Hyperboloid of one sheet

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Introduzione

Questo esercizio rappresenta un esempio applicativo della teoria delle superfici affrontata nel corso di Fisica Matematica del professor Marco Modugno, durante l'anno accademico 2002/2003.

Il lavoro è diviso in due fasi: nella prima si considera una varietà M coincidente con lo *spazio affine Euclideo* E, in cui, definito un sistema di coordinate iperbolico $(f, \vartheta.z)$, si sono calcolati la metrica e i coefficienti della connessione ∇.
Nella seconda parte si considera la sottovarietà Riemanniana Q, di codimensione 1, rappresentata dall' *iperboloide ad una falda*. Pertanto, da un sistema di coordinate adattato, si passa ad un sistema di coordinate indotto $(\vartheta^{\dagger}, z^{\dagger})$.
Ovvero, $f_{|Q} = 0$ è il vincolo.
Dell'iperboloide si sono, dunque, calcolate la metrica, la connessione e le curvature.

Questo studio oltre ad aver contribuito in modo significativo alla comprensione di alcuni argomenti affrontati durante le lezioni, rappresenta un punto di partenza, e un'occasione di stimolo, per l'applicazione di alcune teorie dell'ingegneria strutturale, come, ad esempio, la teoria lineare dei gusci elastici sottili.

E.M.

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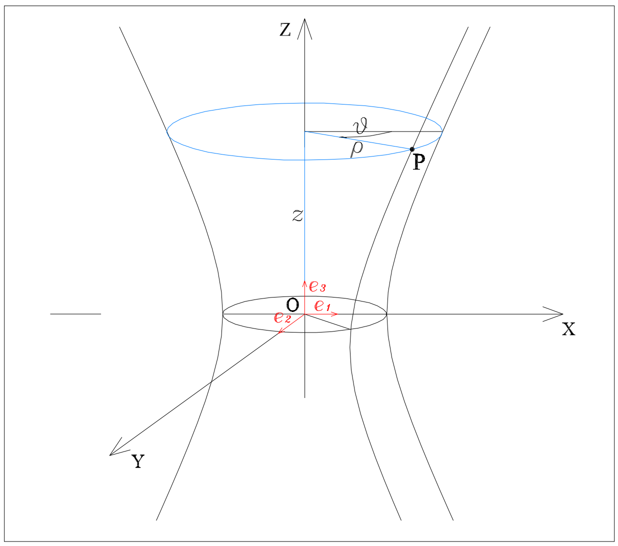
1 Hyperbolic Chart

We shall consider the *hyperbolic chart* $(f, \vartheta, z) : E \to I\!R^3$ which is associated a origin $o \in E$ and an orthonormal basis (e_i) of E.



We will use the coordinate function f to characterize the Hypersurface Q, i.e. $f = 0$, which is defined through the following implicit equation expressed in *cartesian chart*:

$$\frac{x^2+y^2}{a^2} - \frac{z^2}{b^2} - 1 = 0$$

We define the coordinate function f as follow

$$f = \rho - \rho(z)$$

where

$$\rho(z) = \tfrac{a}{b}\sqrt{b^2 + z^2} = \sqrt{x^2 + y^2}$$

So, the transition functions from the *cartesian chart* to *hyperbolic chart* are:

$$\begin{aligned}
x &= (f + \frac{a}{b}\sqrt{b^2 + z^2}) \cos \vartheta \\
y &= (f + \frac{a}{b}\sqrt{b^2 + z^2}) \sin \vartheta \\
z &= z
\end{aligned}$$

2 Metric

The coordinate expression of the covariant metric is:

$$\begin{aligned}
g = \; & df \otimes df + (f + \frac{a}{b}\sqrt{b^2 + z^2})^2 d\vartheta \otimes d\vartheta \\
& + (\frac{a}{b}\frac{z}{\sqrt{b^2 + z^2}})(df \otimes dz + dz \otimes df) \\
& + (\frac{a^2 z^2}{b^2(b^2 + z^2)} + 1) dz \otimes dz
\end{aligned}$$

The coordinate expression of the contravariant metric is:

$$\begin{aligned}
\bar{g} = & (\frac{a^2 z^2}{b^2(b^2 + z^2)} + 1)\partial f \otimes \partial f \\
& - (\frac{a}{b}\frac{z}{\sqrt{b^2 + z^2}})(\partial f \otimes \partial z + \partial z \otimes \partial f) \\
& + \frac{1}{(f + \frac{a}{b}\sqrt{b^2 + z^2})^2}\partial\vartheta \otimes \partial\vartheta + \partial z \otimes \partial z
\end{aligned}$$

Now we can write the coordinate expression of the metric function:

$$G = \frac{1}{2}[\dot{f}^2 + 2\frac{a}{b}\frac{z}{\sqrt{b^2 + z^2}}\dot{f}\dot{z} + (f + \frac{a}{b}\sqrt{b^2 + z^2})^2\dot{\vartheta}^2 + (\frac{a^2 z^2}{b^2(b^2 + z^2)} + 1)\dot{z}^2]$$

3 Connection

Let us to compute the coefficients of the connection ∇ in hyperbolic coordinates, by means *Lagrange formulas*.

Proposition. 3.1. *In hyperbolic coordinates the non-vanishing coefficient of* ∇ *are*

$$\Gamma_{zz}^{\ f} = \frac{ab}{(\sqrt{b^2+z^2})^3}$$

$$\Gamma_{\vartheta\vartheta}^{\ f} = -(f + \frac{a}{b}\sqrt{b^2+z^2})$$

$$\Gamma_{f\vartheta}^{\ \vartheta} = \Gamma_{\vartheta f}^{\ \vartheta} = \frac{1}{f + \frac{a}{b}\sqrt{b^2+z^2}}$$

$$\Gamma_{z\vartheta}^{\ \vartheta} = \Gamma_{\vartheta z}^{\ \vartheta} = \frac{az}{b(f + \frac{a}{b}\sqrt{b^2+z^2})\sqrt{b^2+z^2}}$$

PROOF. The covariant curvature of a curve $c : I\!R \to E$ is given by

$$\begin{aligned}(\nabla dc)_f &= D^2c^f + \frac{a}{b}\frac{c^z}{\sqrt{b^2+(c^z)^2}}D^2c^z + \frac{a}{b}\frac{b^2}{(b^2+(c^z)^2)^{\frac{3}{2}}}(Dc^z)^2 \\ &- (c^f + \frac{a}{b}\sqrt{b^2+(c^z)^2})(Dc^\vartheta)^2\end{aligned}$$

$$\begin{aligned}(\nabla dc)_\vartheta &= (c^f + \frac{a}{b}\sqrt{b^2+(c^z)^2})^2D^2c^\vartheta + 2(c^f + \frac{a}{b}\sqrt{b^2+(c^z)^2}Dc^fDc^\vartheta \\ &+ 2\frac{a}{b}(c^f + \frac{a}{b}\sqrt{b^2+(c^z)^2})\frac{c^z}{\sqrt{b^2+(c^z)^2}}Dc^zDc^\vartheta\end{aligned}$$

$$\begin{aligned}(\nabla dc)_z &= \frac{a^2c^z}{(b^2+(c^z)^2)^2}(Dc^z)^2 + \frac{a}{b}\frac{c^z}{\sqrt{b^2+(c^z)^2}}D^2c^f \\ &+ (\frac{a^2(c^z)^2}{b^2(b^2+(c^z)^2)} + 1)D^2c^z \\ &- (c^f + \frac{a}{b}\sqrt{b^2+(c^z)^2})\frac{a}{b}\frac{c^z}{\sqrt{b^2+(c^z)^2}}(Dc^\vartheta)^2\end{aligned}$$

Hence the contravariant curvature of c is given by

$$
\begin{aligned}
(\nabla dc)^f &= D^2 c^f + \frac{ab}{(b^2+(c^z)^2)^{\frac{3}{2}}}(Dc^z)^2 \\
&- (c^f + \frac{a}{b}\sqrt{b^2+(c^z)^2})(Dc^\vartheta)^2
\end{aligned}
$$

$$
\begin{aligned}
(\nabla dc)^\vartheta &= D^2 c^\vartheta + \frac{2}{(c^f + \frac{a}{b}\sqrt{b^2+(c^z)^2})} Dc^f Dc^\vartheta \\
&+ 2\frac{a}{b}\frac{c^z}{(c^f + \frac{a}{b}\sqrt{b^2+(c^z)^2})\sqrt{b^2+(c^z)^2}} Dc^\vartheta Dc^z
\end{aligned}
$$

$$
(\nabla dc)^z = D^2 c^z
$$

4 Hyperboloid

Now, let us to suppose that the submanifold Q is the Hyperboloid of one sheet characterized by the constraint $\rho = \frac{a}{b}\sqrt{b^2+z^2}$, i.e. $f = 0$. So, from the adapted hyperbolic chart (f, ϑ, z), we obtain the induced chart $(\vartheta^\dagger, z^\dagger)$[1].

4.1 First fundamental form of Q

Proposition. 4.1.1. *The coordinate expression of the covariant and the contravariant induced metric of Q, and the metric function are, respectively*

$$
\begin{aligned}
g^\dagger &= \frac{a^2}{b^2}(b^2+z^2)d\vartheta \otimes d\vartheta + (\frac{a^2 z^2}{b^2(b^2+z^2)} + 1)dz \otimes dz \\
\bar{g}^\dagger &= \frac{b^2}{a^2}\frac{1}{(b^2+z^2)}\partial\vartheta \otimes \partial\vartheta + \frac{b^2(b^2+z^2)}{a^2z^2+b^2(b^2+z^2)}\partial z \otimes \partial z \\
G^\dagger &= \frac{1}{2}[\frac{a^2}{b^2}(b^2+z^2)\dot{\vartheta}^2 + (\frac{a^2z^2}{(b^2+z^2)b^2} + 1)\dot{z}^2]
\end{aligned}
$$

4.2 Connection

Let us to compute the coefficients of the connection $\nabla^\dagger$ in hyperbolic coordinates, by means *Lagrange formulas.*

Proposition. 4.2.1. *The non-vanishing coefficients of $\Gamma^\dagger$ are*

$$
\begin{aligned}
{\Gamma^\dagger}_{z\vartheta}^{\ \ \vartheta} = {\Gamma^\dagger}_{\vartheta z}^{\ \ \vartheta} &= \frac{z}{(b^2+z^2)} \\
{\Gamma^\dagger}_{zz}^{\ \ z} &= \frac{a^2b^2z}{[a^2z^2+b^2(b^2+z^2)](b^2+z^2)} \\
{\Gamma^\dagger}_{\vartheta\vartheta}^{\ \ z} &= -\frac{a^2z(b^2+z^2)}{a^2z^2+b^2(b^2+z^2)}
\end{aligned}
$$

PROOF. The covariant curvature of a curve $c : I\!R \to Q$ is given by

$$
(\nabla dc)_\theta = \frac{a^2}{b^2}(b^2+(c^z)^2)D^2c^\theta + \frac{a^2}{b^2}2c^zDc^zDc^\theta
$$

$$
(\nabla dc)_z = \frac{a^2c^z}{(b^2+(c^z)^2)^2}(Dc^z)^2 + \frac{a^2(c^z)^2+b^2(b^2+(c^z)^2)}{b^2(b^2+(c^z)^2)}D^2c^z - \frac{a^2}{b^2}c^z(Dc^\theta)^2
$$

[1]Later on we'll omit the symbol † on ϑ and z

Hence the contravariant acceleration ∇dc of c is given by

$$(\nabla dc)^{\vartheta} = D^2 c^{\vartheta} + \frac{2c^z}{(b^2 + (c^z)^2)}$$

$$\begin{aligned}(\nabla dc)^{z} = D^2 c^{z} &+ \frac{a^2 b^2 c^z}{[a^2 (c^z)^2 + b^2 (b^2 + (c^z)^2)](b^2 + (c^z)^2)} (Dc^z)^2 \\ &- \frac{a^2 c^z (b^2 + (c^z)^2)}{a^2 (c^z)^2 + b^2 (b^2 + (c^z)^2)} (Dc^{\vartheta})^2\end{aligned}$$

Proposition. 4.2.1. *The unit normal vector field is*

$$n = \sqrt{\frac{a^2 z^2 + b^2 (b^2 + z^2)}{b^2 (b^2 + z^2)}} \partial f - \frac{az}{\sqrt{a^2 z^2 + b^2 (b^2 + z^2)}} \partial z$$

PROOF. Let $\omega : Q \to T_Q M$ be a generic vector field. We can compute the vector field $\omega_n \in TQ^{\perp}$ by means of the orthogonal projection. $\omega_n = \pi^{\perp}(\omega)$

So we obtain

$$n = \frac{\omega_n}{\sqrt{g(\omega_n, \omega_n)}}$$

An alternative way to compute n is by means of the cross product. In this case we have

$$n = \frac{\partial \vartheta \times \partial z}{\|\partial \vartheta \times \partial z\|}$$

Proposition. 4.2.1. *The Weingarten's map and the second fundamental form are*

$$L = \frac{b^2}{a\sqrt{a^2z^2 + b^2(b^2+z^2)}} d\vartheta \otimes \partial\vartheta - \frac{ab^4}{[a^2z^2 + b^2(b^2+z^2)]^{\frac{3}{2}}} dz \otimes \partial z$$

$$\underline{L} = \frac{a(b^2+z^2)}{\sqrt{a^2z^2 + b^2(b^2+z^2)}} d\vartheta \otimes d\vartheta - \frac{ab^2}{(b^2+z^2)\sqrt{a^2z^2 + b^2(b^2+z^2)}} dz \otimes dz$$

Proposition. 4.2.1. *The total curvature (Gauss curvature), and the mean curvature of hyperboloid are, respectively*

$$K = -\frac{b^6}{[a^2z^2 + b^2(b^2+z^2)]^2}$$

$$H = \frac{a^2b^2(z^2-b^2) + b^4(b^2+z^2)}{a[a^2z^2 + b^2(b^2+z^2)]^{\frac{3}{2}}}$$

The principal curvatures are

$$\lambda_1 = \frac{b^2(b^2+z^2)}{a(b^2+z^2)\sqrt{a^2z^2 + b^2(b^2+z^2)}}$$

$$\lambda_2 = -\frac{ab^4}{[a^2z^2 + b^2(b^2+z^2)]^{\frac{3}{2}}}$$

4.3 Curvature

Proposition. 4.3.1. *The curvature tensor, the* Ricci *tensor, the scalar curvature and the covariant curvature tensor are, respectively*

$$R^{\dagger} = -\frac{b^4}{[a^2z^2 + b^2(b^2+z^2)](b^2+z^2)} d\vartheta \otimes dz \otimes \partial\vartheta \otimes dz$$

$$- \frac{a^2b^4(b^2+z^2)}{[a^2z^2 + b^2(b^2+z^2)]^2} dz \otimes d\vartheta \otimes \partial z \otimes d\vartheta$$

$$+ \frac{b^4}{[a^2z^2 + b^2(b^2+z^2)](b^2+z^2)} dz \otimes d\vartheta \otimes \partial\vartheta \otimes dz$$

$$+ \frac{a^2b^4(b^2+z^2)}{[a^2z^2 + b^2(b^2+z^2)]^2} d\vartheta \otimes dz \otimes \partial z \otimes d\vartheta$$

$$Ricci^{\dagger} = -\frac{a^2b^4(b^2+z^2)}{[a^2z^2 + b^2(b^2+z^2)]^2} d\vartheta \otimes d\vartheta - \frac{b^4}{(b^2+z^2)[a^2z^2 + b^2(b^2+z^2)]} dz \otimes dz$$

$$\langle R\rangle^{\dagger} = -2\frac{b^6}{[a^2z^2 + b^2(b^2+z^2)]^2}$$

$$\underline{R}^{\dagger} = 2\langle R\rangle^{\dagger}\eta^{\dagger} \otimes \eta^{\dagger} = -\frac{4a^2b^2}{a^2z^2 + b^2(b^2+z^2)} d\vartheta \wedge dz \otimes d\vartheta \wedge dz$$

where the *volume form* induced by the metric $g^{\dagger}$ and by the orientation of the chosen chart is

$$\eta^{\dagger} = \frac{a\sqrt{a^2z^2 + b^2(b^2+z^2)}}{b^2} d\vartheta \wedge dz$$

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