac{BQ}{QC} = \frac{\text{area}(ASB)}{\text{area}(ASC)}$$

$$\frac{CR}{RA} = \frac{\text{area}(CSB)}{\text{area}(ASB)}$$

moltiplicando i secondi membri si semplifica da cui la tesi

(⇐)



$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RA} = 1$$

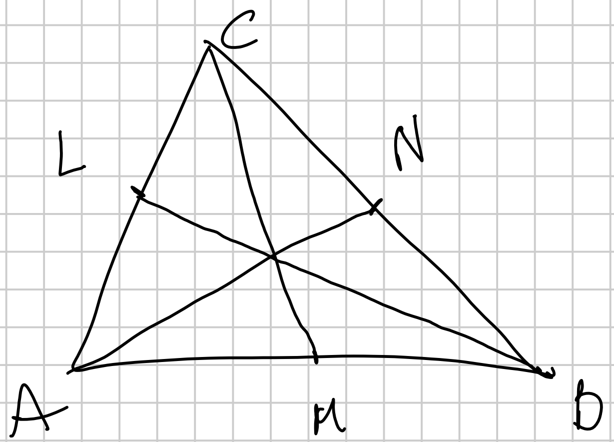
considero R' / BR', AQ, CP siano concorrenti

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR'}{R'A} = 1$$

$$\frac{CR}{RA} = \frac{CR'}{R'A} \qquad R \equiv R'$$

APPLICAZIONI

1) MEDIANE CONCORRENTI



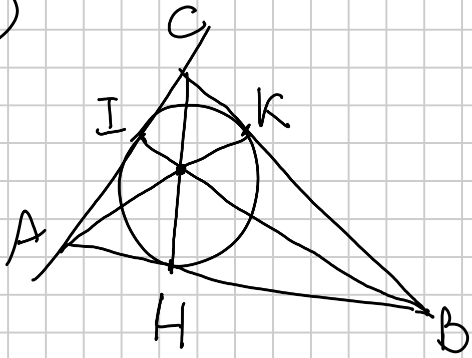
$$\frac{AM}{MB} \cdot \frac{BN}{NC} \cdot \frac{CL}{LA} =$$

$$= 1 \cdot 1 \cdot 1 = 1$$

$\Rightarrow$ sono concorrenti

es

PUNTO di GERGONNE



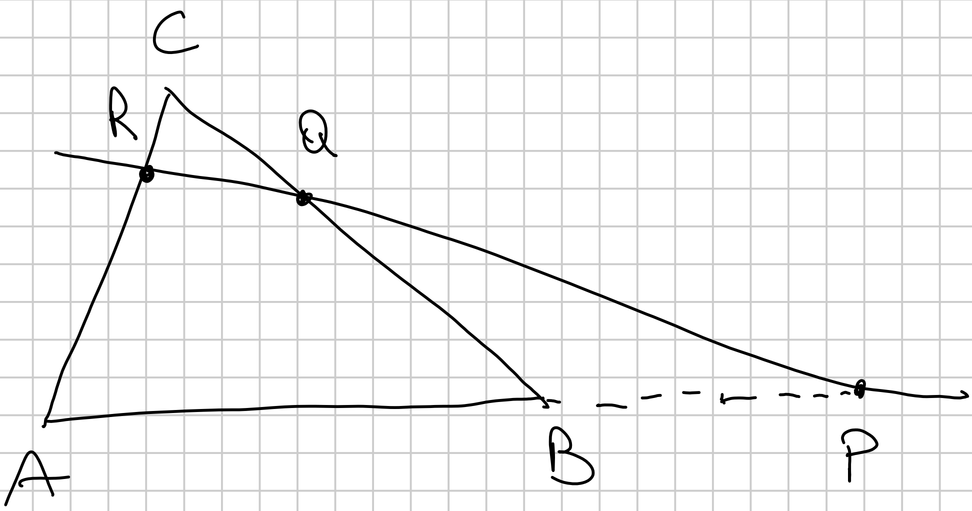
$AI = AH$

$CI = CK$

$BK = BH$

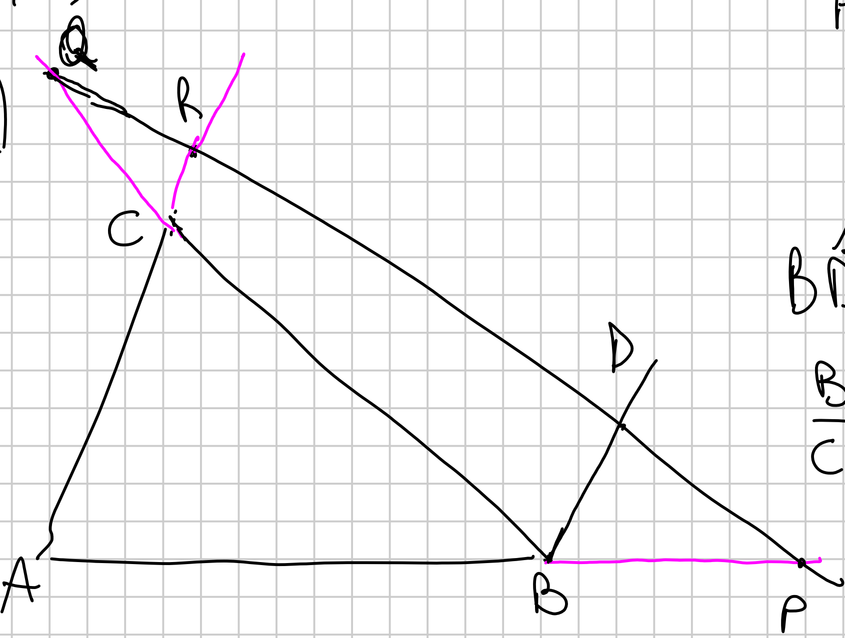
$$\frac{AH}{HB}\cdot\frac{BK}{KC}\cdot\frac{CI}{IA}=\frac{AH}{\cancel{HB}}\cdot\frac{\cancel{BH}}{\cancel{CK}}\cdot\frac{CK}{\cancel{AK}}=1$$

TEOREMA di MENELAO



R, Q, P sono allineati $\Leftrightarrow$ $\frac{AP}{PB}\cdot\frac{BQ}{QC}\cdot\frac{CR}{RA}=-1$

$\Rightarrow$



$BD \parallel AC$

$B\hat{D}Q$ simile $C\hat{R}Q$

$$\frac{BD}{CR}=\frac{QB}{QC}\qquad \boxed{BD=\frac{CR\cdot QB}{QC}}$$

$\overset{\triangle}{ARP}$ simile $\overset{\triangle}{BDP}$

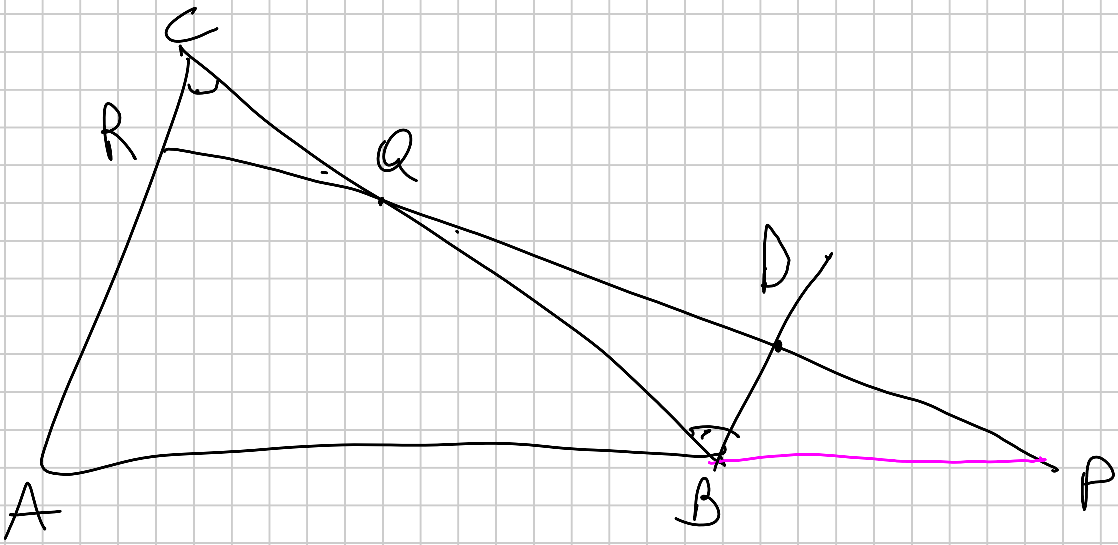
$$\frac{BD}{AR} = \frac{BP}{AP} \qquad \boxed{BD = \frac{BP \cdot AR}{AP}}$$

$$\frac{CR \cdot QB}{QC} = \frac{BP \cdot AR}{AP}$$

$$\frac{AP}{BP} \cdot \frac{QB}{QC} \cdot \frac{CR}{AR} = 1$$

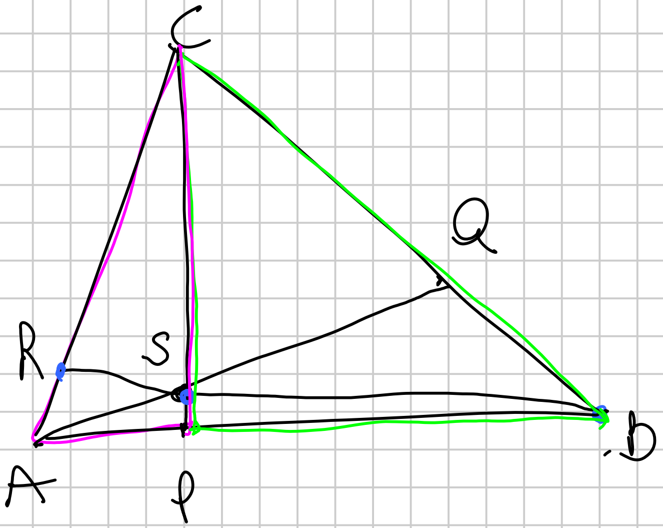
$$\frac{AP}{-PB} \cdot \frac{-BQ}{QC} \cdot \frac{CR}{-RA} = 1$$

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RA} = -1$$



$\boxed{\Leftarrow}$ per assurdo...

CEVA DIMOSTRATO con MENELAO



triangolo $A\overset{\triangle}{P}C$ e i punti R, S, B

$$\left\{\begin{array}{l}\dfrac{AB}{BP}\cdot\dfrac{PS}{SC}\cdot\dfrac{CR}{RA}=-1\\[2ex] \text{triangolo } C\overset{\triangle}{P}B \text{ e i punti } S, Q, A\\[1ex] \dfrac{PA}{AB}\cdot\dfrac{BQ}{QC}\cdot\dfrac{CS}{SP}=-1\end{array}\right.$$

$$\frac{\cancel{AB}}{BP}\cdot\frac{\cancel{PS}}{\cancel{SC}}\cdot\frac{CR}{RA}\cdot\frac{PA}{\cancel{AB}}\cdot\frac{BQ}{QC}\cdot\frac{\overset{-S\cancel{C}}{CS}}{\underset{-(P\cancel{S})}{SP}}=(-1)(-1)$$

$-(AP)$

$$\frac{PA}{BP}\cdot\frac{BQ}{QC}\cdot\frac{CR}{RA}=1 \longrightarrow \text{CEVA}$$

$-(PB)$