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Notes on Geometry of Surfaces

Notes for students
Used in the course of Mathematical Physics for Civil Engineers.

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PREFACE

These brief notes are aimed at sketching a few basic ideas about Riemannian manifolds and submanifolds, with emphasis on the hypersurfaces of a Euclidean three dimensional space.

The reader is supposed to be familiar with the elementary notions concerning linear and multilinear algebra, manifolds, tangent space and the Lie derivative of vector fields and forms.

CONTENTS

INTRODUCTION

All manifolds will be finite dimensional and smooth and all maps between manifolds will be smooth.

We consider a manifold M of dimension m.

We denote the tangent and cotangent spaces of M by TM and T^*M, respectively; moreover, we denote the set of local vector fields $X : M \to TM$ and local forms $\alpha : M \to T^*M$ by $\mathcal{T}(M)$ and $\mathcal{T}^*(M)$, respectively.

We denote the set of local functions $f : M \to \mathrm{I\!R}$ by $\mathcal{F}(M)$.

We denote the k^{th}-tensor power of $\mathcal{T}(M)$ and $\mathcal{T}^*(M)$ by $\mathcal{T}^k(M)$ and $\mathcal{T}^{*k}(M)$, respectively. In particular, we have $\mathcal{T}^0(M) = \mathcal{F}(M) = \mathcal{T}^{*0}(M)$.

Thus, the tensor algebra of M is constituted by the direct sum

$$\mathcal{A}(M) = \bigoplus_{0 \le k \le \infty} \mathcal{T}^{*k}(M) \oplus \bigoplus_{1 \le k \le \infty} \mathcal{T}^k(M) .$$

We shall refer to local charts (x^i) of M.

We denote the local basis of vector fields and forms induced by the above local charts by $(\partial x_i) \subset \mathcal{T}(M)$ and $(dx^i) \subset \mathcal{T}^*(M)$, respectively.

We denote the local charts induced on TM and T^*M, respectively, by $(x^i, \dot{x}^i)$ and $(x^i, \dot{x}_i)$. Thus, for each vector field $X = X^i \, \partial x_i$ and form $\omega = \omega_i \, dx^i$, we can write

$$\dot{x}^i \circ X = \langle dx^i \, , \, X \rangle = X^i \, , \qquad \dot{x}_i \circ \omega = \langle \omega \, , \, \partial x_i \rangle = \omega_i \, .$$

If $p : E \to B$ and $q : F \to B$ are two bundles over the same base space B, then we denote their *fibred product* over B by

$$E \underset{B}{\times} F := \{(e, f) \in E \times F \mid p(e) = q(f)\} \subset E \times F \, .$$

CHAPTER 1

CONNECTIONS ON MANIFOLDS

In this chapter we introduce the notion of linear connection on a manifold M, in terms of the covariant differential ∇, and analyse the torsion T and the curvature R of a linear connection.

Then, we introduce the Riemannian metric g and study the induced linear connection. In this context, we discuss also the relation between the Lagrange formulas and the acceleration of a curve.

1.1 Linear connections

We start by introducing a general linear connection on a manifold and discussing its torsion, curvature tensor and Ricci tensor.

1.1.1 Covariant differential

We introduce the notion of linear connection by means of the associated covariant differential and show its coordinate expression.

1.1.1 Definition. A *linear connection* is defined to be a map

$$\nabla : \mathcal{T}(M) \times \mathcal{A}(M) \to \mathcal{A}(M) : (X, t) \mapsto \nabla_X t\,,$$

which fulfills the following properties:

1) for each $X \in \mathcal{T}(M)$, ∇_X preserves the degree of tensors;

2) ∇ commutes with local restrictions;

3) for each $X, Y \in \mathcal{T}(M)$, $f \in \mathcal{F}(M)$, $t \in \mathcal{A}(M)$,

$$\nabla_{X+Y} t = \nabla_X t + \nabla_Y t\,, \qquad \nabla_{fX} t = f \nabla_X t\,;$$

4) for each $X \in \mathcal{T}(M)$, $t, t' \in \mathcal{A}(M)$,

$$\nabla_X (t + t') = \nabla_X t + \nabla_X t'\,;$$

5) for each $X \in \mathcal{T}(M)$, $t, t' \in \mathcal{A}(M)$,

$$\nabla_X (t \otimes t') = (\nabla_X t) \otimes t' + t \otimes (\nabla_X t')\,;$$

6) for each $X \in \mathcal{T}(M)$, $f \in \mathcal{F}(M)$,

$$\nabla_X f = X.f \equiv \langle df\,,\, X \rangle\,;$$

7) for each $X \in \mathcal{T}(M)$, $Y \in \mathcal{T}(M)$, $\omega \in \mathcal{T}^*(M)$,

$$X.\langle \omega\,,\, Y \rangle = \langle \nabla_X \omega\,,\, Y \rangle + \langle \omega\,,\, \nabla_X Y \rangle\,.$$

For each $X \in \mathcal{T}(M)$, $t \in \mathcal{A}(M)$, we say that $\nabla_X t \in \mathcal{A}(M)$ is the *covariant derivative* of t with respect to X; moreover, we say that the induced tensor $\nabla t \in \mathcal{T}^*(M) \otimes \mathcal{A}(M)$, given by

$$\nabla t : \mathcal{T}(M) \to \mathcal{A}(M) : X \mapsto \nabla_X t\,,$$

is the *covariant differential* of t. $\square$

1.1.2 Proposition. The map ∇ is characterised by its restriction to vector fields

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M) : (X, Y) \mapsto \nabla_X Y\,.$$

Proof. It follows immediately from properties 5) and 7). QED

Thus, let us consider a linear connection ∇ .

1.1.3 Note. From the above definition we obtain immediately the following result. For each $X \in \mathcal{T}(M)$, $f \in \mathcal{F}(M)$, $t \in \mathcal{A}(M)$,

$$\nabla_X(ft) = f\nabla_X t + (X.f)t\,;$$

Hence, in particular, for each $X \in \mathcal{T}(M)$, $k \in \mathrm{I\!R}$, $t \in \mathcal{A}(M)$,

$$\nabla_X(kt) = k\nabla_X t\,.\,\square$$

1.1.4 Proposition. The coordinate expression of ∇ is given by the following formulas.

For each $X \in \mathcal{T}(M)$, $t \in \mathcal{T}^k(M)$,

$$\nabla_X t = X^j\,(\partial_j t^{i_1\dots i_k} + \Gamma_j{}^{i_1}{}_h\, t^{h i_2\dots i_k} + \dots + \Gamma_j{}^{i_k}{}_h\, t^{i_1\dots i_{k-1}h})\,\partial x_{i_1} \otimes \dots \otimes \partial x_{i_k}\,,$$

and, for each $X \in \mathcal{T}(M)$, $t \in \mathcal{T}^{*k}(M)$,

$$\nabla_X t = X^j\,(\partial_j t_{i_1\dots i_k} - \Gamma_j{}^h{}_{i_1}\, t_{h i_2\dots i_k} - \dots - \Gamma_j{}^h{}_{i_k}\, t_{i_1\dots i_{k-1}h})\,dx^{i_1} \otimes \dots \otimes dx^{i_k}\,,$$

where

$$\Gamma_i{}^h{}_j := (\nabla_{\partial x_i}\partial x_j)^h = -(\nabla_{\partial x_i} dx^h)_j\,.$$

Proof. It follows easily from the properties in the definition of ∇ . QED

1.1.2 Torsion

We introduce the notion of torsion tensor of a linear connection and show its coordinate expression.

1.1.5 Lemma. The map

$$\mathsf{T} : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)\,,$$

given by

$$\mathsf{T}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]\,,$$

is a tensor. Moreover, T is antisymmetric.

Proof. Clearly, for each $X, X', Y, Y' \in \mathcal{T}(M)$, we have

$$\mathsf{T}(X+X',Y) = \mathsf{T}(X,Y) + \mathsf{T}(X',Y)\,, \qquad \mathsf{T}(X,Y+Y') = \mathsf{T}(X,Y) + \mathsf{T}(X,Y')\,.$$

Moreover, for each $X, Y \in \mathcal{T}(M)$, $f \in \mathcal{F}(M)$, we have

$$\begin{aligned} \mathsf{T}(fX,Y) &= f\,\nabla_X Y - f\,\nabla_Y X - (Y.f)\,X - f\,[X,Y] + (Y.f)\,X = f\,\mathsf{T}(X,Y)\,,\\ \mathsf{T}(X,fY) &= f\,\nabla_X Y + (X.f)\,Y - f\,\nabla_Y X - f\,[X,Y] - (X.f)\,Y = f\mathsf{T}(X,Y)\,. \end{aligned}$$

Hence, T is a tensor.

Furthermore, we can immediately see that T is antisymmetric. QED

1.1.6 Definition. The tensor T is called the *torsion tensor* of ∇ . □

1.1.7 Proposition. The coordinate expression of the torsion tensor is

$$\begin{aligned}\mathsf{T} &\equiv \mathsf{T}_{ij}{}^{h}\, dx^i \otimes dx^j \otimes \partial x_h \\ &= (\Gamma_i{}^h{}_j - \Gamma_j{}^h{}_i)\, dx^i \otimes dx^j \otimes \partial x_h \\ &= 2\,\Gamma_i{}^h{}_j\, dx^i \wedge dx^j \otimes \partial x_h\,,\end{aligned}$$

with

$$\mathsf{T}_{ij}{}^{h} = \Gamma_i{}^h{}_j - \Gamma_j{}^h{}_i\,. \square$$

1.1.3 Curvature

We introduce the notion of curvature tensor and Ricci tensor of a linear connection and show their coordinate expressions.

1.1.8 Lemma. The map

$$\mathsf{R} : \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)\,,$$

given by

$$\mathsf{R}(X, Y; Z) := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,,$$

is a tensor. Moreover, R is antisymmetric with respect to the first two entries.

Proof. Clearly, for each $X, X', Y, Y', Z, Z' \in \mathcal{T}(M)$, we have

$$\begin{aligned}\mathsf{R}(X + X', Y; Z) &= \mathsf{R}(X, Y; Z) + \mathsf{R}(X', Y; Z)\,, \\ \mathsf{R}(X, Y + Y'; Z) &= \mathsf{R}(X, Y; Z) + \mathsf{R}(X, Y'; Z)\,, \\ \mathsf{R}(X, Y; Z + Z') &= \mathsf{R}(X, Y; Z) + \mathsf{R}(X, Y; Z')\,.\end{aligned}$$

Moreover, for each $X, Y, Z \in \mathcal{T}(M)$, $f \in \mathcal{F}(M)$, we have

$$\begin{aligned}\mathsf{R}(fX, Y; Z) &= f\, \nabla_X \nabla_Y Z - f\, \nabla_Y \nabla_X Z - (Y.f)\, \nabla_X Z - f\, \nabla_{[X,Y]} Z + (Y.f)\, \nabla_X Z \\ &= f\, \mathsf{R}(X, Y; Z)\,, \\ \mathsf{R}(X, fY; Z) &= f\, \nabla_X \nabla_Y Z + (X.Y.f)\, Z - f\, \nabla_Y \nabla_X Z - (Y.X.f)\, Z \\ &\quad - f\, \nabla_{[X,Y]} Z - ([X, Y].f)\, Z \\ &= f\, \mathsf{R}(X, Y; Z)\,.\end{aligned}$$

Hence, R is a tensor.

Moreover, we can immediately see that R is antisymmetric with respect to the first two entries. QED

1.1.9 Definition. The tensor R is called the *curvature tensor* of ∇ . □

1.1.10 Proposition. The coordinate expression of the curvature tensor is

$$\begin{aligned}\mathsf{R} &\equiv \mathsf{R}_{ij}{}^{h}{}_{k}\, dx^i \otimes dx^j \otimes \partial x_h \otimes dx^k \\ &= (\partial_i \Gamma_j{}^h{}_k - \Gamma_i{}^l{}_k\, \Gamma_j{}^h{}_l - \partial_j \Gamma_i{}^h{}_k + \Gamma_j{}^l{}_k\, \Gamma_i{}^h{}_l)\, dx^i \otimes dx^j \otimes \partial x_h \otimes dx^k \\ &= 2\, (\partial_i \Gamma_j{}^h{}_k - \Gamma_i{}^l{}_k\, \Gamma_j{}^h{}_l)\, dx^i \wedge dx^j \otimes \partial x_h \otimes dx^k\,,\end{aligned}$$

with

$$\mathsf{R}_{ij}{}^{h}{}_{k} = \partial_i \Gamma_j{}^h{}_k - \Gamma_i{}^l{}_k\, \Gamma_j{}^h{}_l - \partial_j \Gamma_i{}^h{}_k + \Gamma_j{}^l{}_k\, \Gamma_i{}^h{}_l\,. \square$$

1.1.11 Proposition. The curvature tensor fulfills the identities

$$\mathsf{R}_{ij}{}^{h}{}_{k} = -\mathsf{R}_{ji}{}^{h}{}_{k} \qquad \text{and} \qquad \mathsf{R}_{ij}{}^{h}{}_{k} + \mathsf{R}_{ki}{}^{h}{}_{j} + \mathsf{R}_{jk}{}^{h}{}_{i} = 0\,. \square$$

1.1.12 Corollary. If $\dim M = n$, then the number i_{R} of independent components of the curvature tensor is

$$i_{\mathsf{R}} = \frac{n^2\,(n^2-1)}{3}\,.$$

Proof. It follows by taking into account the symmetry properties of R. We omit a detailed proof (see [15]). QED

1.1.13 Example. We have the following particular cases:
1) if $\dim M = 1$, then $i_{\mathsf{R}} = 0$, hence $\mathsf{R} = 0$;
2) if $\dim M = 2$, then $i_{\mathsf{R}} = 4$;
3) if $\dim M = 3$, then $i_{\mathsf{R}} = 24$. $\square$

1.1.14 Definition. We define the *Ricci tensor* to be the tensor

$$\underline{\mathtt{r}} := C^1_1 \mathsf{R} \in \mathcal{T}^*(M) \otimes \mathcal{T}^*(M)\,,$$

where C^1_1 denotes the contraction of the first contravariant index with the first covariant index. $\square$

1.1.15 Proposition. The coordinate expression of the Ricci tensor is

$$\begin{aligned}\underline{\mathtt{r}} &= \underline{\mathtt{r}}_{ij}\, dx^i \otimes dx^j \\ &= \mathsf{R}_{hi}{}^{h}{}_{j}\, dx^i \otimes dx^j \\ &= (\partial_h \Gamma_i{}^h{}_j - \Gamma_h{}^k{}_j\, \Gamma_i{}^h{}_k - \partial_i \Gamma_h{}^h{}_j + \Gamma_i{}^k{}_j\, \Gamma_h{}^h{}_k)\, dx^i \otimes dx^j\,,\end{aligned}$$

with

$$\underline{\mathtt{r}}_{ij} = \partial_h \Gamma_i{}^h{}_j - \Gamma_h{}^k{}_j\, \Gamma_i{}^h{}_k - \partial_i \Gamma_h{}^h{}_j + \Gamma_i{}^k{}_j\, \Gamma_h{}^h{}_k\,. \square$$

1.2 Riemannian connections

A Riemannian manifold is a manifold whose tangent fibres are equipped with a metric.

The Riemannian metric yields a distinguished linear connection.

The best practical way to compute the coordinate expression of the Riemannian connection is via the Lagrange expression of the covariant acceleration of motions.

The curvature tensor of a Riemannian connection has distinguished properties.

1.2.1 Riemannian metric

We introduce the notion Riemannian metric and analyse the associated algebraic objects.

1.2.1 Definition. A *Riemannian metric* of M is defined to be a symmetric and positive definite bilinear form

$$g : TM \underset{M}{\times} TM \to \mathbb{R} \,. \square$$

Its coordinate expression is

$$g = g_{ij}\, dx^i \otimes dx^j \,.$$

Let us assume that M is equipped with a Riemannian metric g.

The metric g yields the mutually inverse isomorphisms

$$g^\flat : TM \to T^*M : X \to g^\flat(X) \,, \qquad g^\sharp : T^*M \to TM : \omega \to g^\flat(\omega) \,,$$

characterised by

$$\langle g^\flat(X),\, Y\rangle = g(X, Y) \,, \qquad g(g^\sharp(\omega),\, Y) = \langle \omega,\, Y\rangle \,,$$

for each vector field Y.

Their coordinate expressions are

$$g^\flat(X) = g_{ij} X^j dx^i \,, \qquad g^\sharp(\omega) = g^{ij}\omega_j \partial x_i \,,$$

where

$$(g^{ij}) = (g_{hk})^{-1} \,.$$

We denote the contravariant metric by

$$\bar{g} := (g^\sharp \otimes g^\sharp)(g) : T^*M \underset{M}{\times} T^*M \to \mathbb{R} \,.$$

Its coordinate expression is

$$\bar{g} = g^{ij}\, \partial x_i \otimes \partial x_j \,.$$

We define the *metric function* to be the function

$$G : TM \to \mathrm{I\!R} : X \mapsto \tfrac{1}{2}\, g(X, X)\,,$$

with coordinate expression

$$G = \tfrac{1}{2}\, g_{ij}\, \dot{x}^i \dot{x}^j\,.$$

1.2.2 Volume form

We introduce the notion of volume form induced by a Riemannian connection.

1.2.2 Definition. A *volume form* of M is defined to be (at least locally) a section

$$\eta : M \to \Lambda^n T^* M\,,$$

which is identically non vanishing.

The *dual volume form* of η is defined to be the unique section

$$\bar{\eta} : M \to \Lambda T^* M\,,$$

such that

$$\langle \eta, \bar{\eta} \rangle_\wedge = 1\,,$$

where $\langle\, ,\, \rangle_\wedge$ denotes the contraction, in the sense of exterior forms, defined via the interior product i . □

1.2.3 Proposition. The coordinate expression of a volume form and of the dual volume form is of the type

$$\eta = \alpha\, dx^1 \wedge \ldots \wedge dx^n \qquad \text{and} \qquad \bar{\eta} = (1/\alpha)\, \partial x_1 \wedge \ldots \wedge \partial x_n\,,$$

where $\alpha : M \to \mathrm{I\!R}$ is an identically non vanishing function. □

1.2.4 Remark. The standard contraction between η and $\bar{\eta}$ is different from the above contraction. In fact, we have

$$\langle \eta\,,\, \bar{\eta} \rangle = (1/n!)\, \langle \eta, \bar{\eta} \rangle_\wedge = 1/n!\,. \, □$$

1.2.5 Proposition. The Riemannian metric g determines, up to sign, locally a volume form

$$\eta : M \to \Lambda^n T^* M\,,$$

by the condition

$$(\Lambda^n g)\, (\eta, \eta) = 1\,.$$

Moreover, if the manifold M is orientable, then this volume form exists globally.

We have the coordinate expression

$$\eta = \pm\sqrt{\det(g_{ij})}\, dx^1 \wedge \ldots \wedge dx^n \,.$$

In other words, poinwisely, if $(\epsilon^1, \ldots, \epsilon^n)$ is an orthonormal basis of forms, then we can write

$$\eta = \pm\epsilon^1 \wedge \ldots \wedge \epsilon^n \,. \square$$

1.2.6 Corollary. The dual volume form $\bar{\eta}$ of the volume form η induced by the metric turns out to be just the contavariant tensor of η.

In other words, we have

$$\bar{\eta} = (g^\sharp \otimes \ldots \otimes g^\sharp)\, \eta \qquad \text{and} \qquad \eta = (g^\flat \otimes \ldots \otimes g^\flat)\, \bar{\eta} \,. \square$$

1.2.3 Riemannian connection

We introduce the distinguished linear connection induced by the Riemannian metric.

1.2.7 Theorem. *There is a unique linear connection ∇ such that*

$$\nabla g = 0 \,, \qquad \mathsf{T} = 0 \,.$$

Indeed, ∇ is given, for each $X, Y, Z \in \mathcal{T}(M)$, by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X.(g(Y,Z)) + Y.(g(Z,X)) - Z.(g(X,Y)) \\ &\quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X) \,. \end{aligned}$$

Proof. *Uniqueness.* If ∇ exists, then, for each $X, Y, Z \in \mathcal{T}(M)$, we obtain

$$\begin{aligned} X.(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ \nabla_X Y &= \nabla_Y X + [X,Y] \,, \end{aligned}$$

hence, by cyclic rotation of the vector fields,

$$\begin{aligned} +X.(g(Y,Z)) &= +g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ -Z.(g(X,Y)) &= -g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \\ +Y.(g(Z,X)) &= +g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \,, \end{aligned}$$

hence, summing side by side,

$$X.(g(Y,Z)) + Y.(g(Z,X)) - Z.(g(X,Y)) =$$

$$\begin{aligned} &= +\, g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ &\quad +\, g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ &\quad -\, g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \,, \end{aligned}$$

$$= 2\,g(\nabla_X Y, Z) + g(Y, [X, Z]) + g([Y, Z], X) + g(Z, [Y, X])\,.$$

Existence. We can easily prove that the above expression of ∇ fulfills the properties of linear connections. QED

1.2.8 Definition. The unique linear connection ∇ which fulfills the above conditions is called the *Riemannian connection.* □

1.2.9 Proposition. The coordinate expression of the Riemannian connection ∇ is given by

$$\Gamma_i{}^h{}_j = \tfrac{1}{2}\, g^{hk}\,(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})\,.$$

PROOF. The above formula can be obtained as the coordinate expression of the intrinsic formula defining ∇ in the above theorem.

But we can also derive directly the coordinate expression of ∇. In fact, the assumed conditions read, in coordinates, as

$$\partial_h g_{ij} = \Gamma_{hij} + \Gamma_{hji}\,, \qquad \Gamma_{ihj} = \Gamma_{jhi}$$

Then, we obtain

$$\begin{aligned} -\Gamma_{hij} - \Gamma_{hji} &= -\partial_h g_{ij} \\ +\Gamma_{jhi} + \Gamma_{ihj} &= +\partial_j g_{hi} \\ +\Gamma_{ijh} + \Gamma_{jih} &= +\partial_i g_{jh}\,, \end{aligned}$$

hence, summing side by side,

$$2\Gamma_{ihj} = \partial_i g_{jh} + \partial_j g_{ih} - \partial_h g_{ij}\,.$$ QED

1.2.4 Lagrange formulas

A convenient way to compute the coefficients of the Riemannian connection is via the covariant acceleration of curves, expressed through the Lagrange formulas, in the following way.

Let us consider a curve $c : \mathrm{I\!R} \to M$ and its differential

$$dc : \mathrm{I\!R} \to TM\,,$$

with coordinate expression

$$x^i \circ c = c^i\,, \qquad \dot{x}^i \circ dc = Dc^i\,.$$

1.2.10 Lemma. The map

$$\nabla dc := (\nabla_X X) \circ c : \mathrm{I\!R} \to TM\,,$$

where $X : M \to TM$ is an extension of dc, does not depend on the choice of the extension, hence is well defined.

Proof. It follows easily from the coordinate expression of $\nabla_X Y$. QED

1.2.11 Definition. We say that ∇dc is the *curvature* (or the *acceleration*) of c . □

We have the coordinate expression

$$\nabla dc = \big(D^2 c^i + (\Gamma_h{}^i{}_k \circ c)\, Dc^h\, Dc^k\big)\, (\partial x_i \circ c)\,.$$

The *covariant curvature* of c is defined to be the map

$$g^\flat(\nabla dc) : \mathrm{I\!R} \to T^* M\,,$$

with coordinate expression

$$g^\flat(\nabla dc) = g_{ij} \circ c\, \big(D^2 c^j + (\Gamma_h{}^j{}_k \circ c)\, Dc^h\, Dc^k\big)\, (dx^i \circ c)\,.$$

1.2.12 Theorem. [Lagrange formula.] *The covariant curvature of c is given by the following formula*

$$(g^\flat(\nabla dc)) = \mathcal{E}(G, c) := \big(D(\frac{\partial G}{\partial \dot{x}^i} \circ dc) - (\frac{\partial G}{\partial x^i} \circ dc)\big)\, (Dx^i \circ c)\,.$$

Proof. We have

$$\begin{aligned} g^\flat(\nabla dc) &= g_{ij} \circ c\, (D^2 c^j + \Gamma_h{}^j{}_k \circ c\, Dc^h Dc^k) \\ &= g_{ij} \circ c\, D^2 c^j + \tfrac{1}{2}\, (\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk}) \circ c\, Dc^h Dc^k\,. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} D(\frac{\partial G}{\partial \dot{x}^i} \circ dc) - (\frac{\partial G}{\partial x^i} \circ dc) &= D\big((g_{ij} \dot{x}^j) \circ dc\big) - \tfrac{1}{2}\, (\partial_i g_{hk} \dot{x}^h \dot{x}^k) \circ dc \\ &= D(g_{ij} \circ c\, Dc^j) - \tfrac{1}{2}\, (\partial_i g_{hk}) \circ c\, Dc^h Dc^k \\ &= (g_{ij} \circ c) D^2 c^j + \tfrac{1}{2}\, (\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk}) \circ c\, Dc^h Dc^k\,. \text{ QED} \end{aligned}$$

1.2.13 Note. In practice, a quick way to compute the coefficients of ∇ is the following:
- compute the covariant curvature of a generic curve c , through the Lagrange formulas,
- then compute the curvature of c by means of $g^\sharp$,
- eventually extract the coefficients of ∇ . □

1.2.5 Riemannian curvature

We discuss the additional properties of the curvature tensor of the Riemannian connection. In particular, we introduce the Riemannian scalar curvature.

1.2.14 Definition. We define the *Riemannian cuvature tensor* to be the curvature tensor R of the Riemannian connection. □

1.2.15 Proposition. The Riemannian curvature tensor fulfills the following identities

$$\mathsf{R}_{ijhk} = -\mathsf{R}_{jihk}\,, \qquad \mathsf{R}_{ijhk} = -\mathsf{R}_{ijkh}\,, \qquad \mathsf{R}_{ijhk} = \mathsf{R}_{hkij}\,, \qquad \mathsf{R}_{ijhk} + \mathsf{R}_{kihj} + \mathsf{R}_{jkhi} = 0\,.$$

Proof. It follows by a computation in coordinates. We omit a detailed proof. QED

1.2.16 Corollary. If $\dim M = n$, then the number i_{R} of independent components of the Riemannian curvature tensor is

$$i_{\mathsf{R}} = \frac{n^2\,(n^2-1)}{12}\,.$$

Proof. It follows by taking into account the symmetry properties of R. We omit a detailed proof (see [15]). QED

1.2.17 Example. We have the following particular cases:
1) if $\dim M = 1$, then $i_{\mathsf{R}} = 0$, hence $\mathsf{R} = 0$;
2) if $\dim M = 2$, then $i_{\mathsf{R}} = 1$;
3) if $\dim M = 3$, then $i_{\mathsf{R}} = 6$. □

1.2.18 Corollary. The Ricci tensor of the Riemannian connection is symmetric:

$$\underline{\mathrm{r}}_{ij} = \underline{\mathrm{r}}_{ji}\,. \square$$

1.2.19 Proposition. If $\dim M = 1, 2, 3$, then the Ricci tensor completely determines the Riemannian curvature tensor.

Proof. It follows by taking into account the symmetry properties of R. We omit a detailed proof. QED

1.2.20 Definition. We define the *Riemannian scalar curvature* to be the function

$$\langle\underline{\mathrm{r}}\rangle := \bar{g} \lrcorner \underline{\mathrm{r}}\,. \square$$

1.2.21 Proposition. The coordinate expression of the Riemannian scalar curvature is

$$\langle\underline{\mathrm{r}}\rangle = g^{ij}\mathsf{R}_{ki}{}^{k}{}_{j}\,. \square$$

1.2.6 Case when M has dimension 2

In the particular case when $\dim M = 2$, as we have already seen, the Riemannian curvature tensor has only 1 independent component and it is completely determined by the Ricci tensor.

Now, we discuss more explicitly this result, by analysing the expression of the Riemannian curvature tensor. Indeed, we prove that, in this case, the Riemannian curvature tensor and the Ricci tensor are fully determined by the Riemannian scalar curvature.

First, let us observe that the volume form η of any Riemannian manifold M is defined locally up to sign, hence $\eta \otimes \eta$ is uniquely defined globally on M.

1.2.22 Proposition. Let us suppose that $\dim M = 2$.
Then, the covariant Riemannian curvature tensor and the Ricci tensor are given by

$$\underline{R} = 2\,\langle \underline{r} \rangle\, \eta \otimes \eta \qquad \text{and} \qquad \underline{r} = \tfrac{1}{2}\,\langle \underline{r} \rangle\, g\,.$$

In other words, pointwisely, if (ϵ^1, ϵ^2) is an othonormal basis of forms, then we have the expressions

$$\underline{R} = \tfrac{1}{2}\,\langle \underline{r} \rangle\, \big(\epsilon^1 \otimes \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^2 + \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^2 \otimes \epsilon^1 - \epsilon^1 \otimes \epsilon^2 \otimes \epsilon^2 \otimes \epsilon^1 - \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^1 \otimes \epsilon^2\big)\,,$$

and

$$\underline{r} = \tfrac{1}{2}\,\langle \underline{r} \rangle\, \big(\epsilon^1 \otimes \epsilon^1 + \epsilon^2 \otimes \epsilon^2\big)\,.$$

Proof. We observe that all antisymmetric covariant 2–tensors are proportional to η. Therefore, the antisymmetry of $\underline{R}$ with respect to the indices $(1,2)$ and to the indices $(3,4)$ (see Proposition 1.2.15) implies that $\underline{R}$ is of the type

$$\underline{R} = \mu\, \eta \otimes \eta\,, \qquad \text{with} \qquad \mu : M \to \mathbb{R}\,.$$

On the other hand, poinwisely, we have the coordinate expression

$$\begin{aligned}
\eta \otimes \eta &= (\epsilon^1 \wedge \epsilon^2) \otimes (\epsilon^1 \wedge \epsilon^2) \\
&= \tfrac{1}{2}\,(\epsilon^1 \otimes \epsilon^2 - \epsilon^2 \otimes \epsilon^1) \otimes \tfrac{1}{2}\,(\epsilon^1 \otimes \epsilon^2 - \epsilon^2 \otimes \epsilon^1) \\
&= \tfrac{1}{4}\,(\epsilon^1 \otimes \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^2 + \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^2 \otimes \epsilon^1 - \epsilon^1 \otimes \epsilon^2 \otimes \epsilon^2 \otimes \epsilon^1 - \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^1 \otimes \epsilon^2)\,.
\end{aligned}$$

Hence, pointwisely, we can write

$$\underline{R} = \tfrac{1}{4}\,\mu\,(\epsilon^1 \otimes \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^2 + \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^2 \otimes \epsilon^1 - \epsilon^1 \otimes \epsilon^2 \otimes \epsilon^2 \otimes \epsilon^1 - \epsilon^2 \otimes \epsilon^1 \otimes \epsilon^1 \otimes \epsilon^2)\,.$$

(By the way, we observe that the above expression fulfills also the other symmetry properties of $\underline{R}$.)
Then, the above expression yields, pointwisely, the following expression of the the Ricci tensor

$$\underline{r} = \tfrac{1}{4}\,\mu\,(\epsilon^1 \otimes \epsilon^1 + \epsilon^2 \otimes \epsilon^2)\,,$$

which can be read globally as

$$\underline{r} = \tfrac{1}{4}\,\mu\, g\,.$$

Moreover, the above equality yields the following expression of the Riemannian scalar curvature

$$\langle \underline{r} \rangle = \tfrac{1}{2}\,\mu\,.$$

Thus, eventually, we obtain

$$\underline{R} = 2\,\langle \underline{r} \rangle\, \eta \otimes \eta \qquad \text{and} \qquad \underline{r} = \tfrac{1}{2}\,\langle \underline{r} \rangle\, g\,.\,\text{QED}$$

1.2.23 Corollary. Let us suppose that $\dim M = 2$.

If, pointwisely, (e_1, e_2) is an orthonormal basis, then we have the equality

$$\underline{R}(e_1, e_2, e_1, e_2) = \tfrac{1}{2}\,\langle \underline{r} \rangle\,,$$

and, equivalently,

$$\underline{R}(e_1, e_2, e_1, e_2) + \underline{R}(e_2, e_1, e_2, e_2) = \langle \underline{r} \rangle\,. \square$$

1.2.24 Corollary. Let us suppose that $\dim M = 2$.
Then, by denoting the contravariant volume form of M by $\bar{\eta}$, we obtain

$$\langle \underline{R}\,,\, \bar{\eta} \otimes \bar{\eta} \rangle = \tfrac{1}{2}\,\langle \underline{r} \rangle\,.$$

Proof. Let, pointwisely, (e_1, e_2) be an orthonormal basis and (ϵ^1, ϵ^2) the dual basis.
Then, pointwisely, we obtain

$$\begin{aligned}\bar{\eta} \otimes \bar{\eta} &= (e_1 \wedge e_2) \otimes (e_1 \wedge e_2)\\ &= \tfrac{1}{2}\,(e_1 \otimes e_2 - e_2 \otimes e_1) \otimes \tfrac{1}{2}\,(e_1 \otimes e_2 - e_2 \otimes e_1)\\ &= \tfrac{1}{4}\,(e_1 \otimes e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 \otimes e_1 - e_1 \otimes e_2 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1 \otimes e_2)\,.\end{aligned}$$

Hence, pintwisely, we obtain

$$\langle \eta \otimes \eta\,,\, \bar{\eta} \otimes \bar{\eta} \rangle = \tfrac{1}{4}\ .$$

Eventually, we obtain

$$\begin{aligned}\langle \underline{R}\,,\, \bar{\eta} \otimes \bar{\eta} \rangle &= 2\,\langle \underline{r} \rangle\, \langle \eta \otimes \eta\,,\, \bar{\eta} \otimes \bar{\eta} \rangle\\ &= \tfrac{1}{2}\,\langle \underline{r} \rangle\,. \text{QED}\end{aligned}$$

CHAPTER 2

CONNECTIONS AND SUBMANIFOLDS

In this chapter we introduce the notion of submanifold Q of a manifold M.

We discuss and compare two viewpoints for the analysis of geometric structures of the submanifold:

- the viewpoint of the environing manifold,
- the intrinsic viewpoint of the submanifold.

In this context, we analyse the parallel and orthogonal projections of objects of the manifold, with respect to the submanifold. In particular, we study the Gauss splitting of the connection.

Then, we study the hypersurfaces, i.e. the submanifolds of codimension 1.

2.1 Submanifolds

A submanifold of a manifold is defined to be a subset characterised by “regular” constraints. Then, the environing manifold induces a smooth structure on this subset.

2.1.1 Basic definition

Let us consider a manifold M of dimension m.

2.1.1 Definition. A *submanifold* of M is defined to be a subset

$$j : Q \hookrightarrow M\,,$$

which is locally characterised by equations (called *constraints*) of the type

$$x^i = 0\,, \qquad l + 1 \le i \le m\,,$$

where x^i are local functions which belong to a chart (x^j) of M.
Such a chart of M is said to be *adapted* to Q. □

Let us consider a submanifold $Q \subset M$.
We can easily see that Q inherits a smooth structure of manifold with $\dim Q = l$.
The induced atlas is constituted by the charts

$$(x^{\dagger i}) := (x^i)_{|Q}\,, \qquad 1 \le i \le l\,.$$

We shall always refer to adapted charts of M and to the induced charts of Q.

In general, the symbol † will label objects living on the submanifold.

The coordinate expression of the inclusion j is quite simple:

$$(x^i) \circ j = (x^{\dagger i})\,, \quad 1 \le i \le l\,, \qquad (x^i) \circ j = (0)\,, \quad l + 1 \le i \le m\,,$$

2.1.2 Tangent and cotangent spaces

We analyse the basic relations between the tangent and cotangent spaces of the submanifold Q and of the environing manifold M.

We see that there is a natural inclusion $TQ \to T_Q M$ and a natural projection $T^*_Q M \to TQ$, whose coordinate expressions are quite simple.

We denote by

$$T_Q M \subset TM \qquad \text{and} \qquad T^*QM$$

the subspaces of vectors and forms of M whose base point belongs to Q.

2.1.2 Proposition. The map j induces the natural maps

$$Tj : TQ \to T_Q M \qquad \text{and} \qquad T^*j : T^*_Q M \to T^*Q \,,$$

which are, respectively, injective and surjective. □

2.1.3 Note. Indeed, the following interpretations hold.

1) The map Tj allows us to regard naturally the vectors tangent to Q as particular vectors of M.

Accordingly, we shall identify TQ with its image $Tj(TQ) \subset TM$.

2) By definition of the transposition * of the inclusion Tj, the projection T^*j is just the restriction of the forms of M over Q to the vectors tangent to the submanifold Q.

In other words, if ω is a form of M over Q, then, for each vector field X of Q, we have

$$\big(Tj^* \circ (\omega)\big)(X) = \omega(Tj \circ X) \simeq \omega(X) \,. \,\square$$

2.1.4 Proposition. The coordinate expressions of Tj and T^*j are

$$(x^i, \dot{x}^i) \circ Tj = (x^{\dagger i}, \dot{x}^{\dagger i}) \,, \quad 1 \le i \le l \,, \qquad (x^i, \dot{x}^i) \circ Tj = (0, 0) \,, \quad l+1 \le i \le m \,,$$
$$(x^{\dagger i}, \dot{x}^{\dagger}{}_i) \circ T^*j = (x^{\dagger i}, \dot{x}^{\dagger}{}_i) \,, \quad 1 \le i \le l \,.$$

Thus, each vector field $X : Q \to TQ$ of Q can be naturally regarded as a vector field $X : Q \to T_Q M$ of M over Q, according to the coordinate expression

$$X = \sum_{1 \le i \le l} X^i \, \partial x^{\dagger}{}_i = \sum_{1 \le i \le l} X^i \, (\partial x_i) \circ j \,.$$

On the other hand, each form $\omega : Q \to T^*_Q M$ of M over Q can be naturally projected onto a form $\pi(\omega) : Q \to T^*Q$ of Q, according to the coordinate expression

$$\pi(\omega) = \sum_{1 \le i \le l} \omega_i \, dx^{\dagger i} \,. \,\square$$

2.1.5 Remark. We stress that the smooth structure DOES NOT yield a natural linear projection $T_Q M \to TQ$ and a natural linear injection $T^*Q \to T^*_Q M$.

In other words, the smooth structure DOES NOT yield a natural splitting of $T_Q M$ into TQ plus a complementary subspace. Of course, each adapted chart yields locally such a splitting, but different charts yield different splittings.

On the other hand, if M is a Riemannian manifold, then the Riemannian metric g induces a natural splitting as above. □

2.1.3 Induced Riemannian metric

Next, we assume that the environing manifold M be a Riemannian manifold equipped with the Riemannian metric g.

In this case, the submanifold inherits in a natural way a Riemannian metric $g^\dagger$.

From now on, we suppose that M be a Riemannian manifold equipped with the Riemannian metric g.

2.1.6 Proposition. The induced map

$$g^\dagger := j^* g = g \circ (Tj \times Tj) : TQ \underset{Q}{\times} TQ \to \mathrm{I\!R}$$

turns out to be a Riemannian metric of Q.

2.1.7 Definition. The induced metric $g^\dagger$ is said to be the *first fundamental form* of the submanifold Q. □

2.1.8 Note. We observe that, in any adapted chart, the matrix $({g^\dagger}_{ij})$ of $g^\dagger$ coincides with the submatrix of $(g_{ij} \circ j)$ consisting of the first l rows and l columns. □

2.1.4 Parallel and orthogonal projections

The Riemannian metric g of the environing manifold M allows us to split the vectors of M, whose base point belongs to the submanifold Q, into their parallel and orthogonal components with respect to Q.

We show a convenient way to compute the projections into the parallel and orthogonal components.

We denote the parallel and orthogonal projections induced by g by

$$\pi^\parallel : T_Q M \to TQ \subset T_Q M \qquad \text{and} \qquad \pi^\perp : T_Q M \to TQ^\perp \subset T_Q M .$$

Let us compute the coordinate expressions of these projections.

2.1.9 Definition. An adapted chart (x^i) is said to be *special* if

$$(\partial x_i) \circ j : Q \to TQ^\perp , \qquad l+1 \le i \le m .$$

Indeed, the coordinate expressions of the parallel and orthogonal projections are very simple in a special chart.

2.1.10 Proposition. In a special chart, for each vector field X of M over Q, we have the following coordinate expressions

$$\pi^\parallel(X) = \sum_{1 \le i \le l} X^i \, {\partial x^\dagger}_i$$

$$\pi^\perp(X) = \sum_{l+1 \le i \le m} X^i \, (\partial x_i) \circ j .$$

Moreover, in any special chart we have

$$(g^\dagger)^{hk} = (g^{hk}) \circ j\,, \qquad 1 \le h, k \le l\,.\,\square$$

Unfortunately, not all adapted charts are special.

For a general chart, a convenient way to perform the parallel and orthogonal projections is to pass through forms, as follows.

2.1.11 Proposition. The following diagram commutes

$$\begin{tikzcd} T_Q M \arrow[r, "\pi^{\|}"] \arrow[d, "g^\flat"'] & TQ \\ T^*_Q M \arrow[r, "T^* j"'] & T^* Q \arrow[u, "(g^\dagger)^\sharp"'] \end{tikzcd}$$

Proof. Let X be a vector field of M over Q. Then, for each vector field Y of Q, we have

$$\begin{aligned} g(X^{\|}, Y) &= g(X, Y) = \langle g^\flat(X),\, Y\rangle = \langle \pi(g^\flat(X)),\, Y\rangle = g^\dagger\big((g^\dagger)^\sharp(\pi(g^\flat(X))),\, Y\big) \\ &= g^\dagger\big((g^\dagger)^\sharp(\pi(g^\flat(X))),\, Y\big)\,. \end{aligned}$$

Hence, we obtain

$$X^{\|} = (g^\dagger)^\sharp(\pi(g^\flat(X)))\,.\ \text{QED}$$

2.1.12 Corollary. For each vector field X of M over Q, we have the following coordinate expressions, in any adapted chart,

$$\begin{aligned} \pi^{\|}(X) &= \sum_{1\le h\le m}^{1\le i,j\le l} (g^\dagger)^{ij}\, g_{jh}\, X^h\, \partial x^\dagger{}_i \\ \pi^\perp(X) &= \sum_{l+1\le i\le m} X^i\,(\partial x_i)\circ j - \sum_{l+1\le h\le m}^{1\le i,j\le l} (g^\dagger)^{ij}\, g_{jh}\, X^h\, \partial x^\dagger{}_i\,.\,\square \end{aligned}$$

2.1.13 Note. We can easily verify that the coordinate expressions in the above Corollary 2.1.12, valid for any adapted chart, coincide with the expressions in Proposition 2.1.10, valid for a special adapted chart.

In fact, for a special adapted chart, we have, for each $1 \le h \le m$ and $1 \le i \le l$,

$$\sum^{1\le j\le l} (g^\dagger)^{ij}\, g_{jh} = \delta^i_h\,.\,\square$$

2.1.5 Induced Riemannian connection

The submanifold Q inherits in a natural way a Riemannian connection from the environing Riemannian manifold M.

In fact, the Riemannian metric $g^\dagger$, induced on the submanifold Q, yields a Riemannian connection $\nabla^\dagger$ on the submanifold Q.

In Section 1.2.4, we have discussed a convenient way to compute the symbols of a Riemannian connction via the Lagrange formulas.

Clearly, this convenient procedure can be applied also to the induced Riemannian connection $\nabla^\dagger$ by writing the Lagrange formulas for the induced Riemannian metric function $G^\dagger$.

Let $\nabla^\dagger$ be the Riemannian connection of Q induced by $g^\dagger$.

2.1.14 Proposition. According to the general theory, the coefficients of $\nabla^\dagger$ are given by

$$\begin{aligned}\Gamma^\dagger{}_h{}^i{}_k &:= \big(\nabla^\dagger{}_h(\partial x^\dagger{}_k)\big)^i \\ &= \tfrac{1}{2} \sum_{1\le j\le l} (g^\dagger)^{ij}(\partial_h g^\dagger{}_{jk} + \partial_k g^\dagger{}_{jh} - \partial_j g^\dagger{}_{hk})\,, \qquad 1 \le i, h, k \le l\,.\,\square\end{aligned}$$

Next, we refrase the convenient procedure (see Section 1.2.4) for the computation of the symbols of any Riemannian connection ∇ to the case of the induced Riemannian connection $\nabla^\dagger$.

Let us consider a curve $c^\dagger : \mathbb{R} \to Q$ and its differential

$$dc^\dagger : \mathbb{R} \to TQ\,,$$

with coordinate expression

$$x^{\dagger i} \circ c^\dagger = c^{\dagger i}\,, \qquad \dot{x}^{\dagger i} \circ dc^\dagger = Dc^{\dagger i}\,.$$

2.1.15 Lemma. The map

$$\nabla^\dagger dc^\dagger := (\nabla^\dagger{}_{X^\dagger} X^\dagger) \circ c^\dagger : \mathbb{R} \to TQ\,,$$

where $X^\dagger : Q \to TQ$ is an extension of $dc^\dagger$, does not depend on the choice of the extension, hence is well defined. □

2.1.16 Definition. We say that $\nabla^\dagger dc^\dagger$ is the *(intrinsic) curvature* (or the *(intrinsic) acceleration*) of $c^\dagger$. □

We have the coordinate expression

$$\nabla^\dagger dc^\dagger = \big(D^2 c^{\dagger i} + (\Gamma^\dagger{}_h{}^i{}_k \circ c^\dagger)\, Dc^{\dagger h}\, Dc^{\dagger k}\big)\,(\partial x^\dagger{}_i \circ c^\dagger)\,.$$

The *covariant (intrinsic) curvature* of $c^\dagger$ is defined to be the map

$$g^{\dagger\flat}(\nabla^\dagger dc^\dagger) : \mathbb{R} \to T^*Q\,,$$

with coordinate expression

$$g^{\dagger\flat}(\nabla^\dagger dc^\dagger) = g^\dagger{}_{ij} \circ c^\dagger \left(D^2 c^{\dagger j} + (\Gamma^\dagger{}_h{}^j{}_k \circ c^\dagger)\, Dc^{\dagger h}\, Dc^{\dagger k}\right)(dx^{\dagger i} \circ c^\dagger)\,.$$

2.1.17 Theorem. [Lagrange formula.] *The covariant (intrinsic) curvature of* $c^\dagger$ *is given by the following formula*

$$(g^{\dagger\flat}(\nabla^\dagger dc^\dagger)) = \mathcal{E}(G^\dagger, c^\dagger) := \left(D(\frac{\partial G^\dagger}{\partial \dot{x}^{\dagger i}} \circ dc^\dagger) - (\frac{\partial G^\dagger}{\partial x^{\dagger i}} \circ dc^\dagger)\right)(dx^{\dagger i} \circ c^\dagger)\,.\,\square$$

2.1.18 Note. In practice, a quick way to compute the coefficients of $\nabla^\dagger$ is the following:

- compute the covariant curvature of a generic curve $c^\dagger$, through the Lagrange formulas of the submanifold Q ,
- then compute the curvature of $c^\dagger$ by means of $g^{\dagger\sharp}$,
- eventually extract the non vanishing coefficients of $\nabla^\dagger$. □

2.1.6 Gauss splitting of connection

The covariant derivative of a vector field of the submanifold Q with respect to another vector field of the submanifold Q turns out to be a vector field of the environing manifold M , whose base points belong to the submanifold Q . Hence, we can split this vector field into its parallel and orthogonal components with respect to the submanifold Q .

Indeed, these parallel and orthogonal components have very interesting properties.

2.1.19 Lemma. Let $X : Q \to TQ$ and $Y : Q \to TQ$ be vector fields of Q . Then, the map

$$\nabla_X Y := \nabla_{\tilde{X}} \tilde{Y} \circ j : Q \to TM\,,$$

where $\tilde{X} : Q \to TQ$ and $\tilde{Y} : Q \to TQ$ are extensions of X and Y , respectively, does not depend on the choice of these extensions, hence it is well defined.

We have the coordinate expression

$$\begin{aligned}\nabla_X Y = &\sum_{1\le i,j,h\le l} X^i \left(\partial_i Y^j + (\Gamma_i{}^j{}_h \circ j)\, Y^h\right)(\partial x_j \circ j) \\ &+ \sum_{1\le i,h\le l}^{l+1\le r\le n} X^i\, (\Gamma_i{}^r{}_h \circ j)\, Y^h\, (\partial x_r \circ j)\,.\end{aligned}$$

Proof. It follows easily from the coordinate expression of $\nabla_{\tilde{X}} \tilde{Y}$. QED

2.1.20 Lemma. Let $X : Q \to TQ$ and $Y : Q \to TQ$ be vector fields of Q .

Then, we have the splitting

$$\nabla_X Y = \nabla^{\parallel}{}_X Y + \nabla^{\perp}{}_X Y\,,$$

where

$$\nabla^{\|}{}_X Y := \pi^{\|} \circ \nabla_X Y : Q \to TQ \qquad \text{and} \qquad \nabla^{\perp}{}_X Y := \pi^{\perp} \circ \nabla_X Y : Q \to TQ^{\perp} \,.$$

2.1.6.1 Parallel component

Then, we can state a first important result, which concerns the parallel componenent $\nabla^{\|}{}_X Y$ of $\nabla_X Y$.

2.1.21 Theorem. *The map*

$$\nabla^{\|} : \mathcal{T}(Q) \times \mathcal{T}(Q) \to \mathcal{T}(Q) : (X, Y) \mapsto \nabla^{\|}{}_X Y$$

turns out to be the Riemannian connection of Q *.*

Namely, we have

$$\nabla^{\|} = \nabla^{\dagger} \,.$$

Thus, we have the following coordinate expression

$$\begin{aligned} \Gamma^{\dagger}{}_h{}^i{}_k &= (\Gamma_h{}^i{}_k) \circ j + \sum_{1 \le j \le l}^{l+1 \le r \le m} (g^{\dagger})^{ij} \, (g_{jr} \, \Gamma_h{}^r{}_k) \circ j \,, \qquad 1 \le i, h, k \le l \,, \\ &= \tfrac{1}{2} \sum_{1 \le j \le l} (g^{\dagger})^{ij} \, (\partial_h g^{\dagger}{}_{jk} + \partial_k g^{\dagger}{}_{jh} - \partial_j g^{\dagger}{}_{hk}) \,, \qquad 1 \le i, h, k \le l \,. \end{aligned}$$

Proof. We can easily see that the map

$$(X, Y) \mapsto \nabla^{\|}{}_X Y$$

is a linear connection of Q .

Let us prove that $\nabla^{\|}$ is the Riemannian connection of Q , that is that

$$\nabla^{\|} g^{\dagger} = 0 \,, \qquad \nabla^{\|}{}_X Y - \nabla^{\|}{}_X Y - [X, Y] = 0 \,, \quad \forall X, Y \in \mathcal{T}(Q) \,.$$

In fact, we have

$$\begin{aligned} (\nabla^{\|}{}_X g^{\dagger})(Y, Z) &= \nabla^{\|}{}_X \big(g^{\dagger}(Y, Z)\big) - g^{\dagger}(\nabla^{\|}{}_X Y, Z) - g^{\dagger}(Y, \nabla^{\|}{}_X Z) \\ &= \nabla_X \big(g(Y, Z)\big) - g(\nabla^{\|}{}_X Y, Z) - g(Y, \nabla^{\|}{}_X Z) \\ &= 0 \,. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \nabla^{\|}{}_X Y - \nabla^{\|}{}_X Y - [X, Y] &= (\nabla_{\tilde{X}} \tilde{Y})^{\|} - (\nabla_{\tilde{Y}} \tilde{X})^{\|} - [X, Y] \\ &= [\tilde{X}, \tilde{Y}]^{\|} - [X, Y] \\ &= [X, Y] - [X, Y] \,. \end{aligned}$$

Thus, because of the uniqueness of the Riemannian connection, we obtain

$$\nabla^{\|} = \nabla^{\dagger} \,. \text{QED}$$

Now, we can compare the computations of the symbols of the Riemannian connection ∇ in the environing manifold M and of the symbols of the Riemannian connection $\nabla^\dagger$ in the submanifold Q via the Lagrange formulas and find a useful relation between them.

2.1.22 Proposition. Let $c : \mathbb{R} \to Q \subset M$ be a curve. Then, the following diagram commutes

$$\begin{tikzcd}
\cdot \arrow[rr, "\nabla^\dagger dc"] & & \cdot \arrow[d] \\
& T_Q H \arrow[r, "\pi^\parallel"] & TQ \\
\mathbb{R} \arrow[uu] \arrow[dd] \arrow[ur, "\nabla dc"'] \arrow[dr, "g^\flat(\nabla dc)"] & & \\
& T^*_Q M \arrow[r, "T^*j"'] & T^*Q \arrow[uu, "(g^\dagger)^\sharp"'] \\
\cdot \arrow[rr, "(g^\dagger)^\flat(\nabla^\dagger dc)"'] & & \cdot \arrow[u]
\end{tikzcd}$$

Thus, the covariant curvature of c in Q is just the restriction of the covariant curvature of c in M . $\square$

The above result can be interpreted by saying that the restriction to Q of the Lagrange formula for the metric function G of M is just the Lagrange formula of Q for the restricted metric function $G^\dagger$ of Q .

2.1.23 Proposition. The following diagram commutes

$$\begin{tikzcd}
& T^*_Q M \arrow[dd, "Tj^*"] \\
\mathbb{R} \arrow[ur, "\mathcal{E}(G, c) = g^\flat(\nabla dc)"] \arrow[dr, "\mathcal{E}(G^\dagger, c) = (g^\dagger)^\flat(\nabla dc)"'] & \\
& T^*Q
\end{tikzcd}$$

In other words

$$(\mathcal{E}(G))^\dagger = \mathcal{E}(G^\dagger) \, .$$

Proof. The above diagram commutes because it is is a piece of the diagram of the previous theorem.

On the other hand, the same result could be obtained directly from the Lagrange formula, in the following way. The restriction to Q of the partial derivatives of a function (G) of M with respect to adapted coordinates of Q are just the partial derivatives of the restricted function $(G^\dagger)$ of Q with respect to the same coordinates of Q. Hence, we obtain

$$(\mathcal{E}(G))^\dagger = \mathcal{E}(G^\dagger)\,.\,\mathrm{QED}$$

2.1.24 Remark. Indeed, the simple proof of the above Proposition 2.1.23 could also be taken as an alternative direct proof of the above Theorem 2.1.21. □

2.1.25 Note. The above Proposition 2.1.23 provides also a convenient alternative method for computing the symbols of the induced Riemannian connection $\nabla^\dagger$, *when we have already computed the symbols of the Riemannian connection* ∇.

In fact, the non vanishing covariant symbols ${\Gamma^\dagger}_{ihj}$ of the induced Riemannian connection $\nabla^\dagger$ of Q turn out to be just the restrictions to the submanifold Q of the covariant symbols Γ_{ihj} of the Riemannian connection ∇ of M

$${\Gamma^\dagger}_{ihj} = \Gamma_{ihj} \circ j\,, \qquad \text{with} \qquad 1 \le i, h, j \le l\,.$$

Eventually, the non vanishing contravariant symbols ${\Gamma^\dagger}_i{}^h{}_j$ of the induced Riemannian connection $\nabla^\dagger$ of Q can be computed by menas of the metric isomorphism $g^{\dagger\sharp}$ as follows

$${\Gamma^\dagger}_i{}^h{}_j = \sum_{1\le k\le l} g^{hk}\,\Gamma_{ihk}\,.\,\square$$

2.1.6.2 Orthogonal component

Next, we analyse the orthogonal component ${\nabla^\perp}_X Y$ of $\nabla_X Y$.

2.1.26 Theorem. *The map*

$$N \,\equiv\, \nabla^\perp : \mathcal{T}(Q) \times \mathcal{T}(Q) \to \mathcal{T}^\perp(Q) : (X, Y) \mapsto {\nabla^\perp}_X Y$$

turns out to be a symmetric tensor

$$N = TQ \underset{Q}{\times} TQ \to TQ^\perp\,,$$

whose coordinate expression is

$$N = \sum_{1\le h,k\le l} dx^h \otimes dx^k \otimes \big(\pi^\perp\big(\sum_{l+1\le i\le m} ({\Gamma_h}^i{}_k\,\partial_i) \circ j\big)\big)\,.$$

Proof. We have

$$\begin{aligned}\nabla^{\perp}{}_X Y &= (\nabla_{\tilde X} \tilde Y)^{\perp} \circ j \\ &= \pi^{\perp}\big(\sum_{1 \le i,h,k \le m} X^h (\partial_h Y^i + \Gamma_h{}^i{}_k Y^k)\, \partial x_i \big) \circ j \\ &= \sum_{1 \le h,k \le l}^{l+1 \le i \le m} X^h Y^k \pi^{\perp} (\Gamma_h{}^i{}_k \partial_i) \circ j \,. \text{QED}\end{aligned}$$

Thus, we stress that the coordinate expression of $N(X,Y) := \nabla^{\perp}{}_X Y$ does not involve the partial derivatives of the components of Y, but it depends pointwisely (and in a symmetric bilinear way) on the components of both X and Y.

2.1.27 Definition. We call N the *Gauss tensor.* □

2.1.6.3 The splitting

Eventually, we can summarise the above results concerning the parallel and orthogonal components of $\nabla_X Y$ as follows.

2.1.28 Corollary. *[Gauss splitting]*
For each vector fields X, Y of Q, the splitting of the covariant derivative $\nabla_X Y$ into the parallel and orthogonal components to Q reads as

$$\nabla_X Y = \nabla^{\dagger}{}_X Y + N(X,Y) \,. \square$$

2.2 Hypersurfaces

In the particular case when the dimension of the submanifold Q is $\dim Q = \dim M - 1$, we can achieve several further interesting results.

2.2.1 Definition. We say that $Q \subset M$ is a *hypersurface* if $l = m - 1 \,.\square$

From now on, we assume that Q be a hypersurface.

2.2.1 Unit normal vector field

An important feature of the hypersurface depends on its unit normal vector field.

Indeed, this object and the further objects derived from it are "estrinsic" with respect to the hypersurface, as they depend on how the hypersurface Q is embedded in the environing manifold M.

2.2.2 Definition. A *unit normal vector field* is defined to be a vector field

$$n : Q \to TQ^{\perp}$$

such that

$$g(n, n) = 1 \,.\square$$

2.2.3 Proposition. A unit normal vector field can be expressed (up to sign) by the equality

$$\bar{n} := g^{\sharp}(i_{\bar{\eta}_Q} \underline{\eta}_M) : Q \to TQ^{\perp} \,,$$

where

$$\bar{\eta}_Q : Q \to \Lambda^2 TQ \qquad \text{and} \qquad \underline{\eta}_M : Q \to \Lambda^m TM$$

be the volume vector of Q and the contravariant volume form of M.

Thus, we have the coordinate expression (up to sign)

$$\bar{n} = \sum_{1 \le r \le m} \frac{\sqrt{|(g_{hk})|}}{\sqrt{|(g^{\dagger ij})|}} \, g^{rs} \, \partial_r \,, \qquad \text{with} \qquad 1 \le h, k \le m \,, \quad 1 \le i, j \le m - 1 \,, \quad s = m \,.\square$$

2.2.4 Corollary. The above Proposition implies that a unit normal vector field exists at least locally and is unique up to sign.

Moreover, if M and Q are orientable, then a unit normal vector field exists globally and is unique up to sign. $\square$

Now, let us assume that such a unit normal vector field exists globally and let us choose its sign.

So, from now on, we consider a global unit normal vector field n.

2.2.2 Weingarten tensor and second fundamental form

The Weingarten tensor and the associated second fundamental form are further important "extrinsic" objects derived from the unit normal.

2.2.5 Lemma. For each vector field $X : Q \to T_Q M$, we obtain the section

$$\nabla_X n : Q \to TQ \,.$$

PROOF. The identity $g(n, n) = 1$ yields

$$g(\nabla_X n,\, n) = 0 \,.$$

Hence, $\nabla_X n$ is tangent to Q. QED

2.2.6 Definition. We define the following tensors.
We define the *Weingarten tensor* of Q to be the $(1, 1)$–tensor

$$L := \nabla_{\|} n : TQ \to TQ : X \mapsto \nabla_X n \,.$$

We define the *second fundamental form* of Q to be the $(0, 2)$–tensor

$$\underline{L} := \nabla_{\|} \underline{n} : TQ \underset{Q}{\times} TQ \to \mathrm{I\!R} : (X, Y) \mapsto \langle \nabla_X \underline{n},\, Y \rangle \,,$$

where $\underline{n} := g^{\flat}(n) : Q \to T^*Q$. □

2.2.7 Proposition. The second fundamental form turns out to be to be the bilinear form associated with L by the metric $g^{\dagger}$, that is

$$\underline{L} = g^{\dagger\flat}(L) : TQ \underset{Q}{\times} TQ \to \mathrm{I\!R} : (X, Y) \mapsto g^{\dagger}\big(L(X),\, Y\big) \,. \square$$

Hence, the two tensors L and $\underline{L}$ are "equivalent", as they are linked by the mutually inverse metric isomorphisms $g^{\dagger\flat}$ and $g^{\dagger\sharp}$.

We have interesting relations between the second fundamental form, the Riemannian connection ∇ of the environing manifold, the Riemannian connection $\nabla^{\dagger}$ of the hypersurface and the Gauss tensor N, according to the following Proposition and Corollaries.

2.2.8 Proposition. For each vector fields X, Y of Q, we have

$$\underline{L}(X, Y) = -g(\nabla_X Y,\, n) = -g(\nabla_Y X,\, n) \,.$$

Thus, second fundamental form is symmetric:

$$\underline{L}(X, Y) = \underline{L}(Y, X) \,.$$

Proof. We have

$$\begin{aligned}\underline{L}(X,Y) &= g(\nabla_X n, Y)\\ &= X.\big(g(n,Y)\big) - g(n, \nabla_X Y)\\ &= -g(n, \nabla_X Y)\\ &= -g(\nabla_X Y, n)\,.\end{aligned}$$

Analogously, we obtain

$$\underline{L}(Y,X) = -g(\nabla_Y X, n)\,.$$

Moreover, we have

$$\underline{L}(X,Y) = \underline{L}(Y,X)\,,$$

because

$$g(n, \nabla_X Y) = g(n, \nabla_Y X) + g(n, [X,Y]) = g(n, \nabla_Y X)\,.\,\text{QED}$$

2.2.9 Corollary. We have the equality

$$N = -\underline{L} \otimes n = -\nabla_{\|}\underline{n} \otimes n\,,$$

i.e., for each vector fields X, Y of Q,

$$N(X,Y) = -\langle \nabla_X \underline{n}\,,\, Y\rangle\, n = -\langle \nabla_Y \underline{n}\,,\, X\rangle\, n\,,$$

where

$$\underline{n} := g^\flat(n) : Q \to T^*M\,.$$

Proof. We have

$$\begin{aligned}N(X,Y) &= g(\nabla_X Y, n)\, n\\ &= -\underline{L}(X,Y)\, n\\ &= -g(\nabla_X n, Y)\, n\\ &= -\langle \nabla_X \underline{n}\,,\, Y\rangle\, n\,.\,\text{QED}\end{aligned}$$

2.2.10 Corollary. For the hypersurface Q, the Gauss splitting reads as

$$\nabla = \nabla^\dagger - \underline{L} \otimes n\,,$$

i.e., for each vector fields X, Y of Q,

$$\nabla_X Y = {\nabla^\dagger}_X Y - \underline{L}(X,Y)\, n\,.\,\square$$

2.2.11 Remark. We stress that the formulas of the above corollaries do not depend on the sign of n. In fact, $\nabla_{\|}\underline{n} \otimes n$ depends quadratically on n.

This observation implies that the above corollaries hold even if a global n does not exist. In fact, the possible obstruction to the global existence of n is due just to the ambiguity of the sign of n. □

2.2.3 Distinguished points and vectors

There are possible points of the hypersurface and vectors tangent to the hypersurface which have distinguished properties with respect to the Weingarten tensor.

2.2.12 Definition. A point $q \in Q$ is said to be
- *an umbilic point* if

$$L_q = r \, \mathrm{id}_q \,,$$

- *a flat point* if

$$L_q = 0 \,. \square$$

2.2.13 Definition. Non zero vectors $X, Y \in T_q Q$ are said to be *conjugate* if

$$L(X, Y) = 0 \,.$$

A non zero vector $X \in T_q Q$ is said to be *asymptotic* if it is self–conjugate, that is if

$$L(X, X) = 0 \,. \square$$

The Weingarten tensor is a symmetric operator, hence it is diagonalisable.

2.2.14 Definition. We define the *principal curvatures* and the *principal curvature vectors* to be, respectively, the eigenvalues and the eigenvectors of $L \,. \square$

2.2.15 Definition. A 1-dimensional submanifold $c \subset Q$ is said to be a *line of curvature* if its tangent vectors are principal curvature vectors. $\square$

2.2.4 Gauss curvature and mean curvature

The two main invariants of L (i.e. its trace and determinant) play an important role in the theory of hypersurfaces.

2.2.16 Definition. We define the *total curvature (Gauss curvature)* and the *mean curvature* of Q to be, respectively, the functions

$$K := \det L : Q \to \mathrm{I\!R} \qquad \text{and} \qquad H := \mathrm{tr}\, L : Q \to \mathrm{I\!R} \,. \square$$

2.2.17 Note. We have

$$K = \det L = \lambda_1 \dots \lambda_l \qquad \text{and} \qquad H = \mathrm{tr}\, L = \lambda_1 + \dots + \lambda_l \,,$$

where $\lambda_1, \dots, \lambda_l \in \mathrm{I\!R}$ denote the eigenvalues of the Weingarten tensor. $\square$

2.2.18 Note. In the particular case when $\dim Q = 2$ the invariants K and H are the only invariants of L and they characterise $L \,. \square$

2.2.5 Second fundamental form and curvature

We can exhibit interesting relations between the “extrinsic” second fundamental form and the “intrincic” Riemannian curvature tensor of the hypersurface.

2.2.19 Lemma. For each vector fields X, Y, Z of Q, we have

$$\underline{L}\big(X, L(Y)\big) = \underline{L}\big(Y, L(X)\big)\,.$$

Proof. We have the coordinate expression

$$\begin{aligned}
\underline{L}\big(X, L(Y)\big) &= L_{ih}\, X^i\; L_j{}^h\, Y^j \\
&= g^{hk}\, L_{ih}\, L_{jk}\, X^i\, Y^j \\
&= g^{hk}\, L_{jk}\, L_{ih}\, X^i\, Y^j \\
&= g^{hk}\, L_{jh}\, L_{ik}\, X^i\, Y^j \\
&= L_{jh}\, L_i{}^h\, X^i\, Y^j \\
&= \underline{L}\big(Y, L(X)\big)\,.\,\text{QED}
\end{aligned}$$

2.2.20 Proposition. Let us suppose that the Riemannian curvature tensor R of M vanishes. Then, for each vector fields X, Y of Q, we have

$$\nabla^\dagger{}_X\big(L(Y)\big) - \nabla^\dagger{}_Y\big(L(X)\big) - L\big([X, Y]\big) = 0\,.$$

Proof. By recalling the identities

$$L(Y) := \nabla_Y n\,, \qquad L(X) := \nabla_X n\,, \qquad \nabla^\dagger{}_X Y = \nabla_X Y + \underline{L}(X, Y)\, n\,, \qquad \nabla^\dagger{}_Y X = \nabla_Y X + \underline{L}(Y, X)\, n\,,$$

we obtain

$$\begin{aligned}
\nabla^\dagger{}_X\big(L(Y)\big) - \nabla^\dagger{}_Y\big(L(X)\big) - L\big([X, Y]\big) &= \nabla^\dagger{}_X \nabla_Y n - \nabla^\dagger{}_Y \nabla_X n - \nabla_{[X,Y]} n \\
&= \nabla_X \nabla_Y n - \nabla_Y \nabla_X n - \nabla_{[X,Y]} n \\
&\quad + \underline{L}\big(X, L(Y)\big)\, n - \underline{L}\big(Y, L(X)\big)\, n \\
&= \mathsf{R}(X, Y, n) \\
&\quad + \underline{L}\big(X, L(Y)\big)\, n - \underline{L}\big(Y, L(X)\big)\, n \\
&= 0\,.\,\text{QED}
\end{aligned}$$

2.2.21 Proposition. Let us suppose that the Riemannian curvature tensor R of M vanishes. Then, for each vector fields X, Y, Z of Q, we have

$$\mathsf{R}^\dagger(X, Y; Z) = \underline{L}(Y, Z)\, L(X) - \underline{L}(X, Z)\, L(Y)\,.$$

Proof. By recalling the identities

$$L(Y) := \nabla_Y n\,, \qquad L(X) := \nabla_X n\,, \qquad \nabla_X Y = \nabla^\dagger{}_X Y - g\big(L(X), Y\big)\, n\,, \qquad \nabla_Y X = \nabla^\dagger{}_Y X - g\big(L(Y), X\big)\, n\,,$$

we obtain

$$\begin{aligned}
0 &= \mathsf{R}(X,Y;Z)\\
&= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\\[6pt]
&= \nabla_X \big({\nabla^\dagger}_Y Z - g(L(Y),Z)\,n\big)\\
&\quad - \nabla_Y \big({\nabla^\dagger}_X Z - g(L(X),Z)\,n\big)\\
&\quad - {\nabla^\dagger}_{[X,Y]} Z + g(L([X,Y]),Z)\,n\\[12pt]
&= \nabla^\dagger_X {\nabla^\dagger}_Y Z - g(L(X), {\nabla^\dagger}_Y Z)\,n\\
&\quad - g\big({\nabla^\dagger}_X (L(Y)),\, Z)\big)\,n - g(L(Y), {\nabla^\dagger}_X Z)\,n - g(L(Y),Z)\,\nabla_X n\\
&\quad - \nabla^\dagger_Y {\nabla^\dagger}_X Z + g(L(Y), {\nabla^\dagger}_X Z)\,n\\
&\quad + g\big({\nabla^\dagger}_Y (L(X)),\, Z)\big)\,n + g(L(X), {\nabla^\dagger}_Y Z)\,n + g(L(X),Z)\,\nabla_Y n\\
&\quad - {\nabla^\dagger}_{[X,Y]} Z - g\big(L([X,Y]),\, Z\big)\,n\\[6pt]
&= \nabla^\dagger_X {\nabla^\dagger}_Y Z\\
&\quad - g\big({\nabla^\dagger}_X (L(Y)),\, Z)\big)\,n - g(L(Y),Z)\,\nabla_X n\\
&\quad - \nabla^\dagger_Y {\nabla^\dagger}_X Z\\
&\quad + g\big({\nabla^\dagger}_Y (L(X)),\, Z)\big)\,n + g(L(X),Z)\,\nabla_Y n\\
&\quad - {\nabla^\dagger}_{[X,Y]} Z + g\big(L([X,Y]),\, Z\big)\,n\\[6pt]
&= \nabla^\dagger_X {\nabla^\dagger}_Y Z\\
&\quad - g\big({\nabla^\dagger}_X (L(Y)),\, Z)\big)\,n - \underline{L}(Y,Z)\,L(X)\\
&\quad - \nabla^\dagger_Y {\nabla^\dagger}_X Z\\
&\quad + g\big({\nabla^\dagger}_Y (L(X)),\, Z)\big)\,n + \underline{L}(X,Z)\,L(Y)\\
&\quad - {\nabla^\dagger}_{[X,Y]} Z + g\big(L([X,Y]),\, Z\big)\,n\,.
\end{aligned}$$

Next, by considering in the above equality the component tangent to Q, we obtain

$$\begin{aligned}
0 &= \nabla^\dagger_X {\nabla^\dagger}_Y Z - \nabla^\dagger_Y {\nabla^\dagger}_Y Z - {\nabla^\dagger}_{[X,Y]} Z - \underline{L}(Y,Z)\,L(X) + \underline{L}(X,Z)\,L(Y)\\
&= \mathsf{R}(X,Y,Z) - \underline{L}(Y,Z)\,L(X) + \underline{L}(X,Z)\,L(Y)\,.\ \text{QED}
\end{aligned}$$

2.2.22 Note. We might prove the above Lemma contextually to the above Proposition. In fact, in the proof of the above Proposition, the component of

$$0 = \mathsf{R}(X,Y;Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

orthogonal to Q gives the equality

$$\begin{aligned}
0 &= g\big({\nabla^\dagger}_X (L(Y)),\, Z)\big) - g\big({\nabla^\dagger}_Y (L(X)),\, Z)\big) - g\big(L([X,Y]),\, Z\big)\\
&= g\big({\nabla^\dagger}_X (L(Y)) - {\nabla^\dagger}_Y (L(X)) - L([X,Y])\ ,\ \ Z\big)\,,
\end{aligned}$$

which is the staement of the above Lemma. □

2.2.23 Corollary. Let us suppose that the Riemannian curvature tensor R of M vanishes. Then, for each vector fields X, Y, Z of Q, we have

$$\underline{\mathsf{R}}^\dagger(X, Y; Z, W) = \underline{L}(Y, W)\, \underline{L}(X, Z) - \underline{L}(X, W)\, \underline{L}(Y, Z)\,. \square$$

The above Corollary can be reformulated in the following iteresting way.

2.2.24 Theorem. [Gauss theorema egregium]
Let us suppose that the Riemannian curvature tensor R *of* M *vanishes and that* $\dim Q = 2$.
Then, we have

$$\tfrac{1}{2}\, \langle \underline{\mathsf{r}} \rangle^\dagger = \det L \;\equiv\; K\,.$$

In other words, pointwisely, if $(e_1 . e_2)$ *is an othonormal basis, then we obtain*

$$\underline{\mathsf{R}}^\dagger(e_1, e_2, e_1, e_2) = \det L \;\equiv\; K\,.$$

Proof. By recalling the equality

$$\underline{\mathsf{R}}^\dagger(X, Y, Z, W) = \underline{L}(Y, W)\, \underline{L}(X, Z) - \underline{L}(X, W)\, \underline{L}(Y, Z)\,,$$

we obtain

$$\begin{aligned} \underline{\mathsf{R}}(e_1, e_2, e_1, e_2) &= \underline{L}(e_2, e_2)\, \underline{L}(e_1, e_1) - \underline{L}(e_1, e_2)\, \underline{L}(e_2, e_1) \\ &= \det L\,. \end{aligned}$$

On the other hand, we have (see Proposition 1.2.22)

$$\underline{\mathsf{R}}(e_1, e_2, e_1, e_2) = \tfrac{1}{2}\, \langle \underline{\mathsf{r}} \rangle\,. \text{QED}$$

2.2.25 Note. The above Theorem can also be expressed by the equality

$$\langle \underline{\mathsf{R}}^\dagger\,,\, \bar{\eta}^\dagger \otimes \bar{\eta}^\dagger \rangle = \det L \;\equiv\; K\,. \square$$

2.2.26 Remark. We stress that the above results do not depend on the existence of a global n and on its sign and do not depend on the existence of a global $\bar{\eta}^\dagger$ and on its sign. In fact, n and $\bar{\eta}^\dagger$ appear quadratically in the above formula. □

2.2.27 Remark. We stress that, in the equality of the above Theorem, the function $\frac{1}{2}\, \langle \underline{\mathsf{r}} \rangle^\dagger = \langle \underline{\mathsf{R}}^\dagger\,,\, \bar{\eta}^\dagger \otimes \bar{\eta}^\dagger \rangle$ depends only the "intrinsic metric" $g^\dagger$ of the submanifold Q, while the function $\det L$ is defined by means of the "extrinsic" covariant differential ∇n of the normal unit vector of the submanifold Q.

Thus, the above Theorem links "intrinsic" and "extrinsic" objects of the hypersurface Q. □

2.2.28 Corollary. Let consider two hypersurfaces $Q \subset M$ and $Q' \subset M$ and let us suppose that they be *isometric*, that is that there exists a diffeomorphism $f : Q \to Q'$ which preserves the induced metrics $g^{\dagger}$ and $g'^{\dagger}$.

Then, we have

$$K = K' \circ f\,.$$

Proof. In fact, in virtue of the isometry, we have

$$\langle \underline{\mathrm{r}} \rangle^{\dagger} = \langle \underline{\mathrm{r}} \rangle'^{\dagger} \circ f\,,$$

because the the "intrinsic" scalar curvature of a submanifold depends only on the "intrinsic" metric. QED

CHAPTER 3

EXAMPLES

In this chapter, we analyse in detail some distinguished examples.

Indeed, we consider an affine space, a sphere, a cylinder and a paraboloid. With reference to this manifold and these submanifolds, we analyse all general results studied in the above chapters.

3.1 Euclidean spaces

We introduce the notion of Euclidean space, as a simple example of Riemannian manifold.

3.1.1 Definition. We define a *Euclidean space* to be an affine space E, associated with the vector space, equipped with a Euclidean metric of $\bar{E}$

$$\mathsf{g} \in \bar{E}^* \otimes \bar{E}^* \,. \square$$

From now on, we assume a Euclidean space E.

3.1.2 Note. We can regard the Euclidean space E as a Riemannian manifold equipped with the "constant" Riemannian metric

$$g : E \to T^*E \otimes T^*E \;\simeq\; E \times (\bar{E}^* \otimes \bar{E}^*) : e \mapsto \big(e\,,\, \mathsf{g}(e)\big)\,,$$

where we have taken into account the natural isomorphism

$$T^*E = E \times \bar{E}^* \,. \square$$

3.1.1 Distinguished charts

We consider distinguished systems of coordinates, namely, the cartesian, sherical, cylindrical and parabolic coordinates. The computations in parabolic coordinates are due to the student Luca Salvatori (2001).

The Euclidean E space admits a distinguished type of global charts, which reflect in a natural way its affine structure and metric structure.

3.1.3 Definition. A *cartesian chart* is defined to be a chart (x^i) constituted by functions of the type

$$x^i : E \to \mathrm{I\!R} : e \mapsto \mathsf{g}(e - o,\, e_i)\,,$$

where o is a point of E and (e_i) is an orthonormal basis of $\bar{E}$. $\square$

3.1.4 Proposition. In a cartesian chart, the coordinate curves turn out to be the maps

$$x_i : \mathrm{I\!R} \times E \to E : (\lambda, e) \mapsto e + \lambda\, \delta^j_i\, e_j \,.$$

Hence, we obtain

$$\partial x_i = e_i \,. \square$$

From now on, we assume that $\dim E = 3$.

We denote the *cartesian charts* by

$$x \equiv x^1 , \qquad y \equiv x^2 , \qquad z \equiv x^3 .$$

Besides the cartesian charts, we shall be involved with other curvilinear charts.
In particular, we shall consider:
- the *spherical chart* (r, θ, ϕ),
- the *cylindrical chart* (ρ, ϕ, z),
- the *parabolic chart* (ρ, θ, f),
which are associated with a point $o \in E$ and an orthonormal basis (e_i) of $\bar{E}$.

By definition, the transition functions with respect to the cartesian chart are, respectively,

$$\begin{aligned} x &= r \sin\theta \cos\phi , \\ y &= r \sin\theta \sin\phi , \\ z &= r \cos\theta , \end{aligned}$$

$$\begin{aligned} x &= \rho\, \cos\phi , \\ y &= \rho\, \sin\phi , \\ z &= z , \end{aligned}$$

$$\begin{aligned} x &= \rho\, \cos\theta , \\ y &= \rho\, \sin\theta , \\ z &= f + a\, \rho^2 , \quad \text{with} \quad a > 0 . \end{aligned}$$

Hence, we obtain

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} , \\ \rho &= \sqrt{x^2 + y^2} , \\ f &= z - a\, \rho^2 . \end{aligned}$$

In order to help the visibility of formulas in the above charts, we shall denote the indices of components of tensors by the corresponding coordinate function. So, for instance, in a spherical chart, the coordinate expression of a vector field will be written as

$$X = X^r \partial r + X^\theta \partial\theta + X^\phi \partial\phi .$$

3.1.2 Riemannian metric

We compute the expressions of the metric and of the volume form in cartesian, sherical, cylindrical and parabolic coordinates.

The coordinate expression of the covariant and contravariant metrics are

$$\begin{aligned} g &= dx \otimes dx + dy \otimes dy + dz \otimes dz \\ &= dr \otimes dr + r^2 \, d\theta \otimes d\theta + r^2 \, \sin^2\theta \, d\phi \otimes d\phi \\ &= d\rho \otimes d\rho + \rho^2 \, d\phi \otimes d\phi + dz \otimes dz \\ &= (1 + 4\, a^2 \, \rho^2) \, d\rho \otimes d\rho + \rho^2 \, d\theta \otimes d\theta + df \otimes df + 2\, a \, \rho \, (d\rho \otimes df + df \otimes d\rho) \,, \end{aligned}$$

$$\begin{aligned} \bar{g} &= \partial_x \otimes \partial_x + \partial_y \otimes \partial_y + \partial_z \otimes \partial_z \\ &= \partial_r \otimes \partial_r + \frac{1}{r^2} \, \partial_\theta \otimes \partial_\theta + \frac{1}{r^2 \, \sin^2\theta} \, \partial_\phi \otimes \partial_\phi \\ &= \partial_\rho \otimes \partial_\rho + \frac{1}{\rho^2} \, \partial_\phi \otimes \partial_\phi + \partial_z \otimes \partial_z \\ &= \partial_\rho \otimes \partial_\rho + \frac{1}{\rho^2} \, \partial_\theta \otimes \partial_\theta + (1 + 4\, a^2 \, \rho^2) \, \partial_f \otimes \partial_f - 2\, a \, \rho \, (\partial_\rho \otimes \partial_f + \partial_f \otimes \partial_\rho) \,. \end{aligned}$$

Hence, the coordinate expression of the metric function is

$$\begin{aligned} G &= \tfrac{1}{2} \, (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \tfrac{1}{2} \, (\dot{r}^2 + r^2 \, \dot{\theta}^2 + r^2 \, \sin^2\theta \, \dot{\phi}^2) \\ &= \tfrac{1}{2} \, (\dot{\rho}^2 + \rho^2 \, \dot{\phi}^2 + \dot{z}^2) \\ &= \tfrac{1}{2} \, \big((1 + 4\, a^2 \, \rho^2) \, \dot{\rho}^2 + \rho^2 \, \dot{\theta}^2 + \dot{f}^2 + 2\, a \, \rho \, (\dot{\rho} \, \dot{f} + \dot{f} \, \dot{\rho}) \big) \,. \end{aligned}$$

The *volume form* induced by the metric g and by the orientation of the chosen charts has coordinate expression

$$\begin{aligned} \eta &= dx \wedge dy \wedge dz \\ &= r^2 \sin\theta \; dr \wedge d\theta \wedge d\phi \\ &= \rho \; d\rho \wedge d\phi \wedge dz \\ &= \rho \; d\rho \wedge d\theta \wedge df \,. \end{aligned}$$

3.1.3 Riemannian connection

We show that, in a Euclidean space, the Riemannian connection coincides with the standard differential of vector fields.

Then, we compute the coefficients of the connection ∇, in cartesian spherical, cylindrical and parabolic coordinates, by means of the Lagrange formulas.

3.1.5 Proposition. The Riemannian covariant differential ∇ coincides with the standard differential D induced by the affine structure:

$$\nabla = D\,.$$

Proof. In fact, D fulfills all properties of connections; moreover, the torsion tensor of D vanishes in virtue of the Schwartz theorem and $Dg = 0$ because g is constant. QED

3.1.6 Proposition. In cartesian coordinates, all symbols of ∇ vanish

$$\Gamma_i{}^h{}_j = 0\,.\square$$

3.1.7 Proposition. In spherical coordinates the non–vanishing coefficient of ∇ are

$$\begin{aligned}
&\Gamma_\theta{}^r{}_\theta = -r && \Gamma_\phi{}^r{}_\phi = -r\sin^2\theta\\
&\Gamma_r{}^\theta{}_\theta = \Gamma_\theta{}^\theta{}_r = \frac{1}{r} && \Gamma_\phi{}^\theta{}_\phi = -\sin\theta\cos\theta\\
&\Gamma_r{}^\phi{}_\phi = \Gamma_\phi{}^\phi{}_r = \frac{1}{r} && \Gamma_\theta{}^\phi{}_\phi = \Gamma_\phi{}^\phi{}_\theta = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

Proof. The covariant curvature of a curve $c : \mathbb{R} \to E$ is given by

$$\begin{aligned}
(\nabla dc)_r &= D^2c^r - c^r\,(Dc^\theta)^2 - c^r\,\sin^2 c^\theta\,(Dc^\phi)^2\\
(\nabla dc)_\theta &= (c^r)^2\,\big(D^2c^\theta + \frac{2}{c^r}\,Dc^r\,Dc^\theta - \sin c^\theta\,\cos c^\theta\,(Dc^\phi)^2\big)\\
(\nabla dc)_\phi &= (c^r)^2\,\sin^2 c^\theta\big(D^2c^\phi + \frac{2}{c^r}Dc^r Dc^\phi + 2\,\frac{\cos c^\theta}{\sin\theta}\,Dc^\theta\,Dc^\phi\big)\,,
\end{aligned}$$

hence the curvature of c is given by

$$\begin{aligned}
(\nabla dc)^r &= D^2c^r - c^r\,\big((Dc^\theta)^2 - \sin^2 c^\theta\,(Dc^\phi)^2\big)\\
(\nabla dc)^\theta &= D^2c^\theta + \frac{2}{c^r}\,Dc^r\,Dc^\theta - \sin c^\theta\,\cos c^\theta\,(Dc^\phi)^2\\
(\nabla dc)^\phi &= D^2c^\phi + \frac{2}{c^r}\,Dc^r\,Dc^\phi + 2\,\frac{\cos c^\theta}{\sin\theta}\,Dc^\theta\,Dc^\phi\,.\,\text{QED}
\end{aligned}$$

3.1.8 Proposition. In cylindrical coordinates the non–vanishing coefficient of ∇ are

$$\Gamma_\phi{}^\rho{}_\phi = -\rho \qquad \Gamma_\rho{}^\phi{}_\phi = \Gamma_\phi{}^\phi{}_\rho = 1/\rho\,.$$

Proof. The covariant curvature of a curve $c : \mathbb{R} \to E$ is given by

$$\begin{aligned}
(\nabla dc)_\rho &= D^2c^\rho - c^\rho\,(Dc^\phi)^2\\
(\nabla dc)_\phi &= (c^\rho)^2\big(D^2c^\phi + \frac{2}{c^\rho}Dc^\rho Dc^\phi\big)\\
(\nabla dc)_z &= D^2c^z\,,
\end{aligned}$$

hence the curvature of c is given by

$$\begin{aligned}(\nabla dc)^\rho &= D^2 c^\rho - c^\rho (Dc^\phi)^2\\ (\nabla dc)^\phi &= D^2 c^\phi + \frac{2}{c^\rho} Dc^\rho Dc^\phi\\ (\nabla dc)^z &= D^2 c^z\,.\ \text{QED}\end{aligned}$$

3.1.9 Proposition. In parabolic coordinates the non–vanishing coefficient of ∇ are

$$\Gamma_\theta{}^\rho{}_\theta = -\rho\,, \qquad \Gamma_\rho{}^\theta{}_\theta = \Gamma_\theta{}^\theta{}_\rho = 1/\rho\,, \qquad \Gamma_\rho{}^f{}_\rho = 2\,a\,, \qquad \Gamma_\theta{}^f{}_\theta = 2\,a\,\rho^2\,.$$

Proof. The covariant curvature of a curve $c : \mathrm{I\!R} \to E$ is given by

$$\begin{aligned}(\nabla dc)_\rho &= \big(1 + 4\,a^2\,(c^\rho)^2\big)\,D^2 c^\rho + 2\,a\,c^\rho\,D^2 c^f + 4\,a^2\,c^\rho\,(Dc^\rho)^2 - c^\rho\,(Dc^\theta)^2\\ (\nabla dc)_\theta &= (c^\rho)^2\,D^2 c^\theta + 2\,c^\rho\,Dc^\rho\,Dc^\theta\\ (\nabla dc)_f &= 2\,a\,c^\rho\,D^2 c^\rho + D^2 c^f + 2\,a\,(Dc^\rho)^2\,,\end{aligned}$$

hence the curvature of c is given by

$$\begin{aligned}(\nabla dc)^\rho &= D^2 c^\rho - c^\rho (Dc^\theta)^2\\ (\nabla dc)^\theta &= D^2 c^\phi + \frac{2}{c^\rho}\,Dc^\rho Dc^\theta\\ (\nabla dc)^f &= D^2 c^f + 2\,a\,(Dc^\rho)^2 + 2\,a\,(c^\rho)^2\,(Dc^\theta)^2\,.\ \text{QED}\end{aligned}$$

3.1.4 Riemannian curvature

The Riemannian curvature tensor of the Euclidean space vanishes.

3.1.10 Proposition. The Riemannian curvature tensor of ∇ vanishes:

$$\mathsf{R} = 0\,.$$

Proof. In fact, in a cartesian chart the symbols of the connection vanish. QED

3.1.11 Remark. We stress that, if we refer to curvilinear coordinates, then the coefficients of ∇ may be different from zero, because they are not the components of a tensor.

But, also in this curvilinear chart, the components of the Riemannian curvature tensor R still vanish, because, if they are zero in a chart, then they are zero in all charts. □

3.2 Ruled and developable surfaces

In this section we discuss a few notions concerning special types of hypersurfaces of the Euclidean space.

3.2.1 Definition. A *ruled surface* is defined to be a hypersurface Q of E such that through each $q \in Q$ there passes a segment of a straight line lying on Q, which is called a *generator.*

A *developable surface* is defined to be a ruled surface Q such that, for each vector field X tangent to the generators,

$$\nabla_X n = 0\,. \square$$

3.2.2 Remark. A ruled surface is developable if and only if its tangent plane is constant along generators. □

3.2.3 Proposition. If Q is a ruled surface, then

$$K \leq 0\,.$$

If Q is a developable surface, then

$$K = 0\,.$$

Proof. Let X be a unit vector of Q tangent to the generators and Y a unit vector of Q orthogonal to X.

If Q is a ruled surface, then we obtain

$$0 = \nabla_X X = \nabla^{\dagger}{}_X X - \underline{L}(X,X) n\,,$$

which implies

$$\underline{L}(X,X) = 0\,.$$

Then, we obtain

$$K = \underline{L}(X,X)\,\underline{L}(Y,Y) - \underline{L}(X,Y)\,\underline{L}(Y,X) = -\big(\underline{L}(Y,X)\big)^2 \leq 0\,.$$

If Q is a developable surface, then we have additionally

$$0 = \nabla_X n = L(X)\,,$$

hence $K = 0$. QED

Conversely, one can prove the following result (we omit the proof).

3.2.4 Proposition. Let Q be a closed connected ruled surface. Then, Q is developable if and only if

$$K = 0\,. \square$$

3.3 Cylinder

Now, we suppose that the submanifold Q be the circular cylinder C whose axis is the straight line $(o, e_3) \subset E$ and whose radius is $\mathrm{r} > 0$.

We shall refer to the adapted cylindrical chart (ρ, ϕ, z).

3.3.1 Riemannian metric

Let us compute the Riemannian metric and the induced algebraic objects.

3.3.1 Proposition. The coordinate expression of the metric and of the contravariant metric are

$$g^\dagger = g^\dagger = \mathrm{r}^2\, d\phi \otimes d\phi + dz \otimes dz \qquad \text{and} \qquad \bar{g}^\dagger = \frac{1}{\mathrm{r}^2}\, \partial\phi \otimes \partial\phi + \partial z \otimes \partial z\,.$$

The coordinate expression of the metric function is

$$G^\dagger = \tfrac{1}{2}\,(\mathrm{r}^2\, \dot{\phi}^2 + \dot{z}^2)\,.\,\square$$

3.3.2 Proposition. The *volume form* induced by the metric $g^\dagger$ and by the orientation of the chosen chart has coordinate expression

$$\eta^\dagger = \mathrm{r}^2\, d\phi \wedge dz\,.\,\square$$

3.3.2 Extrinsic curvature

Let us compute the unit normal, the Weingarten tensor, the second fundamental form and the Gauss tensor.

3.3.3 Proposition. We have the global unit normal vector field

$$n = \partial\rho\,.\,\square$$

3.3.4 Proposition. The Weingarten map and the second fundamental form are

$$L = \frac{1}{\mathrm{r}}\pi_e\,, \qquad \underline{L} = \frac{1}{\mathrm{r}}{g^\dagger}_e\,,$$

where π_e is the equatorial projection and ${g^\dagger}_e$ is the “equatorial metric”.

Namely, we have the coordinate expressions

$$L = \frac{1}{\mathrm{r}}\, d\theta \otimes \partial\theta\,, \qquad \underline{L} = \mathrm{r}\, d\theta \otimes d\theta\,.\,\square$$

3.3.5 Corollary. The principal curvatures and the corresponding principal eigenvectors are

$$\lambda' = 0 \qquad \text{and} \qquad \lambda'' = \frac{1}{\mathrm{r}}$$

and

$$v' = \partial_\rho \qquad \text{and} \qquad v'' = \partial_\theta \,. \square$$

Thus, the coordinate curves x_ϕ and x_z are curvature lines. $\square$

3.3.6 Corollary. The cylinder is a ruled and developable surface. $\square$

3.3.7 Corollary. The mean curvature and the total curvature are

$$H = \operatorname{tr} L = \frac{1}{\mathrm{r}} \qquad \text{and} \qquad K = \det L = 0 \,. \square$$

3.3.8 Proposition. We have

$$N = -\frac{1}{\mathrm{r}}\, g^{\dagger}{}_{e} \otimes \partial_\rho \,. \square$$

3.3.3 Riemannian connection

Let us compute the symbols of the Riemannian connection by means of the Lagrange formulas.

3.3.9 Proposition. All coefficients of $\nabla^\dagger$ vanish.

PROOF. The covariant curvature of a curve $c : \mathbb{R} \to C$ is given by

$$\begin{aligned}(\nabla dc)_\phi &= \mathrm{r}^2 D^2 c^\phi \,, \\ (\nabla dc)_z &= D^2 c^z \,,\end{aligned}$$

hence the curvature of c is given by

$$\begin{aligned}(\nabla dc)^\phi &= D^2 c^\phi \,, \\ (\nabla dc)^z &= D^2 c^z \,. \text{QED}\end{aligned}$$

3.3.10 Note. We can compute the non vanishing symbols of $\nabla^\dagger$ in an alternative way (see Note 2.1.25).

In fact, the non vanishing symbols $\Gamma^{\dagger}{}_{ihj}$ of $\nabla^\dagger$ are the restrictions to C of the symbols Γ_{ihj} of ∇ , with $i, j, h = \phi, z$.

But, all such symbols ${\Gamma^\dagger}_{ihj}$ vanish

$${\Gamma^\dagger}_{\phi\phi\phi} = {\Gamma^\dagger}_{\phi\phi z} = {\Gamma^\dagger}_{z\phi\phi} = {\Gamma^\dagger}_{z\phi z} = {\Gamma^\dagger}_{\phi z\phi} = {\Gamma^\dagger}_{\phi zz} = {\Gamma^\dagger}_{zz\phi} = {\Gamma^\dagger}_{zzz} = 0\,.\,\square$$

3.3.4 Riemannian curvature

Let us compute the Riemannian curvature tensor, the Ricci tensor and the Riemannian scalar curvature.

3.3.11 Corollary. The Riemannian curvature tensor of $\nabla^\dagger$ vanishes:

$$\mathsf{R}^\dagger = 0\,.\,\square$$

3.3.12 Corollary. The Ricci tensor vanishes

$$\underline{\mathrm{r}}^\dagger = 0\,.$$

3.3.13 Corollary. The Riemannian scalar curvature vanishes

$$\langle \underline{\mathrm{r}} \rangle^\dagger = 0\,.\,\square$$

3.3.14 Note. There is an agreement between the two equalities

$$\tfrac{1}{2}\,\langle \underline{\mathrm{r}} \rangle^\dagger = 0 = K\,.\,\square$$

3.4 Sphere

Now, we suppose that the submanifold Q is the sphere S whose center is $o \in E$ and whose radius is $\mathrm{r} > 0$.

We shall refer to the adapted spherical chart (r, θ, ϕ).

3.4.1 Riemannian metric

Let us compute the Riemannian metric and the induced algebraic objects.

3.4.1 Proposition. The coordinate expression of the metric and of the contravariant metric are

$$g^\dagger = \mathrm{r}^2 \, (d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi) \qquad \text{and} \qquad \bar{g}^\dagger = \frac{1}{\mathrm{r}^2} \, (\partial\theta \otimes \partial\theta + \frac{1}{\sin^2\theta} \, \partial\phi \otimes \partial\phi) \, .$$

The coordinate expression of the metric function is

$$G^\dagger = \tfrac{1}{2} \, \mathrm{r}^2 \, (\dot\theta^2 + \sin^2\theta \, \dot\phi^2) \, . \square$$

3.4.2 Proposition. The *volume form* induced by the metric $g^\dagger$ and by the orientation of the chosen chart has coordinate expression

$$\eta^\dagger = \mathrm{r}^2 \sin\theta \, d\theta \wedge d\phi \, . \square$$

3.4.2 Extrinsic curvature

Let us compute the unit normal, the Weingarten tensor, the second fundamental form and the Gauss tensor.

3.4.3 Proposition. We have the global unit normal vector field

$$n = \partial_r \, . \square$$

3.4.4 Proposition. The Weingarten tensor and the second fundamental form are

$$L = \frac{1}{\mathrm{r}} \, \mathrm{id}_{TS} \qquad \text{and} \qquad \underline{L} = \frac{1}{\mathrm{r}} g^\dagger \, .$$

Namely, we have the coordinate expressions

$$L = \frac{1}{\mathrm{r}} \, (d\theta \otimes \partial\theta + d\phi \otimes \partial\phi) \qquad \text{and} \qquad \underline{L} = \mathrm{r} \, (d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi) \, .$$

Proof. We have

$$\begin{aligned}\nabla\partial_r &= \Gamma_{\theta}{}^{\theta}{}_{r}\, d^{\theta}\otimes\partial_{\theta} + \Gamma_{\phi}{}^{\phi}{}_{r}\, d^{\phi}\otimes\partial_{\phi}\\ &= \frac{1}{\mathrm{r}}\,(d^{\theta}\otimes\partial_{\theta} + d^{\phi}\otimes\partial_{\phi})\,.\ \text{QED}\end{aligned}$$

3.4.5 Corollary. All directions tangent to the sphere are principal directions and all eigenvalues λ are given by

$$\lambda = \frac{1}{\mathrm{r}}\,.\,\square$$

3.4.6 Corollary. The sphere is not a ruled hypersurface (hence it is not a developable hypersurface). □

3.4.7 Corollary. The mean curvature and the total curvature are

$$H = \operatorname{tr} L = \frac{2}{\mathrm{r}} \qquad \text{and} \qquad K = \det L = \frac{1}{\mathrm{r}^2}\,.\,\square$$

3.4.8 Proposition. We have

$$N = -\frac{1}{\mathrm{r}}\, g^{\dagger}\otimes\partial_r\,.$$

Proof. We have

$$N = -\underline{L}\otimes n\,.\ \text{QED}$$

3.4.3 Riemannnian connection

Let us compute the symbols of the Riemannian connection by means of the Lagrange formulas.

3.4.9 Proposition. The non–vanishing coefficients of $\nabla^{\dagger}$ are

$$\Gamma^{\dagger}{}_{\phi}{}^{\theta}{}_{\phi} = -\sin\theta\cos\theta \qquad \text{and} \qquad \Gamma^{\dagger}{}_{\theta}{}^{\phi}{}_{\phi} = \Gamma^{\dagger}{}_{\phi}{}^{\phi}{}_{\theta} = \frac{\cos\theta}{\sin\theta}\,.$$

Proof. The covariant curvature of a curve $c : \mathbb{R}\to S$ is given by

$$\begin{aligned}(\nabla dc)_{\theta} &= \mathrm{r}^2\big(D^2 c^{\theta} - \sin c^{\theta}\,\cos c^{\theta}\,(Dc^{\phi})^2\big)\\ (\nabla dc)_{\phi} &= \mathrm{r}^2\,\sin^2 c^{\theta}\big(D^2 c^{\phi} + 2\,\frac{\cos c^{\theta}}{\sin c^{\theta}}\,Dc^{\theta}\,Dc^{\phi}\big)\,,\end{aligned}$$

hence the curvature of c is given by

$$\begin{aligned}(\nabla dc)^{\theta} &= D^2 c^{\theta} - \sin c^{\theta}\,\cos c^{\theta}\,(Dc^{\phi})^2\\ (\nabla dc)^{\phi} &= D^2 c^{\phi} + 2\,\frac{\cos c^{\theta}}{\sin c^{\theta}}\,Dc^{\theta}\,Dc^{\phi}\,.\ \text{QED}\end{aligned}$$

3.4.10 Note. We can compute the non vanishing symbols of $\nabla^\dagger$ in an alternative way (see Note 2.1.25).

In fact, the non vanishing symbols ${\Gamma^\dagger}_{ihj}$ of $\nabla^\dagger$ are the restrictions to C of the symbols Γ_{ihj} of ∇, with $i, j, h = \theta, \phi$.

All such symbols ${\Gamma^\dagger}_{ihj}$ are

$$
{\Gamma^\dagger}_{\phi\theta\phi} = -\mathrm{r}^2 \sin\theta \, \cos\theta \qquad \text{and} \qquad {\Gamma^\dagger}_{\theta\phi\phi} = {\Gamma^\dagger}_{\phi\phi\theta} = \mathrm{r}^2 \sin\theta \, \cos\theta \,,
$$

which yield

$$
{\Gamma^\dagger}_{\phi}{}^{\theta}{}_{\phi} = -\sin\theta\cos\theta \qquad \text{and} \qquad {\Gamma^\dagger}_{\theta}{}^{\phi}{}_{\phi} = {\Gamma^\dagger}_{\phi}{}^{\phi}{}_{\theta} = \frac{\cos\theta}{\sin\theta} \,. \square
$$

3.4.4 Riemannian curvature

Let us compute the Riemannian curvature tensor, the Ricci tensor and the Riemannian scalar curvature.

3.4.11 Proposition. The coordinate expression of the Riemannian curvature tensor of $\nabla^\dagger$ is

$$
\begin{aligned}
\mathsf{R}^\dagger &= \sin^2\theta \, d\theta \otimes d\phi \otimes \partial_\theta \otimes d\phi - d\theta \otimes d\phi \otimes \partial_\phi \otimes d\theta \\
&\quad - \sin^2\theta \, d\phi \otimes d\theta \otimes \partial_\theta \otimes d\phi - d\phi \otimes d\theta \otimes \partial_\phi \otimes d\theta \\[1ex]
&= 2 \, (\sin^2\theta \, d\theta \wedge d\phi \otimes \partial_\theta \otimes d\phi - d\theta \wedge d\phi \otimes \partial_\phi \otimes d\theta) \,.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
\mathsf{R}^\dagger &= (\partial_\theta \Gamma_\phi{}^\theta{}_j - \Gamma_\theta{}^h{}_j \, \Gamma_\phi{}^\theta{}_h - \partial_\phi \Gamma_\theta{}^\theta{}_j + \Gamma_\phi{}^h{}_j \, \Gamma_\theta{}^\theta{}_h) \, d\theta \otimes d\phi \otimes \partial\theta \otimes dx^j \\
&\quad - (\partial_\theta \Gamma_\phi{}^\theta{}_j - \Gamma_\theta{}^h{}_j \, \Gamma_\phi{}^\theta{}_h - \partial_\phi \Gamma_\theta{}^\theta{}_j + \Gamma_\phi{}^h{}_j \, \Gamma_\theta{}^\theta{}_h) \, d\phi \otimes d\theta \otimes \partial\theta \otimes dx^j \\
&\quad + (\partial_\theta \Gamma_\phi{}^\phi{}_j - \Gamma_\theta{}^h{}_j \, \Gamma_\phi{}^\phi{}_h - \partial_\phi \Gamma_\theta{}^\phi{}_j + \Gamma_\phi{}^h{}_j \, \Gamma_\theta{}^\phi{}_h) \, d\theta \otimes d\phi \otimes \partial\phi \otimes dx^j \\
&\quad - (\partial_\theta \Gamma_\phi{}^\phi{}_j - \Gamma_\theta{}^h{}_j \, \Gamma_\phi{}^\phi{}_h - \partial_\phi \Gamma_\theta{}^\phi{}_j + \Gamma_\phi{}^h{}_j \, \Gamma_\theta{}^\phi{}_h) \, d\phi \otimes d\theta \otimes \partial\phi \otimes dx^j \\[1ex]
&= (\partial_\theta \Gamma_\phi{}^\theta{}_j - \Gamma_\theta{}^h{}_j \, \Gamma_\phi{}^\theta{}_h + \Gamma_\phi{}^h{}_j \, \Gamma_\theta{}^\theta{}_h) \, d\theta \otimes d\phi \otimes \partial\theta \otimes dx^j \\
&\quad - (\partial_\theta \Gamma_\phi{}^\theta{}_j - \Gamma_\theta{}^h{}_j \, \Gamma_\phi{}^\theta{}_h + \Gamma_\phi{}^h{}_j \, \Gamma_\theta{}^\theta{}_h) \, d\phi \otimes d\theta \otimes \partial\theta \otimes dx^j \\
&\quad + (\partial_\theta \Gamma_\phi{}^\phi{}_j - \Gamma_\theta{}^h{}_j \, \Gamma_\phi{}^\phi{}_h + \Gamma_\phi{}^h{}_j \, \Gamma_\theta{}^\phi{}_h) \, d\theta \otimes d\phi \otimes \partial\phi \otimes dx^j \\
&\quad - (\partial_\theta \Gamma_\phi{}^\phi{}_j - \Gamma_\theta{}^h{}_j \, \Gamma_\phi{}^\phi{}_h + \Gamma_\phi{}^h{}_j \, \Gamma_\theta{}^\phi{}_h) \, d\phi \otimes d\theta \otimes \partial\phi \otimes dx^j \\[1ex]
&= (\partial_\theta \Gamma_\phi{}^\theta{}_\phi - \Gamma_\theta{}^h{}_\phi \, \Gamma_\phi{}^\theta{}_h + \Gamma_\phi{}^h{}_\phi \, \Gamma_\theta{}^\theta{}_h) \, d\theta \otimes d\phi \otimes \partial\theta \otimes d\phi \\
&\quad - (\partial_\theta \Gamma_\phi{}^\theta{}_\phi - \Gamma_\theta{}^h{}_\phi \, \Gamma_\phi{}^\theta{}_h + \Gamma_\phi{}^h{}_\phi \, \Gamma_\theta{}^\theta{}_h) \, d\phi \otimes d\theta \otimes \partial\theta \otimes d\phi \\
&\quad + (\partial_\theta \Gamma_\phi{}^\phi{}_\theta - \Gamma_\theta{}^h{}_\theta \, \Gamma_\phi{}^\phi{}_h + \Gamma_\phi{}^h{}_\theta \, \Gamma_\theta{}^\phi{}_h) \, d\theta \otimes d\phi \otimes \partial\phi \otimes d\theta \\
&\quad - (\partial_\theta \Gamma_\phi{}^\phi{}_\theta - \Gamma_\theta{}^h{}_\theta \, \Gamma_\phi{}^\phi{}_h + \Gamma_\phi{}^h{}_\theta \, \Gamma_\theta{}^\phi{}_h) \, d\phi \otimes d\theta \otimes \partial\phi \otimes d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= (\partial_\theta \Gamma_\phi{}^\theta{}_\phi - \Gamma_\theta{}^\phi{}_\phi \, \Gamma_\phi{}^\theta{}_\phi + \Gamma_\phi{}^\theta{}_\phi \, \Gamma_\theta{}^\theta{}_\theta) \, d\theta \otimes d\phi \otimes \partial\theta \otimes d\phi \\
&- (\partial_\theta \Gamma_\phi{}^\theta{}_\phi - \Gamma_\theta{}^\phi{}_\phi \, \Gamma_\phi{}^\theta{}_\phi + \Gamma_\phi{}^\theta{}_\phi \, \Gamma_\theta{}^\theta{}_\theta) \, d\phi \otimes d\theta \otimes \partial\theta \otimes d\phi \\
&+ (\partial_\theta \Gamma_\phi{}^\phi{}_\theta + \Gamma_\phi{}^\phi{}_\theta \, \Gamma_\theta{}^\phi{}_\phi) \, d\theta \otimes d\phi \otimes \partial\phi \otimes d\theta \\
&- (\partial_\theta \Gamma_\phi{}^\phi{}_\theta + \Gamma_\phi{}^\phi{}_\theta \, \Gamma_\theta{}^\phi{}_\phi) \, d\phi \otimes d\theta \otimes \partial\phi \otimes d\theta \\[1ex]
&= (-\cos^2\theta + \sin^2\theta + \frac{\cos\theta}{\sin\theta} \sin\theta \cos\theta)) \, d\theta \otimes d\phi \otimes \partial\theta \otimes d\phi \\
&- (-\cos^2\theta + \sin^2\theta + \frac{\cos\theta}{\sin\theta} \sin\theta \cos\theta)) \, d\phi \otimes d\theta \otimes \partial\theta \otimes d\phi \\
&+ (\frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}) \, d\theta \otimes d\phi \otimes \partial\phi \otimes d\theta \\
&- (\frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}) \, d\phi \otimes d\theta \otimes \partial\phi \otimes d\theta \\[1ex]
&= \sin^2\theta \, d\theta \otimes d\phi \otimes \partial\theta \otimes d\phi - \sin^2\theta \, d\phi \otimes d\theta \otimes \partial\theta \otimes d\phi \\
&- d\theta \otimes d\phi \otimes \partial\phi \otimes d\theta + d\phi \otimes d\theta \otimes \partial\phi \otimes d\theta \, . \text{QED}
\end{aligned}
$$

3.4.12 Corollary. The coordinate expression of the covariant Riemannian curvature tensor is

$$\underline{\mathrm{R}}^\dagger = 4 \, \mathrm{r}^2 \, \sin^2\theta \, d\theta \wedge d\phi \otimes d\theta \wedge d\phi \, .$$

Proof. We have

$$
\begin{aligned}
\underline{\mathrm{R}}^\dagger &= 2 \, \mathrm{r}^2 \, \sin^2\theta \, (d\theta \wedge d\phi \otimes d\theta \otimes d\phi - d\theta \wedge d\phi \otimes d\phi \otimes d\theta) \\
&= 4 \, \mathrm{r}^2 \, \sin^2\theta \, d\theta \wedge d\phi \otimes d\theta \wedge d\phi \, . \text{QED}
\end{aligned}
$$

3.4.13 Corollary. We have the equality

$$
\begin{aligned}
\underline{\mathrm{R}}^\dagger &= \frac{4}{\mathrm{r}^2} \, \eta^\dagger \otimes \eta^\dagger \\
&= \frac{2}{\mathrm{r}^2} \, (2 \, \eta^\dagger \otimes \eta^\dagger) \, . \, \square
\end{aligned}
$$

3.4.14 Corollary. The coordinate expression of the Ricci tensor is

$$\underline{\mathrm{r}}^\dagger = \sin^2\theta \, d^\phi \otimes d^\phi + d^\theta \otimes d^\theta \, . \, \square$$

3.4.15 Corollary. We have the equality

$$
\begin{aligned}
\underline{\mathrm{r}}^\dagger &= \frac{1}{\mathrm{r}^2} \, g^\dagger \\
&= \frac{2}{\mathrm{r}^2} \, (\tfrac{1}{2} \, g^\dagger) \, . \, \square
\end{aligned}
$$

3.4.16 Corollary. The Riemannian scalar curvature is

$$\langle \underline{\mathrm{r}} \rangle^{\dagger} = \frac{2}{\mathrm{r}^2} \,. \square$$

3.4.17 Note. There is an agreement between the two equalities

$$\tfrac{1}{2} \, \langle \underline{\mathrm{r}} \rangle^{\dagger} = \frac{1}{\mathrm{r}^2} = K \,. \square$$

3.5 Paraboloid

Now, we suppose that the submanifold Q is the paraboloid P characterised by the constraint $z = a\,\rho^2$, i.e. $f = 0$.

We shall refer to the adapted parabolical chart (ρ, θ, f).

3.5.1 Riemannian metric

Let us compute the Riemannian metric and the induced algebraic objects.

3.5.1 Proposition. The coordinate expression of the metric and of the contravariant metric are

$$g^{\dagger} = (1 + 4\,a^2 \rho^2)\, d\rho \otimes d\rho + \rho^2\, d\phi \otimes d\phi \qquad \text{and} \qquad \bar{g}^{\dagger} = \frac{1}{1 + 4\,a^2 \rho^2}\, \partial_\rho \otimes \partial_\rho + \frac{1}{\rho^2}\, \partial_\phi \otimes \partial_\phi\,.$$

The coordinate expression of the metric function is

$$G^{\dagger} = \tfrac{1}{2}\left((1 + 4\,a^2\,\rho^2)\,\dot{\rho}^2 + \rho^2\,\dot{\theta}^2\right).\square$$

3.5.2 Proposition. The *volume form* induced by the metric $g^{\dagger}$ and by the orientation of the chosen chart has coordinate expression

$$\eta^{\dagger} = \rho\,\sqrt{1 + 4\,a^2\,\rho^2}\; d\rho \wedge d\theta\,.\square$$

3.5.2 Extrinsic curvature

Let us compute the unit normal, the Weingarten tensor, the second fundamental form and the Gauss tensor.

3.5.3 Proposition. We have the global unit normal vector field

$$n = -\frac{2\,a\,\rho}{\sqrt{1 + 4\,a^2\,\rho^2}}\,\partial r + \sqrt{1 + 4\,a^2\,\rho^2}\,\partial f\,.\square$$

3.5.4 Proposition. The Weingarten tensor and the second fundamental form are

$$L = -\frac{2\,a}{\sqrt{1 + 4\,a^2\,\rho^2}}\,\big(\frac{1}{1 + 4\,a^2\,\rho^2}\,d\rho \otimes \partial_\rho + d\theta \otimes \partial_\theta\big)$$

and

$$\underline{L} = -\frac{2\,a\,\rho}{\sqrt{1 + 4\,a^2\,\rho^2}}\,(d\rho \otimes d\rho + \rho^2\,d\theta \otimes d\theta)\,.\square$$

3.5.5 Corollary. The principal curvatures are

$$\lambda' = -\frac{2\,a}{(1 + 4\,a^2\,\rho^2)^{\frac{3}{2}}} \qquad \text{and} \qquad \lambda'' = -\frac{2\,a}{\sqrt{1 + 4\,a^2\,\rho^2}}\,.$$

The principal vectors are

$$v' = \partial_\rho \qquad \text{and} \qquad v'' = \partial_\theta\,.\,\square$$

Thus, the coordinate curve x_ρ and x_θ are curvature lines. $\square$

3.5.6 Corollary. The paraboloid is not a ruled hypersurface (hence it is not a developable hypersurface). $\square$

3.5.7 Corollary. The total curvature and the mean curvature are

$$K = \frac{4\,a^2}{(1 + 4\,a^2\,\rho^2)^2} \qquad \text{and} \qquad H = -\frac{4\,a\,(1 + 2\,a^2\,\rho^2)}{(1 + 4\,a^2\,\rho^2)^{\frac{3}{2}}}\,.$$

3.5.3 Riemannian connection

Let us compute the symbols of the Riemannian connection by means of the Lagrange formulas.

3.5.8 Proposition. The non–vanishing coefficients of $\nabla^\dagger$ are

$$\Gamma^\dagger{}_\rho{}^\rho{}_\rho = \frac{4\,a^2\,\rho}{1 + 4\,a^2\,\rho^2}\,, \qquad \Gamma^\dagger{}_\theta{}^\rho{}_\theta = -\frac{\rho}{1 + 4\,a^2\,\rho^2}\,, \qquad \Gamma^\dagger{}_\rho{}^\theta{}_\theta = \Gamma^\dagger{}_\theta{}^\theta{}_\rho = \frac{1}{\rho}\,.\,\square$$

3.5.4 Riemannian curvature

Let us compute the Riemannian curvature tensor, the Ricci tensor and the Riemannian scalar curvature.

3.5.9 Proposition. The coordinate expression of the Riemannian curvature tensor of $\nabla^\dagger$ is

$$\mathsf{R}^\dagger = \frac{8\,a^2}{1 + 4\,a^2\,\rho^2}\Big(\frac{\rho^2}{1 + 4\,a^2\,\rho^2}\,d\rho \wedge d\theta \otimes \partial_\rho \otimes d\theta - d\rho \wedge d\theta \otimes \partial_\theta \otimes d\rho\Big)\,.\,\square$$

3.5.10 Corollary. The coordinate expression of the covariant Riemannian curvature tensor is

$$\underline{\mathsf{R}}^\dagger = \frac{16\,a^2\,\rho^2}{1 + 4\,a^2\,\rho^2}\,d\rho \wedge d\theta \otimes d\rho \wedge d\theta\,.\,\square$$

3.5.11 Corollary. We have the equality

$$\begin{aligned}\underline{\mathrm{R}}^{\dagger} &= \frac{16\,a^2}{(1+4\,a^2\,\rho^2)^2}\,\eta^{\dagger}\otimes\eta^{\dagger}\\ &= \frac{8\,a^2}{(1+4\,a^2\,\rho^2)^2}\,(2\,\eta^{\dagger}\otimes\eta^{\dagger})\,.\,\square\end{aligned}$$

3.5.12 Corollary. The coordinate expression of the Ricci tensor is

$$\underline{\mathrm{r}}^{\dagger} = \frac{4\,a^2}{1+4\,a^2\,\rho^2}\,\Big(d\rho\otimes d\rho + \frac{\rho^2}{1+4\,a^2\,\rho^2}\,d\theta\otimes d\theta\Big)\,.\,\square$$

3.5.13 Corollary. We have the equality

$$\begin{aligned}\underline{\mathrm{r}}^{\dagger} &= \frac{4\,a^2}{1+4\,a^2\,\rho^2}\,g^{\dagger}\\ &= \frac{8\,a^2}{1+4\,a^2\,\rho^2}\,(\tfrac{1}{2}\,g^{\dagger})\,.\,\square\end{aligned}$$

3.5.14 Corollary. The Riemannian scalar curvature is

$$\langle\underline{\mathrm{r}}\rangle^{\dagger} = \frac{8\,a^2}{1+4\,a^2\,\rho^2}\,.\,\square$$

3.5.15 Note. There is an agreement between the two equalities

$$\tfrac{1}{2}\,\langle\underline{\mathrm{r}}\rangle^{\dagger} = \frac{4\,a^2}{1+4\,a^2\,\rho^2} = K\,.\,\square$$

CHAPTER 4

SYMBOLS

M	manifold
TM	tangent space
T^*M	cotangent space
$\mathcal{F}(M)$	set of local functions
$\mathcal{T}(M)$	set of local vector fields
$\mathcal{T}^*(M)$	set of local forms
$\mathcal{T}^k(M)$	set of local contravariant tensors of order k
$\mathcal{T}^{*k}(M)$	set of local covariant tensors of order k
$\mathcal{A}(M)$	set of local tensors
(x^i)	coordinate functions
(x_i)	coordinate curves
(dx^i)	base of forms
(∂x_i)	base of vector fields
∂_i	i–th partial derivative
g	Riemannian metric
∇	covariant differential
T	torsion tensor
R	curvature tensor
$\underline{\mathsf{R}}$	covariant curvature tensor
$\underline{\mathsf{r}}$	Ricci tensor
$\langle \underline{\mathsf{r}} \rangle$	scalar curvature function
Q	submanifold
$g^\dagger$	metric induced on the submanifold
$\nabla^\dagger$	connection of the submanifold

N	Gauss tensor
$\mathsf{R}^\dagger$	curvature tensor of the submanifold
$\underline{\mathsf{R}}^\dagger$	covariant curvature tensor of the submanifold
$\underline{\mathsf{r}}^\dagger$	Ricci tensor of the submanifold
$\langle\underline{\mathsf{r}}\rangle^\dagger$	scalar curvature function of the submanifold
n	unit normal vector to an hypersurface
L	Weingarten tensor
$\underline{L}$	second fundamental form
K	determinant of the Weingarten tensor
H	trace Weingarten tensor
E	Euclidean space
(x, y, z)	Cartesian coordinates
(ρ, ϕ, z)	cylindrical coordinates
(r, θ, ϕ)	spherical coordinates
r	radius of the cylinder and of the sphere

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