ELLISSOIDE

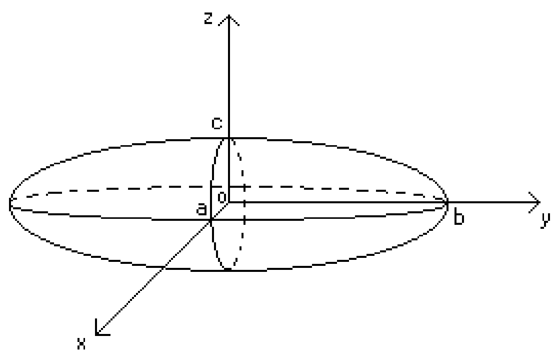
INTRODUZIONE

In geometria, per **ellissoide** si intende il tipo di quadrica che costituisce l'analogo tridimensionale della ellisse nelle due dimensioni.

L'equazione dell'ellissoide standard in un sistema di coordinate cartesiane Oxyz è:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \ .$$

dove a, b e c sono numeri reali positivi fissati che determinano la forma dell'ellissoide.



Se due di questi numeri sono uguali, l'ellissoide si dice sferoide o ellissoide di rotazione; se tutti e tre sono uguali, abbiamo una sfera.

Se ci limitiamo a considerare le possibilità consentite da $a \geq b \geq c$, abbiamo la seguente casistica:

•$a > b > c$, si ha un ellissoide scaleno;

• $a > b = c$, si ha uno sferoide prolato (a forma di sigaro);

•$a = b > c$, si ha uno sferoide oblato (a forma di lenticchia);

•$a \geq b > c = 0$, si ha un ellissoide piatto (due ellissi incollate);

• $a = b = c$, si ha una sfera, come già segnalato.

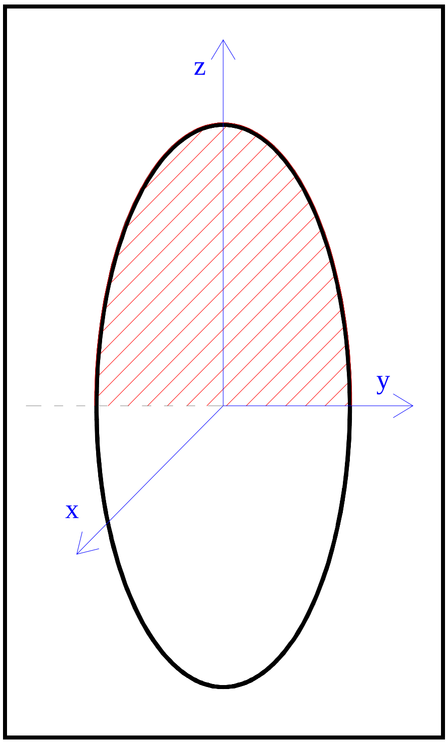
Consideriamo un ***ellissoide di rotazione*** intorno all'asse z,

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1$$

e prendiamo in considerazione la varietà differenziabile Q che consiste nella parte superiore aperta rispetto all'asse z.

L’equazione della sottovarietà Q è dunque

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1\ , \qquad\qquad 0<z\le c\,.$$



SPAZIO AMBIENTE

Sistemi di coordinate

Partiamo da un sistema di coordinate cartesiano x, y, z , poi passiamo ad un sistema di *coordinate cilindriche* ρ, φ, z ed infine arriviamo ad un *sistema di coordinate adattato* φ, z, f .

Ricordiamo che il sistema di coordinate ρ, φ, z ha una singolarità sostanzialmente patologica lungo l’asse z. Nello studio di tutte le formule successive, va escluso l’asse z.

Per studiare i fenomeni geometrici e fisici che avvengono lungo l’asse z, dovremo verificare caso per caso che questi abbiano un limite ben definito quando ci avviciniamo all’asse z.

Definiamo la funzione

$$f=\frac{\rho^2}{a^2}+\frac{z^2}{c^2}\ .$$

La sottovarietà Q è caratterizzata dal vincolo

$$f = 1\,, \qquad 0 < z \leq c\,.$$

Proposizione 1 – La terna φ, z, f è un <u>*sistema di coordinate*</u> .

-Dimostrazione

Una condizione sufficiente è che la matrice Jacobiana

$$J = \begin{pmatrix} \frac{\partial \varphi}{\partial \varphi} & \frac{\partial \varphi}{\partial z} & \frac{\partial \varphi}{\partial \rho} \\ \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} & \frac{\partial z}{\partial \rho} \\ \frac{\partial f}{\partial \varphi} & \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \rho} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial \varphi} & \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \rho} \end{pmatrix}\,,$$

abbia $rango = 3$ su Q.

A tale scopo è sufficiente dimostrare che $\frac{\partial f}{\partial \rho} \neq 0$ su Q.

In effetti, abbiamo:

$$\frac{\partial f}{\partial \rho} = \frac{2\rho}{a^2} \neq 0 \text{ su Q (escluso il vertice)}\ .$$

Pur essendo il vertice un punto singolare rispetto alle coordinate ρ, φ, z, nello stesso, noti $z = c, f = 1$, possiamo ricavarci $\rho = 0$ (quindi questo non dipende da φ).

Sulla nostra sottovarietà Q, la funzione f è definita dappertutto. *QED*

Proposizione 2 – Il sistema di coordinate φ, z, f è un <u>*sistema di coordinate adattato*</u>.

-Dimostrazione

Il vincolo è caratterizzato da $f = 1$ e $df \neq 0$ su Q. *QED*

Proposizione 3 - La coppia φ, z è un sistema di *coordinate lagrangiane* ed f è la *coordinata vincolare*. □

Funzione metrica

Proposizione 4 – La funzione metrica espressa in coordinate cartesiane è data da:

$$G = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \square$$

Proposizione 5 – La funzione metrica in coordinate cilindriche è data da:

$$G = \frac{1}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right).$$

-Dimostrazione -

Tenendo conto delle seguenti funzioni di transizione

$$\begin{cases} x = \rho \ cos\varphi \\ y = \rho \ sen\varphi \\ z = z \end{cases} \qquad \begin{cases} \dot{x} = \dot{\rho} \ cos\varphi - \rho \ sen\varphi \ \dot{\varphi} \\ \dot{y} = \dot{\rho} \ sen\varphi + \rho \ cos\varphi \ \dot{\varphi} \\ \dot{z} = \dot{z} \end{cases} ,$$

otteniamo la seguente espressione

$$G = \frac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) \ . \ QED$$

Proposizione 6 – La funzione metrica in coordinate adattate è data da:

$$G = \frac{1}{2}\left\{\frac{1}{4}\cdot\frac{a^2}{(f-\frac{z^2}{c^2})}\ \dot{f}^2 - \left[\frac{a^2\cdot z}{c^2}\cdot\frac{1}{(f-\frac{z^2}{c^2})}\right]\dot{f}\dot{z} + a^2\left(f-\frac{z^2}{c^2}\right)\dot{\varphi}^2 + \left[\frac{a^2\cdot z^2}{c^4}\cdot\frac{1}{(f-\frac{z^2}{c^2})}+1\right]\dot{z}^2\right\}.$$

-Dimostrazione –

Dobbiamo ricavarci ρ *e* $\dot{\rho}$ dalla relazione $f = \frac{\rho^2}{a^2} + \frac{z^2}{c^2}$.

Abbiamo

$$\rho = a\sqrt{f - \frac{z^2}{c^2}}$$

e quindi :

$$\dot{\rho} = \frac{\partial\rho}{\partial f}\dot{f} + \frac{\partial\rho}{\partial z}\dot{z} = \frac{a}{2\cdot\sqrt{f-\frac{z^2}{c^2}}}\ \dot{f} - \frac{a\cdot z}{c^2}\cdot\frac{1}{\sqrt{f-\frac{z^2}{c^2}}}\ \dot{z} \ .$$

Quindi sostituendo nell'espressione scritta precedentemente, si ottiene

$$G = \frac{1}{2}(\dot{\rho^2} + \rho^2\dot{\varphi^2} + \dot{z^2}) =$$

$$= \frac{1}{2}\left\{\frac{1}{4}\frac{a^2}{(f - \frac{z^2}{c^2})}\dot{f^2} - \left[\frac{a^2 \cdot z}{c^2} \cdot \frac{1}{(f - \frac{z^2}{c^2})}\right]\dot{f}\dot{z} + a^2\left(f - \frac{z^2}{c^2}\right)\dot{\varphi^2} + \left[\frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{(f - \frac{z^2}{c^2})} + 1\right]\dot{z^2}\right\}. QED$$

Matrice della metrica

Proposizione 7 - La matrice della metrica covariante nel sistema di coordinate adattato è

$$\left(g_{ij}\right) = \begin{pmatrix} \frac{1}{4}\cdot\frac{a^2}{(f-\frac{z^2}{c^2})} & 0 & -\frac{a^2\cdot z}{2\,c^2}\cdot\frac{1}{(f-\frac{z^2}{c^2})} \\ 0 & a^2\left(f-\frac{z^2}{c^2}\right) & 0 \\ -\frac{a^2\cdot z}{2\,c^2}\cdot\frac{1}{(f-\frac{z^2}{c^2})} & 0 & \frac{a^2\cdot z^2}{c^4}\cdot\frac{1}{(f-\frac{z^2}{c^2})}+1 \end{pmatrix}. \square$$

Proposizione 8 - La matrice della metrica controvariante nel sistema di coordinate adattato è

$$(g^{hk}) = \begin{pmatrix} \frac{4z^2}{c^4}+\frac{4}{a^2}\left(f-\frac{z^2}{c^2}\right) & 0 & \frac{2z}{c^2} \\ 0 & \frac{1}{a^2}\frac{1}{\left(f-\frac{z^2}{c^2}\right)} & 0 \\ \frac{2z}{c^2} & 0 & 1 \end{pmatrix}.$$

Dimostrazione -

Data una matrice $(A) \in \mathcal{R}^{m\times n}$, con $m = n = 3$, invertibile:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

la sua inversa è la seguente

$$\frac{1}{\det(A)}\begin{pmatrix} +\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} & +\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \\ -\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} & +\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \\ +\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} & +\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \end{pmatrix},$$

dove

$$\begin{vmatrix} A_{ij} & A_{kl} \\ A_{mn} & A_{op} \end{vmatrix} = \det\begin{pmatrix} A_{ij} & A_{kl} \\ A_{mn} & A_{op} \end{pmatrix}.$$

sono i complementi algebrici di (A).

Nel nostro caso abbiamo che il determinante $[g_{ij}]$ della matrice della metrica è

$$[g_{ij}] = det(g_{ij}) = \frac{1}{4}\cdot a^4 \neq 0$$

La matrice della metrica è quindi invertibile.

Perciò abbiamo

$$(g^{hk}) = (g_{ij})^{-1} = \frac{4}{a^4}\begin{pmatrix} \frac{a^4 z}{c^4} + a^2\left(f - \frac{z^2}{c^2}\right) & 0 & \frac{a^4 z}{2c^2} \\ 0 & \frac{1}{4}\frac{a^2}{\left(f - \frac{z^2}{c^2}\right)} & 0 \\ \frac{a^4 z}{2c^2} & 0 & \frac{1}{4}a^4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4z^2}{c^4} + \frac{4}{a^2}\left(f - \frac{z^2}{c^2}\right) & 0 & \frac{2z}{c^2} \\ 0 & \frac{1}{a^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)} & 0 \\ \frac{2z}{c^2} & 0 & 1 \end{pmatrix}. \quad QED$$

Per controllare la correttezza dei calcoli, verifichiamo l'uguaglianza

$$(g_{ij})(g^{hk}) = (I) \ .$$

In effetti, abbiamo

$$\begin{pmatrix} \frac{1}{4}\cdot\frac{a^2}{(f-\frac{z^2}{c^2})} & 0 & -\frac{a^2\cdot z}{2\,c^2}\cdot\frac{1}{(f-\frac{z^2}{c^2})} \\ 0 & a^2\left(f-\frac{z^2}{c^2}\right) & 0 \\ -\frac{a^2\cdot z}{2\,c^2}\cdot\frac{1}{(f-\frac{z^2}{c^2})} & 0 & \frac{a^2\cdot z^2}{c^4}\cdot\frac{1}{(f-\frac{z^2}{c^2})}+1 \end{pmatrix}\begin{pmatrix} \frac{4z^2}{c^4}+\frac{4}{a^2}\left(f-\frac{z^2}{c^2}\right) & 0 & \frac{2z}{c^2} \\ 0 & \frac{1}{a^2}\frac{1}{\left(f-\frac{z^2}{c^2}\right)} & 0 \\ \frac{2z}{c^2} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Inoltre verifichiamo l'uguaglianza

$$[g_{ij}] = \frac{1}{[g^{hk}]} \ .$$

In effetti, abbiamo

$$[g_{ij}] = \frac{1}{4}\cdot a^4 \ , \qquad\qquad [g^{hk}] = \frac{4}{a^4} \ .$$

Tensore metrico

Proposizione 9 - L’espressione tensoriale della metrica covariante è

$$g = \left(\frac{1}{4} \cdot \frac{a^2}{\left(f - \frac{z^2}{c^2}\right)} \right) df \otimes df + \left[a^2 \left(f - \frac{z^2}{c^2} \right) \right] d\varphi \otimes d\varphi$$

$$- \left[\frac{a^2 \cdot z}{2\, c^2} \cdot \frac{1}{\left(f - \frac{z^2}{c^2}\right)} \right] (df \otimes dz + dz \otimes df)$$

$$+ \left[\frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{(f - \frac{z^2}{c^2})} + 1 \right] dz \otimes dz \ . \square$$

Proposizione 10 - L’espressione tensoriale della metrica controvariante è

$$\bar{g} = \left(\frac{4z^2}{c^4} + \frac{4}{a^2} \left(f - \frac{z^2}{c^2} \right) \right) (\partial f \otimes \partial f) - \left(\frac{2z}{c^2} \right) (\partial f \otimes \partial z + \partial z \otimes \partial f) +$$

$$+ \left[\frac{1}{a^2} \frac{1}{\left(f - \frac{z^2}{c^2}\right)} \right] \ (\partial \varphi \otimes \partial \varphi) + (\partial z \otimes \partial z) \ . \ \square$$

Forma volume η

***Proposizione 11* -** La forma volume (in forma covariante) espressa in coordinate adattate è

$$\eta = \frac{1}{2} \cdot a^2 df \wedge d\varphi \wedge dz \,.$$

***Dimostrazione* -**

Abbiamo

$$\eta = \sqrt{det\left(g_{ij}\right)}\; df \wedge d\varphi \wedge dz \;\; ,$$

dove

$$det\left(g_{ij}\right) = \left[g_{ij}\right] = \frac{1}{4} \cdot a^4 \;\; .$$

Quindi

$$\eta = \sqrt{\frac{1}{4} \cdot a^4}\;\; df \wedge d\varphi \wedge dz = \frac{1}{2} \cdot a^2 df \wedge d\varphi \wedge dz \;\; .QED$$

***Proposizione 12* -** La forma volume (in forma controvariante) espressa in coordinate adattate è

$$\eta = \frac{2}{a^2} df \wedge d\varphi \wedge dz \quad .$$

***Dimostrazione* -**

Abbiamo

$$\eta = \sqrt{det(g^{hk})}\; df \wedge d\varphi \wedge dz \;\; ,$$

dove

$$det(g^{hk}) = [g^{hk}] = \frac{4}{a^4} \;\; .$$

Quindi

$$\eta = \sqrt{\frac{4}{a^4}}\;\; df \wedge d\varphi \wedge dz = \frac{2}{a^2} df \wedge d\varphi \wedge dz \;\; .QED$$

Accelerazione covariante

Proposizione 13 – L’accelerazione covariante è

$$a_f = \frac{a^2}{4\left(f - \frac{z^2}{c^2}\right)}\ddot{f} - \frac{a^2}{8\left(f - \frac{z^2}{c^2}\right)^2}\dot{f}^2 + \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)^2}\dot{f} - \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)}\ddot{z}$$

$$-\frac{\left[a^2c^2\left(f - \frac{z^2}{c^2}\right) + a^2z^2\right]}{2c^4\left(f - \frac{z^2}{c^2}\right)^2}\dot{z}^2 - \frac{a^2}{2}\dot{\varphi}^2 \ ,$$

$$a_\varphi = \frac{1}{2}\left\{2a^2\dot{f}\dot{\varphi} - \frac{4a^2\dot{z}}{c^2}\dot{\varphi}\dot{z} + \dot{2}a^2\left(f - \frac{z^2}{c^2}\right)\ddot{\varphi}\right\} \ ,$$

$$a_z = -\frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)}\ddot{f} + \frac{a^2 z}{4c^2\left(f - \frac{z^2}{c^2}\right)^2}\dot{f}^2 + \left[\frac{a^2z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} + \frac{1}{2}\right]\ddot{z} - \frac{a^2z^2}{c^4\left(f - \frac{z^2}{c^2}\right)^2}\dot{f}\dot{z} + \frac{a^2 z}{c^2}\dot{\varphi}^2$$

$$+\left[\frac{a^2c^2z\left(f - \frac{z^2}{c^2}\right) + a^2z^3}{c^6\left(f - \frac{z^2}{c^2}\right)^2}\right]\dot{z}^2 .$$

Dimostrazione -

Secondo le formule di Lagrange, abbiamo

$$a_i = \frac{d}{dt}\frac{\partial G}{\partial \dot{x}^i} - \frac{\partial G}{\partial x^i} \ ,$$

quindi

- $a_f = \frac{d}{dt}\,\frac{\partial G}{\partial \dot{f}} - \frac{\partial G}{\partial f}$,

dove

$$\frac{\partial G}{\partial \dot{f}} = \frac{1}{2}\left\{\frac{1}{2}\frac{a^2}{\left(f - \frac{z^2}{c^2}\right)}f - \left[\frac{a^2 z}{c^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)}\right]\dot{z}\right\} ,$$

$$\frac{d}{dt}\frac{\partial G}{\partial \dot f} = \frac{a^2}{4\left(f-\frac{z^2}{c^2}\right)}\ddot f - \frac{a^2}{4\left(f-\frac{z^2}{c^2}\right)}\dot f^2 + \frac{a^2 z}{c^2\left(f-\frac{z^2}{c^2}\right)^2}\dot f \dot z - \frac{a^2 z}{2c^2\left(f-\frac{z^2}{c^2}\right)}\ddot z$$

$$-\frac{\left(a^2c^2\left(f-\frac{z^2}{c^2}\right)+2a^2z^2\right)}{2c^4\left(f-\frac{z^2}{c^2}\right)^2}\dot z^2 ,$$

$$\frac{\partial G}{\partial f} = -\frac{a^2}{8\left(f-\frac{z^2}{c^2}\right)^2}\dot f^2 + \frac{a^2c^2 z}{2c^4\left(f-\frac{z^2}{c^2}\right)^2}\dot f \dot z + \frac{a^2}{2}\dot\varphi^2 - \frac{a^2c^4z^2}{2c^8\left(f-\frac{z^2}{c^2}\right)^2}\dot z^2 ,$$

$$a_f = \frac{a^2}{4\left(f-\frac{z^2}{c^2}\right)}\ddot f - \frac{a^2}{8\left(f-\frac{z^2}{c^2}\right)^2}\dot f^2 + \frac{a^2 z}{2c^2\left(f-\frac{z^2}{c^2}\right)^2}\dot f - \frac{a^2 z}{2c^2\left(f-\frac{z^2}{c^2}\right)}\ddot z$$

$$-\frac{\left[a^2c^2\left(f-\frac{z^2}{c^2}\right)+a^2z^2\right]}{2c^4\left(f-\frac{z^2}{c^2}\right)^2}\dot z^2 - \frac{a^2}{2}\dot\varphi^2 .$$

- $a_\varphi = \frac{d}{dt}\frac{\partial G}{\partial \dot\varphi} - \frac{\partial G}{\partial \varphi}$,

dove

$$\frac{\partial G}{\partial \dot\varphi} = \frac{1}{2}\left\{2a^2\left(f-\frac{z^2}{c^2}\right)\right\}\dot\varphi ,$$

$$\frac{d}{dt}\frac{\partial G}{\partial \dot\varphi} = \frac{1}{2}\left\{2a^2\dot f\dot\varphi - \frac{4a^2\dot z}{c^2}\dot\varphi z + 2a^2\left(f-\frac{z^2}{c^2}\right)\ddot\varphi\right\} ,$$

$$\frac{\partial G}{\partial \varphi} = 0 ,$$

$$a_\varphi = \frac{1}{2}\left\{2a^2\dot f\dot\varphi - \frac{4a^2\dot z}{c^2}\dot\varphi \dot z + 2a^2\left(f-\frac{z^2}{c^2}\right)\ddot\varphi\right\} .$$

- $a_z = \frac{d}{dt}\frac{\partial G}{\partial \dot z} - \frac{\partial G}{\partial z}$,

dove

$$\frac{\partial G}{\partial \dot z} = \frac{1}{2}\left\{-\frac{a^2 z}{c^2}\frac{1}{\left(f-\frac{z^2}{c^2}\right)}\dot f + \left[\frac{2a^2z^2}{c^4}\frac{1}{\left(f-\frac{z^2}{c^2}\right)}+1\right]\dot z\right\} ,$$

$$\frac{d}{dt}\,\frac{\partial G}{\partial \dot{z}} = -\frac{a^2 z}{2c^2\left(f-\frac{z^2}{c^2}\right)}\ddot{f} + \frac{a^2 z}{2c^2\left(f-\frac{z^2}{c^2}\right)^2}\dot{f}^2 + \left[\frac{a^2 z^2}{c^4\left(f-\frac{z^2}{c^2}\right)} + \frac{1}{2}\right]\ddot{z}$$

$$-\frac{\left(a^2c^2\left(f-\frac{z^2}{c^2}\right)+4a^2z^2\right)}{2c^4\left(f-\frac{z^2}{c^2}\right)^2}\dot{f}\dot{z} + \frac{2a^2c^2z\left(f-\frac{z^2}{c^2}\right)+2a^2z^3}{c^6\left(f-\frac{z^2}{c^2}\right)^2}\dot{z}^2 \ ,$$

$$\frac{\partial G}{\partial z} = \frac{1}{2}\left\{\frac{a^2 z}{2c^2}\frac{1}{\left(f-\frac{z^2}{c^2}\right)^2}\dot{f}^2 - \left[\frac{a^2c^2\left(f-\frac{z^2}{c^2}\right)+2a^2z^2}{c^4\left(f-\frac{z^2}{c^2}\right)^2}\right]\dot{f}\dot{z} - \frac{2a^2z}{c^2}\dot{\varphi}^2\right.$$

$$\left. + \left[\frac{a^2c^4z\left(f-\frac{z^2}{c^2}\right)+a^2c^2z^3}{c^8\left(f-\frac{z^2}{c^2}\right)^2}\right]\dot{z}^2\right\} \ ,$$

$$a_z = -\frac{a^2 z}{2c^2\left(f-\frac{z^2}{c^2}\right)}\ddot{f} + \frac{a^2 z}{4c^2\left(f-\frac{z^2}{c^2}\right)^2}\dot{f}^2 + \left[\frac{a^2z^2}{c^4\left(f-\frac{z^2}{c^2}\right)}+\frac{1}{2}\right]\ddot{z} - \frac{a^2z^2}{c^4\left(f-\frac{z^2}{c^2}\right)^2}\dot{f}\dot{z} + \frac{a^2z}{c^2}\dot{\varphi}^2$$

$$+\left[\frac{a^2c^2z\left(f-\frac{z^2}{c^2}\right)+a^2z^3}{c^6\left(f-\frac{z^2}{c^2}\right)^2}\right]\dot{z}^2 \, . QED$$

Accelerazione controvariante

Proposizione 14 – L'accelerazione controvariante è

$$a^f = \ddot{f} - \frac{1}{2\left(f - \frac{z^2}{c^2}\right)}\dot{f}^2 + \frac{2z}{c^2\left(f - \frac{z^2}{c^2}\right)}\dot{f}\dot{z} - \frac{2\left[c^2\left(f - \frac{z^2}{c^2}\right) + z^2\right]}{c^4\left(f - \frac{z^2}{c^2}\right)}\dot{z}^2 - 2\left(f - \frac{z^2}{c^2}\right)\dot{\varphi}^2 \ ,$$

$$a^\varphi = \ddot{\varphi} - \frac{2z}{c^2\left(f - \frac{z^2}{c^2}\right)}\dot{\varphi}\dot{z} + \frac{1}{\left(f - \frac{z^2}{c^2}\right)}\dot{f}\dot{\varphi} \ ,$$

$$a^z = \ddot{z} \ .$$

Dimostrazione -

Abbiamo

$$a^j = g^{ij}\, a_i \ .$$

Calcolo di

- $a^f = g^{ff} a_f + g^{zf} a_z$;

le seguenti uguaglianze

$$g^{ff} = \frac{4z^2}{c^4} + \frac{4}{a^2}\left(f - \frac{z^2}{c^2}\right) \ , \qquad g^{zf} = \frac{2z}{c^2} \ ,$$

$$a_f = \frac{a^2}{4\left(f - \frac{z^2}{c^2}\right)}\ddot{f} - \frac{a^2}{8\left(f - \frac{z^2}{c^2}\right)^2}\dot{f}^2 + \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)^2}\dot{f} - \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)}\ddot{z}$$
$$- \frac{\left[a^2c^2\left(f - \frac{z^2}{c^2}\right) + a^2z^2\right]}{2c^4\left(f - \frac{z^2}{c^2}\right)^2}\dot{z}^2 - \frac{a^2}{2}\dot{\varphi}^2 \ ,$$

$$a_z = -\frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)}\ddot{f} + \frac{a^2 z}{4c^2\left(f - \frac{z^2}{c^2}\right)^2}\dot{f}^2 + \left[\frac{a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} + \frac{1}{2}\right]\ddot{z} - \frac{a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)^2}\dot{f}\dot{z} + \frac{a^2 z}{c^2}\dot{\varphi}^2$$
$$+\left[\frac{a^2 c^2 z\left(f - \frac{z^2}{c^2}\right) + a^2 z^3}{c^6\left(f - \frac{z^2}{c^2}\right)^2}\right]\dot{z}^2 ,$$

implicano

$$a^f = \ddot{f} - \frac{1}{2\left(f - \frac{z^2}{c^2}\right)}\dot{f}^2 + \frac{2z}{c^2\left(f - \frac{z^2}{c^2}\right)}\dot{f}\dot{z} - \frac{2\left[c^2\left(f - \frac{z^2}{c^2}\right) + z^2\right]}{c^4\left(f - \frac{z^2}{c^2}\right)}\dot{z}^2 - 2\left(f - \frac{z^2}{c^2}\right)\dot{\varphi}^2$$

Calcolo di

- $a^\varphi = g^{\varphi\varphi} a_\varphi$;

le seguenti uguaglianze

$$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)}$$

$$a_\varphi = \frac{1}{2}\left\{2a^2\dot{f}\dot{\varphi} - \frac{4a^2\dot{z}}{c^2}\dot{\varphi}\dot{z} + \dot{2}a^2\left(f - \frac{z^2}{c^2}\right)\ddot{\varphi}\right\}$$

implicano

$$a^\varphi = \ddot{\varphi} - \frac{2z}{c^2\left(f - \frac{z^2}{c^2}\right)}\dot{\varphi}\dot{z} + \frac{1}{\left(f - \frac{z^2}{c^2}\right)}\dot{f}\dot{\varphi}$$

Calcolo di

- $a^z = g^{zz} a_z + g^{fz} a_f$;

le seguenti uguaglianze

$$g^{zz} = 1, \qquad g^{fz} = \frac{2z}{c^2}$$

$$a_z = -\frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)}\ddot{f} + \frac{a^2 z}{4c^2\left(f - \frac{z^2}{c^2}\right)^2}\dot{f}^2 + \left[\frac{a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} + \frac{1}{2}\right]\ddot{z} - \frac{a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)^2}\dot{f}\dot{z} + \frac{a^2 z}{c^2}\dot{\varphi}^2 + \left[\frac{a^2 c^2 z\left(f - \frac{z^2}{c^2}\right) + a^2 z^3}{c^6\left(f - \frac{z^2}{c^2}\right)^2}\right]\dot{z}^2$$

$$a_\varphi = \frac{1}{2}\left\{2a^2\dot{f}\dot{\varphi} - \frac{4a^2\dot{z}}{c^2}\dot{\varphi}\dot{z} + \dot{2}a^2\left(f - \frac{z^2}{c^2}\right)\ddot{\varphi}\right\}$$

implicano

$$a^z = \ddot{z}\,.\ \ QED$$

Simboli di Christoffel

Proposizione 15 – I simboli di Christoffel non nulli sono

$$\Gamma^f_{ff} = -\frac{1}{2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma^f_{\varphi f} = \Gamma^f_{f\varphi} = \frac{1}{2\left(f - \frac{z^2}{c^2}\right)}\left[\frac{-a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} - 1\right] \ ,$$

$$\Gamma^f_{\varphi\varphi} = -2\left(f - \frac{z^2}{c^2}\right) \ ,$$

$$\Gamma^\varphi_{f\varphi} = \Gamma^\varphi_{\varphi f} = \frac{1}{2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma^\varphi_{z\varphi} = \Gamma^\varphi_{\varphi z} = -\frac{z}{c^2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma^f_{zf} = \Gamma^f_{fz} = \frac{z}{c^2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma^f_{zz} = \frac{-2\left[c^2\left(f - \frac{z^2}{c^2}\right) + z^2\right]}{c^4\left(f - \frac{z^2}{c^2}\right)} \ .$$

Dimostrazione -

Applicando la formula generale

$$\boldsymbol{\Gamma^h_{ij} = \frac{1}{2} g^{hk}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)} \ ,$$

otteniamo le seguenti uguaglianze

- $\boldsymbol{\Gamma^f_{ff} = \frac{1}{2} g^{ff}\left(\partial_f g_{ff} + \partial_f g_{ff} - \partial_f g_{ff}\right) + \frac{1}{2} g^{fz}\left(\partial_f g_{fz} + \partial_f g_{fz} - \partial_z g_{ff}\right)}$.

Le seguenti uguaglianze

$$g^{ff} = \frac{4z^2}{c^4} + \frac{4}{a^2}\left(f - \frac{z^2}{c^2}\right) \ , \qquad g_{ff} = \frac{1}{4} \cdot \frac{a^2}{\left(f - \frac{z^2}{c^2}\right)} \ , \qquad g^{fz} = \frac{2z}{c^2} \ ,$$

$$g_{fz} = -\frac{a^2 \cdot z}{2\,c^2} \cdot \frac{1}{(f - \frac{z^2}{c^2})}$$

$$\partial_f g_{ff} = \frac{-a^2}{4\left(f - \frac{z^2}{c^2}\right)^2} \ , \qquad \partial_f g_{fz} = \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)^2} \ , \qquad \partial_z g_{ff} = \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)^2} \ ,$$

implicano

$$\Gamma^f_{ff} = -\frac{1}{2\left(f - \frac{z^2}{c^2}\right)} \ .$$

- $\boldsymbol{\Gamma^{\varphi}_{\varphi\varphi} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_\varphi g_{\varphi\varphi} + \partial_\varphi g_{\varphi\varphi} - \partial_\varphi g_{\varphi\varphi}\right)}$.

Le seguenti uguaglianze

$$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)} \ , \qquad g_{\varphi\varphi} = a^2\left(f - \frac{z^2}{c^2}\right) \ , \qquad \partial_\varphi g_{\varphi\varphi} = 0 \ ,$$

implicano

$$\Gamma^{\varphi}_{\varphi\varphi} = 0 \ .$$

- $\boldsymbol{\Gamma^{z}_{zz} = \frac{1}{2} g^{zz}(\partial_z g_{zz} + \partial_z g_{zz} - \partial_z g_{zz}) + \frac{1}{2} g^{zf}\left(\partial_z g_{zf} + \partial_z g_{zf} - \partial_f g_{zz}\right)}$.

Le seguenti uguaglianze

$$g^{zz} = 1 \ , \qquad g_{zz} = \frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{\left(f - \frac{z^2}{c^2}\right)} + 1 \ , \qquad g^{zf} = \frac{2z}{c^2} , \qquad g_{zf} = -\frac{a^2 \cdot z}{2\,c^2} \cdot \frac{1}{\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\partial_z g_{zz} = \frac{2a^2c^2z\left(f - \frac{z^2}{c^2}\right) + 2a^2z^3}{c^6\left(f - \frac{z^2}{c^2}\right)^2} \ , \quad \partial_z g_{zf} = \frac{-a^2c^2\left(f - \frac{z^2}{c^2}\right) - 2a^2z^2}{2c^4\left(f - \frac{z^2}{c^2}\right)^2}, \qquad \partial_f g_{zz} = \frac{-a^2z^2}{c^4\left(f - \frac{z^2}{c^2}\right)^2}$$

implicano

$$\Gamma^z_{zz} = 0 \ .$$

- $\boldsymbol{\Gamma^{f}_{\varphi f} = \Gamma^{f}_{f\varphi} = \frac{1}{2} g^{ff}\left(\partial_{\varphi} g_{ff} + \partial_{f} g_{ff} - \partial_{f} g_{\varphi f}\right) + \frac{1}{2} g^{fz}\left(\partial_{\varphi} g_{fz} + \partial_{f} g_{\varphi z} - \partial_{z} g_{\varphi z}\right)}$.

Le seguenti uguaglianze

$$g^{ff} = \frac{4z^2}{c^4} + \frac{4}{a^2}\left(f - \frac{z^2}{c^2}\right) \ , \qquad g_{ff} = \frac{1}{4} \cdot \frac{a^2}{(f - \frac{z^2}{c^2})} \ , \qquad g_{\varphi f} = 0 \ ,$$

$$g_{fz} = -\frac{a^2 \cdot z}{2\, c^2} \cdot \frac{1}{(f - \frac{z^2}{c^2})} \ , \qquad g_{\varphi z} = 0 \, , \qquad \partial_f g_{ff} = \frac{-a^2}{4\left(f - \frac{z^2}{c^2}\right)^2} \ ,$$

$$\partial_{\varphi} g_{ff} = 0 \ , \qquad \partial_{\varphi} g_{fz} = 0$$

implicano

$$\Gamma^{f}_{\varphi f} = \Gamma^{f}_{f\varphi} = \frac{1}{2\left(f - \frac{z^2}{c^2}\right)}\left[\frac{-a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} - 1\right].$$

- $\boldsymbol{\Gamma^{f}_{\varphi\varphi} = \frac{1}{2} g^{ff}\left(\partial_{\varphi} g_{\varphi f} + \partial_{\varphi} g_{\varphi f} - \partial_{f} g_{\varphi\varphi}\right) + \frac{1}{2} g^{fz}\left(\partial_{\varphi} g_{\varphi z} + \partial_{\varphi} g_{\varphi z} - \partial_{z} g_{\varphi\varphi}\right)}$.

Le seguenti uguaglianze

$$g^{ff} = \frac{4z^2}{c^4} + \frac{4}{a^2}\left(f - \frac{z^2}{c^2}\right) \ , \qquad g_{\varphi\varphi} = a^2\left(f - \frac{z^2}{c^2}\right) \ , \qquad g_{\varphi f} = g_{\varphi Z} = 0 \ ,$$

$$\partial_f g_{\varphi\varphi} = a^2 \, , \qquad \partial_z g_{\varphi\varphi} = -\frac{2a^2 z}{c^2}$$

implicano

$$\Gamma^{f}_{\varphi\varphi} = -2\left(f - \frac{z^2}{c^2}\right).$$

- $\boldsymbol{\Gamma^{\varphi}_{ff} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_f g_{f\varphi} + \partial_f g_{f\varphi} - \partial_\varphi g_{ff}\right)}$.

Le seguenti uguaglianze

$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)}$, $g_{ff} = \frac{1}{4} \cdot \frac{a^2}{(f - \frac{z^2}{c^2})}$, $g_{f\varphi} = 0$, $\partial_\varphi g_{ff} = 0$,

implicano

$$\Gamma^{\varphi}_{ff} = 0 \ .$$

- $\boldsymbol{\Gamma^{\varphi}_{f\varphi} = \Gamma^{\varphi}_{\varphi f} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_f g_{\varphi\varphi} + \partial_\varphi g_{f\varphi} - \partial_\varphi g_{f\varphi}\right)}$.

Le seguenti uguaglianze

$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)}$, $g_{\varphi\varphi} = a^2\left(f - \frac{z^2}{c^2}\right)$, $g_{f\varphi} = 0$,

$\partial_f g_{\varphi\varphi} = a^2$

implicano

$$\Gamma^{\varphi}_{f\varphi} = \Gamma^{\varphi}_{\varphi f} = \frac{1}{2\left(f - \frac{z^2}{c^2}\right)} \ .$$

- $\boldsymbol{\Gamma^{\varphi}_{z\varphi} = \Gamma^{\varphi}_{\varphi z} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_z g_{\varphi\varphi} + \partial_\varphi g_{z\varphi} - \partial_\varphi g_{z\varphi}\right)}$.

Le seguenti uguaglianze

$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)}$, $g_{\varphi\varphi} = a^2\left(f - \frac{z^2}{c^2}\right)$, $g_{z\varphi} = 0$,

$\partial_z g_{\varphi\varphi} = -\frac{2a^2 z}{c^2}$

implicano

$$\Gamma^{\varphi}_{z\varphi} = \Gamma^{\varphi}_{\varphi z} = -\frac{z}{c^2\left(f - \frac{z^2}{c^2}\right)} \ .$$

- $\boldsymbol{\Gamma^{\varphi}_{zz} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_z g_{z\varphi} + \partial_z g_{z\varphi} - \partial_\varphi g_{zz}\right)}$.

Le seguenti uguaglianze

$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)}$, $\quad g_{zz} = \frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{(f - \frac{z^2}{c^2})} + 1$, $\quad g_{z\varphi} = 0$,

$\partial_\varphi g_{zz} = 0$

implicano

$$\Gamma^{\varphi}_{zz} = 0 \ .$$

- $\boldsymbol{\Gamma^{z}_{\varphi\varphi} = \frac{1}{2} g^{zf}\left(\partial_\varphi g_{\varphi f} + \partial_\varphi g_{\varphi f} - \partial_f g_{\varphi\varphi}\right) + \frac{1}{2} g^{zz}\left(\partial_\varphi g_{\varphi z} + \partial_\varphi g_{\varphi z} - \partial_z g_{\varphi\varphi}\right)}$.

Le seguenti uguaglianze

$g^{zf} = \frac{2z}{c^2}$, $\quad g_{\varphi\varphi} = a^2\left(f - \frac{z^2}{c^2}\right)$, $\quad g^{zz} = 1$, $\quad g_{\varphi f} = g_{\varphi z} = 0$,

$\partial_z g_{\varphi\varphi} = -\frac{2a^2 z}{c^2}$, $\quad \partial_f g_{\varphi\varphi} = a^2$

implicano

$$\Gamma^{z}_{\varphi\varphi} = 0 \, .$$

- $\boldsymbol{\Gamma^{z}_{\varphi z} = \Gamma^{z}_{z\varphi} = \frac{1}{2} g^{zf}\left(\partial_\varphi g_{zf} + \partial_z g_{\varphi f} - \partial_f g_{\varphi z}\right) + \frac{1}{2} g^{zz}\left(\partial_\varphi g_{zz} + \partial_z g_{\varphi z} - \partial_z g_{\varphi z}\right)}$.

Le seguenti uguaglianze

$g^{zf} = \frac{2z}{c^2}$, $\quad g_{zf} = -\frac{a^2 \cdot z}{2\, c^2} \cdot \frac{1}{(f - \frac{z^2}{c^2})}$, $\quad g^{zz} = 1$, $\quad g_{zz} = \frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{(f - \frac{z^2}{c^2})} + 1$,

$g_{\varphi f} = g_{\varphi z} = 0$,

$\partial_\varphi g_{zf} = \partial_\varphi g_{zz} = 0$,

implicano

$$\Gamma^{z}_{\varphi z} = \Gamma^{z}_{z\varphi} = 0 \, .$$

- $\boldsymbol{\Gamma^f_{zf} = \Gamma^f_{fz} = \frac{1}{2} g^{ff}\left(\partial_z g_{ff} + \partial_f g_{zf} - \partial_f g_{zf}\right) + \frac{1}{2} g^{fz}\left(\partial_z g_{fz} + \partial_f g_{zz} - \partial_z g_{zf}\right)}$.

Le seguenti uguaglianze

$$g^{ff} = \frac{4z^2}{c^4} + \frac{4}{a^2}\left(f - \frac{z^2}{c^2}\right) \quad , \qquad g_{ff} = \frac{1}{4} \cdot \frac{a^2}{\left(f - \frac{z^2}{c^2}\right)} \quad , \qquad g^{fz} = \frac{2z}{c^2} \quad , \quad g_{zz} = \frac{a^2 \cdot z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} + 1 \ ,$$

$$\partial_z g_{ff} = \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)^2} \ , \qquad \partial_f g_{zz} = \frac{-a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)^2} \ ,$$

implicano

$$\Gamma^f_{zf} = \Gamma^f_{fz} = \frac{z}{c^2\left(f - \frac{z^2}{c^2}\right)} .$$

- $\boldsymbol{\Gamma^f_{zz} = \frac{1}{2} g^{ff}\left(\partial_z g_{zf} + \partial_z g_{zf} - \partial_f g_{zz}\right) + \frac{1}{2} g^{fz}(\partial_z g_{zz} + \partial_z g_{zz} - \partial_z g_{zz})}$.

Le seguenti uguaglianze

$$g^{ff} = \frac{4z^2}{c^4} + \frac{4}{a^2}\left(f - \frac{z^2}{c^2}\right) \quad , \qquad g_{zf} = -\frac{a^2 z}{2\, c^2\left(f - \frac{z^2}{c^2}\right)} \quad , \qquad g_{zz} = \frac{a^2 \cdot z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} + 1 \ ,$$

$$g^{fz} = \frac{2z}{c^2}$$

$$\partial_z g_{zf} = \frac{-a^2 c^2\left(f - \frac{z^2}{c^2}\right) - 2a^2 z^2}{2c^4\left(f - \frac{z^2}{c^2}\right)^2} , \quad \partial_f g_{zz} = \frac{-a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)^2} \ , \quad \partial_z g_{zz} = \frac{2a^2 c^2 z\left(f - \frac{z^2}{c^2}\right) + 2a^2 z^3}{c^6\left(f - \frac{z^2}{c^2}\right)^2}$$

implicano

$$\Gamma^f_{zz} = \frac{-2\left[c^2\left(f - \frac{z^2}{c^2}\right) + z^2\right]}{c^4\left(f - \frac{z^2}{c^2}\right)} .$$

- $\boldsymbol{\Gamma^z_{fz} = \Gamma^z_{zf} = \frac{1}{2} g^{zf}\left(\partial_f g_{zf} + \partial_z g_{ff} - \partial_f g_{zf}\right) + \frac{1}{2} g^{zz}\left(\partial_f g_{zz} + \partial_z g_{fz} - \partial_z g_{fz}\right)}$.

Le seguenti uguaglianze

$$g^{zf} = \frac{2z}{c^2} \quad , \qquad g_{zf} = -\frac{a^2 \cdot z}{2\,c^2} \cdot \frac{1}{\left(f - \frac{z^2}{c^2}\right)} \quad , \qquad g_{ff} = \frac{1}{4} \cdot \frac{a^2}{\left(f - \frac{z^2}{c^2}\right)} \quad , \qquad g^{zz} = 1 \quad ,$$

$$g_{zz} = \frac{a^2 \cdot z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} + 1 \ ,$$

$$\partial_z g_{ff} = \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)^2} \ , \qquad \partial_f g_{zz} = \frac{-a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)^2} \ ,$$

implicano

$$\Gamma^z_{fz} = \Gamma^z_{zf} = 0\,.$$

- $\boldsymbol{\Gamma^z_{ff} = \frac{1}{2} g^{zf}\left(\partial_f g_{ff} + \partial_f g_{ff} - \partial_f g_{ff}\right) + \frac{1}{2} g^{zz}\left(\partial_f g_{fz} + \partial_f g_{fz} - \partial_z g_{ff}\right)}$.

Le seguenti uguaglianze

$$g^{zf} = \frac{2z}{c^2} \quad , \qquad g_{ff} = \frac{1}{4} \cdot \frac{a^2}{\left(f - \frac{z^2}{c^2}\right)} \quad , \qquad g^{zz} = 1 \quad , \qquad g_{fz} = -\frac{a^2 \cdot z}{2\,c^2} \cdot \frac{1}{\left(f - \frac{z^2}{c^2}\right)} \quad ,$$

$$\partial_f g_{ff} = \frac{-a^2}{4\left(f - \frac{z^2}{c^2}\right)^2} \ , \qquad \partial_f g_{fz} = \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)^2} \ , \qquad \partial_z g_{ff} = \frac{a^2 z}{2c^2\left(f - \frac{z^2}{c^2}\right)^2} \ ,$$

implicano

$$\Gamma^z_{ff} = 0\,.$$

- $\boldsymbol{\Gamma^z_{\varphi f} = \frac{1}{2} g^{zz}\left(\partial_\varphi g_{fz} + \partial_f g_{\varphi z} - \partial_z g_{\varphi f}\right) + \frac{1}{2} g^{zf}\left(\partial_\varphi g_{ff} + \partial_f g_{\varphi f} - \partial_f g_{\varphi f}\right)}$.

Le seguenti uguaglianze

$$g^{zz} = 1 \quad , \qquad g^{zf} = \frac{2z}{c^2} \quad , \qquad g_{ff} = \frac{1}{4} \cdot \frac{a^2}{\left(f - \frac{z^2}{c^2}\right)} \quad , \qquad g_{fz} = -\frac{a^2 \cdot z}{2\,c^2} \cdot \frac{1}{\left(f - \frac{z^2}{c^2}\right)} \quad ,$$

$$g_{\varphi z} = g_{\varphi f} = 0 \ , \qquad \partial_\varphi g_{fz} = \partial_\varphi g_{ff} = 0$$

implicano

$$\Gamma^z_{\varphi f} = 0\,.$$

- $\boldsymbol{\Gamma^{\varphi}_{fz} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_f g_{z\varphi} + \partial_z g_{f\varphi} - \partial_\varphi g_{fz}\right)}$.

Le seguenti uguaglianze

$$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)} \ , \qquad g_{fz} = -\frac{a^2 \cdot z}{2\,c^2}\cdot\frac{1}{\left(f - \frac{z^2}{c^2}\right)} \ , \qquad g_{z\varphi} = g_{f\varphi} = 0 \ , \qquad \partial_\varphi g_{fz} = 0 \ ,$$

implicano

$$\Gamma^{\varphi}_{fz} = 0 \ .$$

- $\boldsymbol{\Gamma^{\varphi}_{zf} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_z g_{f\varphi} + \partial_f g_{z\varphi} - \partial_\varphi g_{zf}\right)}$.

Le seguenti uguaglianze

$$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(f - \frac{z^2}{c^2}\right)} \ , \qquad g_{zf} = -\frac{a^2 \cdot z}{2\,c^2}\cdot\frac{1}{\left(f - \frac{z^2}{c^2}\right)} \ , \qquad g_{z\varphi} = g_{f\varphi} = 0 \ , \qquad \partial_\varphi g_{fz} = 0 \ ,$$

implicano

$$\Gamma^{\varphi}_{zf} = 0 \ .$$

- $\boldsymbol{\Gamma^{z}_{f\varphi} = \frac{1}{2} g^{zz}\left(\partial_f g_{\varphi z} + \partial_\varphi g_{fz} - \partial_z g_{f\varphi}\right) + \frac{1}{2} g^{zf}\left(\partial_f g_{\varphi f} + \partial_\varphi g_{ff} - \partial_f g_{f\varphi}\right)}$.

Le seguenti uguaglianze

$$g^{zz} = 1 \ , \qquad g^{zf} = \frac{2z}{c^2} \ , \qquad g_{fz} = -\frac{a^2 \cdot z}{2\,c^2}\cdot\frac{1}{\left(f - \frac{z^2}{c^2}\right)} \ , \qquad g_{ff} = \frac{1}{4}\cdot\frac{a^2}{\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$g_{\varphi z} = g_{f\varphi} = g_{\varphi f} = 0 \ , \qquad \partial_\varphi g_{fz} = \partial_\varphi g_{ff} = 0 \ ,$$

implicano

$$\Gamma^{z}_{f\varphi} = 0 \ .$$

- $\boldsymbol{\Gamma^{f}_{\varphi z} = \frac{1}{2} g^{ff}\left(\partial_{\varphi} g_{zf} + \partial_{z} g_{\varphi f} - \partial_{f} g_{\varphi z}\right) + \frac{1}{2} g^{fz}\left(\partial_{\varphi} g_{zz} + \partial_{z} g_{\varphi z} - \partial_{z} g_{\varphi z}\right)}$.

Le seguenti uguaglianze

$$g^{ff} = \frac{4z^2}{c^4} + \frac{4}{a^2}\left(f - \frac{z^2}{c^2}\right) \ , \quad g^{fz} = \frac{2z}{c^2} \ , g_{zf} = -\frac{a^2 \cdot z}{2\, c^2} \cdot \frac{1}{\left(f - \frac{z^2}{c^2}\right)} \ , \quad g_{zz} = \frac{a^2 \cdot z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} + 1 \ ,$$

$$g_{\varphi z} = g_{\varphi f} = 0 \ , \qquad \partial_{\varphi} g_{fz} = \partial_{\varphi} g_{zz} = 0 \ ,$$

implicano

$$\Gamma^{f}_{\varphi z} = 0 \ .$$

- $\boldsymbol{\Gamma^{f}_{z\varphi} = \frac{1}{2} g^{ff}\left(\partial_{z} g_{\varphi f} + \partial_{\varphi} g_{zf} - \partial_{f} g_{z\varphi}\right) + \frac{1}{2} g^{fz}\left(\partial_{z} g_{\varphi z} + \partial_{\varphi} g_{zz} - \partial_{z} g_{z\varphi}\right)}$.

Le seguenti uguaglianze

$$g^{ff} = \frac{4z^2}{c^4} + \frac{4}{a^2}\left(f - \frac{z^2}{c^2}\right) \ , \quad g^{fz} = \frac{2z}{c^2} \ , g_{zf} = -\frac{a^2 \cdot z}{2\, c^2} \cdot \frac{1}{\left(f - \frac{z^2}{c^2}\right)} \ , \quad g_{zz} = \frac{a^2 \cdot z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} + 1 \ ,$$

$$g_{z\varphi} = g_{\varphi z} = g_{\varphi f} = 0 \ , \qquad \partial_{\varphi} g_{zf} = \partial_{\varphi} g_{zz} = 0 \ ,$$

implicano

$$\Gamma^{f}_{z\varphi} = 0 \ . \ QED$$

Per dimostrare la correttezza dei calcoli, osserviamo che, facendo riferimento alla _proposizione 14_, abbiamo che l'accelerazione controvariante è della forma

$$a^j = \ddot{x}^j + \Gamma^{j}_{hk}\, \dot{x}^h\, \dot{x}^k \ .$$

Dai risultati ottenuti precedentemente,riguardanti l'accelerazione controvariante possiamo ricavare

$$\Gamma_{ff}^{f} = -\frac{1}{2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma_{\varphi f}^{f} = \Gamma_{f\varphi}^{f} = \frac{1}{2\left(f - \frac{z^2}{c^2}\right)}\left[\frac{-a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} - 1\right] \ ,$$

$$\Gamma_{\varphi\varphi}^{f} = -2\left(f - \frac{z^2}{c^2}\right) \ ,$$

$$\Gamma_{f\varphi}^{\varphi} = \Gamma_{\varphi f}^{\varphi} = \frac{1}{2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma_{z\varphi}^{\varphi} = \Gamma_{\varphi z}^{\varphi} = -\frac{z}{c^2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma_{zf}^{f} = \Gamma_{fz}^{f} = \frac{z}{c^2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma_{zz}^{f} = \frac{-2\left[c^2\left(f - \frac{z^2}{c^2}\right) + z^2\right]}{c^4\left(f - \frac{z^2}{c^2}\right)} \ . \quad QED$$

Tensore di curvatura

Proposizione 16 – Il tensore di curvatura è nullo perché lo spazio ambiente è uno spazio affine. Quindi abbiamo

$$R = 0$$

e

$$R_{ij}{}^{h}{}_{k} = 0 \quad , \qquad i = f, \varphi, z \ , \quad j = f, \varphi, z \ , \quad z = f, \varphi, z \ .$$

Per controllo, verifichiamo, per esempio che

$$R_{\varphi z}{}^{f}{}_{\varphi} = 0 \qquad\qquad R_{f\varphi}{}^{z}{}_{\varphi} = 0$$

- $R_{\varphi z}{}^{f}{}_{\varphi} = \partial_\varphi \Gamma^f_{z\varphi} - \partial_z \Gamma^f_{\varphi\varphi} - \Gamma^f_{\varphi\varphi}\Gamma^f_{zf} + \Gamma^f_{z\varphi}\Gamma^f_{\varphi f} - \Gamma^z_{\varphi\varphi}\Gamma^f_{zz} + \Gamma^z_{z\varphi}\Gamma^f_{\varphi z} - \Gamma^\varphi_{\varphi\varphi}\Gamma^f_{z\varphi} + \Gamma^\varphi_{z\varphi}\Gamma^f_{\varphi\varphi}$

Le seguenti uguaglianze

$$\Gamma^f_{z\varphi} = \Gamma^f_{\varphi z} = \Gamma^z_{\varphi\varphi} = \Gamma^z_{z\varphi} = \Gamma^\varphi_{\varphi\varphi} = 0 \ , \qquad \Gamma^f_{\varphi\varphi} = -2\left(f - \frac{z^2}{c^2}\right) \ , \qquad \Gamma^f_{zf} = \frac{z}{c^2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma^f_{\varphi f} = \frac{1}{2\left(f - \frac{z^2}{c^2}\right)}\left[\frac{-a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} - 1\right] \ , \quad \Gamma^f_{zz} = \frac{-2\left[c^2\left(f - \frac{z^2}{c^2}\right) + z^2\right]}{c^4\left(f - \frac{z^2}{c^2}\right)} \ , \quad \Gamma^\varphi_{z\varphi} = -\frac{z}{c^2\left(f - \frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma^f_{\varphi\varphi} = -2\left(f - \frac{z^2}{c^2}\right) \ , \qquad \partial_\varphi \Gamma^f_{z\varphi} = 0 \, , \qquad \partial_z \Gamma^f_{\varphi\varphi} = \frac{4z}{c^2} \ ,$$

implicano

$$R_{\varphi z}{}^{f}{}_{\varphi} = -\frac{4z}{c^2} + \frac{2z}{c^2} + \frac{2z}{c^2} = 0 \ .$$

- $R_{f\varphi}{}^{z}{}_{\varphi} = \partial_f \Gamma^z_{\varphi\varphi} - \Gamma^f_{f\varphi}\Gamma^z_{\varphi f} - \Gamma^\varphi_{f\varphi}\Gamma^z_{\varphi\varphi} - \Gamma^z_{f\varphi}\Gamma^z_{\varphi z} - \partial_\varphi \Gamma^z_{f\varphi} + \Gamma^f_{\varphi\varphi}\Gamma^z_{ff} + \Gamma^\varphi_{\varphi\varphi}\Gamma^z_{f\varphi} + \Gamma^z_{\varphi\varphi}\Gamma^z_{fz}$

Le seguenti uguaglianze

$$\Gamma^z_{\varphi\varphi} = \Gamma^z_{\varphi f} = \Gamma^z_{f\varphi} = \Gamma^z_{\varphi z} = \Gamma^z_{ff} = \Gamma^\varphi_{\varphi\varphi} = \Gamma^z_{fz} = 0 \ , \qquad \Gamma^f_{f\varphi} = \frac{1}{2\left(f - \frac{z^2}{c^2}\right)}\left[\frac{-a^2 z^2}{c^4\left(f - \frac{z^2}{c^2}\right)} - 1\right] ,$$

$$\Gamma^\varphi_{f\varphi} = \frac{1}{2\left(f - \frac{z^2}{c^2}\right)} \ , \qquad \Gamma^f_{\varphi\varphi} = -2\left(f - \frac{z^2}{c^2}\right) \ ,$$

implicano

$$R_{f\varphi}{}^{z}{}_{\varphi} = 0 \ . \qquad QED$$

Tensore di Ricci

Proposizione 17 – Il tensore di Ricci è nullo perché il tensore di curvatura è nullo, quindi abbiamo

$$\underline{r} = 0 \, ,$$

$$\underline{r}_{ij} = R_{ih}{}^{h}{}_{j} = 0 \, . \; \square$$

Tensore di curvatura scalare

Proposizione 18 – Il tensore di curvatura scalare è nullo perché il tensore di Ricci è nullo, quindi abbiamo

$$\langle \underline{r} \rangle = 0 \, . \; \square$$

Tensore di curvatura covariante

Proposizione 19 – Il tensore di curvatura (in forma covariante) è nullo perché il tensore di curvatura scalare è nullo, quindi abbiamo

$$\underline{R} = 0 \; ,$$

$$R_{ijhk} = g_{lh} R_{ij}{}^{l}{}_{k} = 0 \; . \quad \square$$

SOTTOVARIETA'

Funzione metrica

Proposizione 20 – La funzione metrica è

$$G^{\dagger} = \frac{1}{2}\left\{a^2\left(1-\frac{z^2}{c^2}\right)\dot{\varphi}^2 + \left[\frac{a^2 \cdot z^2}{c^4}\cdot\frac{1}{(1-\frac{z^2}{c^2})}+1\right]\dot{z}^2\right\} .$$

Dimostrazione – Abbiamo le seguenti condizioni:

$$f = 1 \qquad\qquad \dot{f} = 0 .$$

Quindi

$$G^{\dagger} = \frac{1}{2}\left\{a^2\left(1-\frac{z^2}{c^2}\right)\dot{\varphi}^2 + \left[\frac{a^2 \cdot z^2}{c^4}\cdot\frac{1}{(1-\frac{z^2}{c^2})}+1\right]\dot{z}^2\right\} \;. \; QED$$

Matrice della metrica

Proposizione 21 - La matrice della metrica covariante nel sistema di coordinate adattato è

$$\left(g_{ij}{}^{\dagger}\right) = \begin{pmatrix} a^2\left(1-\frac{z^2}{c^2}\right) & 0 \\ 0 & \frac{a^2\cdot z^2}{c^4}\cdot\frac{1}{\left(1-\frac{z^2}{c^2}\right)}+1 \end{pmatrix} . \square$$

Proposizione 22 - La matrice della metrica controvariante nel sistema di coordinate adattato è

$$\left(g^{hk\dagger}\right) = \begin{pmatrix} \frac{1}{a^2}\frac{1}{\left(1-\frac{z^2}{c^2}\right)} & 0 \\ 0 & \frac{\left(1-\frac{z^2}{c^2}\right)}{\left[\frac{a^2z^2}{c^4}+\left(1-\frac{z^2}{c^2}\right)\right]} \end{pmatrix} .$$

Dimostrazione -

Data una matrice $(A) \in \mathcal{R}^{m\times n}$, con $m = n = 2$, invertibile:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

la sua inversa è la seguente

$$\frac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Nel nostro caso abbiamo che il determinante $\left[g_{ij}{}^{\dagger}\right]$ della matrice della metrica è

$$\left[g_{ij}{}^{\dagger}\right] = \, det\left(g_{ij}{}^{\dagger}\right) = a^2\left[\frac{a^2z^2}{c^4}+\left(1-\frac{z^2}{c^2}\right)\right] \neq 0$$

La matrice della metrica è quindi invertibile.

Perciò abbiamo

$$(g^{hk\dagger}) = \left(g_{ij}{}^{\dagger}\right)^{-1} = \frac{1}{a^2\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]}\begin{pmatrix} \frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{(1 - \frac{z^2}{c^2})} + 1 & 0 \\ 0 & a^2\left(1 - \frac{z^2}{c^2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a^2}\frac{1}{\left(1 - \frac{z^2}{c^2}\right)} & 0 \\ 0 & \frac{\left(1 - \frac{z^2}{c^2}\right)}{\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]} \end{pmatrix} \ . \ QED$$

Per controllare la correttezza dei calcoli, verifichiamo l'uguaglianza

$$\left(g_{ij}{}^{\dagger}\right)(g^{hk\dagger}) = (I) \ .$$

In effetti, abbiamo

$$\begin{pmatrix} a^2\left(1 - \frac{z^2}{c^2}\right) & 0 \\ 0 & \frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{(1 - \frac{z^2}{c^2})} + 1 \end{pmatrix}\begin{pmatrix} \frac{1}{a^2}\frac{1}{\left(1 - \frac{z^2}{c^2}\right)} & 0 \\ 0 & \frac{\left(1 - \frac{z^2}{c^2}\right)}{\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Inoltre verifichiamo l'uguaglianza

$$\left[g_{ij}{}^{\dagger}\right] = \frac{1}{[g^{hk\dagger}]} \ .$$

In effetti, abbiamo

$$\left[g_{ij}{}^{\dagger}\right] = a^2\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right] \ , \qquad [g^{hk\dagger}] = \frac{1}{a^2\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]} \ . \ QED$$

Tensore metrico

Proposizione 23 **-** L'espressione tensoriale della metrica covariante è

$$g^{\dagger} = \left[a^2\left(1-\frac{z^2}{c^2}\right)\right] d\varphi \otimes d\varphi + \left[\frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{\left(1-\frac{z^2}{c^2}\right)} + 1\right] dz \otimes dz\ . \square$$

Proposizione 24 **-** L'espressione tensoriale della metrica controvariante è

$$\bar{g}^{\dagger} = \left[\frac{1}{a^2}\frac{1}{\left(1-\frac{z^2}{c^2}\right)}\right] d\varphi \otimes d\varphi\ + \left\{\frac{\left(1-\frac{z^2}{c^2}\right)}{\left(\left[\frac{a^2 z^2}{c^4} + \left(1-\frac{z^2}{c^2}\right)\right]\right)}\right\} dz \otimes dz\ . \square$$

Forma volume η

Proposizione 25 - La forma volume (in forma covariante) espressa in coordinate adattate è

$$\eta^{\dagger} = a\sqrt{\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]}\, d\varphi \wedge dz \ .$$

Dimostrazione -

Abbiamo

$$\eta^{\dagger} = \sqrt{det\left(g_{ij}{}^{\dagger}\right)}\ d\varphi \wedge dz \ ,$$

dove

$$det\left(g_{ij}{}^{\dagger}\right) = \left[g_{ij}{}^{\dagger}\right] = \ a^2\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right] \ .$$

Quindi

$$\eta^{\dagger} = \sqrt{a^2\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]}\, d\varphi \wedge dz = a\sqrt{\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]}\, d\varphi \wedge dz \ . \ \ QED$$

Osservazione 1 -

Nel caso di $a = c = R$, se i calcoli sono corretti, dovremmo trovare la forma volume della sfera che sappiamo valere

$$\eta^{\dagger} = R^2 sin^2\theta\ d\theta \wedge d\varphi \,.$$

A tal proposito abbiamo

$$\eta^{\dagger} \xrightarrow{a=c=R} \mathrm{R}\sqrt{\frac{\mathrm{z}^2}{\mathrm{R}^2} + 1 - \frac{\mathrm{z}^2}{\mathrm{R}^2}}\, d\varphi \wedge dz \ = \mathrm{R}\ d\varphi \wedge dz \ ,$$

$$z = R\ cos\theta \ , \qquad\qquad dz = -R\ sin\theta\ d\theta \ ,$$

quindi

$$\eta^{\dagger} = -R^2 sin^2\theta \; d\varphi \wedge d\theta = \; R^2 sin^2\theta \; d\theta \wedge d\varphi \; . \; QED$$

Osservazione 2 -

Nel caso di $a = \rho >> c$, $e \;\; z \leq c$,se i calcoli sono corretti, dovremmo trovare la forma volume del cilindro che sappiamo valere

$$\eta^{\dagger} = \rho \; d\varphi \wedge dz \, .$$

A tal proposito abbiamo

$$\eta^{\dagger} \xrightarrow{a=\rho\gg c\,,\;\; z\leq c} \rho \;\; d\varphi \wedge dz \;\; ,$$

quindi

$$\eta^{\dagger} \cong \rho \;\; d\varphi \wedge dz \;\; . \; QED$$

Proposizione 26 - La forma volume (in forma controvariante) espressa in coordinate adattate è

$$\bar{\eta}^{\dagger} = \frac{1}{a} \sqrt{\frac{1}{\left[\frac{a^2 z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]}} \, d\varphi \wedge dz \;\; .$$

Dimostrazione -

Abbiamo

$$\bar{\eta}^{\dagger} = \sqrt{det(g^{hk\dagger})} \, d\varphi \wedge dz \;\; ,$$

dove

$$det(g^{hk\dagger}) = [g^{hk\dagger}] = \; \frac{1}{a^2} \frac{1}{\left[\frac{a^2 z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]} \;\; .$$

Quindi

$$\bar{\eta}^{\dagger} = \sqrt{\frac{1}{a^2}\frac{1}{\left[\frac{a^2z^2}{c^4}+\left(1-\frac{z^2}{c^2}\right)\right]}}\,d\varphi\wedge dz = \frac{1}{a}\sqrt{\frac{1}{\left[\frac{a^2z^2}{c^4}+\left(1-\frac{z^2}{c^2}\right)\right]}}\,d\varphi\wedge dz\ .\ \ QED$$

Accelerazione covariante

Proposizione 27 – L'accelerazione covariante è

$$a_\varphi = a^2\left(1-\frac{z^2}{c^2}\right)\ddot{\varphi} - \frac{2a^2 z}{c^2}\dot{\varphi}\dot{z} \ ,$$

$$a_z = \left[\frac{a^2 z^2}{c^4\left(1-\frac{z^2}{c^2}\right)} + 1\right]\ddot{z} + \frac{a^2 c^4 z\left(1-\frac{z^2}{c^2}\right) + a^2 z^3 c^2}{c^8\left(1-\frac{z^2}{c^2}\right)^2}\dot{z}^2 + \frac{a^2 z}{c^2}\dot{\varphi}^2.$$

Dimostrazione -

Secondo le formule di Lagrange, abbiamo

$$a_i = \frac{d}{dt}\frac{\partial G}{\partial \dot{x}^i} - \frac{\partial G}{\partial x^i}.$$

Quindi

- $a_\varphi = \frac{d}{dt}\frac{\partial G}{\partial \dot{\varphi}} - \frac{\partial G}{\partial \varphi}$,

dove

$$\frac{\partial G}{\partial \dot{\varphi}} = a^2\left(1-\frac{z^2}{c^2}\right)\dot{\varphi} \ ,$$

$$\frac{d}{dt}\frac{\partial G}{\partial \dot{\varphi}} = a^2\left(1-\frac{z^2}{c^2}\right)\ddot{\varphi} - \frac{2a^2 z}{c^2}\dot{\varphi}\dot{z} \ ,$$

$$\frac{\partial G}{\partial \varphi} = 0$$

$$a_\varphi = a^2\left(1-\frac{z^2}{c^2}\right)\ddot{\varphi} - \frac{2a^2 z}{c^2}\dot{\varphi}\dot{z}$$

- $a_z = \frac{d}{dt}\frac{\partial G}{\partial \dot{z}} - \frac{\partial G}{\partial z}$,

dove

$$\frac{\partial G}{\partial \dot{z}} = \left[\frac{a^2 z^2}{c^4}\frac{1}{\left(1-\frac{z^2}{c^2}\right)} + 1\right]\dot{z} \ ,$$

$$\frac{d}{dt}\frac{\partial G}{\partial \dot{z}} = \left[\frac{a^2 z^2}{c^4}\frac{1}{\left(1-\frac{z^2}{c^2}\right)} + 1\right]\ddot{z} + \frac{\left[2a^2 z^2 c^4\left(1-\frac{z^2}{c^2}\right) + 2a^2 z^3 c^2\right]}{c^8\left(1-\frac{z^2}{c^2}\right)^2}\dot{z}^2 \ ,$$

$$\frac{\partial G}{\partial z} = \frac{1}{2}\left\{-\frac{2a^2 z}{c^2}\dot{\varphi}^2 + \frac{2a^2 z^2 c^4\left(1-\frac{z^2}{c^2}\right)2a^2 z^3 c^2}{2c^8\left(1-\frac{z^2}{c^2}\right)^2}\dot{z}^2\right\},$$

$$a_z = \left[\frac{a^2 z^2}{c^4\left(1-\frac{z^2}{c^2}\right)} + 1\right]\ddot{z} + \frac{a^2 c^4 z\left(1-\frac{z^2}{c^2}\right) + a^2 z^3 c^2}{c^8\left(1-\frac{z^2}{c^2}\right)^2}\dot{z}^2 + \frac{a^2 z}{c^2}\dot{\varphi}^2 \quad .QED$$

Accelerazione controvariante

Proposizione 28 – L'accelerazione controvariante è

$$a^{\varphi} = \ddot{\varphi} - \frac{2z}{c^2\left(1 - \frac{z^2}{c^2}\right)}\dot{\varphi}\dot{z} \quad ,$$

$$a^{z} = \ddot{z} + \frac{a^2c^2z\left(1 - \frac{z^2}{c^2}\right) + a^2z^3}{c^6\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\left(1 - \frac{z^2}{c^2}\right)\right]}\dot{z}^2 \frac{a^2z\left(1 - \frac{z^2}{c^2}\right)}{c^2\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]}\dot{\varphi}^2 \quad .$$

Dimostrazione -

Abbiamo

$$a^{j} = g^{ij}\, a_i \quad .$$

Calcolo di

- $a^{\varphi} = g^{\varphi\varphi} a_{\varphi}$.

Le seguenti uguaglianze

$$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(1 - \frac{z^2}{c^2}\right)} \quad , \qquad\qquad a^{\varphi} = \ddot{\varphi} - \frac{2z}{c^2\left(1 - \frac{z^2}{c^2}\right)}\dot{\varphi}\dot{z} \quad ,$$

implicano

$$a^{\varphi} = \ddot{\varphi} - \frac{2z}{c^2\left(1 - \frac{z^2}{c^2}\right)}\dot{\varphi}\dot{z} \quad .$$

Calcolo di

- $a^z = g^{zz} a_z$.

Le seguenti uguaglianze

$$g^{zz} = \frac{\left(1 - \frac{z^2}{c^2}\right)}{\left[\frac{a^2 z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]} \quad , \quad a_z = \left[\frac{a^2 z^2}{c^4\left(1 - \frac{z^2}{c^2}\right)} + 1\right] \ddot{z} + \frac{a^2 c^4 z \left(1 - \frac{z^2}{c^2}\right) + a^2 z^3 c^2}{c^8 \left(1 - \frac{z^2}{c^2}\right)^2} \dot{z}^2 + \frac{a^2 z}{c^2} \dot{\varphi}^2 \ ,$$

implicano

$$a^z = \ddot{z} + \frac{a^2 c^2 z \left(1 - \frac{z^2}{c^2}\right) + a^2 z^3}{c^6 \left[\frac{a^2 z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\left(1 - \frac{z^2}{c^2}\right)\right]} \dot{z}^2 \frac{a^2 z \left(1 - \frac{z^2}{c^2}\right)}{c^2 \left[\frac{a^2 z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]} \dot{\varphi}^2 \ . QED$$

Simboli di Christoffel

Proposizione 29 – I simboli di Christoffel non nulli sono

$$\Gamma^{\varphi}_{z\varphi} = \Gamma^{\varphi}_{\varphi z} = -\frac{1}{\left(1-\frac{z^2}{c^2}\right)}\frac{z}{c^2} \ ,$$

$$\Gamma^{z}_{zz} = \frac{a^2c^2z\left(1-\frac{z^2}{c^2}\right)+a^2z^3}{\left[a^2c^2z^2+c^6\left(1-\frac{z^2}{c^2}\right)\right]\left(1-\frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma^{z}_{\varphi\varphi} = \frac{a^2z\left(1-\frac{z^2}{c^2}\right)}{\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]} \ .$$

Dimostrazione -

Applicando la formula generale

$$\boldsymbol{\Gamma^{h}_{ij} = \frac{1}{2}g^{hk}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)} \ ,$$

otteniamo le seguenti uguaglianze

- $\boldsymbol{\Gamma^{\varphi}_{\varphi\varphi} = \frac{1}{2}g^{\varphi\varphi}\left(\partial_\varphi g_{\varphi\varphi} + \partial_\varphi g_{\varphi\varphi} - \partial_\varphi g_{\varphi\varphi}\right)}$.

Le seguenti uguaglianze

$$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(1-\frac{z^2}{c^2}\right)} \ , \qquad g_{\varphi\varphi} = a^2\left(1-\frac{z^2}{c^2}\right) \ , \qquad \partial_\varphi g_{\varphi\varphi} = 0 \ ,$$

implicano

$$\Gamma^{\varphi}_{\varphi\varphi} = 0 \ .$$

- $\boldsymbol{\Gamma^{z}_{\varphi z} = \Gamma^{z}_{z\varphi} = \frac{1}{2} g^{zz}\left(\partial_{\varphi} g_{zz} + \partial_{z} g_{\varphi z} - \partial_{z} g_{\varphi z}\right) = \frac{1}{2} g^{zz}\left(\partial_{\varphi} g_{zz}\right)}$.

Le seguenti uguaglianze

$$g^{zz} = \frac{\left(1-\frac{z^2}{c^2}\right)}{\left[\frac{a^2 z^2}{c^4} + \left(1-\frac{z^2}{c^2}\right)\right]} \ , \qquad g_{zz=}\frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{(1-\frac{z^2}{c^2})} + 1 \ , \qquad \partial_{\varphi} g_{zz} = 0 \ ,$$

implicano

$$\Gamma^{z}_{\varphi z} = \Gamma^{z}_{z\varphi} = 0 \ .$$

- $\boldsymbol{\Gamma^{\varphi}_{zz} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_{z} g_{z\varphi} + \partial_{z} g_{z\varphi} - \partial_{\varphi} g_{zz}\right)}$.

Le seguenti uguaglianze

$$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(1-\frac{z^2}{c^2}\right)} \ , \qquad g_{z\varphi} = 0 \ , \qquad g_{zz} = \frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{(1-\frac{z^2}{c^2})} + 1 \ , \qquad \partial_{\varphi} g_{zz} = 0 \ ,$$

implicano

$$\Gamma^{\varphi}_{zz} = 0 \ .$$

- $\boldsymbol{\Gamma^{\varphi}_{z\varphi} = \Gamma^{\varphi}_{\varphi z} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_{z} g_{\varphi\varphi} + \partial_{\varphi} g_{z\varphi} - \partial_{\varphi} g_{z\varphi}\right)}$.

Le seguenti uguaglianze

$$g^{\varphi\varphi} = \frac{1}{a^2}\frac{1}{\left(1-\frac{z^2}{c^2}\right)} \ , \qquad g_{\varphi\varphi} = a^2\left(1-\frac{z^2}{c^2}\right) \ , \qquad \partial_{z} g_{\varphi\varphi} = -\frac{2a^2 z}{c^2} \ ,$$

implicano

$$\Gamma^{\varphi}_{z\varphi} = \Gamma^{\varphi}_{\varphi z} = -\frac{1}{\left(1-\frac{z^2}{c^2}\right)}\frac{z}{c^2} \ .$$

- $\boldsymbol{\Gamma^{z}_{zz} = \frac{1}{2} g^{zz}(\partial_z g_{zz} + \partial_z g_{zz} - \partial_z g_{zz})}$.

Le seguenti uguaglianze

$$g^{zz} = \frac{\left(1 - \frac{z^2}{c^2}\right)}{\left[\frac{a^2 z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]} \quad , \quad g_{zz} = \frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{\left(1 - \frac{z^2}{c^2}\right)} + 1 \quad , \quad \partial_z g_{zz} = \frac{2a^2 c^2 z \left(1 - \frac{z^2}{c^2}\right) + 2a^2 z^3}{c^6 \left(1 - \frac{z^2}{c^2}\right)^2} \quad ,$$

implicano

$$\Gamma^{z}_{zz} = \frac{a^2 c^2 z \left(1 - \frac{z^2}{c^2}\right) + a^2 z^3}{\left[a^2 c^2 z^2 + c^6 \left(1 - \frac{z^2}{c^2}\right)\right]\left(1 - \frac{z^2}{c^2}\right)} \ .$$

- $\boldsymbol{\Gamma^{z}_{\varphi\varphi} = \frac{1}{2} g^{zz}\left(\partial_\varphi g_{zz} + \partial_\varphi g_{\varphi z} - \partial_z g_{\varphi\varphi}\right)}$.

Le seguenti uguaglianze

$$g^{zz} = \frac{\left(1 - \frac{z^2}{c^2}\right)}{\left[\frac{a^2 z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]} \ , \qquad g_{zz} = \frac{a^2 \cdot z^2}{c^4} \cdot \frac{1}{\left(1 - \frac{z^2}{c^2}\right)} + 1 \ , \qquad g_{\varphi z} = 0 \ ,$$

$$g_{\varphi\varphi} = a^2 \left(1 - \frac{z^2}{c^2}\right) \ , \qquad \partial_\varphi g_{zz} = 0 \ , \qquad \partial_z g_{\varphi\varphi} = -\frac{2a^2 z}{c^2} \ ,$$

implicano

$$\Gamma^{z}_{\varphi\varphi} = \frac{a^2 z \left(1 - \frac{z^2}{c^2}\right)}{\left[\frac{a^2 z^2}{c^2} + c^2 \left(1 - \frac{z^2}{c^2}\right)\right]} \ . \quad QED$$

Tensore di curvatura

Proposizione 30 – Il tensore di curvatura è

$$R^{\dagger} = \frac{\mathrm{a}^2\mathrm{c}^2 - \mathrm{a}^2\mathrm{z}^2}{\left[\frac{\mathrm{a}^2\mathrm{z}^2}{\mathrm{c}^2} + \mathrm{c}^2\left(1 - \frac{z^2}{c^2}\right)\right]^2} dz \otimes d\varphi \otimes dz \otimes d\varphi$$

$$+ \frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2 \left[a^2 z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]} \; dz \otimes d\varphi \otimes d\varphi \otimes dz$$

$$- \frac{\mathrm{a}^2\mathrm{c}^2 - \mathrm{a}^2\mathrm{z}^2}{\left[\frac{\mathrm{a}^2\mathrm{z}^2}{\mathrm{c}^2} + \mathrm{c}^2\left(1 - \frac{z^2}{c^2}\right)\right]^2} d\varphi \otimes dz \otimes dz \otimes d\varphi$$

$$- \frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2 \left[a^2 z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]} \; d\varphi \otimes dz \otimes d\varphi \otimes dz \; .$$

Con

$$R_{z\varphi}{}^{z}{}_{\varphi} = \frac{\mathrm{a}^2\mathrm{c}^2 - \mathrm{a}^2\mathrm{z}^2}{\left[\frac{\mathrm{a}^2\mathrm{z}^2}{\mathrm{c}^2} + \mathrm{c}^2\left(1 - \frac{z^2}{c^2}\right)\right]^2} \; ,$$

$$R_{\varphi z}{}^{z}{}_{\varphi} = -\frac{\mathrm{a}^2\mathrm{c}^2 - \mathrm{a}^2\mathrm{z}^2}{\left[\frac{\mathrm{a}^2\mathrm{z}^2}{\mathrm{c}^2} + \mathrm{c}^2\left(1 - \frac{z^2}{c^2}\right)\right]^2} \; ,$$

$$R_{z\varphi}{}^{\varphi}{}_{z} = \frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2 \left[a^2 z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]} \; ,$$

$$R_{\varphi z}{}^{\varphi}{}_{z} = -\frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2 \left[a^2 z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]} \; .$$

Dimostrazione -

Abbiamo

$$\boldsymbol{R \equiv R_{ij}{}^{h}{}_{k} dx^{i} \otimes dx^{j} \otimes \partial x_{h} \otimes dx^{k}} \;,$$

dove

$$\boldsymbol{R_{ij}{}^{h}{}_{k} = \partial_i \Gamma_j{}^{h}{}_{k} - \Gamma_i{}^{l}{}_{k}\Gamma_j{}^{h}{}_{l} - \partial_j \Gamma_i{}^{h}{}_{k} + \Gamma_j{}^{l}{}_{k}\Gamma_i{}^{h}{}_{l}} \;.$$

Pertanto otteniamo le seguenti uguaglianze

- $R_{z\varphi}{}^{z}{}_{\varphi} = \partial_z \Gamma_{\varphi}{}^{z}{}_{\varphi} - \Gamma_z{}^{z}{}_{\varphi}\Gamma_{\varphi}{}^{z}{}_{z} - \Gamma_z{}^{\varphi}{}_{\varphi}\Gamma_{\varphi}{}^{z}{}_{\varphi} - \partial_{\varphi}\Gamma_z{}^{z}{}_{\varphi} + \Gamma_{\varphi}{}^{z}{}_{\varphi}\Gamma_z{}^{z}{}_{z} + \Gamma_{\varphi}{}^{\varphi}{}_{\varphi}\Gamma_z{}^{z}{}_{\varphi}$.

Le seguenti uguaglianze

$$\Gamma_{\varphi\varphi}^{z} = \frac{a^2 z\left(1-\frac{z^2}{c^2}\right)}{\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]} \;, \qquad \Gamma_{z\varphi}^{z} = \Gamma_{\varphi z}^{z} = 0 \;, \qquad \Gamma_{z\varphi}^{\varphi} = \Gamma_{\varphi z}^{\varphi} = -\frac{1}{\left(1-\frac{z^2}{c^2}\right)}\frac{z}{c^2} \;, \qquad \Gamma_{\varphi\varphi}^{\varphi} = 0 \;,$$

$$\Gamma_{zz}^{z} = \frac{a^2c^2z\left(1-\frac{z^2}{c^2}\right)+a^2z^3}{\left[a^2c^2z^2+c^6\left(1-\frac{z^2}{c^2}\right)\right]\left(1-\frac{z^2}{c^2}\right)} \;,$$

$$\partial_z \Gamma_{\varphi}{}^{z}{}_{\varphi} = \frac{\left(a^2-\frac{3\circ^2 z^2}{c^2}\right)\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]-\left(a^2z-\frac{a^2z^3}{c^2}\right)\left[\frac{2\circ^2 z}{c^2}-2z\right]}{\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]^2} \;,$$

$$\Gamma_{z\varphi}^{\varphi}\Gamma_{\varphi\varphi}^{z} = -\frac{a^2z^2}{c^2\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]} \;, \qquad \Gamma_{\varphi}{}^{z}{}_{\varphi}\Gamma_z{}^{z}{}_{z} = \frac{a^4c^2z^2\left(1-\frac{z^2}{c^2}\right)+a^4z^4}{\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]^2 c^4} \,,$$

implicano

$$R_{z\varphi}{}^{z}{}_{\varphi} = \partial_z \Gamma_{\varphi}{}^{z}{}_{\varphi} - \Gamma_{z\varphi}^{\varphi}\Gamma_{\varphi\varphi}^{z} + \Gamma_{\varphi}{}^{z}{}_{\varphi}\Gamma_z{}^{z}{}_{z} = \frac{a^2c^2-a^2z^2}{\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]^2} \;.$$

- $R_{\varphi z}{}^{z}{}_{\varphi} = \partial_\varphi \Gamma_{z}{}^{z}{}_{\varphi} - \Gamma_{\varphi}{}^{z}{}_{\varphi}\Gamma_{z}{}^{z}{}_{z} - \Gamma_{\varphi}{}^{\varphi}{}_{\varphi}\Gamma_{z}{}^{z}{}_{\varphi} - \partial_z \Gamma_{\varphi}{}^{z}{}_{\varphi} + \Gamma_{z}{}^{z}{}_{\varphi}\Gamma_{\varphi}{}^{z}{}_{z} + \Gamma_{z}{}^{\varphi}{}_{\varphi}\Gamma_{\varphi}{}^{z}{}_{\varphi}$.

Le seguenti uguaglianze

$$\Gamma^{z}_{\varphi\varphi} = \frac{a^2 z\left(1-\frac{z^2}{c^2}\right)}{\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]} \quad , \qquad \Gamma^{\varphi}_{z\varphi} = -\frac{1}{\left(1-\frac{z^2}{c^2}\right)}\frac{z}{c^2} \quad , \qquad \Gamma^{z}_{z\varphi} = \Gamma^{z}_{\varphi z} = \Gamma^{\varphi}_{\varphi\varphi} = 0 \ ,$$

$$\Gamma^{z}_{\varphi\varphi}\Gamma^{\varphi}_{z\varphi} = -\frac{a^2z^2}{c^2\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]} \ , \ \Gamma_{\varphi}{}^{z}{}_{\varphi}\Gamma_{z}{}^{z}{}_{z} = \frac{a^4c^2z^2\left(1-\frac{z^2}{c^2}\right)+a^4z^4}{\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]^2 c^4} \ ,$$

$$\partial_z \Gamma^{z}_{\varphi\varphi} = \frac{\left(a^2 - \frac{3\circ^2 z^2}{c^2}\right)\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right] - \left(a^2z - \frac{a^2z^3}{c^2}\right)\left[\frac{2\circ^2 z}{c^2} - 2z\right]}{\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]^2} \quad ,$$

implicano

$$R_{\varphi z}{}^{z}{}_{\varphi} = -\Gamma_{\varphi}{}^{z}{}_{\varphi}\Gamma_{z}{}^{z}{}_{z} - \partial_z \Gamma_{\varphi}{}^{z}{}_{\varphi} + \Gamma^{\varphi}_{z\varphi}\Gamma^{z}_{\varphi\varphi} = -\frac{a^2c^2 - a^2z^2}{\left[\frac{a^2z^2}{c^2}+c^2\left(1-\frac{z^2}{c^2}\right)\right]^2} \ .$$

- $R_{z\varphi}{}^{\varphi}{}_{z} = \partial_z \Gamma_{\varphi}{}^{\varphi}{}_{z} - \Gamma_{z}{}^{z}{}_{z}\Gamma_{\varphi}{}^{\varphi}{}_{z} - \Gamma_{z}{}^{\varphi}{}_{z}\Gamma_{\varphi}{}^{\varphi}{}_{\varphi} - \partial_\varphi \Gamma_{z}{}^{\varphi}{}_{z} + \Gamma_{\varphi}{}^{z}{}_{z}\Gamma_{z}{}^{\varphi}{}_{z} + \Gamma_{\varphi}{}^{\varphi}{}_{z}\Gamma_{z}{}^{\varphi}{}_{\varphi}$.

Le seguenti uguaglianze

$$\Gamma^{\varphi}_{\varphi z} = \Gamma^{\varphi}_{z\varphi} = -\frac{1}{\left(1-\frac{z^2}{c^2}\right)}\cdot\frac{z}{c^2} \ , \qquad \Gamma^{z}_{zz} = \frac{a^2c^2z\left(1-\frac{z^2}{c^2}\right)+a^2z^3}{\left[a^2c^2z^2+c^6\left(1-\frac{z^2}{c^2}\right)\right]\left(1-\frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma^{z}_{\varphi z} = \Gamma^{\varphi}_{\varphi\varphi} = \Gamma_{z}{}^{\varphi}{}_{z} = 0 \ ,$$

$$\partial z \Gamma^{\varphi}_{\varphi z} = -\frac{c^2\left(1-\frac{z^2}{c^2}\right)+2z^2}{c^4\left(1-\frac{z^2}{c^2}\right)^2} \ ,$$

$$\Gamma_z{}^z{}_z \Gamma_\varphi{}^\varphi{}_z = \frac{-a^2c^2z^2\left(1-\frac{z^2}{c^2}\right)-a^2z^4}{c^2\left[a^2c^2z^2+c^6\left(1-\frac{z^2}{c^2}\right)\right]\left(1-\frac{z^2}{c^2}\right)^2} \ , \qquad \Gamma^{\varphi}_{\varphi z}\Gamma^{\varphi}_{z\varphi} = \frac{z^2}{c^4\cdot\left(1-\frac{z^2}{c^2}\right)^2} \ ,$$

implicano

$$R_{z\varphi}{}^{\varphi}{}_{z} = \partial_z \Gamma_\varphi{}^\varphi{}_z - \Gamma_z{}^z{}_z\Gamma_\varphi{}^\varphi{}_z + \Gamma_\varphi{}^\varphi{}_z\Gamma_z{}^\varphi{}_\varphi = \frac{-c^2+z^2}{\left(1-\frac{z^2}{c^2}\right)^2\left[a^2z^2+c^4\left(1-\frac{z^2}{c^2}\right)\right]} \ .$$

- $R_{\varphi z}{}^{\varphi}{}_{z} = \partial_\varphi \Gamma_z{}^\varphi{}_z - \Gamma_\varphi{}^z{}_z\Gamma_z{}^\varphi{}_z - \Gamma_\varphi{}^\varphi{}_z\Gamma_z{}^\varphi{}_\varphi - \partial_z\Gamma_\varphi{}^\varphi{}_z + \Gamma_z{}^z{}_z\Gamma_\varphi{}^\varphi{}_z + \Gamma_z{}^\varphi{}_z\Gamma_\varphi{}^\varphi{}_\varphi$.

Le seguenti uguaglianze

$$\Gamma^{\varphi}_{\varphi z} = \Gamma^{\varphi}_{z\varphi} = -\frac{1}{\left(1-\frac{z^2}{c^2}\right)}\cdot\frac{z}{c^2} \ , \qquad \Gamma^{z}_{zz} = \frac{a^2c^2z\left(1-\frac{z^2}{c^2}\right)+a^2z^3}{\left[a^2c^2z^2+c^6\left(1-\frac{z^2}{c^2}\right)\right]\left(1-\frac{z^2}{c^2}\right)} \ ,$$

$$\Gamma^{z}_{\varphi z} = \Gamma^{\varphi}_{\varphi\varphi} = \Gamma_z{}^\varphi{}_z = 0 \ ,$$

$$\partial z \Gamma^{\varphi}_{\varphi z} = -\frac{c^2\left(1-\frac{z^2}{c^2}\right)+2z^2}{c^4\left(1-\frac{z^2}{c^2}\right)^2} \ ,$$

$$\Gamma^{\varphi}_{\varphi z}\Gamma^{\varphi}_{z\varphi} = \frac{z^2}{c^4\cdot\left(1-\frac{z^2}{c^2}\right)^2} \ , \qquad \Gamma_z{}^z{}_z \Gamma_\varphi{}^\varphi{}_z = \frac{-a^2c^2z^2\left(1-\frac{z^2}{c^2}\right)-a^2z^4}{c^2\left[a^2c^2z^2+c^6\left(1-\frac{z^2}{c^2}\right)\right]\left(1-\frac{z^2}{c^2}\right)^2} \ ,$$

implicano

$$R_{\varphi z}{}^{\varphi}{}_{z} = -\Gamma_{\varphi}{}^{\varphi}{}_{z}\Gamma_{z}{}^{\varphi}{}_{\varphi} - \partial_z\Gamma_{\varphi}{}^{\varphi}{}_{z} + \Gamma_{z}{}^{z}{}_{z}\Gamma_{\varphi}{}^{\varphi}{}_{z} = -\frac{-c^2+z^2}{\left(1-\frac{z^2}{c^2}\right)^2\left[a^2z^2+c^4\left(1-\frac{z^2}{c^2}\right)\right]}$$

Quindi il tensore di curvatura è

$$R^{\dagger} = \frac{\mathrm{a^2c^2 - a^2z^2}}{\left[\frac{\mathrm{a^2z^2}}{\mathrm{c^2}} + \mathrm{c^2}\left(1-\frac{z^2}{c^2}\right)\right]^2} dz \otimes d\varphi \otimes dz \otimes d\varphi$$

$$+\frac{-c^2+z^2}{\left(1-\frac{z^2}{c^2}\right)^2\left[a^2z^2+c^4\left(1-\frac{z^2}{c^2}\right)\right]}\ dz \otimes d\varphi \otimes d\varphi \otimes dz$$

$$-\frac{\mathrm{a^2c^2 - a^2z^2}}{\left[\frac{\mathrm{a^2z^2}}{\mathrm{c^2}} + \mathrm{c^2}\left(1-\frac{z^2}{c^2}\right)\right]^2} d\varphi \otimes dz \otimes dz \otimes d\varphi$$

$$-\frac{-c^2+z^2}{\left(1-\frac{z^2}{c^2}\right)^2\left[a^2z^2+c^4\left(1-\frac{z^2}{c^2}\right)\right]}\ d\varphi \otimes dz \otimes d\varphi \otimes dz\ .QED$$

Possiamo osservare che, giustamente, il tensore di curvatura trovato è invariante rispetto alla rotazione intorno all'asse z , infatti non dipende da φ.

Per dimostrare la correttezza dei calcoli, verifichiamo le uguaglianze

$$\boldsymbol{R_{ij}{}^{h}{}_{k} = -R_{ji}{}^{h}{}_{k}}$$

$$\boldsymbol{R_{ij}{}^{h}{}_{k} + R_{ki}{}^{h}{}_{j} + R_{jk}{}^{h}{}_{i} = 0}\ .$$

Nel nostro caso, quindi abbiamo

$$\boldsymbol{R_{\varphi z}{}^{z}{}_{\varphi} = -R_{z\varphi}{}^{z}{}_{\varphi}}\ , \qquad \boldsymbol{R_{z\varphi}{}^{z}{}_{\varphi} = -R_{\varphi z}{}^{z}{}_{\varphi}}$$

$$\boldsymbol{R_{z\varphi}{}^{\varphi}{}_{z} = -R_{\varphi z}{}^{\varphi}{}_{z}}\ , \qquad \boldsymbol{R_{\varphi z}{}^{\varphi}{}_{z} = -R_{z\varphi}{}^{\varphi}{}_{z}}$$

$$\boldsymbol{R_{\varphi z}{}^{z}{}_{\varphi} + R_{z\varphi}{}^{z}{}_{\varphi} = 0}$$

$$\boldsymbol{R_{z\varphi}{}^{\varphi}{}_{z} + R_{\varphi z}{}^{\varphi}{}_{z} = 0}\ .\ QED$$

Tensore di Ricci

Proposizione 31 – Il tensore di Ricci è

$$\underline{r}^{\dagger} = \frac{\mathrm{a}^2\mathrm{c}^2 - \mathrm{a}^2\mathrm{z}^2}{\left[\frac{\mathrm{a}^2\mathrm{z}^2}{\mathrm{c}^2} + \mathrm{c}^2\left(1 - \frac{z^2}{c^2}\right)\right]^2}\, d\varphi \otimes d\varphi - \frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2\left[a^2z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]}\, dz \otimes dz \quad .$$

Con

$$\underline{r}_{\varphi\varphi} = \frac{\mathrm{a}^2\mathrm{c}^2 - \mathrm{a}^2\mathrm{z}^2}{\left[\frac{\mathrm{a}^2\mathrm{z}^2}{\mathrm{c}^2} + \mathrm{c}^2\left(1 - \frac{z^2}{c^2}\right)\right]^2},$$

$$\underline{r}_{zz} = -\frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2\left[a^2z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]} \; .$$

Dimostrazione -

$$\underline{\boldsymbol{r}} = \underline{\boldsymbol{r}}_{\boldsymbol{ij}}\, \boldsymbol{dx^i} \otimes \boldsymbol{dx^j} \, .$$

Con

$$\underline{\boldsymbol{r}}_{\boldsymbol{ij}} = \boldsymbol{\partial_h}\boldsymbol{\Gamma_i}^{\mathbf{h}}{}_{\mathbf{j}} - \boldsymbol{\Gamma_h}^{\mathbf{k}}{}_{\mathbf{j}}\boldsymbol{\Gamma_i}^{\mathbf{h}}{}_{\mathbf{k}} - \boldsymbol{\partial_i}\boldsymbol{\Gamma_h}^{\mathbf{h}}{}_{\mathbf{j}} + \boldsymbol{\Gamma_i}^{\mathbf{k}}{}_{\mathbf{j}}\boldsymbol{\Gamma_h}^{\mathbf{h}}{}_{\mathbf{k}} \; .$$

- $\underline{r}_{\varphi\varphi} = R_{z\varphi}{}^{z}{}_{\varphi}$.

$$\underline{r}_{\varphi\varphi} = R_{z\varphi}{}^{z}{}_{\varphi} = \frac{\mathrm{a}^2\mathrm{c}^2 - \mathrm{a}^2\mathrm{z}^2}{\left[\frac{\mathrm{a}^2\mathrm{z}^2}{\mathrm{c}^2} + \mathrm{c}^2\left(1 - \frac{z^2}{c^2}\right)\right]^2} \; .$$

- $\underline{r}_{zz} = R_{\varphi z}{}^{\varphi}{}_{z}$.

$$\underline{r}_{zz} = R_{\varphi z}{}^{\varphi}{}_{z} = -\frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2\left[a^2z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]} .$$

Quindi il tensore di Ricci è

$$\underline{r}^{\dagger} = \frac{\mathrm{a}^2\mathrm{c}^2 - \mathrm{a}^2\mathrm{z}^2}{\left[\frac{\mathrm{a}^2\mathrm{z}^2}{\mathrm{c}^2} + \mathrm{c}^2\left(1 - \frac{z^2}{c^2}\right)\right]^2}\ d\varphi \otimes d\varphi$$

$$- \frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2 \left[a^2 z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]}\ dz \otimes dz \quad .QED$$

Possiamo osservare che, giustamente, il tensore di Ricci trovato è invariante rispetto alla rotazione intorno all'asse z, infatti non dipende da φ.

Tensore di curvatura scalare

Proposizione 32 – La curvatura scalare è

$$\langle \underline{r}^{\dagger} \rangle = \frac{[(a^2 - c^2)z^2 + 2c^4]}{a^4 c^2}.$$

Dimostrazione -

$$\langle \underline{r}^{\dagger} \rangle = \langle \underline{r}^{\dagger}, \overline{g}^{ij\dagger} \rangle .$$

Con

$$\underline{r}^{\dagger} = \frac{a^2c^2 - a^2z^2}{\left[\frac{a^2z^2}{c^2} + c^2\left(1 - \frac{z^2}{c^2}\right)\right]^2}\, d\varphi \otimes d\varphi - \frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2 \left[a^2z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]}\, dz \otimes dz \quad .$$

$$\bar{g}^{\dagger} = \left[\frac{1}{a^2}\frac{1}{\left(1 - \frac{z^2}{c^2}\right)}\right] \partial\varphi \otimes \partial\varphi \; + \left\{\frac{\left(1 - \frac{z^2}{c^2}\right)}{\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]}\right\} \partial z \otimes \partial z .$$

Quindi la curvatura scalare è

$$\langle \underline{r}^{\dagger} \rangle = \left\{\frac{a^2c^2 - a^2z^2}{\left[\frac{a^2z^2}{c^2} + c^2\left(1 - \frac{z^2}{c^2}\right)\right]^2}\right\}\left[\frac{1}{a^2}\frac{1}{\left(1 - \frac{z^2}{c^2}\right)}\right]$$

$$- \left\{\frac{-c^2 + z^2}{\left(1 - \frac{z^2}{c^2}\right)^2 \left[a^2z^2 + c^4\left(1 - \frac{z^2}{c^2}\right)\right]}\right\}\left\{\frac{\left(1 - \frac{z^2}{c^2}\right)}{\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]}\right\} =$$

$$= \frac{[(a^2 - c^2)z^2 + 2c^4]}{a^4 c^2} \;\; . \; QED$$

Osservazione 1 -

Nel caso di $z = c$, il tensore di curvatura scalare tende ad un valore ben definito, in quanto questo dipende dalla base scelta.
A tal proposito abbiamo

$$\langle \underline{r}^{\dagger} \rangle \xrightarrow{z=c} \frac{[(a^2 - c^2)z^2 + 2c^4]}{a^4 c^2} = \frac{a^2 + c^2}{a^4} \quad .QED$$

Osservazione 2 -

Nel caso di $a = c$, se i calcoli sono corretti, dovremmo trovare il tensore di curvatura scalare della sfera che sappiamo valere

$$\langle \underline{r}^{\dagger} \rangle = \frac{2}{c^2} \ .$$

Infatti abbiamo

$$\langle \underline{r}^{\dagger} \rangle \xrightarrow{a=c} \frac{[(a^2 - c^2)z^2 + 2c^4]}{a^4 c^2} = \frac{2c^4}{c^6} = \frac{2}{c^2} \quad .QED$$

Osservazione 3 -

Nel caso di $z \to 0$, abbiamo

$$\langle \underline{r}^{\dagger} \rangle \xrightarrow{z\to 0} \frac{[(a^2 - c^2)z^2 + 2c^4]}{a^4 c^2} = \frac{2c^2}{a^4} \quad .$$

Nel caso di $a \gg c$, abbiamo

$$\langle \underline{r}^{\dagger} \rangle \xrightarrow{a\gg c} \frac{[(a^2 - c^2)z^2 + 2c^4]}{a^4 c^2} = 0 \quad .$$

Nel caso di $c \gg a$, abbiamo

$$\langle \underline{r}^{\dagger} \rangle \xrightarrow{c\gg a} \frac{[(a^2 - c^2)z^2 + 2c^4]}{a^4 c^2} = \infty \quad .QED$$

Tensore di curvatura covariante

Proposizione 33 – Il tensore di curvatura (in forma covariante) è

$$\underline{R}^{\dagger} = \left\{\frac{\mathrm{a^4z^4 + 3a^2c^4z^2 - 2a^2c^2z^4 - 3c^6z^2 + c^4z^4 + 2c^8}}{a^2c^6}\right\} d\varphi \wedge dz \otimes \; d\varphi \wedge dz \; .$$

Dimostrazione -

$$\underline{R}^{\dagger} = 2\, \langle \underline{r}^{\dagger} \rangle \, \eta \otimes \eta \; ,$$

$$\langle \underline{r}^{\dagger} \rangle \; = \frac{\mathrm{[(a^2 - c^2)z^2 + 2c^4]}}{a^4c^2} \; ,$$

$$\eta = a \sqrt{\left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right]} \, d\varphi \wedge dz \; ,$$

$$\underline{R}^{\dagger} = 2 \left\{\frac{\mathrm{[(a^2 - c^2)z^2 + 2c^4]}}{a^4c^2}\right\} a^2 \left[\frac{a^2z^2}{c^4} + \left(1 - \frac{z^2}{c^2}\right)\right] d\varphi \wedge dz \otimes \; d\varphi \wedge dz$$

$$= \left\{\frac{\mathrm{a^4z^4 + 3a^2c^4z^2 - 2a^2c^2z^4 - 3c^6z^2 + c^4z^4 + 2c^8}}{a^2c^6}\right\} d\varphi \wedge dz \otimes \; d\varphi \wedge dz \; .$$