A short introduction to

Linear Shell Theory

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Abstract

This short introduction to shell theory aims at providing civil engineering students with some natural applications of the theoretical background developed during the course of Mathematical Physics taught by Prof. Marco Modugno. It is worth pointing out, in fact, how the whole mechanics of thin shell structures is basically referable to the geometry of the mid–surface.

After a brief introduction to recall the basic notions on the theory of surfaces, first the deformation and then the equilibrium of shell continua will be presented. The linear constitutive law will be just mentioned. At the end, applications to some typical shell geometries subjected to a membrane state of stress will also be proposed.

These notes collect the content of two seminars delivered on June 2009 by the Author for the course of Mathematical Physics at School of Civil Engineering, University of Florence.

A special thanks is devoted to Prof. Marco Modugno for the constructive discussions.

The text has not been thoroughly revised and may contain typing mistakes and mismatching references.

Contents

1 Introduction

1.1 Surfaces

Let E be the affine Euclidean space. The submanifold $Q \subset E$ is a surface if $dim\, Q = 2$.

Suppose $Q \subset E$ is a surface which can be described by an induced coordinate system of dimension $q = m - k$, where m is the dimension of E and k denotes the number of constraints (codimension of Q). Since Q is a surface $m = 3$, $k = 1$ and $q = 2$. The induced coordinate system is given by

$$X^\dagger : Q \to I\!R^q : p \mapsto x^\alpha(p) \tag{1}$$

From now on the quantities living on Q will be distinguished by the symbol $\dagger$ and the components will be written using superscripts and subscripts, running from 1 to 2, in Greek letters. The Latin indices will denote components of quantities that are applied on Q but lie out, namely belonging to the vector space $\bar{T_Q}E$.

The unit normal vector is defined as follows

$$\bar{n} : Q \to \bar{TQ}^\perp \text{ so that } g\,(\bar{n}, \bar{n}) = 1. \tag{2}$$

where g is the metric tensor defined on $\bar{TE}$ and $\bar{TQ}^\perp$ is the orthogonal space.

Analogously, on the surface Q it is possible to define the induced metric as

$$g^\dagger : \bar{TQ} \times \bar{TQ} \to I\!R$$

that in components[1] becomes

$$g^\dagger = g_{\alpha\beta} \underline{\mathrm{d}}^\alpha \otimes \underline{\mathrm{d}}^\beta$$

Given two vectorial fields $\bar{u} : Q \to \bar{TQ}$ and $\bar{v} : Q \to \bar{TQ}$, the covariant derivative of $\bar{v}$ with respect to $\bar{u}$ can be split as follows

$$\nabla_{\bar{u}} \bar{v} = \nabla^{\|}_{\bar{u}} \bar{v} + \nabla^{\perp}_{\bar{u}} \bar{v} \tag{3}$$

where

$$\nabla^{\|} : \bar{TQ} \times \bar{TQ} \to \bar{TQ} \tag{4}$$

$$\nabla^{\perp} : \bar{TQ} \times \bar{TQ} \to \bar{TQ}^\perp \tag{5}$$

The application $\nabla^{\|}$ is called *second fundamental form* of the surface. For further details see [1] and [2].

[1] In some books the covariant components of the metric tensor $g^\dagger$ are also denoted as $g_{11} = E, \quad g_{12} = F, \quad g_{22} = G.$

The *Weingarten*[2]map L is defined as the following endomorphism

$$L := \nabla \bar{n} : T\bar{Q} \rightarrow T\bar{Q} : \bar{u} \mapsto \nabla_{\bar{u}} \bar{n} \tag{6}$$

In addition to that, the *total curvature (Gauss curvature)* K and the *mean curvature* H of a surface Q are defined as follows

$$K := \det L : Q \rightarrow I\!R \tag{7}$$

$$H := \operatorname{tr} L : Q \rightarrow I\!R \tag{8}$$

Finally, eigenvalues of L are called *principal curvatures*. See [1].

Let $\underline{L}$ be the second order covariant tensor related to the Weingarten endomorphism L by the metric tensor $g^{\dagger}$, so that

$$\underline{L} := \nabla \underline{n} : T\bar{Q} \times T\bar{Q} \rightarrow I\!R : (\bar{u}, \bar{v}) \mapsto g\left(L\left(\bar{u}\right), \bar{v}\right) = \nabla_{\bar{u}} \bar{n} \cdot \bar{v} \tag{9}$$

where $\underline{n} = g^{\flat}\left(\bar{n}\right)$.

The following differentiation

$$0 = \nabla_{\bar{u}}\left(g\left(\bar{v}, \bar{n}\right)\right) = g\left(\nabla_{\bar{u}} \bar{v}, \bar{n}\right) + g\left(\bar{v}, \nabla_{\bar{u}} \bar{n}\right) \Rightarrow \tag{10}$$

$$g\left(\nabla_{\bar{u}} \bar{v}, \bar{n}\right) = -g\left(\bar{v}, \nabla_{\bar{u}} \bar{n}\right) \tag{11}$$

proves that the scalar quantity $\underline{L}\left(\bar{u}, \bar{v}\right)$ represents the normal component to the surface Q of the covariant derivative, namely

$$\nabla_{\bar{u}} \bar{v} = \nabla^{\parallel}_{\bar{u}} \bar{v} - \underline{L}\left(\bar{u}, \bar{v}\right) \bar{n} \tag{12}$$

Dealing with mechanics of shell continuums, equation (12) will be often used. For this reason in the following it is worth expanding its expression in components.

Suppose $\{\bar{\partial}_{\alpha}\}$, $\alpha = 1, 2$ is a basis related to the induced coordinate system describing the surface, then

$$\nabla_{\bar{\partial}_{\beta}} \bar{\partial}_{\alpha} = \nabla^{\dagger}_{\bar{\partial}_{\beta}} \bar{\partial}_{\alpha} - \underline{L}\left(\bar{\partial}_{\beta}, \bar{\partial}_{\alpha}\right) \bar{n} \tag{13}$$

and for both right hand terms we have, respectively

$$\nabla^{\dagger}_{\bar{\partial}_{\beta}} \bar{\partial}_{\alpha} = \underline{\mathrm{d}}^{\gamma}\left(\bar{\partial}_{\beta}\right) \left(\partial_{\gamma}\left(\underline{\mathrm{d}}^{\omega}\left(\bar{\partial}_{\alpha}\right)\right) + \Gamma^{\omega}_{\gamma\lambda} \underline{\mathrm{d}}^{\lambda}\left(\bar{\partial}_{\alpha}\right)\right) \bar{\partial}_{\omega} \tag{14}$$

$$= \delta^{\gamma}_{\beta}\left(\Gamma^{\omega}_{\gamma\lambda} \delta^{\lambda}_{\alpha}\right) \bar{\partial}_{\omega} = \Gamma^{\omega}_{\beta\alpha} \bar{\partial}_{\omega} \tag{15}$$

$$\underline{L}\left(\bar{\partial}_{\beta}, \bar{\partial}_{\alpha}\right) = \left(L\left(\bar{\partial}_{\beta}\right) \cdot \bar{\partial}_{\alpha}\right) = \nabla_{\bar{\partial}_{\beta}} \bar{n} \cdot \bar{\partial}_{\alpha} \tag{16}$$

$$= L^{\omega}_{\beta} \bar{\partial}_{\omega} \cdot \bar{\partial}_{\alpha} = L^{\omega}_{\beta} g_{\omega\alpha} = L_{\beta\alpha} \tag{17}$$

[2]Julius Weingarten (March 2, 1836 Berlin - June 16, 1910 Freiburg) was a German mathematician.

Finally, equation (13) in components becomes

$$\nabla_\beta \bar{\partial}_\alpha = \Gamma^\omega_{\beta\alpha} \bar{\partial}_\omega - L_{\beta\alpha} \bar{n} \tag{18}$$

Note that in the remainder of this book, for the sake of brevity, we will use $\nabla_\beta \cdot$ instead of $\nabla_{\bar{\partial}_\beta} \cdot$.

Analogously, for an element of the contravariant basis, recalling the general equation for covariant derivatives, and considering the above Gauss splitting, we have the following expression

$$\nabla_\beta \underline{\mathrm{d}}^\alpha = -\Gamma^\alpha_{\beta\lambda} \underline{\mathrm{d}}^\lambda - L^\alpha_\beta \underline{n} \tag{19}$$

Often in the following shell theory we will deal with vector fields which do not belong to the tangent space, so it is useful to present an example of derivative of vectors applied in Q but lying out of the tangent space. Namely, suppose that $\bar{v} \in \bar{T_Q E}$. We can decompose the field $\bar{v}$ into the tangent and orthogonal component as follows

$$\bar{v} = \bar{v}^{\|} + \bar{v}^{\perp} \tag{20}$$

that in components is written as

$$\bar{v} = v^\alpha \bar{\partial}_\alpha + v^\xi \bar{n} \tag{21}$$

Hence, given $\bar{u} \in \bar{TQ}$ the derivative of $\bar{v}$ with respect to $\bar{u}$ is

$$\nabla_{\bar{u}} \bar{v} = \nabla_{\bar{u}} \bar{v}^{\|} + \nabla_{\bar{u}} \bar{v}^{\perp} = \nabla^\dagger_{\bar{u}} \bar{v}^{\|} - \underline{L}\left(\bar{u}, \bar{v}^{\|}\right) \bar{n} + \nabla_{\bar{u}} \bar{v}^{\perp} \tag{22}$$

that in components turns into

$$\nabla_{\bar{u}} \bar{v} = u^\beta \left(\partial_\beta v^\alpha + \Gamma^\alpha_{\beta\gamma} v^\gamma + v^\xi L^\alpha_\beta \right) \bar{\partial}_\alpha + u^\beta \left(v^\xi_{,\beta} - \underline{L}_{\alpha\beta} v^\alpha \right) \bar{n} \tag{23}$$

In the same way, the dual form $\underline{v} \in T^*_Q E$ can be differentiated as follows

$$\nabla_{\bar{u}} \underline{v} = \nabla_{\bar{u}} \underline{v}^{\|} + \nabla_{\bar{u}} \underline{v}^{\perp} = \nabla^\dagger_{\bar{u}} \underline{v}^{\|} - \underline{L}(\bar{u}, \underline{v}^{\|}) \underline{n} + \nabla_{\bar{u}} v^{\perp} \tag{24}$$

that in components becomes

$$\nabla_{\bar{u}} \underline{v} = u^\beta \left(\partial_\beta v_\alpha - \Gamma^\gamma_{\alpha\beta} v_\gamma + v_\xi \underline{L}_{\beta\gamma} \right) \underline{\mathrm{d}}^\gamma + u^\beta \left(v_{\xi,\beta} - L^\alpha_\beta v_\alpha \right) \underline{n} \tag{25}$$

Examples of surfaces will be provided in appendix 5, where, within the application of the shell theory, the above results will be applied to some well known geometries.

1.2 Shell continuum

We define a shell–shaped region modeled on a surface Q and with thickness 2ϵ as a continuous medium $G(\epsilon)$ embedded in the Euclidean space E each point of which is determined through a coordinate system $\{x^\alpha, \xi\} : G(\epsilon) \to \mathbb{R}^3$. Therefore, given $p^\star \in G(\epsilon)$ it is defined by its position p normally projected on Q - by using the surface coordinate system introduced in (1) - and by the normal coordinate ξ taken along the unit normal vector $\bar{n}$. In fact we have

$$p^\star \mapsto (x^\alpha(p), \xi(p)) \tag{26}$$

The basis induced by the coordinate system $\{x^\alpha, \xi\}$ is $\{\bar{\partial}_\alpha, \bar{n}\}$.

It is worthwhile pointing out that mechanics of shells - by virtue of such above statements - is traced back to the theory of surfaces, in fact vectors and tensors fields belonging to $T_Q^- E$ will always be split into the parallel and normal components.

Note also that the symbol $\star$ denotes quantities belonging to the shell continuum.

1.3 General assumptions

The shell theory here introduced is based on the following hypotheses

Hypothesis 1 *The shell is sufficiently thin, so that*

$$\frac{2\epsilon}{L} \ll 1 \qquad L = \min\{R_{\min}, L_{\min}\} \tag{27}$$

where $R_{\min}$ and $L_{\min}$ are the minimum radius and a typical dimension of the shell structure, respectively.

Hypothesis 2 (Linear theory) *Displacements are infinitesimally small such that their products can be neglected in the kinematic expressions. This assumption allows us to write the equilibrium equations in the unstrained shell configuration.*

Hypothesis 3 *The material filaments along the coordinate ξ remain straight throughout the deformation and no length change is allowed. Namely, the distance between $p^\star \in G(\epsilon)$ and the surface Q is unaltered*

$$\xi = \text{const.} \tag{28}$$

Hypothesis 4 (Kirchhoff–Love theory) *The line elements initially normal to the shell's mid–surface remain normal to it during the deformation.*

$$\bar{g}\left(\bar{\partial}_{\alpha_d}, \bar{n}_d\right) = 0 \tag{29}$$

where the subscript d is denotes quantities related to deformed configuration.

Note that the last hypothesis is nothing but the extension to a two–dimensional model of the Bernoulli theory for beams.

2 Strain tensor

A generic point $p^\star \in G(\epsilon)$ is determined by the vector $\bar{r}^\star$ referred to the global Cartesian axes, so that

$$\bar{r}^\star = \bar{r} + \xi \bar{n} \tag{30}$$

where $\xi \in (-\epsilon, \epsilon)$. See figure 1.

Let us suppose now that a quasi–static motion produces a deformed shell configuration points of which are univocally determined by the vector

$$\bar{r}_d^\star = \bar{r}_d + \xi_d \bar{n}_d \tag{31}$$

where $\xi_d \in (-\epsilon, \epsilon)$.

The displacement field is obtained by subtracting equations (30) and (31), so that

$$\bar{r}_d^\star - \bar{r}^\star = \bar{r}_d - \bar{r} + \xi \left(\bar{n}_d - \bar{n} \right) \tag{32}$$

where we have made use of hypothesis 3. Equation (32) allows us to define the positional field as a function of two vector fields

$$\bar{v} = \bar{r}_d - \bar{r} \quad \bar{v} \in T_{\bar{Q}}E \tag{33}$$

$$\bar{w} = \bar{n}_d - \bar{n} \quad \bar{w} \in T\bar{Q} \tag{34}$$

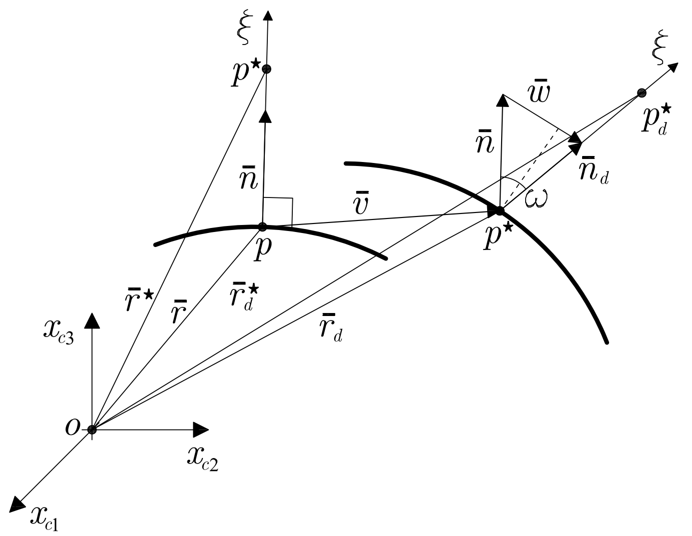

Figure 1: Two dimensional sketch of the displacement field for Kirchhoff–Love shells.

To obtain the strain tensor no more theoretical concepts are required. We already know the definition and we just need to compute the metric tensors associated to the coordinate systems in the strained and the original configurations, so we have

$$\gamma_{ij} = \begin{pmatrix} \gamma_{\alpha\beta} & \gamma_{\alpha 3} \\ \gamma_{3\alpha} & \gamma_{33} \end{pmatrix}$$

where

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(g^{\star}_{\alpha\beta_d} - g^{\star}_{\alpha\beta}\right) \tag{35}$$

$$\gamma_{\alpha 3} = \gamma_{3\alpha} = \frac{1}{2}\left(g^{\star}_{\alpha 3_d} - g^{\star}_{\alpha 3}\right) \tag{36}$$

$$\gamma_{33} = \frac{1}{2}\left(\bar{n}_d \cdot \bar{n}_d - \bar{n} \cdot \bar{n}\right) = 0 \tag{37}$$

According to equation (**??**) we have

$$g^{\star}_{\alpha\beta_d} = \bar{\partial}^{\star}_{\alpha_d} \cdot \bar{\partial}^{\star}_{\beta_d} \tag{38}$$

and

$$g^{\star}_{\alpha\beta} = \bar{\partial}^{\star}_{\alpha} \cdot \bar{\partial}^{\star}_{\beta} \tag{39}$$

so that

$$\begin{aligned}
\gamma_{\alpha\beta} &= \frac{1}{2}\left[\bar{\partial}^{\star}_{\alpha_d} \cdot \bar{\partial}^{\star}_{\beta_d} - \bar{\partial}^{\star}_{\alpha} \cdot \bar{\partial}^{\star}_{\beta}\right] = \\
&= \frac{1}{2}\left[\left(\bar{\partial}_{\alpha_d} + \xi\nabla_{\alpha}\bar{n}_d\right) \cdot \left(\bar{\partial}_{\beta_d} + \xi\nabla_{\beta}\bar{n}_d\right)\right] + \\
&- \frac{1}{2}\left[\left(\bar{\partial}_{\alpha} + \xi\nabla_{\beta}\bar{n}\right) \cdot \left(\bar{\partial}_{\beta} + \xi\nabla_{\alpha}\bar{n}\right)\right] = \\
&= \frac{1}{2}\left[\bar{\partial}_{\alpha_d} \cdot \bar{\partial}_{\beta_d} + \bar{\partial}_{\alpha_d} \cdot \xi\nabla_{\beta}\bar{n}_d + \bar{\partial}_{\beta_d} \cdot \xi\nabla_{\alpha}\bar{n}_d + \xi^2\nabla_{\alpha}\bar{n}_d \cdot \nabla_{\beta}\bar{n}_d\right] \\
&- \frac{1}{2}\left[\bar{\partial}_{\alpha} \cdot \bar{\partial}_{\beta} + \bar{\partial}_{\alpha} \cdot \xi\nabla_{\beta}\bar{n} + \bar{\partial}_{\beta} \cdot \xi\nabla_{\alpha}\bar{n} + \xi^2\nabla_{\alpha}\bar{n} \cdot \nabla_{\beta}\bar{n}\right]
\end{aligned} \tag{40}$$

where we realize that the tensor $\gamma_{\alpha\beta}$ can be split in three parts as follows

$$\gamma_{\alpha\beta} = \alpha_{\alpha\beta} + \xi\omega_{\alpha\beta} + \xi^2\varphi_{\alpha\beta} \tag{41}$$

We define the *stretching strain tensor* as

$$\alpha_{\alpha\beta} = \frac{1}{2}\left[\bar{\partial}_{\alpha_d} \cdot \bar{\partial}_{\beta_d} - \bar{\partial}_{\alpha} \cdot \bar{\partial}_{\beta}\right] = \frac{1}{2}\left(g_{\alpha\beta_d} - g_{\alpha\beta}\right) \tag{42}$$

next, the *first bending strain tensor* as

$$\omega_{\alpha\beta} = \frac{1}{2}\left[\bar{\partial}_{\alpha_d} \cdot \nabla_{\beta}\bar{n}_d + \bar{\partial}_{\beta_d} \cdot \nabla_{\alpha}\bar{n}_d - \bar{\partial}_{\alpha} \cdot \nabla_{\beta}\bar{n} - \bar{\partial}_{\beta} \cdot \nabla_{\alpha}\bar{n}\right] \tag{43}$$

and the *second bending strain tensor* as

$$\varphi_{\alpha\beta} = \frac{1}{2}\left[\nabla_{\alpha}\bar{n}_d \cdot \nabla_{\beta}\bar{n}_d - \nabla_{\alpha}\bar{n} \cdot \nabla_{\beta}\bar{n}\right] \tag{44}$$

Considering now that the displacements are small enough to be negligible the second order terms

$$\nabla_\alpha \bar{v} \cdot \nabla_\beta \bar{v} \simeq 0$$
$$\nabla_\alpha \bar{v} \cdot \nabla_\beta \bar{w} \simeq 0$$

and recalling equations (33) and (34), the stretching and the bending strain tensors become, respectively

$$\alpha_{\alpha\beta} = \frac{1}{2}\left(\bar{\partial}_\alpha \cdot \nabla_\beta \bar{v} + \bar{\partial}_\beta \cdot \nabla_\alpha \bar{v}\right) = \frac{1}{2}\left(v_{\alpha|\beta} + v_{\beta|\alpha} + 2v^\xi L_{\alpha\beta}\right) \tag{45}$$

$$\begin{aligned}\omega_{\alpha\beta} = & \frac{1}{2}\left(\bar{\partial}_\alpha \cdot \nabla_\beta \bar{w} + \bar{\partial}_\beta \cdot \nabla_\alpha \bar{w}\right) + \\ & + \frac{1}{2}\left(\nabla_\alpha \bar{v} \cdot \nabla_\beta \bar{n} + \nabla_\beta \bar{v} \cdot \nabla_\alpha \bar{n}\right) = \\ & = \frac{1}{2}\left(w_{\alpha|\beta} + w_{\beta|\alpha} + v^\gamma_{|\alpha} L_{\gamma\beta} + v^\gamma_{|\beta} L_{\gamma\alpha}\right) + \\ & + \frac{1}{2}\left(v^\xi \left(L^\gamma_\alpha L_{\gamma\beta} + L^\gamma_\beta L_{\gamma\alpha}\right)\right)\end{aligned} \tag{46}$$

$$\varphi_{\alpha\beta} = \frac{1}{2}\left(w^\gamma_{|\alpha} L_{\gamma\beta} + w^\gamma_{|\beta} L_{\gamma\alpha}\right) \tag{47}$$

where we have put

$$\nabla_\alpha \bar{v} = \left(v^\gamma_{|\alpha} + v^\xi L^\gamma_\alpha\right)\bar{\partial}_\gamma + \left(v^\xi_{,\alpha} - v^\gamma L_{\alpha\gamma}\right)\bar{n} \tag{48}$$

$$v^\gamma_{|\alpha} = v^\gamma_{,\alpha} + v^\omega \Gamma^\gamma_{\alpha\omega} \tag{49}$$

and

$$\nabla_\alpha \bar{w} = w^\gamma_{|\alpha} \bar{\partial}_\gamma - w^\gamma L_{\alpha\gamma} \bar{n} \tag{50}$$

$$w^\gamma_{|\alpha} = w^\gamma_{,\alpha} + w^\omega \Gamma^\gamma_{\alpha\omega} \tag{51}$$

and

$$\nabla_\alpha \bar{n} \cdot \nabla_\beta \bar{w} = L^\gamma_\alpha \bar{\partial}_\gamma \cdot \left(w^\omega_{|\beta} d\bar{e}r_\omega - w^\omega L_{\beta\omega} \bar{n}\right) = L_{\omega\alpha} w^\omega_{|\beta} \tag{52}$$

Finally, the strain tensor assumes the following form

$$\begin{aligned}\gamma_{\alpha\beta} = & \frac{1}{2}\left(v_{\alpha|\beta} + v_{\beta|\alpha} + 2v^\xi L_{\alpha\beta}\right) + \\ & + \frac{1}{2}\xi\left(w_{\alpha|\beta} + w_{\beta|\alpha} + v^\gamma_{|\alpha} L_{\gamma\beta} + v^\gamma_{|\beta} L_{\gamma\alpha}\right) + \\ & + \frac{1}{2}\left(v^\xi\left(L^\gamma_\alpha L_{\gamma\beta} + L^\gamma_\beta L_{\gamma\alpha}\right)\right) + \\ & + \frac{1}{2}\xi^2\left(w^\gamma_{|\alpha} L_{\gamma\beta} + w^\gamma_{|\beta} L_{\gamma\alpha}\right)\end{aligned} \tag{53}$$

The stretching strain tensor does not depend on the thickness, in fact it describes the deformation of the mid–surface Q. The bending strain tensors describe the deformation along the thickness.

The transversal components of the strain are

$$\gamma_{3\alpha} = \gamma_{\alpha 3} = \frac{1}{2}\left(\bar{n}_d \cdot \bar{\partial}_{\alpha_d} - \bar{n} \cdot \bar{\partial}_\alpha\right) = v^\xi_{,\alpha} - v^\gamma L_{\alpha\gamma} + w_\alpha \tag{54}$$

2.1 Kirchhoff–Love strain theory

If we take into account the *Kirchhoff-Love* hypothesis, see hypothesis 4, we have

$$\bar{\partial}_{\alpha_d} \cdot \bar{n}_d = 0 \Rightarrow (\bar{n} + \bar{w}) \cdot \left(\bar{\partial}_\alpha + \nabla_\alpha \bar{v}\right) = 0 \Rightarrow \tag{55}$$

$$\bar{w} \cdot \partial_\alpha + \bar{n} \cdot \nabla_\alpha \bar{v} = 0 \Rightarrow w_\alpha = v^\gamma L_{\alpha\gamma} - v^\xi_{,\alpha} \tag{56}$$

and we observe that the variables reduce just to the field $\bar{v}$. Thus, the strain tensor turns into

$$\alpha_{\alpha\beta} = \frac{1}{2}\left(v_{\alpha|\beta} + v_{\beta|\alpha} + 2v^\xi L_{\alpha\beta}\right) \tag{57}$$

$$\omega_{\alpha\beta} = v^\gamma_{|\alpha} L_{\gamma\beta} + v^\gamma_{|\beta} L_{\gamma\alpha} + v^\gamma L_{\gamma\alpha|\beta} - v^\xi_{,\alpha\beta} + v^\xi L^\gamma_\alpha L_{\gamma\beta} \tag{58}$$

$$\begin{aligned} 2\varphi_{\alpha\beta} = {} & \xi^2 \left(v^\delta_{|\alpha} L_{\delta\gamma} L^\gamma_\beta + v^\delta L_{\delta\gamma|\alpha} L^\gamma_\beta - v^\xi_{\gamma\alpha} L^\gamma_\beta\right) + \\ & + \xi^2 \left(v^\delta_{|\beta} L_{\delta\gamma} L^\gamma_\alpha + v^\delta L_{\delta\gamma|\beta} L^\gamma_\alpha - v^\xi_{\gamma\beta} L^\gamma_\alpha\right) \end{aligned} \tag{59}$$

In the linear theory the second bending strain tensor can be neglected because ξ is very small and its square makes the contribution of $\varphi_{\alpha\beta}$ insignificant.

Finally, we have

$$\gamma_{33} = \gamma_{\alpha 3} = \gamma_{3\alpha} = 0 \tag{60}$$

Consider now a Cartesian coordinate system where all the Christoffel symbols vanish, we immediately realize the well known expression of the strain tensor for bending plates

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(v_{\alpha,\beta} + v_{\beta,\alpha} - 2\xi v^\xi_{,\alpha\beta}\right) \tag{61}$$

3 Stress in shell continuums

3.1 Shifters

Before reasoning upon the stress state characterizing a shell continuum it is worth introducing some geometrical relations linking points belonging to the mid–surface Q with corresponding points belonging to the shell thought as a three–dimensional continuum.

Therefore, let us recall the relation already met to compute the components of the metric tensor $g^{\star}_{\alpha\beta}$, see equation (40) on page 8, between the basis in $p^{\star} \in G(\epsilon)$ and the basis in p projection of $p^{\star}$ on Q along the normal coordinate curve ξ. So we have

$$\bar{\partial}^{\star}_{\alpha} = \bar{\partial}_{\alpha} + \xi L^{\beta}_{\alpha} \bar{\partial}_{\beta} \tag{62}$$

$$\bar{n} = \bar{n}^{\star} \tag{63}$$

which in a short notation assumes the following form

$$\bar{\partial}^{\star}_{i} = \mathsf{S}^{h}_{i} \bar{\partial}_{h} \tag{64}$$

Hence, with respect to the basis associated to the coordinate system $\{x^{\alpha}, \xi\}$ the tensor S has the following components

$$\mathsf{S}^{h}_{i} = \begin{pmatrix} 1+\xi L^{1}_{1} & \xi L^{2}_{1} & 0 \\ \xi L^{1}_{2} & 1+\xi L^{2}_{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the superficial part of S can be expressed by the following tensor product

$$\mathsf{S}^{\dagger} = \underline{\mathrm{d}}^{\gamma} \otimes \bar{\partial}^{\star}_{\gamma} \tag{65}$$

so that

$$\mathsf{S}^{\dagger}\left(\bar{\partial}_{\beta}\right) = \left(\underline{\mathrm{d}}^{\gamma} \otimes \bar{\partial}^{\star}_{\gamma}\right)\left(\bar{\partial}_{\beta}\right) = \bar{\partial}^{\star}_{\beta} \tag{66}$$

In the same way we define $\mathsf{F}^{\dagger}$ as follows

$$\mathsf{F}^{\dagger} = \bar{\partial}_{\gamma} \otimes \underline{\mathrm{d}}^{\star\gamma} \tag{67}$$

so that

$$\mathsf{F}^{\dagger}\left(\underline{\mathrm{d}}^{\beta}\right) = \left(\bar{\partial}_{\gamma} \otimes \underline{\mathrm{d}}^{\star\gamma}\right)\left(\underline{\mathrm{d}}^{\beta}\right) = \underline{\mathrm{d}}^{\star\beta} \tag{68}$$

Tensors $\mathsf{S}^{\dagger}$ and $\mathsf{F}^{\dagger}$ are called *shifter* tensors.

3.2 Contraction of surface forces

Consider now a curve $c : \mathbb{R} \to Q$ representing the intersection of the surface Q_c normal to Q which splits the shell continuum $G(\epsilon)$ into two portions.

Let $\bar{\nu} \in \bar{TQ}$ be the unit normal vector applied in p outward pointing from c and let $\bar{l} \in \bar{TQ}$ be the unit vector tangent to c applied in the same point. Then the three unit vectors $\left\{\bar{\nu}, \bar{l}, \bar{n}\right\}$ form a local basis in p. A similar triplet of vectors can be defined in $p^{\star}$ as $\left\{\bar{\nu}^{\star}, \bar{l}^{\star}, \bar{n}\right\}$. Note that the symbol $\star$ denotes as usual quantities belonging to the shell thickness. See figure 2.

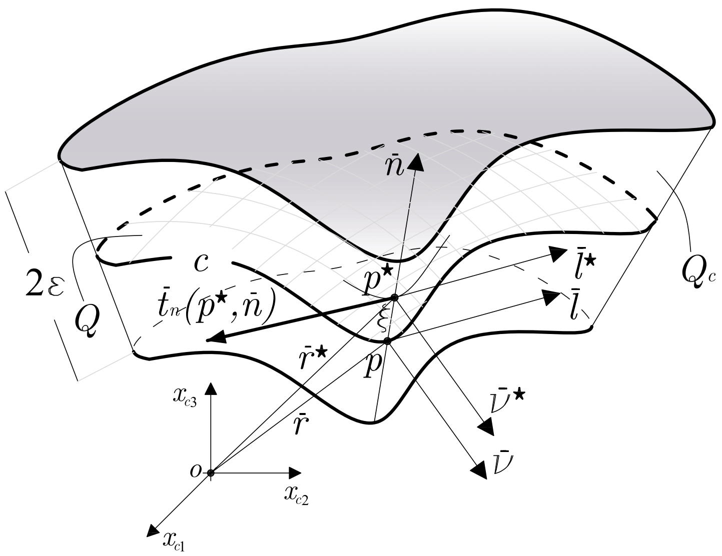

Figure 2: Local bases in $G(\epsilon)$ and in Q.

In order to ensure the equilibrium condition, the portion of the shell included by Q_c must exert on the remaining part of the continuum a tension such as for each point $p^{\star}$ is entirely described by the stress vector $\bar{t}^{\star}$. Moreover the stress vector $t^{\star}$ can be equivalently expressed by Cauchy stress tensor as follows

$$\bar{t}^{\star}\left(p^{\star}, \underline{\nu}^{\star}\right) = \sigma^{\star}\left(p^{\star}\right) \underline{\nu}^{\star} \tag{69}$$

where $\sigma^{\star}$ is the contravariant form of the stress tensor defined in $p^{\star}$. For the sake of brevity hereafter $\sigma^{\star}\left(p^{\star}\right)$ will be denoted simply by σ.

Now our goal is to establish a relation between the stress state distributed along the surface Q_c and the stress state along the boundary of the mid–surface of the shell. This can be done by means of a reduction, i.e. a contraction, of the stress per unit area to a stress per unit line.

Therefore, let us define two vector fields $\mathbf{n}$ and $\mathbf{m}$ such as

$$\int_c \mathbf{n}(p, \underline{\nu}) dl = \int_{Q_c} \bar{t}^{\star}(p^{\star}, \underline{\nu}^{\star}) d\mathcal{A}^{\star} \tag{70}$$

$$\int_c \mathbf{m}(p, \underline{\nu}) dl = \int_{Q_c} ((p^{\star} - p) \times \bar{t}^{\star}(p^{\star}, \underline{\nu}^{\star})) d\mathcal{A}^{\star} \tag{71}$$

Equalities (70) and (71) guarantee that the stress system $\mathbf{n}$ and $\mathbf{m}$ is statically equivalent to the stress system $\bar{t}^{\star}$ along the fiber ξ passing through p.

The oriented elemental area in equations (70) and (71) with respect to the local basis $\left\{\bar{\nu}^{\star}, \bar{l}^{\star}, \bar{n}\right\}$ is given by the following vectorial product

$$\underline{\nu}^{\star} d\mathcal{A}^{\star} = dl\bar{l}^{\star} \times d\xi\bar{n} \tag{72}$$

and since $dl\bar{l}^\star = dl^\alpha \bar{\partial}^\star_\alpha$, equation (72) can be equivalently expressed as follows

$$\underline{\nu} d\mathcal{A}^\star = dl^\alpha \bar{\partial}^\star_\alpha \times d\xi \bar{n} = \eta^\star_{\alpha\beta} dl^\alpha d\xi \underline{\mathrm{d}}^{\star\beta} = \epsilon_{\alpha\beta}\sqrt{g^\star} dl^\alpha d\xi \underline{\mathrm{d}}^{\star\beta} \tag{73}$$

where $g^\star = \det\left(g^\star_{\alpha\beta}\right)$.

Moreover, back to the mid–surface we notice it is possible to write

$$dl\bar{l} \times \bar{n} = \underline{\nu} dl \tag{74}$$

which in the coordinate system $\{x^\alpha, \xi\}$ becomes

$$dl^\alpha \bar{\partial}_\alpha \times \bar{n} = \eta_{\alpha\beta} dl^\alpha \underline{\mathrm{d}}^\beta = \epsilon_{\alpha\beta}\sqrt{g} dl^\alpha \underline{\mathrm{d}}^\beta \tag{75}$$

where $g = \det\left(g_{\alpha\beta}\right)$.

Equation (69) and (73) allow us to rewrite equations (70) and (71) as follows

$$\int_c \mathbf{n}(p, \underline{\nu}) dl = \int_{Q_c} \sigma \epsilon_{\alpha\beta}\sqrt{g^\star} dl^\alpha d\xi \underline{\mathrm{d}}^{\star\beta} \tag{76}$$

$$\int_c \mathbf{m}(p, \underline{\nu}) dl = \int_{Q_c} (p^\star - p) \times \sigma \epsilon_{\alpha\beta}\sqrt{g^\star} dl^\alpha d\xi \underline{\mathrm{d}}^{\star\beta} \tag{77}$$

Next, by virtue of the *shifter* $\mathsf{F}^\dagger$, the latter equations become

$$\int_c \mathbf{n}(p, \underline{\nu}) dl = \int_{Q_c} \sigma \epsilon_{\alpha\beta}\sqrt{g^\star} dl^\alpha d\xi \left(\bar{\partial}_\gamma \otimes \underline{\mathrm{d}}^{\star\gamma}\right) \underline{\mathrm{d}}^\beta \tag{78}$$

$$\int_c \mathbf{m}(p, \underline{\nu}) dl = \int_{Q_c} \xi \bar{n} \times \sigma \epsilon_{\alpha\beta}\sqrt{g^\star} dl^\alpha d\xi \left(\bar{\partial}_\gamma \otimes \underline{\mathrm{d}}^{\star\gamma}\right) \underline{\mathrm{d}}^\beta \tag{79}$$

which, taking into account equations (74) and (75), become

$$\int_c \mathbf{n}(p, \underline{\nu}) dl = \int_c \int_{-\epsilon}^{+\epsilon} \sqrt{\frac{g^\star}{g}} \sigma \left(\bar{\partial}_\gamma \otimes \underline{\mathrm{d}}^{\star\gamma}\right) \underline{\nu} dl d\xi \tag{80}$$

$$\int_c \mathbf{m}(p, \underline{\nu}) dl = \int_c \int_{-\epsilon}^{+\epsilon} \xi \bar{n} \times \sqrt{\frac{g^\star}{g}} \sigma \left(\bar{\partial}_\gamma \otimes \underline{\mathrm{d}}^{\star\gamma}\right) \underline{\nu} dl d\xi \tag{81}$$

and finally

$$\mathbf{n}(p, \underline{\nu}) = \int_{-\epsilon}^{+\epsilon} \mathsf{g} \sigma \left(\bar{\partial}_\gamma \otimes \underline{\mathrm{d}}^{\star\gamma}\right) \underline{\nu} d\xi \tag{82}$$

$$\mathbf{m}(p, \underline{\nu}) = \bar{n} \times \int_{-\epsilon}^{+\epsilon} \xi \mathsf{g} \sigma \left(\bar{\partial}_\gamma \otimes \underline{\mathrm{d}}^{\star\gamma}\right) \underline{\nu} d\xi \tag{83}$$

where we have put $\mathsf{g} = \sqrt{g^\star / g}$

Both integrands in (82) and (83) can be further simplified just substituting $\sigma = \sigma^{ij}\left(\bar{\partial}^{\star}_i \otimes \bar{\partial}^{\star}_j\right)$ and $\underline{\nu} = \nu_\alpha \underline{\mathrm{d}}^\alpha$ as follows

$$\mathbf{n}(p,\underline{\nu}) = \left(\int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha j}\bar{\partial}^{\star}_j d\xi\right)\nu_\alpha \tag{84}$$

$$\mathbf{m}(p,\underline{\nu}) = \bar{n} \times \left(\int_{-\epsilon}^{+\epsilon} \mathsf{g}\xi\sigma^{\alpha j}\bar{\partial}^{\star}_j d\xi\right)\nu_\alpha \tag{85}$$

and using once again equations (62) and (63) they assume the following form

$$\mathbf{n}(p,\underline{\nu}) = \left(\int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha\gamma} d\xi + \int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha\beta}\xi d\xi L^{\gamma}_{\beta}\right)\bar{\partial}_\gamma \nu_\alpha + \\ + \left(\int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha\xi} d\xi\right)\bar{n}\nu_\alpha \tag{86}$$

$$\mathbf{m}(p,\underline{\nu}) = \bar{n} \times \left(\int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha\gamma}\xi d\xi + \int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha\gamma}\xi^2 d\xi L^{\gamma}_{\beta}\right)\bar{\partial}_\gamma \nu_\alpha \tag{87}$$

where we can finally define two tensors N and M

$$N = N^{\alpha\beta}\left(\bar{\partial}_\alpha \otimes \bar{\partial}_\beta\right) + N^{\alpha\xi}\left(\bar{\partial}_\alpha \otimes \bar{n}\right) \tag{88}$$

$$M = M^{\alpha\beta}\left(\bar{\partial}_\alpha \otimes \bar{\partial}_\beta\right) \tag{89}$$

respectively as

$$N^{\alpha\beta} = \int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha\beta} d\xi + \int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha\gamma}\xi d\xi L^{\beta}_{\gamma} \tag{90}$$

$$N^{\alpha\xi} = \int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\star\alpha\xi} d\xi \tag{91}$$

and

$$M^{\alpha\beta} = \int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha\beta}\xi d\xi + \int_{-\epsilon}^{+\epsilon} \mathsf{g}\sigma^{\alpha\gamma}\xi^2 d\xi L^{\beta}_{\gamma} \tag{92}$$

such as

$$\mathbf{n}(p,\underline{\nu}) = N\underline{\nu} = N^{\alpha\beta}\nu_\alpha\bar{\partial}_\beta + N^{\alpha\xi}\nu_\alpha\bar{n} \tag{93}$$

$$\mathbf{m}(p,\underline{\nu}) = \bar{n} \times M\underline{\nu} = \bar{n} \times M^{\alpha\beta}\nu_\alpha\bar{\partial}_\beta \tag{94}$$

Two fields $\mathbf{n}$ and $\mathbf{m}$ are called *surface stress vector* and *surface couple vector* respectively; while the fields N and M are termed *surface stress tensor* and *surface couple tensor*.

From the above results it is immediate to notice that the surface stress vector $\mathbf{n}$ belongs to $T_{\bar{Q}}E$, consequently it can be split into a superficial part and an orthogonal part as follows

$$\mathbf{n} = \mathbf{n}^{\parallel} + \mathbf{n}^{\perp} \tag{95}$$

where

$$\mathbf{n}^{\|} = N^{\alpha\beta}\nu_\alpha\bar{\partial}_\beta \tag{96}$$

$$\mathbf{n}^{\perp} = N^{\alpha\xi}\nu_\alpha\bar{n} \tag{97}$$

while the surface couple vector $\mathbf{m}$ belongs to $\bar{TQ}$ so that

$$\mathbf{m} = \mathbf{m}^{\|} \tag{98}$$

As the last remak we point out that the coefficient g involved in the integration of Cauchy stress tensor along the thickness depends only on the geometrical features of the mid–surface Q, in fact it is easy to prove the following expression

$$\mathsf{g} = \det\left(\mathsf{S}_i^h\right) = 1 + \xi H + \xi^2 K \tag{99}$$

where H and K are the *mean curvature* and the *total curvature* of the surface Q defined in equations (8) and (7).

3.3 Body forces and load density

Suppose the the curve $c : I\!R \to Q$ is closed in such a way as to capture a surface portion $Q' \subset Q$ bounded by $\partial Q \equiv c$. Assuming c to be a directrix, that is a curve through which a line generating a given ruled surface always passes, the generatrices directed along $\bar{n}$ define a cylinder $G_c(\epsilon) \subset G(\epsilon)$ with thickness 2ϵ and also bounded by the surface $Q_c \cup Q^\epsilon \cup Q_{-\epsilon}$.

We assume that the volume forces acting at every point belonging to the cylinder $G_c(\epsilon)$ and the load density acting at every point on the upper and lower surfaces Q^ϵ and $Q_{-\epsilon}$ can be integrated along the thickness to yield a new force system defined on the mid–surface Q' as follows

$$\bar{q} : Q' \to \bar{T_{Q'}}E \tag{100}$$

$$\bar{s} : Q' \to \bar{TQ'} \tag{101}$$

where $\bar{q} = q^\beta\bar{\partial}_\beta + q^\xi\bar{n}$ represents the load vector and $\bar{s} = \bar{n} \times s^\beta\bar{\partial}_\beta$ represents the load–moment vector.

See [3] for details.

3.4 Eulero's equations

The equilibrium equations for the mid surface portion Q' can be written as follows

$$\int_{\partial Q'} \mathbf{n}(p, \underline{\nu})dl + \int_{Q'} \bar{q}dQ' = 0 \tag{102}$$

$$\int_{\partial Q'} \left(\mathbf{m}(p, \underline{\nu}) + \bar{r} \times \mathbf{n}(p, \underline{\nu})\right) dl + \int_{Q'} (\bar{r} \times \bar{q} + \bar{s})dQ' = 0 \tag{103}$$

which yield

$$\int_{\partial Q'} N\underline{\nu} dl + \int_{Q'} \bar{q} dQ' = 0 \tag{104}$$

$$\int_{\partial Q'} (\bar{n} \times M\underline{\nu} + \bar{r} \times N\underline{\nu}) dl + \int_{Q'} (\bar{r} \times \bar{q} + \bar{s}) dQ' = 0 \tag{105}$$

Making use of the divergence theorem, and due to the arbitrariness of Q', the above equations become

$$\mathrm{div} N + \bar{q} = 0 \tag{106}$$

$$\mathrm{div}(\bar{n} \times M^{\alpha h} \bar{\partial}_h + \bar{r} \times N^{\alpha h} \bar{\partial}_h) + \bar{r} \times \bar{q} + \bar{s} = 0 \tag{107}$$

Details on the divergence of vector and tensor field in curvilinear coordinate systems can be found in [4].

Equations (106) and (107) can be written in components as follows

$$\nabla^{\dagger}_{\alpha} N^{\alpha\beta} + L^{\beta}_{\alpha} N^{\alpha\xi} + q^{\beta} = 0 \tag{108}$$

$$\nabla_{\alpha} N^{\alpha\xi} + L_{\alpha\gamma} N^{\alpha\gamma} + q^{\xi} = 0 \tag{109}$$

$$\nabla^{\dagger}_{\alpha} M^{\beta\alpha} - N^{\xi\beta} + s^{\beta} = 0 \tag{110}$$

$$\eta_{\alpha\beta} \left(L^{\alpha}_{\gamma} M^{\beta\gamma} - N^{\alpha\beta} \right) = 0 \tag{111}$$

where equations (108) assure the translational equilibrium in the tangent plane, while (109) represents the translational equilibrium along the normal direction. Next, two equations in (110) impose the rotational equilibrium about the surface axes, respectively. Finally, the last equilibrium condition (111) gives the symmetry to the tensor $L^{\alpha}_{\gamma} M^{\beta\gamma} - N^{\alpha\beta}$.

PROOF

Here we want to show all steps we made to pass from the equilibrium equations (106) and (107) to the corresponding expressions in components (108) to (111).

Let us start form equation (106). We invoke the definition of divergence for second order contravariant tensors already used in equation (**??**), so we have

$$\begin{gathered}
(\mathrm{div} N)^h = N^{\alpha h}_{,\alpha} + \Gamma^{\alpha}_{\alpha\gamma} N^{\gamma h} + \Gamma^{h}_{\alpha t} N^{\alpha t} = \\
= N^{\alpha\beta}_{,\alpha} + N^{\alpha\xi}_{,\alpha} + \Gamma^{\alpha}_{\alpha\gamma} N^{\gamma\beta} + \Gamma^{\alpha}_{\alpha\gamma} N^{\gamma\xi} + \Gamma^{\beta}_{\alpha t} N^{\alpha t} + \Gamma^{\xi}_{\alpha t} N^{\alpha t} = \\
= N^{\alpha\beta}_{,\alpha} + N^{\alpha\xi}_{,\alpha} + \Gamma^{\alpha}_{\alpha\gamma} N^{\gamma\beta} + \Gamma^{\alpha}_{\alpha\gamma} N^{\gamma\xi} + \\
+ \Gamma^{\beta}_{\alpha\gamma} N^{\alpha\gamma} + \Gamma^{\beta}_{\alpha\xi} N^{\alpha\xi} + \Gamma^{\xi}_{\alpha\gamma} N^{\alpha\gamma} + \Gamma^{\xi}_{\alpha\xi} N^{\alpha\xi}
\end{gathered}$$

Now we just need to separate the tangential and normal components as follows

$$(\mathrm{div} N)^{\beta} = N^{\alpha\beta}_{,\alpha} + \Gamma^{\alpha}_{\alpha\gamma} N^{\gamma\beta} + \Gamma^{\beta}_{\alpha\gamma} N^{\alpha\gamma} + \Gamma^{\beta}_{\alpha\xi} N^{\alpha\xi} \tag{112}$$

$$(\mathrm{div} N)^{\xi} = N^{\alpha\xi}_{,\alpha} + \Gamma^{\alpha}_{\alpha\gamma} N^{\gamma\xi} + \Gamma^{\xi}_{\alpha\xi} N^{\alpha\xi} + \Gamma^{\xi}_{\alpha\gamma} N^{\alpha\gamma} \tag{113}$$

By virtue of the the identity $(\nabla_\alpha \bar{n})^\beta = L_\alpha^\beta = \Gamma_{\alpha\xi}^\beta$ equation (112) becomes

$$(\mathrm{div} N)^\beta = \nabla_\alpha^\dagger N^{\alpha\beta} + L_\alpha^\beta N^{\alpha\xi} \tag{114}$$

where we have just collected the surface divergence[3] terms into

$$\nabla_\alpha^\dagger N^{\alpha\beta} = N_{,\alpha}^{\alpha\beta} + \Gamma_{\alpha\gamma}^\alpha N^{\gamma\beta} + \Gamma_{\alpha\gamma}^\beta N^{\alpha\gamma} \tag{115}$$

Equation (114) proves the in–plane translational equilibrium expressed in (108).

Concerning equation (113), the translational equilibrium along the normal direction is readily proved remembering both $\Gamma_{\alpha\gamma}^\xi = L_{\alpha\gamma}$ and[4]

$$\nabla_\alpha N^{\alpha\xi} = N_{,\alpha}^{\alpha\xi} + \Gamma_{\alpha\gamma}^\alpha N^{\gamma\xi} + \Gamma_{\alpha\xi}^\xi N^{\alpha\xi} \tag{116}$$

Hence we obtain

$$(\mathrm{div} N)^\xi = \nabla_\alpha N^{\alpha\xi} + L_{\alpha\gamma} N^{\alpha\gamma} \tag{117}$$

which finally proves equation (109)

In order to prove equations (110) and (111), first we simplify equation (107) by taking into account equation (106). So it becomes

$$\mathrm{div}\bar{n} \times M^{\alpha h}\bar{\partial}_h + \bar{n} \times \mathrm{div}\left(M^{\alpha h}\bar{\partial}_h\right) + \mathrm{div}\bar{r} \times N^{\alpha h}\bar{\partial}_h + \bar{s} = 0 \tag{118}$$

We can split the divergence of the tensor $M^{\alpha h}$ in accordance with the results in (114) and (117), thus we have

$$\begin{aligned} &\nabla_\alpha \bar{n} \times M^{\alpha h}\bar{\partial}_h + \bar{n} \times \left(\nabla_\alpha^\dagger M^{\alpha\beta} + L_\gamma^\beta M^{\gamma\xi}\right)\bar{\partial}_\beta + \\ &+ \bar{n} \times \left(\nabla_\alpha^\dagger M^{\alpha\xi} + L_{\alpha\gamma} M^{\alpha\gamma}\right)\bar{n} + \bar{r}_{,\alpha} \times N^{\alpha h}\bar{\partial}_h + \bar{s} = 0 \end{aligned} \tag{119}$$

which after further algebra becomes

$$\begin{aligned} &L_\alpha^\gamma \bar{\partial}_\gamma \times M^{\alpha\omega}\bar{\partial}_\omega + L_\alpha^\gamma \bar{\partial}_\gamma \times M^{\alpha\xi}\bar{n} + \bar{n} \times \left(\nabla_\alpha^\dagger M^{\alpha\beta} + L_\gamma^\beta M^{\gamma\xi}\right)\bar{\partial}_\beta + \\ &+ \bar{\partial}_\alpha \times N^{\alpha\omega}\bar{\partial}_\omega + \bar{\partial}_\alpha \times N^{\alpha\xi}\bar{n} + \bar{n} \times s^\beta \bar{\partial}_\beta = 0 \end{aligned} \tag{120}$$

Collecting the normal and tangential terms we obtain the following three scalar equations

$$\eta_{\gamma\omega}\left(L_\alpha^\gamma M^{\alpha\omega} + N^{\gamma\omega}\right) = 0 \tag{121}$$

and

$$\bar{n} \times \left(\nabla_\alpha^\dagger M^{\alpha\beta} - N^{\beta\xi} + s^\beta\right)\bar{\partial}_\beta = 0 \tag{122}$$

which finally proves the rotational equilibrium (110) about the surface axes.$\diamondsuit$

Usually a new variable is introduced to make easier possible further calculations; in fact we define the **pseudo-stress tensor** the symmetric tensor

$$\tilde{N}^{\alpha\beta} = N^{\alpha\beta} - L_\gamma^\alpha M^{\beta\gamma} \tag{123}$$

It is straightforward to notice that $\tilde{N} \equiv N$ only when either a membrane stress state holds or for flat shells, namely when *Weingarten*'s tensor is identically zero.

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[3]In literature the divergence of the surface tensor $N^{\alpha\beta}$ is often denoted by $N_{|\alpha}^{\alpha\beta}$.

[4]In literature the divergence $\nabla_\alpha^\dagger N^{\alpha\xi}$ is often denoted by $N_{|\alpha}^{\alpha\xi}$.

3.5 Membrane state of stress

In this last section we introduce an hypothesis on the state of the stress that enables us to derive a closed form solution for several shell geometries without invoking the constitutive law. Examples of these closed form solutions will be provided in appendix 5.

A shell continuum is subjected to a membrane stress state when both the following condition hold

$$N^{\alpha\xi} = 0 \tag{124}$$

$$M^{\alpha\beta} = 0 \tag{125}$$

Hence, the equilibrium equations become

$$\nabla_\alpha N^{\alpha\beta} + q^\beta = 0 \tag{126}$$

$$L_{\alpha\gamma} N^{\alpha\gamma} + q^\xi = 0 \tag{127}$$

$$\eta_{\alpha\beta} N^{\alpha\beta} = 0 \tag{128}$$

where equation (126) represents the translational equilibrium along the tangent plane; equation (127) represents the equilibrium along $\bar{n}$ and finally equation (128) states the rotational equilibrium about $\bar{n}$ and establishes the symmetry of N.

4 Constitutive equation for shell continuums

The Kirchhoff–Love hypothesis and the inextensibility of material fibers along $\bar{n}$ allows one to consider the shear stress components $N^{\xi\alpha}$ unrelated to strains, so that the constitutive problem can be solved through the plane stress model. Thus, components $N^{\xi\alpha}$ are found only by means of the equilibrium equations. The analytical derivation of the constitutive equations is beyond the scope of this book, so we will just present the final equations that will be used in the appendix 5 in order to solve some case studies. However, readers can find thorough discussions in [2] and [5].

Suppose a membrane state of stress, the constitutive equations are the following

$$\tilde{N}^{\alpha\beta} = D H^{\alpha\beta\lambda\mu} \alpha_{\lambda\mu} \tag{129}$$

$$M^{\alpha\beta} = B H^{\alpha\beta\lambda\mu} \omega_{\lambda\mu} \tag{130}$$

where

$$H^{\alpha\beta\lambda\mu} = \frac{1-\nu}{2} \left(g^{\alpha\lambda} g^{\beta\mu} + g^{\alpha\mu} g^{\beta\lambda} + \frac{2\nu}{1-\nu} g^{\alpha\beta} g^{\lambda\mu} \right) \tag{131}$$

The fourth–order tensor $H^{\alpha\beta\lambda\mu}$ has the following symmetries

$$H^{\alpha\beta\lambda\mu} = H^{\beta\alpha\lambda\mu} = H^{\alpha\beta\mu\lambda} = H^{\lambda\mu\alpha\beta} \tag{132}$$

Finally, coefficients D and B are the in–plane and the bending stiffness, respectively, defined as

$$D = \frac{E(2\varepsilon)}{1-\nu^2} \tag{133}$$

$$B = \frac{E(2\varepsilon)^3}{12(1-\nu^2)} \tag{134}$$

5 Applications of the shell theory

Here some applications of the above theory are presented. For all cases the external loads ensure a membrane state of stress and consequently analytical closed–form solutions can be reached. See also [4] for more details.

5.1 Spherical dome

5.1.1 Geometry

The spherical dome is a shell modeled on a portion of sphere having radius r and aperture $\pi/2$ (hemisphere). Given the geometry, the first step is to identify the simplest coordinate system able to describe such a geometry. Of course it is a spherical system, see section **??** on page **??**.

Let X be the spherical coordinate system[5] so that

$$X = (\varphi, \vartheta, \rho) : E \to \mathbb{R}^3 \tag{135}$$

where E is the affine Euclidean space in which the surface Q is embedded. The origin of the system is located at the center of the hemisphere. With respect to a Cartesian coordinate system, the following transformations hold

$$x = \rho \sin\varphi \sin\vartheta \tag{136}$$

$$y = \rho \sin\varphi \cos\vartheta \tag{137}$$

$$z = \rho \cos\varphi \tag{138}$$

The adapted coordinate system X induces the surface coordinate system $X^\dagger$ by imposing the constraint $\rho = r$. Therefore, the induced coordinate system is

$$X^\dagger = (\varphi^\dagger, \vartheta^\dagger) : Q \to \mathbb{R}^2 \tag{139}$$

The covariant and contravariant expressions of the metric tensor $g^\dagger$ associated with the induced coordinate system are, respectively

$$\underline{g} = r^2 \underline{\mathrm{d}}^\varphi \otimes \underline{\mathrm{d}}^\varphi + r^2 \sin^2\varphi \underline{\mathrm{d}}^\vartheta \otimes \underline{\mathrm{d}}^\vartheta \tag{140}$$

$$\bar{g} = \frac{1}{r^2} \bar{\partial}_\varphi \otimes \bar{\partial}_\varphi + \frac{1}{r^2 \sin^2\varphi} \bar{\partial}_\vartheta \otimes \bar{\partial}_\vartheta \tag{141}$$

[5]Note that this coordinate system has been slightly changed compared with that depicted in figure **??**.

The nonvanishing Christoffel symbols on Q are

$$\Gamma^{\varphi}_{\vartheta\vartheta} = -\sin\varphi\cos\varphi$$
$$\Gamma^{\vartheta}_{\varphi\vartheta} = \Gamma^{\vartheta}_{\vartheta\varphi} = \frac{\cos\varphi}{\sin\varphi}$$

The unit normal vector of Q is

$$\bar{n} = \bar{\partial}_{\rho} \tag{142}$$

The Weingarten tensor and the second fundamental form for Q are, respectively

$$L = \frac{1}{r}(\underline{\mathrm{d}}^{\varphi} \otimes \bar{\partial}_{\varphi} + \underline{\mathrm{d}}^{\vartheta} \otimes \bar{\partial}_{\vartheta}) \tag{143}$$
$$\underline{L} = r(\underline{\mathrm{d}}^{\varphi} \otimes \underline{\mathrm{d}}^{\varphi} + \sin^2\varphi \underline{\mathrm{d}}^{\vartheta} \otimes \underline{\mathrm{d}}^{\vartheta}) \tag{144}$$

5.1.2 Displacements and strains

To compute the in–plane state of stress only the stretching strain tensor α is required

$$\alpha_{\varphi\varphi} = v_{\varphi},_{\varphi} + rv^{\xi} \tag{145}$$
$$\alpha_{\theta\theta} = v_{\vartheta},_{\vartheta} + \sin\varphi\cos\varphi + r\sin^2\varphi v^{\xi} \tag{146}$$
$$\alpha_{\vartheta\varphi} = \frac{1}{2}(v_{\varphi},_{\theta} + v_{\vartheta},_{\phi}) - \frac{\cos}{\sin\varphi}v_{\vartheta} \tag{147}$$

5.1.3 Equilibrium and constitutive law

The equilibrium equations (126) to (128) for a spherical shell assume the following form

$$N^{\varphi\varphi},_{\varphi} + \cot\varphi N^{\varphi\varphi} - \sin\varphi\cos\varphi N^{\vartheta\vartheta} + q^{\varphi} = 0 \tag{148}$$
$$-N^{\varphi\varphi}r - N^{\vartheta\vartheta}r\sin^2\varphi + q^{\xi} = 0 \tag{149}$$
$$N^{\vartheta\varphi},_{\varphi} + 3\cot\varphi N^{\vartheta\varphi} + q^{\vartheta} = 0 \tag{150}$$

The constitutive equations are

$$\begin{aligned} N^{\varphi\varphi} &= D\frac{1}{r^4}\left(v_{\varphi},_{\varphi} + rv^{\xi}\right) + \\ &+ D\left(\frac{\nu}{r^4\sin^2\varphi}(v_{\vartheta},_{\vartheta} + \sin\varphi\cos\varphi v_{\varphi} + r\sin^2\varphi v^{\xi})\right) \end{aligned} \tag{151}$$
$$\begin{aligned} N^{\vartheta\vartheta} &= D\frac{1}{r^4\sin^4\varphi}\left(v_{\vartheta},_{\vartheta} + \sin\varphi\cos\varphi v_{\varphi} + r\sin^2\varphi v^{\xi}\right) + \\ &+ D\frac{\nu}{r^4\sin^2\varphi}\left(v_{\varphi},_{\varphi} + rv^{\xi}\right) \end{aligned} \tag{152}$$
$$N^{\vartheta\varphi} = D\left(\frac{1-\nu}{r^4\sin^2\varphi}\frac{1}{2}\left(v_{\varphi},_{\theta} + v_{\vartheta},_{\varphi}\right) - \frac{\cos\varphi}{\sin\varphi}v_{\vartheta}\right) \tag{153}$$

5.1.4 *Load case: self weight*

The dead load due to the self weight provides, of course, a symmetrical action so that the expected solution will not depend on ϑ.

Suppose the load per unit area is $\bar{q}$, uniformly distributed throughout the shell. The vector has only the vertical component

$$\bar{q} = -q^z \bar{e}_z \tag{154}$$

whereas, with respect to the basis $\{\bar{\partial}_\varphi, \bar{\partial}_\vartheta, \bar{n}\}$ the vector load $\bar{q}$ is written follows

$$q^{<>} = -q^z \cos\varphi \bar{n} + q^z \sin\varphi \bar{\partial}_\varphi \tag{155}$$

By multiplying equation (148) by $\sin\varphi$ we obtain

$$(\sin\varphi N^{\varphi\varphi})_{,\varphi} - \sin^2\varphi \cos\varphi N^{\vartheta\vartheta} + \sin\varphi q^\varphi = 0 \tag{156}$$

Let us introduce now the physical components of the stress tensor N, so that

$$N^{<\alpha\beta>} = \frac{N^{\alpha\beta}}{|\underline{\mathrm{d}}^\alpha||\underline{\mathrm{d}}^\beta|} = N^{\alpha\beta}|\bar{\partial}_\alpha||\bar{\partial}_\beta| \tag{157}$$

Hence, equation (156) becomes

$$(\sin\varphi N^{<\varphi\varphi>})_{,\varphi} - \cos\varphi N^{<\vartheta\vartheta>} + r\sin\varphi q^{<\varphi>} = 0 \tag{158}$$

Analogously, by multiplying equation (150) by $\sin^2\varphi$, considering the physical components and noticing that $q^\vartheta = 0$, we obtain

$$(\sin\varphi N^{<\vartheta\varphi>})_{,\varphi} + \cos\varphi N^{<\vartheta\varphi>} = 0 \tag{159}$$

The remaining equilibrium equation becomes

$$-\frac{N^{<\varphi\varphi>}}{r} - \frac{N^{<\vartheta\vartheta>}}{r} + q^{<\xi>} = 0 \tag{160}$$

where, resolving equation (160) for $N^{<\vartheta\vartheta>}$, equation (158) turns into

$$(\sin^2\varphi N^{<\varphi\varphi>})_{,\varphi} = (q^{<\xi>} r\cos\varphi - q^{<\varphi>} r\sin\varphi)\sin\varphi \tag{161}$$

which can be integrated as follows

$$\sin^2\varphi N^{<\varphi\varphi>} = \int_{\bar{\varphi}}^{\varphi} r\big(q^{<\xi>}(\phi)\cos\phi - q^{<\varphi>}(\phi)\sin\phi\big)\sin\phi d\phi + K \tag{162}$$

Equation (162) represents the equilibrium of a spherical cap included by latitude $\bar{\varphi}$ and $\varphi \in [\bar{\varphi}, \pi/2]$. In particular the quantity $2\pi r K$, excepting the sign, equilibrates the resultant acting on the cap identified by the aperture $\bar{\varphi}$.

Considering now equation (155)

$$\sin^2\varphi N^{<\varphi\varphi>} = -rq^z\ \big[-\cos\phi\,\big]_{\bar{\varphi}}^{\varphi} \tag{163}$$

for the latitude φ the whole meridian stress when $\bar{\varphi}=0 \Rightarrow K=0$ is

$$N^{<\varphi\varphi>} = -\frac{rq^z(1-\cos\varphi)}{\sin\varphi} = -\frac{rq^z}{1+\cos\varphi} \tag{164}$$

so that equation (160) becomes

$$N^{<\vartheta\vartheta>} = rq^z\big(\frac{\sin^2\varphi-\cos\varphi}{1+\cos\varphi}\big) \tag{165}$$

The third equilibrium equation does not depend on the two latter results, therefore, since $q^{\vartheta}=0$, we have

$$N^{<\vartheta\varphi>} = 0 \tag{166}$$

5.1.5 Load case: uniform load on the horizontal projection of the shell

This load case keeps unaltered the simplifications regarding the symmetry already discussed in the preceding case. Indeed, here too we are looking for a solution not depending on ϑ.

The load q^z is now projected on the horizontal plane

$$q = -q^z\cos\varphi\bar{e}_z \tag{167}$$

therefore with respect to the local basis, the physical components are

$$q^{<>} = -q^z\cos^2\varphi + q^z\bar{n}\sin\varphi\cos\varphi\bar{\partial}_{\varphi} \tag{168}$$

By means of a procedure similar to that formerly used we obtain that equation (162) now becomes

$$\begin{aligned}\sin^2\varphi N^{<\varphi\varphi>} &= \int_{\bar{\varphi}}^{\varphi} r\big(q^{<\xi>}(\phi)\cos\phi - q^{<\varphi>}(\phi)\sin\phi\big)\sin\phi d\phi + K \\ &= \int_{\bar{\varphi}}^{\varphi} -rq^z\sin\varphi\cos\varphi + K\end{aligned} \tag{169}$$

from which

$$\sin^2\varphi N^{<\varphi\varphi>} = -\frac{1}{2}[\cos^2\varphi]_{\bar{\varphi}}^{\varphi} \tag{170}$$

Next, if $\bar{\varphi}=0 \Rightarrow K=0$, the whole meridian stress is

$$N^{<\varphi\varphi>} = -\frac{1}{2}rq^z \tag{171}$$

Finally, from equation (160) we obtain

$$N^{<\vartheta\vartheta>} = -\frac{1}{2}rq^z\cos 2\varphi \tag{172}$$

5.2 Cylindrical shell

In this example we want to compute the stress state for a cylindrical shell subjected to some of the most typical load conditions, e.g. uniform pressure, dead weight, hydrostatic pressure.

5.2.1 Geometry

Obviously we choose as an adapted coordinate system a cylindrical one with a little rearrangement compared with the one introduced in section **??** on page **??**,

$$X = (\vartheta, z, \rho) : E \to I\!R^3 \tag{173}$$

where, as usual, E is the affine Euclidean space in which the cylindrical surface Q is embedded. The relationships between the Cartesian system, with the origin along the axis of the cylinder, and the cylindrical coordinates are

$$x = \rho sen\vartheta \tag{174}$$

$$y = \rho cos\theta \tag{175}$$

$$z = z \tag{176}$$

The above adapted coordinate system induces the surface system $X^\dagger$ due to the constraint $\rho = r$, where r is the radius of the cylinder. So we have

$$X^\dagger = (\theta^\dagger, z^\dagger) : Q \to I\!R^2 \tag{177}$$

The covariant and contravariant forms of the surface induced metric are, respectively

$$\underline{g} = r^2 \underline{\mathrm{d}}^\vartheta \otimes \underline{\mathrm{d}}^\vartheta + \underline{\mathrm{d}}^z \otimes \underline{\mathrm{d}}^z \tag{178}$$

$$\bar{g} = \frac{1}{r^2}\bar{\partial}_\vartheta \otimes \bar{\partial}_\vartheta + \bar{\partial}_z \otimes \bar{\partial}_z \tag{179}$$

All Christoffel symbols vanish on Q.
The unit normal vector of Q is

$$\bar{n} = \bar{\partial}_\rho \tag{180}$$

The Weingarten tensor and the second fundamental form are, respectively

$$L = \frac{1}{r}\mathrm{d}^\vartheta \otimes \bar{\partial}_\vartheta \tag{181}$$

$$\underline{L} = r\underline{\mathrm{d}}^\vartheta \otimes \underline{\mathrm{d}}^\vartheta \tag{182}$$

5.2.2 *Displacements and strains*

To compute the in–plane state of stress only the stretching strain tensor α is required

$$\alpha_{\vartheta\vartheta} = v_{\vartheta,\vartheta} + r v_{\xi} \tag{183}$$

$$\alpha_{\vartheta z} = \frac{1}{2}(v_{\vartheta},_{z} + v_{z},_{\vartheta}) \tag{184}$$

$$\alpha_{zz} = v_{z},_{z} \tag{185}$$

5.2.3 *Equilibrium and constitutive law*

For a cylindrical shell subjected to a membrane state of stress the equilibrium equations in the scalar form are

$$N^{\vartheta\vartheta},_{\vartheta} + N^{\vartheta z},_{z} + p^{\vartheta} = 0 \tag{186}$$

$$N^{z,\vartheta} + N^{zz},_{z} + p^{z} = 0 \tag{187}$$

$$-N^{\vartheta\vartheta} L_{\vartheta\vartheta} + p^{\xi} = 0 \tag{188}$$

$$N^{\vartheta z} = N^{z\vartheta} \tag{189}$$

The constitutive equations assume the following form

$$N^{\vartheta\vartheta} = \frac{D}{r^2}\left(\frac{1}{r^2}(v_{\vartheta},_{\vartheta} + r v_{\xi}) + v_{z},_{z}\right) \tag{190}$$

$$N^{\vartheta z} = D\left(\frac{1-\nu}{2r^2}(v_{\vartheta}, z + v_{z},_{\vartheta}\right) \tag{191}$$

$$N^{zz} = D\left(\frac{\nu}{r^2}(v_{\vartheta},_{\vartheta} + r v_{\xi}) + v_{z},_{z}\right) \tag{192}$$

5.2.4 *Load case: uniform pressure and self weight*

This load condition is characterized by two load components, namely q^{ξ} and q^{z}. The symmetry around the z–axis permits to delate all terms containing the derivatives with respect to ϑ.

The equilibrium equations become accordingly

$$N^{\vartheta\theta} = \frac{q^{\xi}}{r} \tag{193}$$

$$N^{\vartheta z},_{z} = 0 \tag{194}$$

$$N^{zz},_{z} + p^{z} = 0 \tag{195}$$

Next, taking into account the boundary conditions (at $z = 0$) related to the particular load condition and using the physical components, we obtain

$$N^{<\vartheta\theta>} = q^{\xi} r \tag{196}$$

$$N^{<\vartheta z>}{}_{,z} = 0 \Rightarrow N^{<\vartheta z>} = 0 \tag{197}$$

$$\begin{aligned} N^{zz}{}_{,z} + q^{z} = 0 \Rightarrow N^{zz} = \int_0^z -q^z d\zeta + K \Rightarrow \\ N^{zz} = N^{<zz>} = -q^z \left(z - h\right) \end{aligned} \tag{198}$$

Thus, the only nonzero components of $\bar{v}$ are those along ξ and z due to the self load and to the Poisson effect, which are respectively

$$v^{\xi} = \frac{r^2 q^{\xi} + r\nu q^z (z-h)}{E(2\epsilon)} \tag{199}$$

$$v^{z} = \frac{1}{E(2\epsilon)} \left(-q^z \left(\frac{z^2}{2} - hz\right) - \nu r q^{\xi} z\right) \tag{200}$$

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