1 Hyperbolic Chart

We shall consider the *hiperbolic chart* $(x, y, f) : E \longrightarrow R$, which are associated a point $o \in E$ and an orthonormal basis (e_i) of $\overline{E}$.
By definition, the transition function with respect to the cartesian chart are:

$$\begin{aligned} x &= x \\ y &= y \\ z &= f + ax^2 - ay^2 \quad \text{with} \quad a > 0 \end{aligned}$$

Hence, we obtain

$$f = z - a(x^2 - y^2).$$

2 Metric

The coordinate expression of the covariant and controvariant metrics are

$$\begin{aligned} g &= (1 + 4a^2x^2)dx \otimes dx + (1 + 4a^2y^2)dy \otimes dy + df \otimes df - 4a^2xy(dx \otimes dy + dy \otimes dx) + \\ &\quad +2ax(dx \otimes df + df \otimes dx) - 2ay(dy \otimes df + df \otimes dy) \end{aligned}$$

$$\begin{aligned} \overline{g} &= \partial x \otimes \partial x + \partial y \otimes \partial y + (1 + 4a^2x^2 + 4a^2y^2)\partial f \otimes \partial f - 2ax(\partial x \otimes \partial f + \partial f \otimes \partial x) + \\ &\quad +2ay(\partial y \otimes \partial f + \partial f \otimes \partial y) \end{aligned}$$

Hence, the coordinate expression of the *metric function* is

$$G = \frac{1}{2}\Big((1 + 4a^2x^2)\dot{x}^2 - 8a^2xy\dot{x}\dot{y} + 4ax\dot{x}\dot{f} + (1 + 4a^2y^2)\dot{y}^2 - 4ay\dot{y}\dot{f} + \dot{f}^2\Big)$$

The *volume forme* induced by the metric g and by orientation of the chosen charts has coordinate expression

$$\eta = dx \wedge dy \wedge df$$

3 Connection

Let us compute the coefficient of the connection ∇, in hyperbolic coordinates, by means of the *Lagrange formulas*.

3.1 Proposition.

The non-vanishing coefficient of ∇ are.

$$
\begin{aligned}
\Gamma^{\dagger}{}_{x}{}^{f}{}_{x} &= 2a \\
\Gamma^{\dagger}{}_{y}{}^{f}{}_{y} &= -2a
\end{aligned}
$$

PROOF. The covariant curvature of a curve $c : R \longrightarrow E$ is given by

$$
\begin{aligned}
(\nabla dc)_x &= (1+4a^2x^2)D^2c^x + 4a^2x(Dc^x)^2 - 4a^2xyD^2c^y - 4a^2x(Dc^y)^2 + 2axD^2c^f \\
(\nabla dc)_y &= (1+4a^2y^2)D^2c^y + 4a^2y(Dc^y)^2 - 4a^2xyD^2c^x - 4a^2y(Dc^x)^2 - 2ayD^2c^f \\
(\nabla dc)_f &= D^2c^f + 2a(Dc^x)^2 - 2a(Dc^y)^2 + 2axD^2c^x - 2ayD^2c^y
\end{aligned}
$$

hence the curvature of c is given by

$$
\begin{aligned}
(\nabla dc)^x &= D^2c^x \\
(\nabla dc)^y &= D^2c^y \\
(\nabla dc)^f &= D^2c^f + 2a(Dc^x)^2 - 2a(Dc^y)^2
\end{aligned}
$$

4 Hyperboloid

Now, we supposed that the submanifold Q is the hyperboloid H characterised by the constraint $z = ax^2 - ay^2$, i.e. $f = 0$.
We shall refer to the adapted hyperbolic chart (x, y, f).

4.1 Metric

4.1.1 Proposition.

The coordinate expression of the covariant and controvariant metric, of the metric function and of the volume form of H are

$$
\begin{aligned}
g^{\dagger} &= (1+4a^2x^2)dx\otimes dx + (1+4a^2y^2)dy\otimes dy - 4a^2xy(dx\otimes dy + dy\otimes dx) \\
\overline{g^{\dagger}} &= \frac{1+4a^2y^2}{1+4a^2x^2+4a^2y^2}\partial x\otimes\partial x + \frac{1+4a^2x^2}{1+4a^2x^2+4a^2y^2}\partial y\otimes\partial y +
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{4a^2xy}{1+4a^2x^2+4a^2y^2}(\partial x\otimes\partial y+\partial y\otimes\partial x) \\
G^\dagger &= \frac{1}{2}\Big((1+4a^2x^2)\dot{x}^2-8a^2xy\dot{x}\dot{y}+(1+4a^2x^2)\dot{y}^2\Big) \\
\eta^\dagger &= (1+4a^2x^2+4a^2y^2)^{\frac{1}{2}}dx\wedge dy
\end{aligned}
$$

4.2 Connection

let us the compute the coefficient of the connection $\nabla^\dagger$, in the adapted coordinates, by means of the Lagrange formulas.

4.2.1 Proposition.

The non-vanishing coefficient of $\nabla^\dagger$ are

$$
\Gamma^{\dagger}{}_x{}^x{}_x = \frac{4a^2x}{1+4a^2x^2+4a^2y^2} \qquad \Gamma^{\dagger}{}_y{}^x{}_y = -\frac{4a^2x}{1+4a^2x^2+4a^2y^2}
$$

$$
\Gamma^{\dagger}{}_x{}^y{}_x = -\frac{4a^2y}{1+4a^2x^2+4a^2y^2} \qquad \Gamma^{\dagger}{}_y{}^y{}_y = \frac{4a^2x}{1+4a^2x^2+4a^2y^2}
$$

PROOF. The covariant curvature of a curve $c: R \longrightarrow Q$ is given by

$$
\begin{aligned}
(\nabla dc)_x &= (1+4a^2x^2)D^2c^x+4a^2x(Dc^x)^2-4a^2xyD^2c^y-4a^2x(Dc^y)^2 \\
(\nabla dc)_y &= (1+4a^2y^2)D^2c^y+4a^2y(Dc^y)^2-4a^2xyD^2c^x-4a^2x(Dc^x)^2
\end{aligned}
$$

hence the curvature of c is given by

$$
\begin{aligned}
(\nabla dc)^x &= D^2c^x+\frac{4a^2x}{1+4a^2x^2+4a^2y^2}(Dc^x)^2-\frac{4a^2x}{1+4a^2x^2+4a^2y^2}(Dc^y)^2 \\
(\nabla dc)^y &= D^2c^y+\frac{4a^2y}{1+4a^2x^2+4a^2y^2}(Dc^y)^2-\frac{4a^2y}{1+4a^2x^2+4a^2y^2}(Dc^x)^2
\end{aligned}
$$

4.2.2 Proposition.

The normal versor is

$$n = -\frac{2ax}{(1+4a^2x^2+4a^2y^2)^{\frac{1}{2}}}\partial x + \frac{2ay}{(1+4a^2x^2+4a^2y^2)^{\frac{1}{2}}}\partial y + (1+4a^2x^2+4a^2y^2)^{\frac{1}{2}}\partial f$$

4.3 Second fundamental form

4.3.1 Proposition.

The Weingarten's map and the second fundamental form are

$$L = \frac{2a}{(1+4a^2x^2+4a^2y^2)^{\frac{3}{2}}} \Big(-(1+4a^2y^2)dx\otimes\partial x - 4a^2xydx\otimes\partial y + \\ +4a^2xydx\otimes\partial x + (1+4a^2x^2)dy\otimes\partial y\Big)$$

$$\underline{L} = -\frac{2a}{(1+4a^2x^2+4a^2y^2)} \Big(dx\otimes dx - dy\otimes dy\Big).$$

4.3.2 Corollary.

We have

$$N = -\frac{2a}{1+4a^2x^2+4a^2y^2} \Big(-\frac{2ax}{(1+4a^2x^2+4a^2y^2)^{\frac{1}{2}}}(dx\otimes dx - dy\otimes dy)\otimes\partial x + \\ +\frac{2ay}{(1+4a^2x^2+4a^2y^2)^{\frac{1}{2}}}(dx\otimes dx - dy\otimes dy)\otimes\partial y + \\ +(1+4a^2x^2+4a^2y^2)^{\frac{1}{2}}(dx\otimes dx - dy\otimes dy)\otimes\partial f\Big)$$

4.3.3 Corollary.

The total and the mean curvature are

$$K = detL = -\frac{4a^2}{(1+4a^2x^2+4a^2y^2)^2} \qquad H = trL = \frac{8a^3(x^2-y^2)}{(1+4a^2x^2+4a^2y^2)^{\frac{3}{2}}}$$

4.3.4 Corollary.

The hyperboloid H is a ruled surface and not developable.

4.3.5 Corollary.

The principal curvature and the principal vector are

$$\lambda' = \frac{2a}{(1+4a^2x^2+4a^2y^2)^{\frac{3}{2}}}\left((2a^2x^2-2a^2y^2)-\left((2a^2x^2-2a^2y^2)^2+1+4a^2x^2+4a^2y^2\right)^{\frac{1}{2}}\right)$$

$$\lambda'' = \frac{2a}{(1+4a^2x^2+4a^2y^2)^{\frac{3}{2}}}\left((2a^2x^2-2a^2y^2)+\left((2a^2x^2-2a^2y^2)^2+1+4a^2x^2+4a^2y^2\right)^{\frac{1}{2}}\right)$$

$$v' = \partial x - \frac{4a^2xy}{1+2a^2x^2+2a^2y^2+\left((2a^2x^2-2a^2y^2)^2+1+4a^2x^2+4a^2y^2\right)^{\frac{1}{2}}}\partial y$$

$$v'' = -\frac{4a^2xy}{1+2a^2x^2+2a^2y^2+\left((2a^2x^2-2a^2y^2)^2+1+4a^2x^2+4a^2y^2\right)^{\frac{1}{2}}}\partial x + \partial y$$

4.3.6 Corollary.

The vector conjugate and the vector asymptotic are

$$\begin{aligned} u' &= \alpha\partial x + \beta\partial y \\ u'' &= \partial x + \frac{\alpha}{\beta}\partial y \\ w' &= \partial x + \partial y \\ w'' &= \partial x - \partial y \end{aligned}$$

4.4 Curvature

4.4.1 Proposition.

The curvature tensor is

$$R^{\dagger} = \frac{8a^2}{(1+4a^2x^2+4a^2y^2)^2}\Big(4a^2xy dx\wedge dy\otimes\partial x\otimes dx - (1+4a^2y^2)dx\wedge dy\otimes\partial x\otimes dy + $$
$$+(1+4a^2x^2)dx\wedge dy\otimes\partial y\otimes dx - 4a^2xy dx\wedge dy\otimes\partial y\otimes dy\Big).$$

4.4.2 Corollary.

The covariant curvature tensor is

$$\underline{R}^\dagger = -\frac{16a^2}{(1+4a^2x^2+4a^2y^2)^2}\Big((1+4a^2x^2+4a^2y^2)dx\wedge dy\otimes dx\wedge dy\Big) = 2\langle R^\dagger\rangle \eta\otimes\eta.$$

4.4.3 Corollary.

The Ricci tensor is

$$\begin{aligned} Ricci^\dagger = -\frac{4a^2}{(1+4a^2x^2+4a^2y^2)^2}\Big(& -4a^2xydy\otimes dx + (1+4a^2x^2)dx\otimes dx + \\ & +(1+4a^2y^2)dy\otimes dy - 4a^2xydx\otimes dy\Big). = \\ & = -\frac{4a^2}{(1+4a^2x^2+4a^2y^2)^2}g^\dagger = \frac{1}{2}\langle R^\dagger\rangle g^\dagger. \end{aligned}$$

4.4.4 Corollary.

The scalar curvature is

$$\langle R^\dagger\rangle = -\frac{8a^2}{(1+4a^2x^2+4a^2y^2)^2} = 2K.$$