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Notes on Differential Geometry

Notes for students
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CHAPTER 1

LINEAR ALGEBRA

In this chapter we briefly review the fundamental notions of linear algebra that are used throughout the book.

1.1 Vector spaces

Here, we give the definition and main properties of vector spaces. We introduce the concept of linear map and bilinear map, and study the direct sum splittings. The subject is developed for arbitrary vector spaces; only in the final section we specialise the above notions to the case of finitely generated vector spaces, which is the most important case to our purposes.

1.1.1 Vector spaces

1.1.1 Definition. A *vector space* is defined to be a 3plet $(V, +, \cdot)$, where V is a non–empty set, and

$$+ : V \times V \to V \qquad \text{and} \qquad \cdot : \mathbb{R} \times V \to V\,,$$

are two maps which fulfill the following properties:

1. $\forall v, w, z \in V$, we have $(v + w) + z = v + (w + z)$,
2. $\exists 0 \in V$ such that, $\forall v \in V$, we have $v + 0 = 0$,
3. $\forall v \in V$, $\exists -v \in V$, such that $v + (-v) = 0$,
4. $\forall v, w \in V$, we have $v + w = w + v$,
5. $\forall v, w \in V$, $\lambda \in \mathbb{R}$, we have $\lambda(v + w) = \lambda v + \lambda w$,
6. $\forall v \in V$, $\lambda, \mu \in \mathbb{R}$, we have $\lambda(\mu v) = (\lambda\mu)v$,
7. $\forall v \in V$, $\lambda, \mu \in \mathbb{R}$, we have $(\lambda + \mu)v = \lambda v + \mu v$,
8. $\forall v \in V$, we have $1v = v$. $\square$

1.1.2 Note. As a consequence of the axioms, we obtain

$$0\,v = 0\,, \qquad \text{for all} \qquad v \in V\,.\square$$

From now on, we will denote a vector space $(V, +, \cdot)$ by V, by abuse of notation.

1.1.3 Note. Let V be a vector space. A non–empty subset $V' \subset V$ is said to be a *vector subspace* of V if

$$v + w \in V' \quad \text{and} \quad \lambda v \in V'\,, \qquad \text{for all} \qquad v, w \in V'\,,\ \lambda v \in V'\,.$$

So, a vector subspace $V' \subset V$ turns out to be a vector space, whose operations $+$ and $\cdot$ coincide with the restriction of the corresponding operations of V. □

1.1.4 Note. Let V be a vector space and I a non empty set. A vector $v \in V$ is said to be a *linear combination* of the vectors $\{b_i\}_{i\in I}$ if there exists a finite family $\{v^i\}_{i\in J}$ of elements of $\mathbb{R}$, where $J \subset I$ is a finite subset, such that v is equal to the following finite sum of elements of $\{b_i\}_{i\in I}$

$$v = \sum_{i\in J} v^i\, b_i\,.$$

The family $\{b_i\}_{i\in I}$ is said to be *independent* if the vector 0 is a linear combination of $\{b_i\}_{i\in I}$ in a unique way, namely, the trivial one.

In other words, if $J \subset I$ is a finite subset and $\{v_j\}_{j\in J}$ is a family of real numbers, then

$$0 = \sum_{j\in J} v^j\, b_j \qquad \Rightarrow \qquad v^j = 0\,, \quad \forall\, j \in J\,.$$

Given a non empty subset $S \subset V$, we define the *linear span* of S in V to be the following vector subspace of V

$$\operatorname{span}(S) := \Big\{ \sum_{j\in J} v^j\, x_j \mid \{v^j\}_{j\in J} \subset \mathbb{R}\,,\ \{x_j\}_{j\in J} \subset S\,,\ J \text{ finiteset} \Big\}\,.$$

It can be proved that $\operatorname{span}(S)$ is the smallest vector subspace (with respect to the inclusion) of V containing S. □

1.1.5 Note. Now, suppose that V, W be vector subspaces of a vector space Z. We observe that $V \cap W$ is a subspace of Z, while, in general, $V \cup W$ is not a subspace of Z.

We define the following subspace of Z

$$V + W := \operatorname{span}(V \cup W) = \big\{ \lambda v + \mu w \ \mid\ v \in V\,,\ w \in W\,,\ \lambda, \mu \in \mathbb{R} \big\}\,.$$

It can be easily shown that

1. $V \cap W$ is the greatest subspace of Z contained both in V and in W;

2. $V + W$ is the smallest subspace of Z containing $V \cup W$. □

1.1.6 Definition. A non empty subset $G \subset V$ is said to be a set of *generators* of V if any vector of V is a linear combination of the vectors of G, or, equivalently, if $V = \text{span}(G)$.

A non empty subset $B \subset V$ is said to be a *basis* if B is a set of independent generators. □

It can be shown that any vector space admits a basis. More precisely, the following Theorem holds.

1.1.7 Theorem. *Let V be a vector space, $G \subset V$ a set of generators and $I \subset G$ an independent set. Then, there exists a basis B such that*

$$I \subset B \subset G. \square$$

1.1.8 Note. If B is a basis of a vector space V, then each vector of V can be expressed as a linear combination of vectors of B.

Moreover, in virtue of the independence of B, such a linear combination turns out to be unique. Namely, if $B = \{b_i\}_{i \in I}$ and $v \in V$, then v can be uniquely written as $v = \sum_{i \in J} v^i \, b_i$, where $J \subset I$ is a finite subset. □

1.1.9 Note. It can be shown that any two bases of the same vector space have the same cardinality (i.e., the same "number of elements"). Such a cardinality is said to be the *dimension* of the vector space.

In particular, a vector space is said to be
– *finite dimensional* if it admits a basis with finite cardinality;
– *infinite dimensional* if it admits a basis with infinite cardinality.
In this book, we are mostly concerned with finite dimensional vector spaces. □

1.1.10 Note. Any subspace of a finite dimensional vector space Z is a finite dimensional vector space, whose dimension is less than or equal to the dimension of the space Z.

Suppose that V and W be subspaces of a finite dimensional vector space Z. Then, we have the *Grassmann's formula*

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W). \square$$

We have distinguished examples of vector spaces.

1.1.11 Example. The set of real numbers $\mathbb{R}$ is endowed with sum and product which make $\mathbb{R}$ a vector space.

We have the distinguished basis $\{1\}$ of $\mathbb{R}$; hence, $\dim \mathbb{R} = 1$. □

1.1.12 Example. Let V and W be two vector spaces. Then, the *cartesian product* $V \times W$ is endowed with a natural structure of vector space.

Namely, for all $x, y \in V$, $z, t \in W$ and $\lambda \in \mathbb{R}$, we set

$$+ : (V \times W) \times (V \times W) \to V \times W : ((x, z), (y, t)) \mapsto (x + y, z + t)\,,$$
$$\cdot : \mathbb{R} \times (V \times W) \to V \times W : (\lambda, (x, z)) \mapsto (\lambda x, \lambda z)\,.$$

It is easily proved that, if B and C are bases of V and W respectively, then $B \times C$ is a basis of $V \times W$. □

1.1.13 Example. Let V be a vector space and $W \subset V$ a subspace.
Then, we introduce an equivalence relation in V . Namely, if $v, v' \in V$, then we set

$$v \sim v' \qquad \Leftrightarrow \qquad v - v' \in W\,.$$

The quotient set V/W has a natural structure of vector space. We say that V/W is the *quotient vector space.* □

1.1.14 Example. Let S be a set and V a vector space, and consider the set of maps

$$\mathrm{Map}(S, V) := \{f \mid f : S \to V\}\,.$$

The set $\mathrm{Map}(S, V)$ is endowed with a natural structure of vector space, with the operations $+$ and $\cdot$ defined as follows. For all $f, g \in \mathrm{Map}(S, V)$ and all $\lambda \in \mathbb{R}$ we set

$$f + g : S \to V : x \mapsto f(x) + g(x)\,, \qquad \text{for all} \qquad f, g \in \mathrm{Map}(S, V)\,,$$
$$\lambda f : S \to V : x \mapsto \lambda f(x)\,, \qquad \text{for all} \qquad \lambda \in \mathbb{R}\,,\ f \in \mathrm{Map}(S, V)\,.$$

The properties of $+$ and $\cdot$ can be easily checked. □

1.1.15 Example. Let S be a set. We can define a vector space naturally generated by S , in the following way.
We define $\mathrm{Free}(S)$ to be the set of formal linear combinations of elements of S

$$\sum_{j\in J} v^j\, x_j\,, \qquad \text{where} \qquad J \quad \text{is a finite set,} \quad \{v^i\}_{i\in J} \subset \mathbb{R}\,, \quad \{x_i\}_{i\in J} \subset S\,.$$

In other words, more precisely, we define $\mathrm{Free}(S)$ to be the set

$$\mathrm{Free}(S) := \{f : S \to \mathbb{R} \mid f^{-1}(\mathbb{R} \setminus \{0\}) \text{ finite set}\} \subset \mathrm{Map}(S, \mathbb{R})\,.$$

The set $\mathrm{Free}(S)$ turns out to be equipped with the natural operations of vector space. Even more, we can easily seen that the set $\mathrm{Free}(S)$ is a vector subspace of the vector space $M(S, \mathbb{R})$.
Moreover, if $x \in S$, we define $f_x \in \mathrm{Free}(S)$ to be the map

$$f_x : S \to \mathbb{R} : y \mapsto \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}\,.$$

Then, we obtain a natural injection

$$S \to \mathrm{Free}(S) : x \mapsto f_x .$$

We call $\mathrm{Free}(S)$ the *free vector space generated by* S.

One can easily prove that the set S is a basis of $\mathrm{Free}(S)$. □

1.1.2 Linear maps

1.1.16 Definition. Let V, W be vector spaces, and $f : V \to W$ a map. Then, f is said to be *linear* if the following properties hold, for all $v, w \in V$ and $\lambda \in \mathbb{R}$,

$$f(v + w) = f(v) + f(w) \qquad \text{and} \qquad f(\lambda v) = \lambda f(v) . \square$$

The following result ensures the existence of non trivial linear maps.

1.1.17 Theorem. *Let V, W be vector spaces, $B \subset V$ a basis of V and $f : B \to W$ a map. Then, f can be uniquely extended to a linear map.*

Namely, if $B = \{b_i\}_{i \in I}$ and $v \in V$, then v can be uniquely written as $v = \sum_{i \in J} v^i \, b_i$, where $J \subset I$ is a finite subset and $\{v_j\}_{j \in J}$ is a family of real numbers, hence the unique linear extension $\hat{f}$ of f is determined on v by setting

$$\hat{f}(v) = v^i f(b_i) . \square$$

1.1.18 Note. Let V and W be vector spaces. With each map $f : V \to W$ we can associate the two subsets

$$\ker f := \{v \in V \mid f(v) = 0\} \subset V \quad \text{and} \quad \operatorname{im} f := \{w \in W \mid v \in V : f(v) = w\} \subset W .$$

The sets $\ker f$ and $\operatorname{im} f$ are said to be, respectively, the *kernel* and the *image* of f .

If f is linear, then $\ker f$ is a vector subspace of V and $\operatorname{im} f$ is a vector subspace of W .

A remarkable property of the kernel of a linear map f is that f is injective if and only if $\ker f = \{0\}$.

A linear map $f : V \to W$ is said to be an *isomorphism* if it is bijective. In this case,

$$\ker f = \{0\} \qquad \text{and} \qquad \operatorname{im} f = W ,$$

and f^{-1} turns out to be linear. □

The following important property holds for finite dimensional vector spaces.

1.1.19 Proposition. Let V and W be two finite dimensional vector spaces.

For each linear map $f : V \to W$, we have

$$\dim V = \dim \ker f + \dim \operatorname{im} f . \square$$

1.1.20 Corollary. Two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.

PROOF. It follows from the above Proposition 1.1.19 and Theorem 1.1.17. QED

1.1.21 Note. If Z is a vector space, and $f : V \to W$ and $g : W \to Z$ are two linear maps, then the composition

$$f \circ g : V \to Z : v \mapsto f(g(v))$$

turns out to be a linear map. □

1.1.22 Note. The subset

$$L(V, W) := \{f : V \to W \mid f \text{ linear}\} \subset M(V, W)$$

turns out to be a vector subspace. □

1.1.23 Definition. An $f \in L(V, V)$ is said to be an *endomorphism* of V. We set

$$\mathrm{End}(V) := L(V, V). \;\square$$

1.1.24 Definition. An isomorphism $f \in \mathrm{End}(V, V)$ is said to be an *automorphism* of V. We set

$$\mathrm{Aut}(V) := \{f \in \mathrm{End}(V) \mid f \text{ automorphism}\}\,. \;\square$$

1.1.25 Remark. The subset $\mathrm{Aut}(V) \subset \mathrm{End}(V)$ is not a vector subspace.

On the other hand, $\mathrm{Aut}(V)$ is endowed with the natural structure of (non abelian) group with respect to the composition of maps. □

1.1.26 Definition. We define the *dual space* of v to be the vector space

$$V^* := L(V, \mathbb{R})\,. \;\square$$

1.1.27 Definition. We define the *transpose* of the linear map $f \in L(V, W)$ to be the linear map

$$f^* : W^* \to V^* : \alpha \mapsto \alpha \circ f\,.$$

We define the *transposition* to be the linear map

$$* : L(V, W) \to L(W^*, V^*) : f \mapsto f^*\,. \;\square$$

1.1.28 Note. If Z is a vector space, and $f : V \to W$ and $g : W \to Z$ are two linear maps, then we have

$$(g \circ f)^* = f^* \circ g^*\,.$$

Hence, the transpose of an injective map is surjective and the transpose of a surjective map is injective. □

1.1.29 Note. We have the natural linear map

$$V \to V^{**} : v \mapsto v^{**} ,$$

where

$$v^{**} : V^{*} \to \mathbb{R} : \alpha \mapsto \alpha(v) .$$

The above map turns out to be injective.
We will see that in the finite dimensional case this map is an isomorphism. □

1.1.30 Note. We have the natural isomorphism

$$L(\mathbb{R}, V) \to V : f \mapsto f(1) ,$$

which allows us to make the important identification

$$L(\mathbb{R}, V) \simeq V . \square$$

1.1.3 Multilinear maps

Now, we generalise the definition of linear map. Namely, we consider maps defined on a cartesian product of vector spaces, and define a multilinear map to be a map which is linear with respect to every factor.

We start by introducing bilinear maps; then, the definition can be easily generalised to cartesian products of several vector spaces.

Let us consider three vector spaces V, W, Z .

1.1.31 Definition. A map

$$f : V \times W \to Z$$

is said to be *bilinear* if the following properties hold for each $v, v' \in V$, $w, w' \in W$, $\lambda, \mu \in \mathbb{R}$,

$$f(\lambda v + \mu v', w) = \lambda f(v, w) + \mu f(v', w) ,$$
$$f(v, \lambda w + \mu w') = \lambda f(v, w) + \mu f(v, w') . \square$$

1.1.32 Note. The above definition admits a further straightforward generalisation to cartesian products of p vector spaces. In this case, we speak of tp–linear map, or of multilinear map. □

1.1.33 Remark. If $f : V \times W \to Z$ is a bilinear map, then f is NOT a linear map between the vector spaces $V \times W$ and Z . □

1.1.34 Note. Let $f : V \times W \to Z$ be a bilinear map. If $v \in V$ and $w \in W$, then we set

$$f(v, \cdot) : W \to Z : w \mapsto f(v, w) \qquad \text{and} \qquad f(\cdot, w) : V \to Z : v \mapsto f(v, w) .$$

Clearly, the maps $f(v,\cdot)$ and $f(\cdot,w)$ are linear.
Hence, we obtain the linear maps

$$f_V : V \to L(W,Z) : v \mapsto f(v,\cdot) \qquad \text{and} \qquad f_W : W \to L(V,Z) : w \mapsto f(\cdot,w)\,.\square$$

The following result ensures the existence of non trivial bilinear maps.

1.1.35 Theorem. *Let V, W, Z be vector spaces, $B \subset V$ a basis of V, $C \subset W$ a basis of W, and $f : B \times C \to Z$ a map. Then, f can be uniquely extended to a bilinear map.*

Proof. In fact, we recall that $B \times C$ turns out to be a basis of $V \times W$. If $B = \{b_i\}_{i\in I}$ and $C = \{c_j\}_{j\in J}$, then a vector $(v,w) \in V \times W$ can be written as a finite linear combination

$$(v,w) = (v^i b_i,\, w^j c_j)\,,$$

and the unique bilinear extension $\hat{f}$ of f is determined on (v,w) by setting

$$\hat{f}(v,w) = v^i w^j\, f(b_i, c_j)\,. \text{QED}$$

1.1.36 Definition. Let us consider a bilinear map $f : V \times W \to Z$.
We define the *radicals* of f to be the following subspaces of V and W

$$N_V(f) := \{v \in V \mid f(v,\cdot) = 0\} \;\equiv\; \ker f_V\,,$$
$$N_W(f) := \{w \in W \mid f(\cdot,w) = 0\} \;\equiv\; \ker f_W\,.$$

The map f is said to be *non degenerate* if

$$N_V(f) = 0 \qquad \text{and} \qquad N_W(f) = 0\,.$$

A non degenerate bilinear map $f : V \times W \to \mathbb{R}$ with values in $\mathbb{R}$ is said to be a *duality.* $\square$

1.1.37 Note. A duality $f : V \times W \to \mathbb{R}$ yields the linear injections

$$f_V : V \to W^* : v \mapsto f_v \qquad \text{and} \qquad f_W : W \to V^* : w \mapsto f_w\,.\square$$

1.1.38 Definition. Let us consider a duality $f : V \times W \to \mathbb{R}$ and suppose that $\dim V = \dim W$. Let $(b_i)_{i\in I}$ and $(c_i)_{i\in I}$ be two bases of V and W, respectively.
We say that (b_i) and (c_i) are *dual bases* (*with respect to* f) if

$$f(b_i, c_j) = \begin{cases} 1 & \text{if} \quad i = j\,, \\ 0 & \text{if} \quad i \neq j\,.\square \end{cases}$$

1.1.39 Note. We denote the subset of bilinear maps $V \times W \to Z$ by

$$L^2(V,W;\,Z) := \{f : V \times W \to Z \;\mid\; f \;\text{ is bilinear}\} \subset M(V \times W, Z)\,.$$

Clearly, this subset turns out to be a vector subspace. □

1.1.40 Example. A remarkable example of bilinear map is provided by the *contraction*

$$\langle\, , \rangle : V^* \times V \to \mathbb{R} : (\alpha, v) \mapsto \alpha(v)\,. \square$$

1.1.41 Note. The contraction is a non degenerate bilinear map.

PROOF. 1) If $\alpha \in V^*$ and $f(\alpha, \cdot) = 0$, then $\alpha = 0$, hence $N_{V^*} = 0$.
2) If $0 \neq v \in V$, then there exists a basis B of V such that $v \in B$.
Then, we define the map $\alpha_v : B \to \mathbb{R}$ such that $\alpha(w) = 1$ if $w = v$ and $\alpha(w) = 0$ otherwise.
We can extend α_v to a unique element of V^*, which turns out to be non zero.
This proves that $N_V = 0$. QED

1.1.42 Example. The composition of linear maps yields the bilinear map

$$\circ : L(V, W) \times L(W, Z) \to L(V, Z) : (f, g) \mapsto g \circ f\,. \square$$

Analogously, if $V_1, \ldots, V_p$ and Z are vector spaces, then we denote the subset of *p–linear (or multilinear) maps* $V_1 \times \ldots \times V_p \to Z$ by

$$L^p(V_1, \ldots, V_p; Z) \subset M(V_1 \times \ldots \times V_p;\, Z)\,.$$

In the particular case when $V = V_1 = \cdots = V_p$, we set

$$L^p(V;\, Z) := L^p(V, \ldots, V; Z)\,.$$

1.1.4 Algebras

Besides sum and scalar product, it is very useful to consider further algebraic operations on a vector space. To this aim, we give the following definition.

1.1.43 Definition. Let V be a vector space, and $\cdot : V \times V \to V$ a bilinear map. Then, we say the pair $(V, \cdot)$ to be an *algebra*.

An algebra $(V, \cdot)$ is said to be

1. *associative* if, for any $v, w, z \in V$, we have
$$(v \cdot w) \cdot z = v \cdot (w \cdot z)\,;$$
2. *with unity* if there exists an element $1_V \in V$ such that, for any $v \in V$, we have $1_V \cdot v = v \cdot 1_V = v$;
3. *commutative* if for any $v, w \in V$, we have $v \cdot w = w \cdot v$;
4. *anticommutative* if, for any $v, w \in V$, we have $v \cdot w = -w \cdot v$;

5. a *Lie Algebra* if it is anticommutative and, for any $v, w, z \in V$, we have

$$[v,[w,z]]+[z,[v,w]]+[w,[z,v]]=0\,.\square$$

Lie algebras are of fundamental importance in many fields of mathematical physics. In the second part of this book, we will see concrete examples of Lie algebras in mechanics.

By analogy with the concept of vector subspace, we can introduce the concept of *subalgebra*. Namely, if $(V,\cdot)$ is an algebra, then a vector subspace $W \subset V$ is said to be a *subalgebra* if the bilinear map $\cdot$ restricts to a bilinear map $\cdot : W \times W \to W$.

Given two algebras, we introduce the class of linear maps between them which "preserve" the algebra operations.

1.1.44 Definition. Let $(V,(\,,\,))$, $(W,\{\,,\,\})$ be an algebra, and $f \in L(V,W)$. We say that f is an *algebra morphism* if, for any $v, v' \in V$, we have

$$f((v,v'))=\{f(v),f(v')\}\,.\square$$

The set of algebra morphisms $f \in L(V,W)$ is a vector subspace of $L(V,W)$. Of course, we say that $f \in \mathrm{End}(V)$ is an *algebra endomorphism* if it is an algebra morphism.

Given an algebra, one can consider two further sets of linear endomorphism which "preserve" the algebra operation in a different way from the one above.

1.1.45 Definition. Let $(V,(\,,\,))$ be an algebra and $f \in \mathrm{End}(V)$. We say that f is

1. a *derivation* of $(V,(\,,\,))$ if, for any $v, v' \in V$, we have

$$f((v,v'))=(f(v),v')+(v,f(v'))\,;$$

2. an *antiderivation* of $(V,(\,,\,))$ if, for any $v, v' \in V$, we have

$$f((v,v'))=(f(v),v')-(v,f(v'))\,.\square$$

The set of derivations (antiderivations) $f \in \mathrm{End}(V)$ is a vector subspace of $\mathrm{End}(V)$.

1.1.46 Example. If $(V,\cdot)$ is an algebra, then we define a new bilinear map

$$[\,,\,]: V \times V \to V : (v,w) \mapsto [v,w] := v\cdot w - w\cdot v\,.$$

It turns out that $(V,[\,,\,])$ is a Lie algebra. The map $[\,,\,]$ is said to be the *commutator*, due to the fact that, if $(V,\cdot)$ is commutative, then $[\,,\,]=0$. $\square$

1.1.47 Example. The composition $\circ$ endows the vector space $\mathrm{End}(V)$ with the structure of associative algebra with the unity id_V. This algebra is nor commutative neither anticommutative. Anyway, $\mathrm{End}(V)$ can be endowed with the commutator, hence $(\mathrm{End}(V),[\,,\,])$ is a Lie algebra. $\square$

1.1.48 Example. The space $M(V, \mathbb{R})$ has a natural structure of associative and commutative algebra with unity. Namely, we define

$$\cdot : M(V, \mathbb{R}) \times M(V, \mathbb{R}) \to M(V, \mathbb{R}) : (f, g) \mapsto f \cdot g$$

where $f \cdot g : V \to \mathbb{R} : v \mapsto f(v)g(v)$. □

1.1.5 Symmetric and antisymmetric multilinear maps

We introduce the definition and main properties of symmetric and antisymmetric multilinear maps. We begin to deal with bilinear maps.

1.1.49 Definition. Let $f \in L^2(V; \mathbb{R})$. We say f to be *symmetric* if, for all $v, v' \in V$, we have

$$f(v, v') = f(v', v) .$$

We denote the subset of symmetric bilinear maps by

$$S^2(V, \mathbb{R}) \subset L^2(V, \mathbb{R}) . \square$$

1.1.50 Note. The subset $S^2(V, \mathbb{R}) \subset L^2(V, \mathbb{R})$ turns out to be a vector subspace. □

1.1.51 Note. Let $f : V \times V \to \mathbb{R}$ be a symmetric bilinear map.
We define the linear map

$$f^\flat : V \to V^* : v \mapsto f(\cdot, v) \equiv f(v, \cdot) .$$

If f is non degenerate, then the map $f^\flat : V \to V^*$ is injective. □

1.1.52 Definition. Let $q : V \to \mathbb{R}$ be a map. Then, q is said to be *quadratic* if, for all $v, w \in V$ and $\lambda \in \mathbb{R}$,

$$q(\lambda v) = \lambda^2 q(v) ,$$
$$q(v + w) - q(v) - q(w) = 2b_q(v, w) ,$$

where $b_q : V \times V \to \mathbb{R}$ is a symmetric bilinear map. The set of quadratic maps is a vector subspace of the vector space $M(V, \mathbb{R})$. □

1.1.53 Theorem. [Carnot] *Let $f : V \times V \to \mathbb{R}$ be a symmetric bilinear map. Then, the map*

$$f^\diamond : V \to \mathbb{R} : v \mapsto f(v, v)$$

is quadratic.

Indeed, the map $f \mapsto f^\diamond$ is a bijection between the set of symmetric bilinear maps and the set of quadratic maps. The inverse isomorphism is the map $q \mapsto f$ which associates with every quadratic map $q : V \to \mathbb{R}$ the bilinear form

$$f : V \times V \to \mathbb{R} : (v, w) \mapsto \tfrac{1}{2} \big(q(v + w) - q(v) - q(w)\big) . \square$$

1.1.54 Definition. A symmetric bilinear map $f \in S^2(V, \mathrm{I\!R})$ is said to be *positive definite* if, for all $v \in V$, with $v \neq 0$, we have

$$f(v, v) > 0 \,. \square$$

1.1.55 Note. Clearly, a positive definite bilinear map is non degenerate.

Note that the set of positive definite maps is not a vector space (even if the sum of two positive definite maps is a positive definite map). $\square$

1.1.56 Definition. Let $f \in L^2(V; \mathrm{I\!R})$. We say f to be *antisymmetric* if, for all $v, v' \in V$, we have

$$f(v, v') = -f(v', v) \,.$$

We denote the subset of antisymmetric bilinear maps by

$$A^2(V, \mathrm{I\!R}) \subset L^2(V, \mathrm{I\!R}) \,. \square$$

1.1.57 Note. The subset $A^2(V, \mathrm{I\!R}) \subset L^2(V, \mathrm{I\!R})$ turns out to be a vector subspace. $\square$

Now, we generalise the definition of symmetric and antisymmetric map also to maps in $L^p(V; \mathrm{I\!R})$.

Let us denote by $\mathfrak{S}_p$ the group of permutations of the set of integers $(1, \dots, p)$.

A basic example of permutation is an exchange of two elements of $(1, \dots, p)$. Each permutation $\sigma \in \mathfrak{S}_p$ can be written as the composition of either an even or an odd number of exchanges. Accordingly, we say a permutation to be *even* or *odd*.

If $\sigma \in \mathfrak{S}_p$, then we set

$$|\sigma| := \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases} \,.$$

1.1.58 Definition. We say a map $f \in L^p(V, \mathrm{I\!R})$ to be, respectively, *symmetric*, or *antisymmetric*, if, for all $v_1, \dots, v_p \in V$ and $\sigma \in \mathfrak{S}_r$,

$$f\big(\sigma(v_1), \dots, \sigma(v_p)\big) = f(v_1, \dots, v_p) \,,$$
$$f\big(\sigma(v_1), \dots, \sigma(v_p)\big) = |\sigma| f(v_1, \dots, v_p) \,.$$

We denote the subsets of symmetric and antisimmetric bilinear maps, respectively, by by

$$S^p(V, \mathrm{I\!R}) \subset L^p(V, \mathrm{I\!R}) \qquad \text{and} \qquad A^p(V, \mathrm{I\!R}) \subset L^p(V, \mathrm{I\!R}) \,. \square$$

1.1.59 Note. The subsets $S^p(V, \mathrm{I\!R}) \subset L^p(V, \mathrm{I\!R})$ and $A^p(V, \mathrm{I\!R}) \subset L^p(V, \mathrm{I\!R})$ turn out to be a vector subspaces. $\square$

1.1.6 Direct sums

1.1.60 Definition. Let Z be a vector space, and $V, W \subset Z$ two vector subspaces. We say that Z *splits into the direct sum* of V and W if

$$Z = V + W \quad \text{and} \quad V \cap W = \{0\}\,.$$

In such a case, we write

$$Z = V \oplus W\,.\square$$

We can characterise $Z = V \oplus W$ in the two following equivalent ways.

1.1.61 Theorem. *Let Z be a vector space, and $V, W \subset Z$ two vector subspaces. Then, the following conditions are equivalent.*

1. *$Z = V \oplus W$.*
2. *To every $z \in Z$ there exists a unique $v \in V$ and a unique $w \in W$ such that $z = v + w$.*
3. *There exist two maps*

$$\pi_1 : Z \to V\,, \qquad \pi_2 : Z \to W\,,$$

which fulfill

$$\pi_1|_V = \mathrm{id}_V\,, \qquad \pi_1|_W = 0\,, \qquad \pi_1 + \pi_2 = \mathrm{id}_Z\,, \qquad \pi_2|_V = 0\,, \qquad \pi_2|_W = \mathrm{id}_W\,.$$

PROOF. 1) $\Leftrightarrow$ 2). Suppose that $Z = V \oplus W$ and $z \in Z$, $v, v' \in V$, $w, w' \in W$, such that $z = v + w = v' + w'$. Then, $v - v', w - w' \in V \cap W$, so that $v = v'$ and $w = w'$.

Conversely, let $z \in Z$, $v \in V$ and $w \in W$, such that $z = v + w$. If $x \in V \cap W$, then $z = (v - x) + (w + x)$ is another decomposition of z. It turns out that $x = 0$.

2) $\Rightarrow$ 3). If the second condition holds, then we define the maps π_1 and π_2 as the unique maps such that, for any $z \in Z$, we have $z = \pi_1(z) + \pi_2(z)$. We can prove that π_1 and π_2 are linear, and fulfill the third condition.

3) $\Rightarrow$ 2). Conversely, if the third condition holds and $z \in Z$, then we have $z = \pi_1(z) + \pi_2(z)$. If $v \in V$ and $w \in W$ such that $z = v + w$, then $\pi_1(z) - v, w - \pi_2(z) \in V \cap W$. Hence, we have

$$\pi_1(\pi_1(z) - v) = \pi_1(z) - v = 0\,, \qquad \pi_2(w - \pi_2(z)) = w - \pi_2(z) = 0\,.\ \text{QED}$$

The maps π_1 and π_2 are said to be the *projections* of the splitting $Z = V \oplus W$.

We have some consequences of the above Theorem together with Theorem 1.1.7.

1.1.62 Corollary. Let $Z = V \oplus W$. If B is a basis of V and C is a basis of W, then $B \cup C$ is a basis of Z. $\square$

A basis of $Z = V \oplus W$ of the above type is said to be a *adapted* to the splitting $Z = V \oplus W$.

If $Z = V \oplus W$ is a finite dimensional vector space, then the Grassmann's formula yields

$$\dim(V \oplus W) = \dim V + \dim W\,.$$

1.1.63 Corollary. Let Z be a vector space and $V \subset Z$ a vector subspace. Then, there exists a vector subspace $W \subset Z$ such that

$$Z = V \oplus W\,.\,\square$$

1.1.64 Remark. We stress that the above subspace W above is not uniquely determined. $\square$

It is possible to introduce splittings into direct sums of a finite number of vector spaces by means of a straightforward generalisation of our definition. We will make use of such splittings in next section.

1.1.65 Example. Let V and W be two vector spaces.

We have the two distinguished subspaces of $V \times W$

$$\widetilde{V} := \{(v, 0) \mid v \in V\} \qquad \text{and} \qquad \widetilde{W} := \{(0, w) \mid w \in W\}\,.$$

Clearly, $\widetilde{V}$ and $\widetilde{W}$ are naturally isomorphic, respectively, to V and W, and we have the splitting

$$V \times W = \widetilde{V} \oplus \widetilde{W}\,.$$

We remark that, in the more general case, in which V and W are just sets, the cartesian product $V \times W$ has no distinguished subsets. $\square$

1.1.66 Example. We have the splitting

$$L^2(V, \mathbb{R}) = S^2(V, \mathbb{R}) \oplus A^2(V, \mathbb{R})\,.$$

In fact, we have the two surjective linear maps

$$S : L^2(V, \mathbb{R}) \to S^2(V, \mathbb{R}) : f \mapsto S(f)\,,$$
$$A : L^2(V, \mathbb{R}) \to A^2(V, \mathbb{R}) : f \mapsto A(f)\,,$$

where $S(f)$ and $A(f)$ are defined to be the maps

$$S(f) : V \times V \to \mathbb{R} : (v, v') \mapsto \tfrac{1}{2}\left(f(v, v') + f(v', v)\right),$$
$$A(f) : V \times V \to \mathbb{R} : (v, v') \mapsto \tfrac{1}{2}\left(f(v, v') - f(v', v)\right).$$

Each map $f \in L^2(V, \mathbb{R})$ can be written in a unique way as the sum $f = S(f) + A(f)$.

It is worth noting that such a splitting does not hold for $p > 2$. $\square$

1.1.67 Example. We will meet several examples of algebras which have the structure of a direct sum. More precisely, suppose that $(V, \cdot)$ is an algebra, where

$$V \equiv \oplus_{n\in\mathbb{N}} V_n$$

and

$$v \in V_n,\ v' \in V_m \qquad \Rightarrow \qquad v \cdot v' \in V_{n+m}\,.$$

Then, $(V, \cdot)$ is said to be a *graded algebra*. Elements of V_n are said to have *degree n*.

It is possible to introduce the notions of graded associative algebras, graded commutative algebras, graded anticommutative algebras and graded Lie algebras. Moreover, it is possible to introduce the concept of graded algebra morphism and graded derivation. □

1.2 Finite dimensional vector spaces

In this section, we specialise the definitions and results of the above section to the case of finite dimensional vector spaces.

In particular, we give the definition and main properties of matrices. The conventions on indices and sums that we adopt are often used in mathematical physics and geometry.

Then, we introduce the matrix representations of finite dimensional vector spaces, and give the matrix representations of some examples from the above section. The conventions on indices allow us to know the transformation properties of any index of every matrix representation with respect to any change of basis.

1.2.1 Matrices

In this subsection we give a general definition of matrix, by which we recover the usual one.

It is convenient to adopt a positional notation for matrix indexes. The position (upper or lower) of an index will label a transformation property of the index under a change of basis. In concrete examples and computations, this notation will be quite useful.

We introduce an important notation on sums, which allows us to skip the summation symbol, making formulas more compact and readable. If the same index appears both as a subscript and as a superscript, a sum over the range of the index is understood. For example

$$\sum_{i \in J} v^i \, b_i \qquad \text{is denoted by} \qquad v^i b_i \,.$$

Let $n \in \mathbb{N} \setminus \{0\}$. We set

$$\underline{n} := \{1, \dots, n\}$$

1.2.1 Definition. Let $n, m, r, s \in \mathbb{N} \setminus \{0\}$.

A *matrix* with r *contravariant indices* in $\underline{n}$ and s *covariant indexes* in $\underline{m}$ is defined to be a (partially ordered) family of real numbers

$$(x^{i_1 \dots i_r}{}_{j_1 \dots j_s}) \,, \qquad \text{where} \qquad i_1, \dots, i_r \in \underline{n} \,, \quad j_1, \dots j_r \in \underline{m} \,.$$

We denote by

$$\mathcal{M}^{n \dots n}{}_{m \dots m}$$

the set of matrices with r contravariant indexes in $\underline{n}$ and s covariant indexes in $\underline{m}$. □

1.2.2 Note. The set $\mathcal{M}^{n \dots n}{}_{m \dots m}$ has a natural structure of vector space. Namely, if

$$(x^{i_1 \dots i_r}{}_{j_1 \dots j_s}), \ (y^{h_1 \dots h_r}{}_{k_1 \dots k_s}) \in \mathcal{M}^{n, \dots, n}{}_{m, \dots, m}$$

and $\lambda \in \mathrm{I\!R}$, then we define the operations $+$ and $\cdot$ as follows

$$(x^{i_1\dots i_r}{}_{j_1\dots j_s}) + (y^{h_1\dots h_r}{}_{k_1\dots k_s}) := (x^{i_1\dots i_r}{}_{j_1\dots j_s} + y^{i_1\dots i_r}{}_{j_1\dots j_s}),$$
$$\lambda\,(x^{i_1\dots i_r}{}_{j_1\dots j_s}) := (\lambda x^{i_1\dots i_r}{}_{j_1\dots j_s})\,.\,\square$$

1.2.3 Note. More formally, a matrix $(x^{i_1\dots i_r}{}_{j_1\dots j_s}) \in \mathcal{M}^{n\dots n}{}_{m\dots m}$ can be regardeded as the map

$$x : \underline{n} \times \dots \times \underline{n} \times \underline{m} \times \dots \times \underline{m} \;\equiv\; \underline{n}^r \times \underline{m}^s \to \mathrm{I\!R} : (i_1, \dots, i_r, j_1, \dots, j_s) \mapsto x^{i_1\dots i_r}{}_{j_1\dots j_s}\,.$$

In this way, we have the vector space isomorphism

$$\mathcal{M}^{n\dots n}{}_{m\dots m} \to M(\underline{n} \times \dots \times \underline{m}, \mathrm{I\!R})$$

(see Example 1.1.14). $\square$

1.2.4 Definition. We define the map

$$\delta : \underline{n}^r \times \underline{n}^r \to \mathrm{I\!R} :$$
$$((a_1, \dots, a_r), (b_1, \dots, b_r)) \mapsto \delta^{a_1\dots a_r}_{b_1\dots b_r} \;\equiv\; \begin{cases} 1 & (a_1, \dots, a_r) = (b_1, \dots, b_r) \\ 0 & \text{otherwise} \end{cases}\,.$$

If $r = 1$, then δ^i_j is said to be the *Kronecker's symbol.*
Indeed, we have $\delta^{a_1\dots a_r}_{b_1\dots b_r} = \delta^{a_1}_{b_1} \dots \delta^{a_r}_{b_r}$. $\square$

1.2.5 Note. We have the distinguished subset of matrices of $\mathcal{M}^{n\dots n}{}_{m\dots m}$, consisting of the matrices whose entries vanish all except one.

More precisely, for any $i_1, \dots, i_r, \in \underline{n}$, $j_1, \dots, j_s \in \underline{m}$, we set

$$\mathfrak{M}_{i_1\dots i_r}{}^{j_1\dots j_s} := (\delta^{h_1\dots h_r}_{i_1\dots i_r}\,\delta^{j_1\dots j_s}_{k_1\dots k_s}),$$

The subset of the above matrices turns out to be a natural basis of $\mathcal{M}^{n\dots n}{}_{m\dots m}$. In fact, if $x := (x^{i_1\dots i_r}{}_{j_1\dots j_s}) \in \mathcal{M}^{n\dots n}{}_{m\dots m}$, then we can write x in a unique way as a linear combination of the vectors $\mathfrak{M}_{i_1\dots i_r}{}^{j_1\dots j_s}$, namely

$$x = x^{i_1\dots i_r}{}_{j_1\dots j_s}\,\mathfrak{M}_{i_1\dots i_r}{}^{j_1\dots j_s}\,.$$

Hence, we have

$$\dim \mathcal{M}^{n\dots n}{}_{m\dots m} = n^r m^s\,.\,\square$$

1.2.6 Note. It is possible to introduce more general sets of matrices with mixed covariant and contravariant indexes with different ranges.

For example, the set of matrices $(x_i{}^j{}_k)$, with i and k covariant and j contravariant, and $1 \le i \le n$, $1 \le j \le m$, $1 \le k \le p$, is denoted by $\mathcal{M}_n{}^m{}_p$.

We leave to the reader the task of generalising the above definition. $\square$

1.2.7 Note. The vector spaces of matrices are endowed with further natural operations, which we define only in particular cases, leaving to the reader the task of generalising them.

1. If $x \in \mathcal{M}^m{}_n$, and $x \in \mathcal{M}^n{}_p$, then we define the (generalised) *matrix product*

   $$xy \in \mathcal{M}^m{}_n{}^n{}_p$$

   to be

   $$((xy)^i{}_j{}^h{}_k) := (x^i{}_j \cdot y^h{}_k)\,.$$

   The matrix product can be regarded as a bilinear map on the cartesian product $\mathcal{M}^m{}_n \times \mathcal{M}^n{}_p$. It is easy to extend the above product to a bilinear map defined on the cartesian product of any kind of set of matrices.

2. Let $z \in \mathcal{M}^m{}_n{}^n{}_p$, and $i \in \underline{r}$, $j \in \underline{s}$. We define the *contraction* $C^2_1(x)$ of x to be

   $$C^2_1(x) \in \mathcal{M}^m{}_p\,, \qquad \text{where} \qquad C^2_1(x) := (x^i{}_j{}^j{}_k).$$

   The contraction can be seen as a linear map which is defined, more generally, on sets of matrices with at least a contravariant index and a covariant index ranging on the same set. □

1.2.8 Example. We have two kinds of set of matrices with one index, namely $\mathcal{M}^n$ and $\mathcal{M}_n$.

If $x \equiv (x^i) \in \mathcal{M}^n$, then we can write x with respect to the natural basis as

$$x = x^i\,\mathfrak{M}_i\,.$$

Equivalently, we write x as the array

$$\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix},$$

which is said to be a *column vector.*

If $y \equiv (y_i) \in \mathcal{M}_n$, then we can write y with respect to the natural basis as

$$y = y_i\,\mathfrak{M}^i\,.$$

Equivalently, we write y as the array

$$y = (y_1, \dots, y_n)\,,$$

which is said to be a *row vector.*

The matrix sets $\mathcal{M}^n$ and $\mathcal{M}_n$ will be used in the matrix representation of a vector space and its dual, respectively. □

1.2.2 Square matrices

Now, we discuss the distinguished particular case of sets of matrices with two indices.

1.2.9 Definition. We define a *square matrix* M to be an element of one of the following four vector spaces

$$\mathcal{M}^{nm}\,, \qquad \mathcal{M}^{n}{}_{m}\,, \qquad \mathcal{M}_{n}{}^{m}\,, \qquad \mathcal{M}_{nm}\,.\square$$

We first consider the vector space $\mathcal{M}^{n}{}_{m}$, introducing here some operations.

1.2.10 Note. Let $x \equiv (x^{i}{}_{j}) \in \mathcal{M}^{n}{}_{m}$.
Then, we can write x with respect to the natural basis as

$$x = x^{i}{}_{j}\,\mathfrak{M}_{i}{}^{j}\,.$$

Equivalently, we write the matrix x as a rectangular array, where the first index of the matrix is said to be the *row index*, and the second index is said to be the *column index*

$$\begin{pmatrix} x^{1}{}_{1} & \dots & x^{1}{}_{m} \\ \vdots & \dots & \vdots \\ x^{n}{}_{1} & \dots & x^{n}{}_{m} \end{pmatrix}$$

We define the *transposition* to be the linear map

$$^{t} : \mathcal{M}^{n}{}_{m} \to \mathcal{M}_{m}{}^{n} : (x^{i}{}_{j}) \mapsto (x^{i}{}_{j})^{t} \equiv (x_{i}{}^{j})\,.$$

The trasposition turns out to be a linear isomorphism. The transposition can be defined analogously for $\mathcal{M}^{nm}$, $\mathcal{M}_{nm}$.
We define the *trace* to be the linear map

$$C^{1}_{1} : \mathcal{M}^{n}{}_{n} \to \mathrm{I\!R} : (x^{i}{}_{j}) \mapsto x^{i}{}_{i}\,.$$

The trace can be defined also for $\mathcal{M}_{n}{}^{n}$.
We define the *(square) matrix product* to be the linear map

$$\mathcal{M}^{n}{}_{m} \times \mathcal{M}^{m}{}_{p} \to \mathcal{M}^{n}{}_{p} : ((x^{i}{}_{j}), (y^{h}{}_{k})) \mapsto (x^{i}{}_{j} y^{j}{}_{k})\,.$$

which is the composition of the (generalised) matrix product with the contraction C^{2}_{1}.
The above map coincides with the standard matrix product. $\square$

One can do an analogous construction also for $\mathcal{M}_{n}{}^{m}$.

1.2.11 Note. In the case $m = n$, the above map endows $\mathcal{M}^{n}{}_{n}$ with the structure of an associative algebra with the unit matrix (δ^{i}_{j}).

We denote by

$$\mathcal{I}^n{}_n \subset \mathcal{M}^n{}_n$$

the subset consisting of invertible elements, i.e. the subset of matrices $(x^{i'}{}_j) \in \mathcal{M}^n{}_n$ such that there exists a matrix $x^{-1} \equiv (x^i{}_{j'}) \in \mathcal{M}^n{}_n$ which fulfills

$$x^{i'}{}_j \, x^j{}_{j'} = \delta^{i'}{}_{j'} \qquad \text{and} \qquad x^i{}_{j'} \, x^{j'}{}_j = \delta^i_j \, . \square$$

1.2.12 Note. One can show that, for a given matrix, the maximal number of independent columns is equal to the maximal number of independent rows. Such a number is defined to be the *rank* of the matrix.

It can be shown that a matrix in $\mathcal{M}^n{}_n$ is invertible if and only if it is of rank n. The subset $\mathcal{I}^n{}_n$ is endowed, by the restriction of matrix product, with a group structure. $\square$

1.2.3 Symmetric and antisymmetric matrices

First of all, we consider the vector space $\mathcal{M}_{nn}$.

1.2.13 Definition. We define the map

$$\varsigma : \underline{n}^r \times \underline{n}^r \to \mathrm{I\!R} : ((a_1, \ldots, a_r), (b_1, \ldots, b_r)) \mapsto \varsigma^{a_1 \ldots a_r}_{b_1 \ldots b_r} := \sum_{\sigma \in \mathfrak{S}_r} \delta^{a_1 \ldots a_r}_{b_{\sigma(1)} \ldots b_{\sigma(r)}} \, . \square$$

1.2.14 Note. In particular, let us consider the cases $r = 1, 2$.
If $r = 1$, then $\varsigma^i_j = \sigma^i_j$.
If $r = 2$, then

$$\varsigma^{ij}_{hk} = \begin{cases} 1 & \text{if } (i,j) = (h,k) \text{ or } (i,j) = (k,h) \\ 0 & \text{if } (i,j) \neq (h,k) \text{ and } (i,j) \neq (k,h) \, . \square \end{cases}$$

1.2.15 Definition. We define the map

$$\epsilon : \underline{n}^r \times \underline{n}^r \to \mathrm{I\!R} : ((a_1, \ldots, a_r), (b_1, \ldots, b_r)) \mapsto \epsilon^{a_1 \ldots a_r}_{b_1 \ldots b_r} := \sum_{\sigma \in \mathfrak{S}_r} |\sigma| \delta^{a_1 \ldots a_r}_{b_{\sigma(1)} \ldots b_{\sigma(r)}} \, . \square$$

1.2.16 Note. In particular, let us consider the cases $r = 1, 2$ and $r = n$.
If $r = 1$, then $\epsilon^i_j = \sigma^i_j$.
If $r = 2$, then

$$\varsigma^{ij}_{hk} = \begin{cases} 1 & \text{if } (i,j) = (h,k) \\ -1 & \text{if } (i,j) = (k,h) \\ 0 & \text{if } (i,j) \neq (h,k) \text{ and } (i,j) \neq (k,h) \, . \end{cases}$$

If $r = n$, then we set $\epsilon_{i_1 \ldots i_n} := \epsilon^{1 \ldots n}_{i_1 \ldots i_n}$. $\square$.

1.2.17 Definition. We define a matrix $(x^{ij}) \in \mathcal{M}_{nn}$ to be *symmetric*, or *antisymmetric*, if, respectively,

$$x^{ij} = x^{ji}\,, \qquad \text{or} \qquad x^{ij} = -x^{ji}\,.\,\square$$

1.2.18 Note. The subsets of symmetric and antisymmetric matrices

$$\mathcal{S}_{nn} \subset \mathcal{M}_{nn} \qquad \text{and} \qquad \mathcal{A}_{nn} \subset \mathcal{M}_{nn}$$

are vector subspaces.

A basis of $\mathcal{S}_{nn}$ and a basis of $\mathcal{A}_{nn}$ are provided, respectively, by the families of matrices

$$\mathfrak{S}_{ij} = (\varsigma^{hk}_{ij})\,, \qquad 1 \le i \le j \le n\,,$$
$$\mathfrak{A}_{ij} = (\epsilon^{hk}_{ij})\,, \qquad 1 \le i < j \le n\,.$$

Thus, we have

$$\dim \mathcal{S}_{nn} = \tfrac{1}{2}\,(n^2 + n) \qquad \text{and} \qquad \dim \mathcal{A}_{nn} = \tfrac{1}{2}\,(n^2 - n)\,.$$

We have the splitting

$$\mathcal{M}_{nn} = \mathcal{S}_{nn} \oplus \mathcal{A}_{nn}\,.$$

In fact, we have the linear maps

$$S : \mathcal{M}_{nn} \to \mathcal{S}_{nn} : (x_{ij}) \mapsto S(x_{ij}) \qquad \text{and} \qquad A : \mathcal{M}_{nn} \to \mathcal{A}_{nn} : (x_{ij}) \mapsto A(x_{ij})\,,$$

where

$$S(x_{ij}) := \tfrac{1}{2}\,(x_{ij} + x_{ji})\,, \qquad \text{and} \qquad A(x_{ij}) := \tfrac{1}{2}\,(x_{ij} - x_{ji})\,.$$

Hence,

$$S(x_{ij}) + A(x_{ij}) = (x_{ij})\,.\,\square$$

Now, we generalise the definition of symmetric and antisymmetric square matrices to matrices in $\mathcal{M}_{n\dots n}$, with r covariant indices.

1.2.19 Definition. We say a matrix $(x_{i_1 \dots i_r}) \in \mathcal{M}_{n\dots n}$ to be

1. *symmetric* if, for any $\sigma \in \mathfrak{S}_r$, we have

$$x_{\sigma(i_1)\dots\sigma(i_r)} = x_{i_1\dots i_r}\,;$$

2. *antisymmetric* if, for any $\sigma \in \mathfrak{S}_r$, we have

$$x_{\sigma(i_1)\dots\sigma(i_r)} = |\sigma|\, x_{i_1\dots i_r}\,.\,\square$$

1.2.20 Note. The subsets of symmetric and antisymmetric matrices

$$\mathcal{S}_{n\dots n} \subset \mathcal{M}_{n\dots n} \qquad \text{and} \qquad \mathcal{A}_{n\dots n} \subset \mathcal{M}_{n\dots n}$$

are vector subspaces.

A basis of $\mathcal{S}_{n\dots n}$ and a basis of $\mathcal{A}_{n\dots n}$ are provided, respecctively, by the families of matrices

$$\begin{aligned} \mathfrak{S}_{i_1\dots i_r} &= (\varsigma^{h_1\dots h_r}_{i_1\dots i_r})\,, \qquad 1 \le i_1 \le \dots \le i_r \le n\,, \\ \mathfrak{A}_{i_1\dots i_r} &= (\epsilon^{h_1\dots h_r}_{i_1\dots i_r})\,, \qquad 1 \le i_1 < \dots < i_r \le n\,. \end{aligned}$$

Thus,

$$\dim \mathcal{S}_{n\dots n} = \binom{n+r-1}{r} \qquad \text{and} \qquad \dim \mathcal{A}_{n\dots n} = \binom{n}{r}\,.$$

Hence,
1) $\dim \mathcal{S}_{n\dots n}$ is an increasing sequence in r converging to $+\infty$;
2) $\dim \mathcal{A}_{n\dots n} = 1$, for $r = n$, and $\dim \mathcal{A}_{n\dots n} = 0$, for $r > n$.
Moreover, for $r = n$, we have

$$\mathfrak{A}_{1\dots n} = \epsilon_{i_1\dots i_n}\, \mathfrak{M}^{i_1\dots i_n}\,.\square$$

An analogous construction can be done for the space $\mathcal{M}^{n\dots n}$.

1.2.21 Remark. We stress that the splitting of $\mathcal{M}_{n\dots n}$ into the direct sum of symmetric and antisymmetric matrices holds only in the case $r = 2$. □

1.2.4 Determinant

We give the definition of the determinant of a square matrices.

1.2.22 Note. We identify $\mathcal{M}^n{}_n$ with $\mathcal{M}^n \times \dots \times \mathcal{M}^n$ by means of the following natural isomorphism

$$\mathcal{M}^n{}_n \to \mathcal{M}^n \times \dots \times \mathcal{M}^n : (x^i{}_j) \mapsto (x^i{}_1, \dots, x^i{}_n)\,.$$

The matrix $\mathfrak{A}_{1\dots n} = (\epsilon_{i_1\dots i_n})$ yields the antisymmetric n–linear map

$$\mathcal{M}^n \times \dots \times \mathcal{M}^n \to \mathrm{I\!R} : (x^{i_1}, \dots, x^{i_n}) \mapsto \epsilon_{i_1\dots i_n}\, x^{i_1} \cdots x^{i_n}\,.\square$$

1.2.23 Definition. We define the *determinant* to be the antisymmetric n–linear map

$$\det : \mathcal{M}^n{}_n \to \mathrm{I\!R} : (x^i{}_j) \mapsto \epsilon_{i_1\dots i_n}\, x^{i_1}{}_1, \dots, x^{i_n}{}_n \ \square$$

The determinant has the following characterisation.

1.2.24 Proposition. The determinant is the unique antisymmetric n–linear map of the type

$$\det : \mathcal{M}^n{}_n \to \mathrm{I\!R}\,,$$

such that

$$\det(\delta^i_j) = 1\,.\,\square$$

PROOF. The vector space of antisymmetric n–linear maps $\mathcal{M}^n \times \ldots \times \mathcal{M}^n \to \mathrm{I\!R}$ has dimension 1, hence such a map is characterised by its value on an n–tuple of non–zero vectors. QED

1.2.25 Corollary. If $A, B \in \mathcal{M}^n{}_n$, then we have $\det(AB) = \det A\, \det B$.

PROOF. The two antisymmetric n–linear maps

$$\det(A\cdot) : \mathcal{M}^n{}_n \to \mathrm{I\!R} : B \mapsto \det(AB) \qquad \text{and} \qquad \det(A)\det\cdot : \mathcal{M}^n{}_n \to \mathrm{I\!R} : B \mapsto \det(A)\,\det(B)$$

assume the same value on the matrix $B = (\delta^i{}_j))$, as

$$\det(A\,(\delta^i{}_j)) = \det(A)\,\det(\delta^i{}_j)\,.$$

Hence, the two n–linear maps $\det(A\cdot)$ and $\det(A)\det\cdot$ coincide. QED

Analogously, we can define the determinant for elements of $\mathcal{M}_{nn}$, or $\mathcal{M}^{nn}$, or $\mathcal{M}_n{}^n$.

1.2.5 Matrix representations

In this subsection, we show that, for each finite dimensional vector space, the choice of a basis yields an isomorphism with a vector space of matrices. This isomorphism is said to be a *matrix representation* of the vector space.

We show also that a matrix representation of a vector space yields also matrix representations of the dual space, of the space of endomorphisms, and so on.

Let us consider the *finite dimensional* vector spaces V, W, Z, respectively, with dimensions

$$\dim V = n\,, \qquad \dim W = p\,, \qquad \dim Z = q\,.$$

Moreover, we assume the ordered bases

$$(b_i)_{1\le i\le n} \subset V\,, \qquad (c_j)_{1\le j\le p} \subset W\,, \qquad (d_k)_{1\le k\le q} \subset Z\,.$$

1.2.26 Proposition. We have the matrix representation

$$V \to \mathcal{M}^n : v^i\, b_i \mapsto (v^i)\,.$$

PROOF. It is an obvious consequence of Theorem 1.1.17 and Proposition 1.1.19. QED

We stress that a change of the basis, or even a change of the order of a basis, yields a different matrix representation.

A matrix representation of V induced by (b_i) yields a matrix representation of V^*.

1.2.27 Lemma. For $1 \le i \le n$, let us denote by $\beta^i \in V^*$ the unique linear extension of the map
$$\beta^i : (b_j) \to \mathbb{R} : b_j \mapsto \beta^i(b_j) = \delta^i_j \,.$$
Then, (β^i) turns out to be a basis of V^*.

Proof. In fact, any linear map $\alpha : V \to \mathbb{R}$ is uniquely determined by its restriction to the subset $(b_i) \subset V$.
Hence, we can write $\alpha = \alpha(b_i)\,\beta^i$, where the components $\alpha(b_i)$ are uniquely determined. QED

1.2.28 Proposition. We have the matrix representation of V^*
$$V^* \to \mathcal{M}_n : \alpha \mapsto (\alpha_i) \;\equiv\; \big(\alpha(b_i)\big)\,.$$
Hence, $\dim V = \dim V^*$. Therefore, V and V^* are (not naturally) isomorphic. □

1.2.29 Note. If V is finite dimensional, then we have
$$\dim V = \dim V^{**}$$
and a natural linear isomorphism (see Note 1.1.29)
$$V \to V^{**}\,. \square$$

In what follows, we denote by $(\gamma^j)_{1\le j\le p}$ and $(\delta^k)_{1\le k\le q}$ the dual bases of W and Z, respectively.

1.2.30 Proposition. We have the matrix representation
$$L(V,W) \to \mathcal{M}^p{}_n : f \mapsto \big(\gamma^j(f(b_i))\big)\,.$$
Hence, $\dim L(V,W) = \dim V \cdot \dim W$. □

1.2.31 Corollary. Let $f \in L(V,W)$ and $h \in L(W,Z)$. Moreover, suppose that $(f^j{}_i) \in \mathcal{M}^p{}_n$ and $(h^k{}_j) \in \mathcal{M}^q{}_p$ be their matrix representations.
Then the matrix representation of $h \circ f$ is the matrix product of $(h^k{}_j)$ with $(f^j{}_i)$, i.e.
$$((h\circ f)^k{}_i) = (h^k{}_j\, f^j{}_i)\,. \square$$

1.2.32 Note. In particular, we observe that the Kronecker's symbol is the matrix representation of $\mathrm{id}_V \in \mathrm{End}\, V$.
Hence, the matrix representation
$$\mathrm{End}(V) \to \mathcal{M}^n{}_n$$
is an isomorphism of associative algebras with unity, which restricts to a group isomorphism
$$\mathrm{Aut}(V) \to \mathcal{I}^n{}_n\,. \square$$

1.2.33 Definition. We define the *rank* of $f \in L(V, W)$ to be the dimension of the vector subspace $\operatorname{im} f \subset W$. □

1.2.34 Note. The rank of $f \in L(V, W)$ is equal to the rank of the matrix which corresponds to f with respect to the given bases in V and W. □

1.2.35 Note. If $f \in L(V, W)$, then the matrix representation of f induces the linear map

$$\widetilde{f} : \mathcal{M}^n \to \mathcal{M}^p : (v^i) \mapsto (v^i \, f^j{}_i)\,,$$

which is the unique linear map which makes the following diagram commute

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ \downarrow & & \downarrow \\ \mathcal{M}^n & \xrightarrow{(\widetilde{f})} & \mathcal{M}^p \end{array}$$

1.2.36 Note. Let $f \in L(V, W)$. Then, $f^* : W^* \to V^*$ is represented, with respect to the dual bases, by the matrix $(f_i{}^j) \in \mathcal{M}_p{}^n$, which is the transpose of the matrix $(f^j{}_i)$. □

1.2.6 Transitions of matrix representations

Next, we discuss the effect of a change of the basis on the matrix representation of vectors and linear maps.

Let us consider a vector space V with finite dimension $\dim V = n$.
Moreover, let us consider two ordered bases of V and the dual bases of V^*

$$(b_i)_{1 \le i \le n} \quad \text{and} \quad (b'_j)_{1 \le j \le n}\,,$$
$$(\beta^i)_{1 \le i \le n} \quad \text{and} \quad (\beta'^j)_{1 \le j \le n}\,.$$

1.2.37 Note. We can express, in a unique way, the vectors of each of the two bases, and of their dual bases, as linear combination of the vectors of the other basis as follows

$$b'_i = B'^j_i \, b_j \quad \text{and} \quad b_i = B_i{}^j \, b'_j\,,$$
$$\beta'^i = C'^i{}_j \, \beta^j \quad \text{and} \quad \beta^i = C^i{}_j \, \beta'^j\,. \ \square$$

1.2.38 Definition. We say (B'^j_i), $(B_i{}^j)$, $(C'^i{}_j)$ and $(C^i{}_j)$ to be the *transition matrices* between, respectively the bases (b_i) and (b'_i) and the dual bases (β^i) and (β'^i). □

1.2.39 Proposition. The transition matrices are invertible and we have

$$(B_i{}^j)^{-1} = (B'^j_i) = (C^i{}_j) \qquad \text{and} \qquad (C^i{}_j)^{-1} = (C'^i{}_j) = (B_i{}^j)\,. \ \square$$

1.2.40 Corollary. The transition from the basis (b_i) to the basis $(b_{i'})$ induces the following transitions of matrix representations of V and V^*

$$v'^i = v^j \, B_j{}^i \qquad \text{and} \qquad v^i = v'^j \, B'_j{}^i \,,$$
$$\alpha'_i = B'_i{}^j \, \alpha_j \qquad \text{and} \qquad \alpha_i = B_i{}^j \, \alpha'_j \,. \square$$

1.2.41 Remark. Let us consider the matrix representation of $\mathrm{End}(V)$

$$\mathrm{End}(V) \to \mathcal{M}^n{}_n : f \mapsto (f^j{}_i) \,.$$

induced by the basis (b_i) in the domain and the basis (b'_i) in the codomain.

This is an isomorphism of vector spaces, but it is not an isomorphism of associative algebras with unity.

In fact, it does not preserve the product of matrices and the matrix representation of id_V is

$$\mathrm{id}_V \mapsto B_i{}^j \,. \square$$

1.2.42 Note. The following practical rule about change of bases holds for matrix representations of any vector space:

- *contravariant indices* transform by multiplication with the matrix of the change of basis,

- *covariant indices* transform by multiplication with the transpose of the inverse matrix of the change of basis.

This rule shows the advantages coming from our notation, making very easy to evaluate the effect of a change of basis on a matrix representation. □

1.2.7 Matrix representation of bilinear maps

We end this section with the matrix representation of bilinear maps.

Let us consider the *finite dimensional* vector spaces V, W, Z, respectively with dimensions

$$\dim V = n \,, \qquad \dim W = p \,, \qquad \dim Z = q \,.$$

Moreover, we assume the ordered bases and their dual bases

$$(b_i)_{1 \le i \le n} \subset V \,, \qquad (c_j)_{1 \le j \le p} \subset W \,, \qquad (d_k)_{1 \le k \le q} \subset Z \,,$$
$$(\beta^i)_{1 \le i \le n} \subset V^* \,, \qquad (\gamma^j)_{1 \le j \le p} \subset W^* \,, \qquad (\delta^k)_{1 \le k \le q} \subset Z^* \,.$$

Then, the induced basis of the cartesian product vector space $V \times W$ is

$$(b_i, c_j) \qquad 1 \le i \le n \,, \quad 1 \le j \le p \,.$$

Thus, we have $\dim V \times W = \dim V \, \dim W$.

1.2.43 Proposition. We have the matrix representation

$$L^2(V,W;Z) \to \mathcal{M}^q{}_{np} : f \mapsto (f^k{}_{ij}) = \delta^k\big(f(b_i,c_j)\big)\,.$$

Thus, we have $\dim L(V,W;Z) = \dim V\ \dim W\ \dim Z\,.\,\square$

1.2.44 Example. We have the matrix representation

$$L^2(V^*,V;\mathrm{I\!R}) \to \mathcal{M}^n{}_n : f \mapsto (f^j{}_i) := \big(f(\beta^j,b_i)\big)\,.$$

The contraction (see Example 1.1.40) turns out to have the matrix representation

$$(\delta^j_i) = (\langle\beta^j,b_i\rangle) \in \mathcal{M}^n{}_n\,.\,\square$$

1.2.45 Example. We have the matrix representation

$$L^2(V;\mathrm{I\!R}) \to \mathcal{M}_{nn} : f \mapsto (f_{ij}) := \big(f(b_i,b_j)\big)\,.$$

Moreover, f is symmetric (antisymmetric) if and only if

$$f_{ij} = f_{ji} \qquad (f_{ij} = -f_{ji})\,.$$

Hence, we have the matrix representations

$$S^2(V;\mathrm{I\!R}) \to \mathcal{S}_{nn} \qquad \text{and} \qquad A^2(V;\mathrm{I\!R}) \to \mathcal{A}_{nn}\,,$$

and, more generally, we have the matrix representations

$$S^r(V;\mathrm{I\!R}) \to \mathcal{S}_{n\dots n} \qquad \text{and} \qquad A^r(V;\mathrm{I\!R}) \to \mathcal{A}_{n\dots n}\,.\,\square$$

1.3 Tensors

Tensor products are used in mechanics and field theory because they provide a very convenient way of representing linear and multilinear maps between finite dimensional vector spaces. By means of tensor products it is easy to perform operations in a coordinate free way, such that contractions and determinant.

In this section, we give a brief outline of the general construction of tensor products between (possibly infinite dimensional) vector spaces. Then, we show how tensor products yield a way of representing spaces of linear and multilinear maps.

Using this representation we identify several vector spaces of maps which are naturally isomorphic to a single tensor product of vector spaces. Hence, tensor products will be distinguished representatives in any class of naturally isomorphic spaces of linear and multilinear maps.

1.3.1 Tensor product of two vector spaces

In this subsection, we give the definition of tensor product of two vector spaces V, W as a vector space arising naturally from the cartesian product $V \times W$.

Then, we show that the tensor product is uniquely characterised by means of a “universal property”.

Let us consider two vector spaces V, W.
We consider the free vector space generated by $V \times W$ (see Example 1.1.15)

$$\mathrm{Free}(V \times W) := \{f : V \times W \to \mathrm{I\!R}\} := \mathrm{Map}(V \times W, \mathrm{I\!R}) .$$

Moreover, let us consider the natural vector subspace

$$N(V \times W) \subset \mathrm{Free}(V \times W)$$

generated by the elements of the type

$$f(\lambda x_1 + \mu x_2, y) - \lambda f(x_1, y) - \mu f(x_2, y) \quad \text{and} \quad f(x, \lambda y_1 + \mu y_2) - \lambda f(x, y_1) - \mu f(x, y_2) ,$$

for each $x, x_1, x_2 \in V$, $y, y_1, y_2 \in W$ and $\lambda, \mu \in \mathrm{I\!R}$.

1.3.1 Definition. We define the *tensor product vector space* of the vector spaces V and W to be the quotient space

$$V \otimes W := \mathrm{Free}(V \times W) / N(V \times W) .$$

Moreover, we define the *tensor product map* of the vector spaces V and W to be the quotient map

$$\otimes : V \times W \to V \otimes W : (x, y) \mapsto x \otimes y := [(x, y)] . \square$$

1.3.2 Proposition. The following facts hold.

The tensor product $V \otimes W := \mathrm{Free}(V \times W) \big/ N(V \times W)$ turns out to be a vector space.

The tensor product $\otimes : V \times W \to V \otimes W$ turns out to be a bilinear map.

The image $\mathrm{im} \otimes \subset V \otimes W$ of $V \times W$ consists of the distinguished elements, called *decomposable*, of the type $v \otimes w \in V \times W$, with $v \in V$ and $w \in W$.

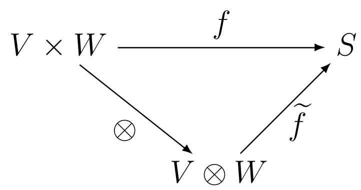
The image $\mathrm{im} \otimes \subset V \otimes W$ of $V \times W$ is not a vector subspace of $V \otimes W$.

The map $\otimes : V \times W \to V \otimes W$ is not surjective.

The tensor product $V \otimes W$ is generated by the image $\mathrm{im} \otimes \subset V \otimes W$ of $V \times W$, i.e., by the set of decomposable elements. □

1.3.3 Theorem. *The tensor product* $(V \otimes W, \otimes)$ *fulfills the following* universal property.

To each vector space S *and to each bilinear map* $f : V \times W \to S$, *there is a unique linear map* $\widetilde{f} : V \otimes W \to S$, *such that the diagram commutes*



Proof. Suppose that S be a vector space and $f : V \times W \to S$ a bilinear map.

Then, we define the linear map $\widetilde{F} : \mathrm{Free}(V \times W) \to S$ to be the unique linear extension of the map

$$F : (V \times W) \to S : x \otimes y \mapsto f(x, y) .$$

The bilinearity of f implies that $\widetilde{F}|_{N(V \times W)} = 0$, hence the map

$$\widetilde{f} : V \otimes W \to S : x \mapsto \widetilde{F}(x)$$

turns out to be well defined and linear.

Suppose that $\widetilde{f}' : V \otimes W \to S$ be another map making the diagram of the statement commutative. Then, for each $(v, w) \in V \times W$, we have $\widetilde{f}(v \otimes w) = \widetilde{f}'(v \otimes w)$.

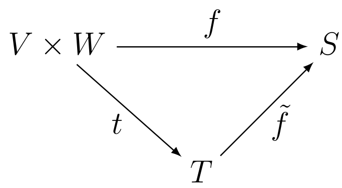
So, f and f' coincide on a set of generators of $V \otimes W$ (i.e. , the set of decomposable elements), hence $\widetilde{f} = \widetilde{f}'$. QED

1.3.4 Corollary. We have the natural mutually inverse isomorphisms

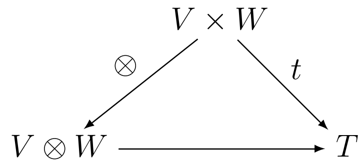
$$\begin{aligned} L^2(V, W; S) &\to L(V \otimes W, S) : f \mapsto \widetilde{f} , \\ L(V \otimes W, S) &\to L^2(V, W; S) \;\; : f \mapsto f \circ \otimes \; . \square \end{aligned}$$

The universal property uniquely characterises the tensor product, as is shown in the following Theorem.

1.3.5 Theorem. *Let* T *be a vector space and* $t : V \times W \to T$ *a bilinear map such that to each vector space* S *and to each bilinear map* $f : V \times W \to S$ *there is a unique* *linear map* $\widetilde{f} : T \to S$ *such that the following diagram commutes*



Then, there is a natural isomorphism between $V \otimes W$ *and* T *such that the following diagram commutes*



Proof. The Theorem follows easily from the universal property of $(V \otimes W, \otimes)$ to (T, t) and *viceversa.* QED

1.3.6 Note. The universal property is the fundamental feature of tensor products, allowing us to pass from bilinear maps to linear maps.

In the following, we use very frequently the universal property, but even if we not give an explicit mention of this.

More precisely, a linear map on a tensor product is well defined by the assignement of its value on decomposable elements in terms of a bilinear map.

Accordingly, given a bilinear map $f : V \times W \to S$, the linear map $\widetilde{f} : V \otimes W \to S$ is well defined by the assignement

$$\widetilde{f} : V \otimes W \to S : x \otimes y \mapsto f(x, y) , \qquad \text{for each} \qquad x \in V , \quad y \in V . \square$$

1.3.7 Example. As an example of application of the universal property, we exhibit the natural isomorphism

$$\mathbb{R} \otimes V \to V : r \otimes v \to r\, v . \square$$

1.3.8 Proposition. Let $v \in V$ and $w \in W$. Then

$$v \otimes w = 0 \qquad \Rightarrow \qquad v = 0 , \quad \text{or} \quad w = 0 .$$

Proof. Let us suppose that $v \neq 0$, $w \neq 0$ and that $v \otimes w = 0$.

Then, the 1st hypothesis implies that there is a bilinear map $f : V \times W \to \mathbb{R}$, such that $f(v, w) \neq 0$, and the 2nd hypothesis implies that $\widetilde{f}(v \otimes w) = 0$.

But, in this way, we obtain a contradiction, because $\widetilde{f}(v \otimes w) = f(v, w)$. QED

1.3.9 Proposition. Let $(b_i)_{i \in I}$ and $(c_j)_{j \in J}$ be bases of V and W, respectively. Then the subset

$$\{b_i \otimes c_j\}_{(i,j) \in I \times J} \subset V \otimes W$$

turns out to be a basis of $V \otimes W$.

If V and W are finite dimensional, then $V \otimes W$ is finite dimensional and

$$\dim V \otimes W = \dim V \, \dim W \, .$$

PROOF. Let us consier any family of real numbers $\{r_{ij}\}_{(i,j) \in I \times J}$.

In virtue of a property of bilinear maps, there is a unique bilinear map $f : V \times W \to \mathrm{I\!R}$, such that $f(v_i, v_j) = r_{ij}$.

Hence, in virtue of the universal property of the tensor product, there is a unique linear map $\widetilde{f} : V \otimes W \to \mathrm{I\!R}$, such that $\widetilde{f}(v_i \otimes v_j) = r_{ij}$.

Therefore, in virtue of a property of linear maps, $\{b_i \otimes c_j\}_{(i,j) \in I \times J}$ turns out to be a basis of $V \otimes W$. QED

1.3.10 Corollary. Let V and W be finite dimensional, and let (b_i) and (c_j) be bases of V and W , respectively. Then, we have the matrix representation

$$V \otimes W \to \mathcal{M}^{nm} : z^{ij} \, b_i \otimes c_j \mapsto (z^{ij}) \, . \square$$

1.3.11 Note. The tensor product could be regarded as a commutative operation in the set of vector spaces. In fact, we have the natural linear isomorphism, called the *tensor transposition*,

$$V \otimes W \to W \otimes V : v \otimes w \to w \otimes v \, . \square$$

1.3.12 Definition. We define the *tensor product* of two linear maps $f : V \to Z$ and $g : W \to T$ to be the linear map

$$f \otimes g : V \otimes W \to Z \otimes T : v \otimes w \mapsto f(v) \otimes g(w) \, . \square$$

1.3.13 Note. If $f : V \to Z$ and $g : W \to T$ are two linear maps, then we have

$$\operatorname{im}(f \otimes g) = \operatorname{im} f \otimes \operatorname{im} g \qquad \text{and} \qquad \ker(f \otimes g) = \ker f \otimes W + V \otimes \ker g \, . \square$$

1.3.14 Note. If V and W split as $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$, then we have

$$V \oplus W = (V_1 \otimes W_1) \oplus (V_1 \otimes W_2) \oplus (V_2 \otimes W_1) \oplus (V_2 \otimes W_2) \, . \square$$

1.3.2 Tensor product of several vector spaces

The tensor product can be easily generalised to several vector spaces.

1.3.15 Note. We can introduce the tensor product of a finite number of vector spaces in two ways.

1. By using multilinear maps, instead of bilinear maps, in the universal property.

2. By induction, according to the following procedure.

There is a natural isomorphism

$$(V_1 \otimes \ldots \otimes V_n) \otimes V_{n+1} \simeq (V_1 \otimes \ldots V_{n-1}) \otimes (V_n \otimes V_{n+1}) :$$
$$(v_1 \otimes \ldots \otimes v_n) \otimes v_{n+1} \simeq (v_1 \otimes \ldots v_{n-1}) \otimes (v_n \otimes v_{n+1}) .$$

This isomorphism suggests to set

$$V_1 \otimes \ldots \otimes V_{n+1} := (V_1 \otimes \ldots \otimes V_n) \otimes V_{n+1} \simeq (V_1 \otimes \ldots V_{n-1}) \otimes (V_n \otimes V_{n+1}) .$$

We can easily prove that the above definitions are equivalent. Hence, tensor product can be regarded as an associative operation in the set of vector spaces.

Indeed, the above induction procedure can be equivalently performed by focusing the associative rule on different factors. □

1.3.16 Corollary. We have the natural mutually inverse isomorphisms

$$L^p(V_1 \times \ldots \times V_n \,; S) \to L(V_1 \otimes \ldots \otimes V_n \,, S) \;: f \mapsto \widetilde{f} ,$$
$$L(V_1 \otimes \ldots \otimes V_n \,, S) \to L^p(V_1 \times \ldots \times V_n \,; S) : f \mapsto f \circ \otimes \;. \square$$

1.3.3 Tensor representations

Here we give the isomorphism between the tensor product of two vector spaces and a certain space of linear maps. In the finite dimensional case, this yields a way for representing any space of linear maps with a tensor product of two vector spaces. This tensor product is the same for any pair of naturally isomorphic spaces of linear maps. Then, we generalise this procedure to spaces of multilinear maps.

Let V and W be two vector spaces.

1.3.17 Theorem. *We have a natural linear injection*

$$R_1 : V^* \otimes W \to L(V, W) : \alpha \otimes w \mapsto f[\alpha \otimes w] ,$$

where we have set

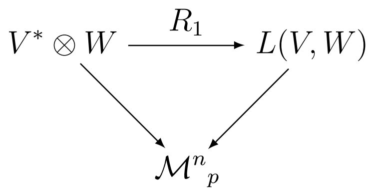
$$f[\alpha \otimes w] : V \to W : v \mapsto \alpha(v) \cdot w .$$

If V and W a re finite dimensional, then R_1 is an isomorphism.

Proof. The injectivity of R_1 comes from $f[\alpha \otimes w] = 0$ if and only if $\alpha = 0$ or $w = 0$.
In the finite dimensional case, the surjectivity comes from the equality

$$\dim(V^* \otimes W) = \dim L(V, W) . \text{QED}$$

1.3.18 Corollary. The matrix representations of $V^* \otimes W$ and $L(V, W)$ are related by the commutative diagram



The choice of the representation of $V^* \otimes W$ is the unique one that makes the diagram commuting. □

1.3.19 Remark. If V or W are not finite dimensional, then the representation R_1 is valued into the subspace of $L(V, W)$ such that its matrix representation (which is a matrix with an infinite number of entries) is a matrix with a finite number of non vanishing entries. □

In the finite dimensional case, the natural isomorphism of remark **??** yields the following result.

1.3.20 Corollary. If V and W are finite dimensional, then

$$V \otimes W \simeq V^{**} \otimes W \simeq L(V^*, W) \,. \square$$

1.3.21 Theorem. *We have the natural linear injection*

$$R_2 : V^* \otimes W^* \to L(V, W; I\!R) : \alpha \otimes \beta \mapsto f[\alpha \otimes \beta] \,,$$

where we have set

$$f[\alpha \otimes \beta] : V \times W \to I\!R : (v, w) \mapsto \alpha(v) \cdot \beta(w) \,.$$

If V and W are finite dimensional, then R_2 is an isomorphism.

Proof. The map R_2 is clearly injective, because $f[\alpha \otimes \beta] = 0$ implies $\alpha = 0$ or $\beta = 0$, but this implies $\alpha \otimes \beta = 0$.

In the finite dimensional case, it is also surjective because

$$\dim V^* \otimes W^* = \dim L(V, W; \mathrm{I\!R}). \text{ QED}$$

1.3.22 Corollary. The natural isomorphism $L(V \otimes W, \mathrm{I\!R}) \simeq L(V, W; \mathrm{I\!R})$ of corollary **??** together with R_2 yields the natural linear injection

$$V^* \otimes W^* \simeq L^2(V, W; \mathrm{I\!R}) \,.$$

If V and W are finite dimensional, then the above map is an isomorphism. □

1.3.23 Remark. It is easy to construct tensor representations for any space of linear and multilinear map by using the above representations R_1 and R_2 as building blocks. For example, if V is finite dimensional, then we have the isomorphisms

$$V^* \otimes V \simeq L(V^*, V^*) \simeq L(V, V) \simeq L(V, V)^* \,,$$
$$V \otimes V^* \otimes V \otimes V \simeq L(L(V, V), L(V^*, V)) \simeq L(L(V, V)^*, L(V, V^*)) \,.$$

This shows how tensor products yield distinguished representatives in the class of naturally isomorphic spaces of linear and multilinear maps. □

Tensor products prove to be very useful also in coordinate computations. In fact, by means of tensor products, it is easy to give the rule of change of basis for the matrix representation of linear maps.

Let (b_i), $(b_{i'})$ be two bases of V, and (c_j), $(c_{j'})$ be two bases of W, (β^i), $(\beta^{i'})$ (γ^j), $(\gamma^{j'})$ be the corresponding dual bases, and $(B^i{}_{i'})$, $(C^j{}_{j'})$ be the matrices of the change of bases, whose inverses are denoted by $(B^{i'}{}_i)$, $(C^{j'}{}_j)$.

1.3.24 Proposition. Let $f \in V^* \otimes W$, with

$$f = f^j{}_i \beta^i \otimes c_j = f^{j'}{}_{i'} \beta^{i'} \otimes c_{j'} \,.$$

Then, we have

$$f^{j'}{}_{i'} = C^{j'}{}_j f^j{}_i B_{i'}{}^i \,,$$

where $(B_{i'}{}^i) = (B^i{}_{i'})^t$.

Proof. Due to Proposition **??** we have

$$f = f^j{}_i \beta^i \otimes c_j = f^j{}_i (B_{i'}{}^i \beta^{i'}) \otimes (C^{j'}{}_j c_{j'}) \,. \text{QED}$$

1.3.4 Tensor algebra

In this section we deal with the spaces of tensors arising from a given vector space V. We introduce the operations on the spaces of tensors, namely contractions and interior products. We stress that the definitions hold for any vector space, and we also give the matrix representations in the finite dimensional case.

If V is finite dimensional, then we choose a basis (b_i) of V, and (β^i) denotes the dual basis.

1.3.25 Definition. We define the *space of r–contravariant and s–covariant tensors* (or the space of tensors of degree (r, s)) of V to be the tensor product of r copies of V and s copies of V^*

$$\otimes^r{}_s V := V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^* \,. \square$$

If V is finite dimensional, then we have the natural isomorphism

$$\otimes^r{}_s V \to L^{r+s}(V^*, \dots, V^*, V, \dots, V; \mathbb{R})$$

and the natural basis

$$(b_{i_1} \otimes \dots \otimes b_{i_r} \otimes \beta^{j_1} \otimes \dots \otimes \beta^{j_s})$$

of $\otimes^r{}_s V$, with $1 \leq i_1, \dots, i_r, j_1, \dots, j_s \leq n$, so $\dim(\otimes^r{}_s V) = n^{r+s}$.

The above basis yields the matrix representation

$$\otimes\, {}^r{}_s V \to \mathcal{M}^{n\dots n}{}_{n\dots n} :$$
$$t^{i_1\dots i_r}{}_{j_1\dots j_s} b_{i_1} \otimes \dots \otimes b_{i_r} \otimes \beta^{j_1} \otimes \dots \otimes \beta^{j_s} \mapsto (t^{i_1\dots i_r}{}_{j_1\dots j_s}) \equiv t^{i_1\dots i_r}{}_{j_1\dots j_s} M_{i_1\dots i_r}{}^{j_1\dots j_s} .$$

1.3.26 Remark. It is possible to introduce tensor spaces with mixed covariant and contravariant indexes, like $V \otimes V^* \otimes V$. This turns out to be very useful in some concrete applications (for example, the matrix representation of the curvature tensor). □

We can perform several operations with tensor spaces. We give only the basic constructions, together with their matrix representations.

We define the *tensor product* to be the bilinear map

$$\otimes^r V \times \otimes^{r'} V \to \otimes^{r+r'} V : (t, t') \mapsto t \otimes t'$$

In the finite dimensional case, if $t \in \otimes^r V$ and $t' \in \otimes^{r'} V$ such that

$$t = t^{i_1\dots i_r} b_{i_1} \otimes \dots \otimes b_{i_r} , \qquad t' = t'^{i_1\dots i_{r'}} b_{i_1} \otimes \dots \otimes b_{i_{r'}} ,$$

then

$$t \otimes t' = t^{i_1\dots i_r} t'^{i_{r+1}\dots i_{r'}} b_{i_1} \otimes \dots \otimes b_{i_r} \otimes b_{i_{r+1}} \otimes \dots \otimes b_{i_{r+r'}} .$$

Hence, the matrix representation of the tensor product is the matrix product of the matrix representations of the tensors.

1.3.27 Remark. The tensor product endows the vector spaces

$$\otimes V := \oplus_{n \in \mathbb{N}} \otimes^n V , \qquad \otimes V^* := \oplus_{n \in \mathbb{N}} \otimes_n V^* ,$$

with the structure of associative graded algebras with unity (see example 1.1.67). □

Next, we define the contractions. The starting point is the trace map.

1.3.28 Definition. We define the *trace* to be the unique linear map $\mathrm{tr} : V^* \otimes V \to \mathbb{R}$ corresponding to the contraction $\langle , \rangle : V^* \times V \to \mathbb{R}$; equivalently,

$$\mathrm{tr} : V^* \otimes V \to \mathbb{R} : (\alpha, v) \mapsto \langle \alpha, v \rangle \equiv \alpha(v) .$$

In the finite dimensional case, we have the matrix representation

$$\operatorname{tr}(f^j{}_i\beta^i \otimes b_j) \mapsto f^i{}_i\,.$$

We generalise the above definition by introducing the contractions.

Let $1 \le h \le r$ and $1 \le k \le s$. We define the *contraction* to be the linear map

$$\begin{aligned} C^h_k : \otimes^r{}_s V \to \otimes^{r-1}{}_{s-1} V : v_1 \otimes \ldots \otimes v_r \otimes \alpha^1 \otimes \ldots \otimes \alpha^s \\ \mapsto \alpha^k(v_h) \cdot v_1 \otimes \ldots \otimes \widehat{v_h} \otimes \ldots \otimes v_r \otimes \alpha^1 \otimes \ldots \otimes \widehat{\alpha^k} \otimes \ldots \otimes \alpha^s\,, \end{aligned}$$

where $\widehat{v_h}$ and $\widehat{\alpha^k}$ are omitted.

Hence, as in the case of tr, the contractions are defined by means of the universal property. In the case $r = s = 1$, we have $C^1_1 = \operatorname{tr}$.

In the finite dimensional case, if $t \in \otimes^r{}_s V$ with

$$t = t^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} b_{i_1} \otimes \ldots \otimes b_{i_r} \otimes \beta^{j_1} \otimes \ldots \otimes \beta^{j_s}\,,$$

then

$$\begin{aligned} C^h_k t = & t^{i_1 \ldots i_{h-1} i i_{h+1} \ldots i_r}{}_{j_1 \ldots j_{k-1} i j_{k+1} \ldots j_s} \\ & b_{i_1} \otimes \ldots b_{i_{h-1}} \otimes \widehat{b_{i_h}} \otimes b_{i_{h+1}} \otimes b_{i_r} \otimes \beta^{j_1} \otimes \ldots \otimes \beta^{j_{k-1}} \otimes \widehat{\beta^{j_k}} \otimes \beta^{j_{k+1}} \otimes \beta^{j_s}\,, \end{aligned}$$

where $\widehat{b_{i_h}}$ and $\widehat{\beta^{j_k}}$ are suppressed.

Hence, the matrix representation of the contraction of a tensor is the contraction of the matrix representation of the tensor.

Let $r \le s$. We define the *interior product* to be the bilinear map

$$\begin{aligned} & \lrcorner : \otimes^r V \times \otimes_s V^* \to \otimes_{s-r} V^* : \\ & (v_1 \otimes \ldots \otimes v_r, \alpha^1 \otimes \ldots \otimes \alpha^s) \mapsto \alpha^1(v_1)\ \ldots \alpha^r(v_r) \otimes \alpha^{r+1} \otimes \ldots \otimes \alpha^s\,. \end{aligned}$$

In the finite dimensional case, if $t \in \otimes^r V$ and $\tau \in \otimes_s V^*$ with

$$t = t^{i_1 \ldots i_r} b_{i_1} \otimes \ldots \otimes b_{i_r}\,, \qquad \tau = \tau_{j_1 \ldots j_s} \beta^{j_1} \otimes \ldots \otimes \beta^{j_s}\,,$$

then

$$t \lrcorner \tau = t^{i_1 \ldots i_r} \tau_{i_1 \ldots i_r j_{r+1} \ldots j_s} \beta^{j_{r+1}} \otimes \ldots \otimes \beta^{j_s}\,.$$

Hence, the matrix representation of the interior product of two tensors t, τ is the product followed by r contractions of the matrix representations of t, τ.

1.3.29 Example. Let $v \in V$. Then the map

$$v \lrcorner : \otimes V^* \to \otimes V^*$$

is a graded derivation of the tensor algebra $\otimes V^*$ of degree -1, i.e. it restricts on elements of degree s to a linear map

$$v \lrcorner : \otimes_s V^* \to \otimes_{s-1} V^*$$

and for any $\alpha \in \otimes_s V^*$, $\beta \in \otimes_{s'} V^*$ we have

$$v \lrcorner (\alpha \otimes \beta) = (v \lrcorner \alpha) \otimes \beta + \alpha \otimes (v \lrcorner \beta) \,. \square$$

1.3.30 Example. In the finite dimensional case, if $f \in L(V, W)$ and $g \in L(W, Z)$, then $g \circ f \in L(V, Z)$ can be obtained via $f \otimes g \in V^* \otimes W \otimes W^* \otimes Z$ by the contraction $C^1_2(f \otimes g)$. $\square$

1.3.5 Antisymmetric tensors

The symmetric tensors that we meet in this book are only metric tensors, for which $r = 2$. On the contrary, we shall deal with a wide variety of antisymmetric tensors. So, in this subsection, we give the definitions of symmetric and antisymmetric tensor, then we describe some basic facts on antisymmetric tensors.

We consider a finite dimensional vector space V, with $\dim V = n$, then we fix a basis (b_i) of V and denote by (β^i) the dual basis of V^*.

1.3.31 Definition. A tensor $t \in \otimes^r V$ is said to be *symmetric* (respectively, *antisymmetric*) if the corresponding r–linear map $t : V^* \times \ldots \times V^* \to \mathbb{R}$ is symmetric (antisymmetric). $\square$

So, $t = t^{i_1 \ldots i_r} b_{i_1} \otimes \ldots \otimes b_{i_r}$ is symmetric (antisymmetric) if and only if for any $\sigma \in \mathfrak{S}_r$ we have

$$t^{\sigma(i_1)\ldots\sigma(i_r)} = t^{i_1\ldots i_r} \qquad (t^{\sigma(i_1)\ldots\sigma(i_r)} = |\sigma| t^{i_1\ldots i_r}) \,.$$

Hence, the matrix representation of a symmetric (antisymmetric) tensor is a symmetric (antisymmetric) matrix in $\mathcal{M}^{n\ldots n}$.

We denote the vector subspace of antisymmetric r-tensors by

$$\overset{r}{\wedge} V \subset \otimes^r V \,.$$

In the finite dimensional case we have the matrix representation $\overset{r}{\wedge} V \to \mathcal{A}^{n\ldots n}$, hence

$$\dim \overset{r}{\wedge} V = \dim \mathcal{A}^{n\ldots n} = \binom{n}{r} \,.$$

The *wedge product* is defined to be the antisymmetrised tensor product

$$V \times \ldots \times V \to \overset{r}{\wedge} V : (v_1, \ldots, v_r) \mapsto v_1 \wedge \ldots \wedge v_r \,,$$

where

$$v_1 \wedge \ldots \wedge v_r := \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} |\sigma| v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(r)} \,.$$

1.3.32 Remark. The set

$$(b_{i_1} \wedge \ldots \wedge b_{i_r})_{1 \le i_1 < \cdots < i_r \le r} \subset \overset{r}{\wedge} V$$

is a basis of $\overset{r}{\wedge} V$; more precisely, it is the basis which corresponds to the basis

$$(A^{i_1 \ldots i_r})_{1 \le i_1 < \cdots < i_r \le r}$$

of $\mathcal{A}^{n \ldots n}$ via the matrix representation of $\overset{r}{\wedge} V$. □

1.3.33 Remark. Each element $w \in \overset{r}{\wedge} V$ can be decomposed in each of the following ways

$$\begin{aligned} w &= \sum_{1 \le i_1, \ldots, i_r \le r} w^{i_1 \ldots i_r} b_{i_1} \otimes \ldots \otimes b_{i_r} \\ &= \sum_{1 \le i_1, \ldots, i_r \le r} w^{i_1 \ldots i_r} b_{i_1} \wedge \ldots \wedge b_{i_r} \\ &= r! \sum_{1 \le i_1 < \cdots < i_r \le r} w^{i_1 \ldots i_r} b_{i_1} \wedge \ldots \wedge b_{i_r} \,. \square \end{aligned}$$

1.3.34 Proposition. There is a unique bilinear map

$$\overset{r}{\wedge} V \times \overset{s}{\wedge} V \to \overset{r+s}{\wedge} V : (w, z) \mapsto w \wedge z \,,$$

such that, for each $v_1, \ldots, v_{r+s} \in V$,

$$(v_1 \wedge \ldots \wedge v_r) \wedge (v_{r+1} \wedge \ldots \wedge v_{r+s}) = v_1 \wedge \ldots \wedge v_{r+s} \,. \square$$

We have the component expression

$$w \wedge z = \frac{1}{(r+s)!} \sum_{\sigma} w^{i_{\sigma(1)} \ldots i_{\sigma(r)}} z^{i_{\sigma(r+1)} \ldots i_{\sigma(r+s)}} b_{i_1} \wedge \ldots \wedge b_{i_{r+s}} \,.$$

The above bilinear map $\wedge$ is said to be the *exterior product.*

1.3.35 Remark. The exterior product endows the vector space

$$\wedge V := \oplus_{n \in \mathbb{N}} \overset{n}{\wedge} V$$

with the structure of an associative graded algebra with unity. We stress that, even if $\wedge V \subset \otimes V$, $\wedge V$ is not a subalgebra of $\otimes V$. Hence, a derivation of $\otimes V$ needs not to restrict to a derivation of $\otimes V$.

The algebra $\wedge V$ is a graded anticommutative algebra, i.e. if $v \in \overset{n}{\wedge} V$ and $v' \in \overset{m}{\wedge} V$ then

$$v \wedge v' = (-1)^{nm} v' \wedge v\,.$$

We observe that, in the finite dimensional case, $\dim \wedge V = 2^n$. □

Let W be a finite–dimensional vector space, and $f : V \to W$ be a linear map. Then we define the *exterior product* $\overset{r}{\wedge} f$ to be the linear map

$$\overset{r}{\wedge} f : \overset{r}{\wedge} V \to \overset{r}{\wedge} W : v_1 \wedge \ldots \wedge v_r \mapsto f(v_1) \wedge \ldots \wedge f(v_r)\,. \tag{1.3.1}$$

If (c_j) is a basis of W, and f has the matrix representation $(f^j{}_i)$, then we have

$$f(t^{i_1 \ldots i_r} b_{i_1} \wedge \ldots \wedge b_{i_r}) = t^{i_1 \ldots i_r} f^{j_1}{}_{i_1} \ldots f^{j_r}{}_{i_r} c_{j_1} \wedge \ldots \wedge c_{j_r}$$

Now, let us consider the dual space V^*. We recall that the restriction of the natural isomorphism

$$(\otimes^r V)^* \simeq \otimes^r V^*$$

to the subspace of antisymmetric tensors yields the natural isomorphism

$$(\overset{r}{\wedge} V)^* \simeq \overset{r}{\wedge} V^* \tag{1.3.2}$$

If we restrict the standard contraction of tensors to antisymmetric tensors, we obtain the following contraction map.

For each $r \geq 1$, we have the bilinear *contraction map*

$$\overset{r}{\wedge} V^* \times V \to \overset{r-1}{\wedge} V^* : (\omega, X) \mapsto X \lrcorner\, \omega\,,$$

characterised by the following formula, for each $\alpha_1, \ldots, \alpha_r \in V^*$,

$$\begin{aligned} X \lrcorner\, (\alpha_1 \wedge \ldots \wedge \alpha_r) &\equiv \frac{1}{r!} X \lrcorner\, \big(\sum_\sigma |\sigma| \alpha_{\sigma(1)} \otimes \ldots \otimes \alpha_{\sigma(r)} \big) \\ &:= \frac{1}{r} \sum_{1 \leq i \leq r} (-1)^{i-1} \alpha_i(X) \alpha_1 \wedge \ldots \wedge \hat{\alpha}_i \wedge \ldots \wedge \alpha_r\,, \end{aligned}$$

where the hat on '$\hat{\alpha}_i$' denotes that we have omitted this factor.

By iteration, we obtain the multilinear map, which is antisymmetric with respect to vectors,

$$\overset{r}{\wedge} V^* \times (V \times \ldots \times V) \to \mathbb{R} : (\omega; X_1, \ldots, X_r) \mapsto X_r \lrcorner \ldots \lrcorner X_1 \lrcorner\, \omega\,,$$

characterised by the following formula, for each $\alpha_1, \ldots, \alpha_r \in V^*$,

$$X_r \lrcorner \ldots \lrcorner X_1 \lrcorner\, (\alpha_1 \wedge \ldots \wedge \alpha_r) = \frac{1}{r!} \det \big(\alpha_i(X_j) \big)\,.$$

This map yields a duality

$$\overset{r}{\wedge} V^* \times \overset{r}{\wedge} V \to \mathbb{R} : (\omega; w) \mapsto \langle \omega, \, w \rangle \,,$$

characterised by the following formula, for each $\alpha_1, \ldots, \alpha_r \in V^*, X_1, \ldots, X_r$,

$$\langle \alpha_1 \wedge \ldots \wedge \alpha_r, \, X_1 \wedge \ldots \wedge X_r \rangle = \frac{1}{r!} \det \big(\alpha_i(X_j)\big) \,.$$

This duality yields the isomorphism $(\overset{r}{\wedge} V)^* \simeq \overset{r}{\wedge} V^*$ (1.3.2).

The above contraction has a disadvantage: it is a derivation of the tensor product (example 1.3.29), but it is no longer a derivation of the exterior product. We are going to give a new kind of contraction which is a graded antiderivation of the exterior product.

For each $1 \leq r$, we define the *interior product* to be the bilinear map

$$\overset{r}{\wedge} V^* \times V \to \overset{r-1}{\wedge} V^* : (\omega, X) \mapsto i_X \omega \,,$$

characterised by the following formula, for each $\alpha_1, \ldots, \alpha_r \in V^*$,

$$i_X(\alpha_1 \wedge \ldots \wedge \alpha_r) := \sum_{1 \leq i \leq r} (-1)^{i-1} \alpha_i(X) \alpha_1 \wedge \ldots \wedge \hat{\alpha}_i \wedge \ldots \wedge \alpha_r \,.$$

By iteration, we obtain the multilinear map, which is antisymmetric with respect to vectors,

$$\overset{r}{\wedge} V^* \times (V \times \ldots \times V) \to \mathbb{R} : (\omega; X_1, \ldots, X_r) \mapsto i_{X_r} \ldots i_{X_1} \omega \,,$$

characterised by the following formula, for each $\alpha_1, \ldots, \alpha_r \in V^*$,

$$i_{X_r} \ldots i_{X_1} (\alpha_1 \wedge \ldots \wedge \alpha_r) = \det \big(\alpha_i(X_j)\big) \,.$$

This map yields a duality

$$\overset{r}{\wedge} V^* \times \overset{r}{\wedge} V \to \mathbb{R} : (\omega; w) \mapsto \langle \omega \,|\, w \rangle \,,$$

characterised by the following formula, for each $\alpha_1, \ldots, \alpha_r \in V^*, X_1, \ldots, X_r$,

$$\langle \alpha_1 \wedge \ldots \wedge \alpha_r \,|\, X_1 \wedge \ldots \wedge X_r \rangle = \det \big(\alpha_i(X_j)\big) \,.$$

This duality yields an isomorphism $(\overset{r}{\wedge} V)^* \simeq \overset{r}{\wedge} V^*$ which is no longer the restriction of the isomorphism $(\otimes^r V)^* \simeq \otimes^r V^*$, but it is more appropriate in several respects, as we are going to see.

1.3.36 Corollary. The interior product differs from the contraction for a factor

$$i_X \omega = r \omega(X) \qquad \langle \omega \,|\, w \rangle = r! \langle \omega, \, w \rangle \,. \square$$

1.3.37 Corollary. The interior product is a graded antiderivation of the exterior product of degree -1, i.e.

$$i_X : \wedge V^* \to \wedge V^*$$

restricts to a linear map

$$i_X : \overset{s}{\wedge} V^* \to \overset{s-1}{\wedge} V^*$$

and, if $X \in V$ and $\alpha \in \overset{s}{\wedge} V^*$, $\beta \in \overset{s'}{\wedge} V^*$, then

$$i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^s \alpha \wedge i_X\beta \,. \square$$

1.3.38 Corollary. In the finite dimensional case, the bases

$$(b_{i_1} \wedge \ldots \wedge b_{i_r})_{1 \le i_1 < \cdots < i_r \le n} \in \overset{r}{\wedge} V$$
$$(\beta^{i_1} \wedge \ldots \wedge \beta^{i_r})_{1 \le i_1 < \cdots < i_r \le n} \in \overset{r}{\wedge} V^*$$

are dual with respect to the contraction $\langle \,|\, \rangle$. $\square$

1.4 Euclidean spaces

Here, we recall some basic facts about Euclidean vector spaces. We describe only the basic facts which will be used throughout the book.

Throughout this section, V will denote a finite dimensional vector space, with $\dim V = n$. We fix a basis (b_i) of V , and denote the dual basis by (β^i) .

1.4.1 Euclidean spaces

1.4.1 Definition. A *metric* on V is defined to be a symmetric positive definite bilinear function

$$g : V \times V \to \mathrm{I\!R} \,.$$

An *Euclidean space* is defined to be a pair (V, g) , in which g is a metric on V . □

A metric can be given, equivalently, as a symmetric tensor

$$g \in V^* \otimes V^* \,,$$

whose associated bilinear map is positive definite.

We recall that $g^\flat : V \to V^*$ is an isomorphism, due to non degeneracy of g and finite dimensionality of V . Hence, we set

$$f^\sharp := f^{\flat -1} : V^* \to V : \alpha \mapsto (f^\flat)^{-1}(\alpha) \,.$$

The above isomorphism $g^\flat$ yields a metric $\overline{g} \in V \otimes V$ on the vector space V^* . More precisely,

$$\overline{g} := g \circ (g^\sharp, g^\sharp) : V^* \times V^* \to \mathrm{I\!R} \,.$$

We have the expressions

$$g = g_{ij} \beta^i \otimes \beta^j \,, \qquad \overline{g} = g^{ij} b_i \otimes b_j$$

where $g^{ij} g_{jk} = \delta^i{}_k$ and $g_{ij} g^{jk} = \delta_i{}^k$.

Accordingly, the matrix representation of $g^\flat$ and $g^\sharp$ are (g_{ij}) , (g^{ij}) , respectively.

We define the *length function* to be the function

$$V \to \mathrm{I\!R} : v \mapsto \|v\| := \sqrt{g^\diamond(v)} \equiv \sqrt{g(v, v)} \,.$$

If $v = v^i b_i$, then $\|v\| = \sqrt{g_{ij} v^i v^j}$.

In the rest of the section we suppose that V is endowed with a Euclidean metric g .

1.4.2 Remark. The metric g induces a metric on $\otimes^r{}_s V$, namely

$$\begin{aligned} \otimes^r{}_s g : \otimes^r{}_s V \times \otimes^r{}_s V &\to \mathrm{I\!R} \\ (t_1 \otimes \ldots \otimes t_r \otimes t^1 \otimes \ldots \otimes t^s, q_1 \otimes \ldots \otimes q_r \otimes q^1 \otimes \ldots \otimes q^s) &\mapsto g(t_1, q_1) \ldots \overline{g}(t^s, q^s) \,. \end{aligned}$$

If

$$t = t^{i_1 \dots i_r}{}_{j_1 \dots j_s} b_{i_1} \otimes \dots \otimes b_{i_r} \otimes \beta^{j_1} \otimes \dots \otimes \beta^{j_s} \,,$$
$$s = s^{h_1 \dots h_r}{}_{k_1 \dots k_s} b_{h_1} \otimes \dots \otimes b_{h_r} \otimes \beta^{k_1} \otimes \dots \otimes \beta^{k_s} \,,$$

then

$$\otimes^r{}_s g(t, s) = g_{i_1 h_1} \dots g_{i_r h_r} g^{j_1 k_1} \dots g^{j_s k_s} t^{i_1 \dots i_r}{}_{j_1 \dots j_s} s^{h_1 \dots h_r}{}_{k_1 \dots k_s} \,.$$

Moreover, the metric $\otimes^r{}_0 g$ induces the metric $\overset{r}{\wedge} g$ on $\overset{r}{\wedge} V$ by restriction. □

Let $W \subset V$ be a subspace. In general, there is no natural choice of a vector subspace $Z \subset V$ such that $V = W \oplus Z$. We are going to show that the metric g yields such a natural choice.

Let $W \subset V$ be a subset, with $l = \dim W > 0$. Then, we define the following subset

$$W^\perp := \{v \in V \mid g(v, w) = 0 \; \forall \, w \in W\} \subset V \,.$$

It turns out that $W^\perp$ is a vector subspace of V. We are going to prove that the following splitting holds

$$V = W \oplus W^\perp \,.$$

We observe that $W \cap W^\perp = 0$.

We denote the inclusions of W and $W^\perp$ respectively by

$$\iota_W : W \hookrightarrow V \,, \qquad \iota_{W^\perp} : W \hookrightarrow V \,,$$

so that ι_W^* and $\iota_{W^\perp}^*$ are surjective in virtue of Note 1.1.28.

For practical reasons, we shall adopt the following convention, which will be necessary for the correct understanding of our formulas:

- indices i, j, h, k will run from 1 to $n = \dim V$;
- indices a, b, c, d will run from 1 to $l = \dim W$;
- indices r, s, t will run from l to 3.

We suppose that the basis (b_i) of V is *semi–adapted* to the splitting of V; namely, we suppose that the subset

$$(b_a)_{1 \le a \le l} \subset (b_i)_{1 \le i \le n}$$

is a basis of W. We note that we do not start, at this stage, with a basis of V which is adapted to both W and $W^\perp$.

We have the following expressions

$$\iota_W(v^a) = (v^a, 0) \,, \qquad \iota_W^*(\alpha^i) = (\alpha^a) \,.$$

1.4.3 Proposition. The vector subspaces $W, W^\perp \subset V$ can be endowed with two metrics induced naturally by g, namely

$$g^\dagger \equiv g \circ (\iota_W, \iota_W) : W \times W \to \mathrm{I\!R}\,,$$
$$g^\perp \equiv g \circ (\iota_{W^\perp}, \iota_{W^\perp}) : W^\perp \times W^\perp \to \mathrm{I\!R}\,,$$

We have the matrix representation

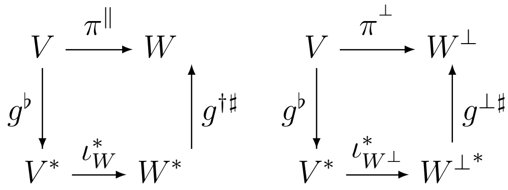
$$g^\dagger = g_{ab} \beta^a \otimes \beta^b\,.$$

We denote by $(g^{\dagger ab})$ the matrix representation of $\overline{g^\dagger} \in W \otimes W$.

We define the maps

$$\pi^\parallel : V \to W \qquad \pi^\perp : V \to W^\perp\,,$$

by requiring the commutativity of the diagrams



1.4.4 Theorem. *We have*

$$\pi^\parallel|_W = \mathrm{id}_W\,, \qquad \pi^\perp|_{W^\perp} = \mathrm{id}_{W^\perp}\,, \qquad \pi^\parallel|_{W^\perp} = 0\,, \qquad \pi^\perp|_W = 0\,,$$
$$\pi^\parallel + \pi^\perp = \mathrm{id}_V\,,$$

hence, by theorem 1.1.61,

$$V = W \oplus W^\perp\,.$$

Proof. In fact, the linear projections are characterised by the following condition. For each $v \in V$, $w \in W$, $w' \in W^\perp$

$$g(v, w) = g\big(\pi^\parallel(v), w\big) \qquad g(v, w') = g\big(\pi^\perp(v), w'\big)\,.\ \text{QED}$$

1.4.5 Corollary. For each $v \in V$ such that $v = v^i b_i$, we have

$$\begin{aligned}\pi^\parallel(v) &= \sum_{1\le i\le n} \sum_{1\le a,b\le l} v^i\, g_{ib}\, g^{\dagger ba}\, b_a \\ &= \sum_{1\le a,b\le l} \sum_{l+1\le r\le n} (v^a + v^r\, g_{rb}\, g^{\dagger ba}) b_a\,, \\ \pi^\perp(v) &= \sum_{1\le a,b\le l} \sum_{l+1\le r\le l} v^r (b_r - g_{rb}\, g^{\dagger ba}\, b_a)\,.\ \square\end{aligned}$$

The above formulas become very simple in the particular case when (b_i) is adapted to the splitting of V $((g_{rb}) = 0)$.

We shall be also involved with the covariant counterpart of the above splitting. We can achieve it in the following way.

Let us consider the subspaces

$$W_\perp := \{\alpha \in V^* \mid \alpha|_W = 0\} \subset V^* , \qquad W_\| := g^\flat(W) \subset V^* .$$

1.4.6 Proposition. We have the direct sum splitting

$$V^* = W_\| \oplus W_\perp . \square \tag{1.4.1}$$

However, we observe that, in several respects, the above splitting is not convenient. So, we introduce an isomorphism of V^* with a cartesian product of two vector spaces.

1.4.7 Theorem. *We have the isomorphism*

$$g^\flat|_{W^\perp} \circ g^{\perp\sharp} : W^{\perp *} \to W_\perp ,$$

which yields, by a composition with $\iota^*_{W^\perp}$, *the linear map*

$$\pi_\perp : V^* \to W_\perp ,$$

and the linear isomorphism

$$(\pi, \pi_\perp) : V^* \to W^* \times W_\perp .$$

Proof. It comes from the isomorphism

$$V^* \simeq (W \oplus W^\perp)^* \simeq W^* \times W^{\perp *} . \text{QED}$$

We stress that nor W^* neither $W^{\perp *}$ are defined as vector subspaces of V^* . For each $\alpha \in V^*$ we have the expression

$$\pi_\perp(\alpha) = \sum_{1\le a,b\le l} \sum_{l+1\le r\le n} (\alpha_r - \alpha_a\, g^{\dagger ab}\, g_{br})\, \beta^r .$$

The above splitting of the dual space is sufficient and appropriate for our needs. This splitting has two important features: namely, its first component is the space of forms which live on the subspace $W \subset V$; moreover, the first projection can be easily computed.

Finally, we show that the metric g yields a distinguished subset of the set of bases on V .

A family $(e_i)_{i\in I}$ of vectors of V is said to be

1. *orthogonal*, if, for any $i, j \in I$ such that $i \neq j$, we have $g(e_i, e_j) = 0$;

2. *orthonormal*, if it is orthogonal and $\|e_i\| = 1$ for all $i \in I$.

1.4.8 Lemma. Let $(e_i)_{i\in I}$ be an orthogonal family of V. Then $(e_i)_{i\in I}$ is an independent set.

PROOF. Suppose that $v = v^i e_i = 0$. Then

$$0 = g(v, x_i) = v^i \,. \text{QED}$$

So, if $(e_i)_{i\in I}$ is an orthogonal family of V, then the cardinality of I is less than or equal to $\dim V$.

1.4.9 Proposition. There exists an orthonormal basis (e_i) of V.

PROOF. We pick $v \in V$ and set $e_1 := v/\|v\|$. Then, one can repeat this procedure on $\operatorname{span}(e_1)^{\perp}$, obtaining an orthonormal family (e_i) whose cardinality is $\dim V$. QED

1.4.2 Orthogonal maps

1.4.10 Definition. Let (V, g) be a Euclidean space. A map $f : V \to V$ is said to be

1. *orthogonal* (with respect to g) if

$$g(f(v), f(w)) = g(v, w) \qquad \forall v, w \in V\,,$$

2. *length preserving* (with respect to g) if:

$$\|f(v)\| = \|v\| \qquad \forall\, v \in V\,. \square$$

1.4.11 Remark. Carnot's Theorem implies that a map is orthogonal if and only if it is length preserving. □

In order to show the main properties of orthogonal maps, we introduce the transposition operator.

1.4.12 Lemma. Let (V, g) be a Euclidean space, and $f : V \to V$ a linear map. Then, there exists a unique linear map $f^t : V \to V$ such that

$$g(f(v), w) = g(v, f^t(w)) \qquad \forall\, v, w \in V\,.$$

PROOF. In fact, if we define the linear endomorphism $\cdot^t : \operatorname{End}(V) \to \operatorname{End}(V)$ as the unique map making the following diagram commute

$$\begin{array}{ccccc}
V^* \otimes V & \xrightarrow{g^\sharp \otimes g^\flat} & V \otimes V^* & \xrightarrow{\simeq} & V^* \otimes V \\
\simeq\big\downarrow & & & & \big\downarrow\simeq \\
\operatorname{End}(V) & & \xrightarrow{\quad \cdot^t \quad} & & \operatorname{End}(V)
\end{array}$$

where $\simeq$ is the natural isomorphism of Theorem 1.3.17, then the result is obtained by setting $f^t := \cdot^t (f)$. QED

If $(f^i{}_j)$ is the matrix representation of f, then we have

$$(f^t)^i{}_j = g_{hj} f^h{}_k g^{ik} .$$

We say the above map f^t to be the *transpose map* of f (with respect to g). We have

$$(f^t)^t = f , \qquad (f \circ g)^t = g^t \circ f^t \qquad \mathrm{id}_V^t = \mathrm{id}_V .$$

1.4.13 Remark. There is an analogy between the transpose map and the dual map. In fact, if W is a vector space and $f \in L(V, W)$, then $f^* \in L(W^*, V^*)$ is the unique map such that, for each $\alpha \in W^*$ and $v \in V$

$$\langle f^*(\alpha), v\rangle = \langle \alpha, f(v)\rangle . \square$$

1.4.14 Lemma. Let (V, g) be a Euclidean space, and $f : V \to V$ an orthogonal map. Then, f is linear.

PROOF. Let (e_i) be an orthonormal basis of V. Then, being f orthogonal, $(f(e_i))$ is an orthonormal set, hence an orthonormal basis (proposition 1.4.9). Moreover, for any $v \in V$ we have

$$f(v) = f(v)^i f(e_i) = \sum_i g(f(v), f(e_i)) f(e_i) = \sum_i g(v, e_i) f(e_i) = v^i f(e_i) ,$$

hence f is linear. QED

1.4.15 Proposition. Let (V, g) be a Euclidean space, and $f : V \to V$. The following facts are equivalent.

1. The map f is orthogonal.
2. The map f is linear and invertible, and $f^{-1} = f^t$. $\square$

The set of orthogonal maps $f : V \to V$ with respect to g is denoted by $O(V, g)$.

1.4.16 Corollary. The set $O(V, g)$ is a subgroup

$$O(V, g) \subset \mathrm{Aut}(V) . \square$$

1.5 Volume forms

In this section, we deal with antisymmetric tensors of degree n of a fixed vector space V, with $\dim V = n$. This space is quite important because it is 1-dimensional. We also fix a basis (b_i) of V.

1.5.1 Volume forms

We have

$$\dim \overset{n}{\wedge} V = 1\,.$$

The elements

$$b_1 \wedge \ldots \wedge b_n \in \overset{n}{\wedge} V \qquad \beta^1 \wedge \ldots \wedge \beta^n \in \overset{n}{\wedge} V^*$$

are dual bases with respect to the contraction $\langle\,|\,\rangle$.

The component expression of $\omega \in \overset{n}{\wedge} V^*$ is

$$\omega = \sum_{1<i_1,\ldots,i_n<n} \omega_{i_1\ldots i_n} \beta^{i_1} \wedge \ldots \wedge \beta^{i_n} = n! \omega_{1\ldots n} \beta^1 \wedge \ldots \wedge \beta^n\,,$$

with

$$\omega_{i_1\ldots i_n} = \frac{1}{n!} \langle \omega \,|\, b_{i_1} \wedge \ldots \wedge b_{i_1} \rangle\,.$$

Moreover, we can write

$$\omega_{i_1\ldots i_n} = \epsilon_{i_1\ldots i_n}\, \omega_{1\ldots n}\,.$$

If $w \in \overset{n}{\wedge} V$ is non vanishing, then it is a basis of $\overset{n}{\wedge} V$. Moreover, there is a unique element $\omega \in \overset{n}{\wedge} V^*$, such that

$$\langle \omega \,|\, w \rangle = 1\,.$$

This ω is just the dual basis of w, with respect to the duality $\langle\,|\,\rangle$.

1.5.1 Definition. We say $w \in (\overset{n}{\wedge} V \setminus \{0\})$, or equivalently, the dual element $\omega \in (\overset{n}{\wedge} V \setminus \{0\})$, to be a *volume form.* □

1.5.2 Orientation

We denote the set of ordered bases of V by B. If $(b_{i'}) \in \mathsf{B}$, then we denote with $(B^i{}_{i'})$ the matrix of the change of basis.

We can define the following equivalence relation in B

$$(b'_1, \ldots, b'_n) \sim (b_1, \ldots, b_n) \qquad \Longleftrightarrow \qquad \det(B^i{}_{i'}) > 0\,.$$

Clearly, there are exactly two equivalence classes; two elements of B belong to the same class of to different classes according with the positive or negative sign of the associated $\det(B^i{}_{i'})$.

1.5.2 Definition. We say each one of the two equivalence classes of the above relation to be an *orientation* of V . □

1.5.3 Note. The choice of an orientation is equivalent to the choice of one connected component of the one–dimensional vector space $\overset{n}{\wedge} V$.

In fact, if $(b_1, \dots, b_n)$, $(b'_1, \dots, b'_n) \in \mathsf{B}$, then

$$b'_1 \wedge \dots \wedge b'_n = \det(B^i{}_{i'}) b_1 \wedge \dots \wedge b_n \,.$$

Moreover, $b'_1 \wedge \dots \wedge b'_n$ and $b_1 \wedge \dots \wedge b_n$ belong to the same connected component if and only if $\det(B^i{}_{i'}) > 0$.

In virtue of the duality, the choice of an orientation of V determines a choice of the orientation of V^* , and conversely. □

A map $f \in \mathrm{End}(V)$ is said to be either *orientation preserving*, or *orientation reversing* if either it preserves the equivalence classes of B , or it sends each equivalence class onto the other.

Let us fix an orientation of V and assume that V is equipped with a Euclidean metric g . We have the metric $\overset{n}{\wedge}\overline{g}$ on $\overset{n}{\wedge} V^*$ which is induced by g (remark 1.4.2).

There are only two forms $\nu \in \overset{n}{\wedge} V^*$ such that

$$\overset{n}{\wedge}\overline{g}(\nu, \nu) = 1 \,;$$

they differ by the sign.

Hence, there is only one oriented form ν fulfilling the above equation. If (b_i) is oriented and (β^i) is the dual basis, then

$$\nu = \sqrt{\det(g_{ij})}\, \beta^1 \wedge \dots \wedge \beta^n \,,$$

hence, if (e_i) is oriented and orthonormal, and (ϵ^i) is the dual basis,

$$\eta = \epsilon^1 \wedge \dots \wedge \epsilon^n \,.$$

1.5.4 Definition. We say ν to be the *unitary volume form.* □

1.5.3 Determinant

We introduce the determinant $\det(f)$ of a map $f \in \mathrm{End}(V)$.

Suppose that a vector space W has dimension 1 . Then, the vector space $W^* \otimes W$ has dimension 1 . Moreover, it has the natural basis $\mathrm{id}_W \in W^* \otimes W$. Hence, we have the natural isomorphism

$$W^* \otimes W \to \mathbb{R} \,.$$

It is easy to realise that the above isomorphism coincides with tr .

1.5.5 Definition. We define the *determinant* to be the (non linear) map

$$\det : \mathrm{End}(V) \to \mathbb{R} : f \mapsto \mathrm{tr}(\overset{n}{\wedge} f)\,.\square$$

If $f \in \mathrm{End}(V)$ and $(f^j{}_i)$ is the matrix representation of f , then we have

$$\overset{n}{\wedge}(f)(b_1 \wedge \ldots \wedge b_n) = \det(f^j{}_i)\, b_1 \wedge \ldots \wedge b_n\,,$$

where $\det(f^j{}_i)$ has been defined for matrices of $\mathcal{M}^n{}_n$, hence

$$\det(f) = \det(f^j{}_i)\,.$$

By the way, we observe that, for each $f, g \in \mathrm{End}(V)$, we have

$$\det(f \circ g) = \det(f) \cdot \det(g)\,, \qquad \det(\mathrm{id}_V) = 1 \qquad \det(f^*) = \det(f)\,.$$

1.5.6 Lemma. We have the equality

$$\mathrm{Aut}(V) = \{f \in \mathrm{End}(V) \mid \det(f) \neq 0\}\,.$$

Proof. It follows from

$$\det(f \circ f^{-1}) = \det(f) \cdot \det(f^{-1}) = \det(\mathrm{id}_V) = 1\,,$$

and from the definition of determinant. QED

Let us consider the subset

$$\mathrm{SAut}(V) := \{f \in \mathrm{Aut}(V) \mid \det(f) > 0\}\,.$$

It turns out that $\mathrm{SAut}(V) \subset \mathrm{Aut}(V)$ is a subgroup. We say $\mathrm{SAut}(V)$ to be the group of *special automorphisms* of V .

1.5.7 Lemma. If V is oriented, then the set of orientation preserving maps of V is the group $\mathrm{SAut}(V)$. $\square$

From now on, in the rest of the section, we suppose that (V, g) is a Euclidean space.

1.5.8 Lemma. Let $f \in O(V, g)$. Then we have $\det(f) = \det(f^t) = \pm 1$.

Proof. In fact, if $f \in O(V, g)$, then $\overset{n}{\wedge}(f)$ sends the dual of the unitary volume form η into the dual of $\pm\eta$, because f preserves the orientation or reverses the orientation. The fact that $\det(f) = \det(f^t)$ comes from the fact that $f^t \in O(V, g)$. $\square$

We define the group of *special orthogonal maps* to be the subgroup

$$SO(V, g) := O(V, g) \cap \mathrm{SAut}(V)\,;$$

hence, $SO(V, g)$ is the subgroup of $O(V, g)$ whose element have positive determinant.

1.5.4 Hodge's isomorphism

We stress that

$$\dim \overset{r}{\wedge} V = \dim \overset{n-r}{\wedge} V \,.$$

1.5.9 Remark. We define the *Hodge* map to be the linear isomorphism

$$* : \overset{r}{\wedge} V^* \to \overset{n-r}{\wedge} V^* : \omega \mapsto i_{g^\sharp(\omega)} \eta \,.$$

In particular, we have

$$* : \overset{n}{\wedge} V^* \to \mathbb{R}$$
$$* : \overset{n-1}{\wedge} V^* \to V^*$$
$$* : V^* \to \overset{n-1}{\wedge} V^* \,.$$

For

$$\psi \in \overset{n}{\wedge} V^* \qquad \omega \in \overset{n-1}{\wedge} V^* \qquad \alpha \in V^* \,,$$

we have the expressions

$$*\psi = n! \sqrt{\det(g_{ij})}\, \psi^{1\dots n}$$
$$*\omega = (n-1)!(-1)^{n-i} \sqrt{\det(g_{ij})} \sum_{1 \le i \le n} \omega^{1\dots\hat{i}\dots n} \beta^i$$
$$*\alpha = (-1)^{i-1} \sqrt{\det(g_{ij})} \sum_{1 \le i \le n} \alpha^i \beta^1 \wedge \dots \hat{\beta}^i \wedge \dots \wedge \beta^n \,. \square$$

If we exchange the role of V and V^*, then we obtain analogous maps, which will be denoted by the same symbol $*$.

1.5.10 Remark. By means of a positively oriented orthonormal basis, we can easily prove the following properties of $*$, for each $\omega \in \overset{r}{\wedge} V^*, \psi \in \overset{s}{\wedge} V^*$,

$$* * \omega = (-1)^{r(n-r)} \omega$$
$$(*\omega) \wedge \psi = (-1)^{s(n+1)} * \big(i_{g^\sharp(\psi)} \omega\big) \qquad s \le r$$
$$*(\omega \wedge \psi) = i_{g^\sharp(\psi)} * \omega \qquad s \le r \,. \square$$

1.5.5 Cross product

Now, we specialise the above results to the case when $\dim V = 3$.

Thus, let us consider a 3-dimensional vector space V equipped with a Euclidean metric g and let us choose an orientation.

Let

$$(b_1, b_2, b_3)\,, \quad (\beta^1, \beta^2, \beta^3) \quad \text{and} \quad (e_1, e_2, e_3)\,, \quad (\epsilon^1, \epsilon^2, \epsilon^3)$$

be a positively oriented basis and its dual, a positively oriented orthonormal basis and its dual, respectively.

1.5.11 Remark. The expression of the unitary volume form is

$$\eta = \sqrt{\det(g_{ij})}\,\beta^1 \wedge \ldots \wedge \beta^n = \epsilon^1 \wedge \ldots \wedge \epsilon^n\,.$$

For

$$\psi \in \overset{3}{\wedge} V^* \qquad \omega \in \overset{2}{\wedge} V^* \qquad \alpha \in V^*\,,$$

we have the expressions

$$*\psi = 3!\sqrt{\det(g_{ij})}\psi^{123}$$
$$*\omega = 2!\sqrt{\det(g_{ij})}(\omega^{12}\beta^3 + \omega^{31}\beta^2 + \omega^{23}\beta^1)$$
$$*\alpha = \sqrt{\det(g_{ij})}(\alpha^1\beta^2 \wedge \beta^3 + \alpha^3\beta^1 \wedge \beta^2 + \alpha^2\beta^3 \wedge \beta^1)\,.\square$$

1.5.12 Definition. The *cross product* is defined to be the bilinear map

$$V \times V \to V : (X, Y) \mapsto X \times Y \equiv g^\sharp\big(i_{X\wedge Y}\eta\big)\,.\square$$

1.5.13 Remark. In a positively oriented orthonormal basis, we have the following component expression, for each $X, Y \in V$,

$$X \times Y = (X^1Y^2 - X^2Y^1)e_3 + (X^3Y^1 - X^1Y^3)e_2 + (X^2Y^3 - X^3Y^2)e_1\,.\square$$

The properties of the $*$ isomorphism (see) yield the following properties of the cross product.

1.5.14 Remark. The cross product is characterised by the following properties.
For each $X, Y \in V$,

$$\|X \times Y\|^2 = \|X\|^2\|Y\|^2 - g(X, Y)^2\,,$$
$$g(X \times Y,\, X) = 0 = g(Y \times Y,\, X)\,,$$

and, if $X, Y \in V$ are independent, then $(X, Y, X \times Y)$ is positively oriented.
It follows that, if $X, Y \in V$ are independent, then

$$X \wedge Y \wedge (X \times Y) \in \overset{3}{\wedge} V^*$$

is non vanishing and positively oriented. □

1.5.15 Remark. For each $X, Y, Z \in V$, we have

$$g(X \times Y,\, Z) = \langle \eta \,|\, X \wedge Y \wedge Z \rangle$$
$$(X \times Y) \times Z = g(X,\, Z) Y - g(Y,\, Z) X \,. \square$$

1.5.16 Remark. The cross product makes the vector space V a Lie algebra. Namely, we have the following properties.

For each $X, Y, Z \in V, k \in \mathbb{R}$,

$$(X + Y) \times Z = X \times Z + Y \times Z \qquad X \times (Y + Z) = X \times Y + X \times Z$$
$$(kX) \times Y = k(X \times Y) = X \times (kY)$$
$$X \times Y = -Y \times X$$
$$(X \times Y) \times Z + (Z \times X) \times Y + (Y \times Z) \times X = 0. \square$$

1.5.17 Remark. If $\omega \in \overset{2}{\wedge} V^*$, then there is a unique $\Omega \in V$, such that, for each $X \in V$, we have

$$\omega(X) = \Omega \times X \,.$$

Namely, such an Ω is given by

$$\Omega = \frac{1}{2} i_\omega \bar{\eta} \,,$$

where $\bar{\eta} \in \overset{3}{\wedge} V$ is the dual basis of $\eta \in \overset{3}{\wedge} V^*$.

Hence, we have the component expression

$$\Omega = \frac{1}{\sqrt{\det(g_{ij})}} (\omega_{23} b_1 + \omega_{12} b_3 + \omega_{31} b_2) \,. \square$$

1.6 Affine spaces

Affine spaces are important for classical mechanics because they offer a geometrical model of the basic configuration spaces.

Even more, we observe that affine spaces constitute the appropriate framework carrying the primitive differential analysis, which is extended to manifolds later.

We will introduce the standard definition of affine space associated with a vector space in a form which will allow a straightforward generalisation. Indeed, only the group properties of the vector space are involved in such a definition. Hence, we can abstract this definition to a definition of affine space associated with a (possibly non abelian) group.

We will see that the configuration space of a rigid system has a natural structure of affine space associated with a group. This has very interesting consequences on the equations of motion of the system. This is one of the new features of our approach to rigid systems.

1.6.1 Action of a group on a set

Let G be a group and S a set.

A *right action* of G on S is defined to be a map

$$a : S \times G \to S : (s, g) \mapsto sg\,,$$

such that, for each $g, g' \in G$, $s \in S$,

$$s(gg') = (sg)g' \qquad s1_G = s\,.$$

A *left action* of G on S is defined to be a map

$$a : G \times S \to S : (g, s) \mapsto gs\,,$$

such that, for each $g, g' \in G$, $s \in S$,

$$(g'g)s = g'(gs) \qquad 1_G s = s\,.$$

Of course, the maps $(s, g) \mapsto sg$ and $(g, s) \mapsto gs$ should not be confused, in general, with the operation in G.

Moreover, a right (left) action is said to be

1. *free* if no $g \in G$, except 0_G, has fixed points,
2. *transitive* if, for each $s, s' \in S$ there is a $g \in G$ such that $s' = sg$ ($s' = gs$).

1.6.1 Remark. If a right (left) action is free and transitive, then for $s, s' \in S$ the element $g \in G$ such that $s' = sg$ ($s' = gs$) is unique. We denote this element by

$$g = s^{-1}s' \qquad (g = s's^{-1})\,.$$

This abuse of notation is quite useful and does not create any inconsistency. In fact, it well cooperates with the standard notation of groups, according to the formulas

$$s' = s(s^{-1}s') \qquad (s^{-1}s')(s'^{-1}s'') = s^{-1}s''$$

(and analogously for a left action). However, we stress that the symbol $s^{-1}s'$ ($s's^{-1}$) must be taken as a whole and s^{-1} has no meaning alone.

Thus, if the right action is free and transitive, then the choice of an 'origin' $o \in S$ yields the bijection

$$a_o : G \to S : g \mapsto og$$

and the inverse map

$$\delta_o : S \to G : s \mapsto o^{-1}s$$

(and analogously for a left action). □

1.6.2 Remark. Let a be a transitive action. If $s \in S$, then the set

$$H_s := \{h \in G \mid a(s,h) = s\} \subset G$$

is a subgroup of G.

We say H_m to be the *isotropy subgroup* of the action a of G at m. □

1.6.2 Affine spaces associated with vector spaces

Here, following the standard convention, we will consider right actions of a vector space (as an abelian group) on a set. Hence, we will adopt the additive notation for the group operation and the action. The following definitions and result can be restated for left actions of a vector space; we leave this task to the reader.

1.6.3 Definition. An *affine space* is defined to be a triple (P, DP, τ), where P is a set, DP is a vector space and

$$\tau : P \times DP \to P : (p, v) \mapsto p + v := \tau(p, v)$$

is a free and transitive action (called *translation*). □

We recall (remark 1.6.1) that the choice of an 'origin' $o \in P$ yields the bijections τ_o and δ_o. If $p \in P$, then we denote by $(p - o) \in V$ the unique element such that $p = o + (p - o)$. So, we have

$$\tau_o : V \to P : g \mapsto o + g\,, \qquad \delta_o : P \to V : s \mapsto s - o\,.$$

Of course, each vector space turns out to be an affine space associated with itself in a natural way.

The *dimension* of an affine space is defined to be the dimension of the associated vector space. In this book we mainly deal with finite dimensional affine spaces.

Let us consider an affine space (P, DP, τ). For the sake of simplicity, we often denote the affine space just by P, omitting to mention explicitly the associated vector space DP and the translation τ.

1.6.4 Lemma. Let P, P' be affine spaces. Let $f : P \to P'$ be a map which fulfills, for a certain $o \in P$,

$$f(a) = f(o) + Df(a - o), \quad \forall a \in P\,,$$

where $Df \in L(V, V')$ is a linear map. Then $Df : V \to V'$ is the unique linear map fulfilling the above properties. Moreover, Df is independent from the choice of $o \in P$.

PROOF. Let $D'f \in L(V, V')$ be such that $f(a) = f(o) + D'f(a - o)$ for any $a \in P$. Then, we have

$$0 = f(a) - f(a) = f(o) + Df(a - o) - f(o) - D'f(a - o)\,,$$

hence $Df = D'f$. Moreover, if $o' \in P$ such that $f(a) = f(o') + \tilde{D}f(a - o')$ for any $a \in P$, where $\tilde{D}f \in L(V, V')$, then we have

$$\begin{aligned} 0 &= f(a) - f(a) \\ &= f(o) + Df(a - o) - f(o') - \tilde{D}f(a - o') \\ &= Df(a - o + o - o') - \tilde{D}f(a - o')\,, \end{aligned}$$

hence $Df = \tilde{D}f$. QED

1.6.5 Definition. Let P, P' be affine spaces. Then an *affine map* is defined to be a map $f : P \to P'$ which fulfills, for a certain $o \in P$,

$$f(a) = f(o) + Df(a - o), \quad \forall a \in P,$$

where $Df \in L(V, V')$ is a linear map. □

1.6.6 Remark. Each constant map $a' : P \to P'$ between affine spaces is affine and

$$Da' = 0_{DP'}\,.$$

The identity map $\mathrm{id}_P : P \to P$ is affine and

$$D\,\mathrm{id}_P = \mathrm{id}_{DP}\,.$$

If $f : P \to P'$ and $f' : P' \to P''$ are affine maps, then the composite map $f' \circ f : P \to P''$ is affine and

$$D(f' \circ f) = Df' \circ Df\,. \;\square$$

1.6.7 Example. Let P be an affine space. For each $o \in P$, the maps

$$\tau_o : DP \to P \qquad \delta_o : P \to DP$$

are affine and invertible, and $\tau_o^{-1} = \delta_o$. Moreover, their derivatives are

$$D\tau_o = D\delta_o = \mathrm{id}_{DP} \,. \square$$

1.6.8 Example. If $P_1, \ldots, P_n$ are affine spaces, then

$$P \equiv P_1 \times \ldots \times P_n$$

turns out to be naturally an affine space associated with the vector space

$$DP \equiv DP_1 \times \ldots \times DP_n$$

by means of the translation map

$$\tau : P \times DP \to P : (a_1, \ldots, a_n; v_1, \ldots, v_n) \mapsto (a_1 + v_1, \ldots, a_n + v_n) \,. \square$$

1.6.9 Remark. Let P, P' be affine spaces. The set of affine maps between P and P'

$$\mathcal{A}(P, P') := \{f : P \to P' \mid f \text{ affine}\}$$

turns out to be an affine space associated with the vector space

$$D\mathcal{A}(P, P') := \{\phi : P \to DP' \mid \phi \text{ affine}\}$$

according to the natural translation

$$\mathcal{A} \times D\mathcal{A} \to \mathcal{A} : (f, \phi) \mapsto f + \phi \,.$$

We set $\mathrm{End}(P) := \mathcal{A}(P, P)$. $\square$

1.6.10 Remark. Let P be an affine space. The subset of affine invertible maps

$$\mathrm{Aut}(P) := \{f : P \to P \mid f \text{ affine invertible}\} \subset \mathrm{End}(P)$$

turns out to be a group. $\square$

1.6.11 Definition. A *Euclidean affine space* is defined to be an affine space P associated with a Euclidean vector space (DP, g).

A *rigid map* of P is defined to be a map $f : P \to P$ such that

$$\|f(p) - f(q)\| = \|p - q\| \qquad \forall\, p, q \in P \,. \square$$

Hence, a Euclidean affine space is finite dimensional.

1.6.12 Theorem. *Let P be a Euclidean affine space associated with (DP, g), and $f : P \to P$ a map. Then, f is a rigid transformation if and only if f is an affine map whose derivative Df is an element of $O(DP, g)$.*

Proof. Let us suppose that f be a rigid transformation. Then, if $p \in P$, the map

$$f_p : DP \to DP : v \mapsto f(p+v) - f(p)$$

is orthogonal, hence linear. For each $q \in P$ we have

$$f(q) = f(p) + f_p(q-p),$$

hence f is affine. QED

Let P be a Euclidean affine space. We denote by $R(P)$ the set of rigid transformations of P. It turns out that $R(P)$ is a subgroup

$$R(P) \subset \mathrm{Aut}(P).$$

We have the natural injective group morphism map

$$+ : DP \to R(P) : v \to \tau_v,$$

where $\tau_v : P \to P : p \mapsto p+v$ is the translation by v.

Moreover, we have the natural surjective group morphism

$$D : R(P) \to O(DP, g) : f \mapsto Df.$$

1.6.13 Theorem. *We have a sequence of group morphisms*

$$DP \xrightarrow{+} R(P) \xrightarrow{D} O(DP, g),$$

where $+$ *is injective,* p *is surjective and* $\mathrm{im}\,+ = \ker p$. □

1.6.14 Corollary. Let $o \in P$. Then, we have the group isomorphism

$$R(P) \to DP \times O(DP, g) : f \mapsto (\tau_{f(p)},\ Df). \square$$

We set

$$SR(P) := D^{-1}(SO(DP, g)).$$

The elements of $SR(P)$ are said to be *special rigid transformations.*

Moreover, we have the natural inclusion

$$DP \to SR(P) : v \mapsto f_v,$$

where f_v is the translation by v. It turns out that

$$DP = D^{-1}(\{\mathrm{id}_P\}).$$

Finally, we point out that there is a natural free right action of DP on $R(P)$, namely

$$R(P) \times DP \to R(P) : (f, f_v) \mapsto f + v,$$

where

$$f + v : P \to P : p \mapsto f(p) + v\,.$$

We can develop differential calculus starting from finite dimensional affine spaces, according to the following scheme.

1.6.15 Remark. A map $f : P \to P'$ between finite dimensional affine spaces is said to be *differentiable* if it approximates an affine map at first order, i.e. if we can write

$$f(a+h) = f(a) + Df_a(h) + 0_a(h), \quad \forall a \in P, \forall h \in DP\,,$$

where $Df_a : DP \to DP'$ is a linear map and the map $O_a : DP \to DP'$ is infinitesimal of order greater than 1[1]. Then, we can easily prove that the map

$$df : P \times DP \to DP' : (a,h) \mapsto Df_a(h)$$

is unique.

We can easily prove the *chain rule* for differentiable maps.

Of course, affine maps are differentiable.

M oreover, by a simple induction procedure we can define the differentiability of any order $1 < k < \infty$ and the corresponding k-differential. If a map is k-differentiable for any $1 < k < \infty$, then it is said to be C^∞ or *smooth.* □

1.6.3 Affine spaces associated with groups

One can generalise the standard concept of affine space by replacing vector spaces with (possibly non abelian) groups. The resulting setting is quite similar to that of standard affine spaces. This generalisation is very useful for the description of the configuration space of rigid systems.

For a notational convenience, we carry on this generalisation by using left group actions, rather than right group actions. We leaveto the reader the task of repeating the following construction for right actions, and to recover the definition and some results of the previous section.

1.6.16 Definition. A *(left) affine space* is defined to be a triple (C, DC, l), where C is a set, DC is a group and

$$l : DC \times C \to C$$

is a free and transitive left action. □

Of course, a group G is an affine space associated to itself with respect to the multiplication map.

[1]There is no need to fix a norm on the vector spaces DP, DP'; in fact, we recall that all norms on finite dimensional vector spaces are equivalent.

Let us consider an affine space (C, DC, l). For the sake of simplicity, we often denote the affine space just by C, omitting to mention explicitly the associated group DC and the left action l.

1.6.17 Definition. An *affine map* is defined to be a map $f : C \to C'$ between affine spaces such that, for a certain $o \in C$,

$$f(a) = Df(ao^{-1})f(o), \qquad \forall a \in C,$$

where

$$Df : DC \to DC'$$

is a group morphism. □

As in the previous section we can easily prove that, if such a Df exists, then it is unique and independent from the choice of o. We say Df to be the *(generalised) derivative* of f.

1.6.18 Remark. Each constant map $a' : C \to C'$ between affine spaces is affine and

$$Da' = 1_{DC'} .$$

The identity map $\mathrm{id}_C : C \to C$ is affine and

$$D\,\mathrm{id}_C = \mathrm{id}_{DC} .$$

If $f : C \to C'$ and $f' : C' \to C''$ are affine maps, then the composite map $f' \circ f : C \to C''$ is affine and

$$D(f' \circ f) = Df' \circ Df . \square$$

1.6.19 Example. Let C be an affine space. For each $o \in C$, the maps

$$l_o : DC \to C \qquad \delta_o : C \to DC$$

are affine and invertible, and $l_o^{-1} = \delta_o$. Moreover, their derivatives are

$$Dl_o = D\delta_o = 1_{DC} . \square$$

1.6.20 Remark. Let C, C' be affine spaces associated with groups. The set of affine maps between C and C'

$$\mathcal{A}(C, C') := \{f : C \to C' \mid f \text{ affine}\}$$

turns out to be an affine space associated with the group

$$D\mathcal{A}(C, C') := \{\phi : C \to DC' \mid \phi \text{ affine}\}$$

according to the natural left action

$$D\mathcal{A}(C,C') \times \mathcal{A}(C,C') \to \mathcal{A}(C,C') : (\phi, f) \mapsto \phi f \,. \square$$

CHAPTER 2

DIFFERENTIAL GEOMETRY

This chapter is devoted to a brief outline of the basic facts about manifolds, bundles and connections, and Lie groups. The purpose is, as in the previous chapter, both to introduce the notation and to serve as a reference for applications in mechanics.

2.1 Manifolds

In this section we recall a few basic facts concerning manifolds. Manifolds provide the natural background where many important definition and results of analysis can be extended. Namely, concepts from analysis having local character can be reformulated for topological spaces which are locally homeomorphic to a finite dimensional vector space, i.e. which admit 'local coordinate systems'. The reformulation does not involve any distinguished choice of such a coordinate system at each point. For example, the definition of differential, the rank theorem and the implicit function theorem, the existence and uniqueness theorem for the solution of ordinary differential equations can be restated for manifolds in a coordinate free way.

We stress that in this book we will only deal with finite dimensional manifolds.

2.1.1 Definition. A *topological space* is defined to be a set M together with a family $\mathcal{T}$ of subsets (called the *topology*, or the family of *open subsets*), which fulfills the following properties

1. if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$;
2. if $\mathcal{T}' \subset \mathcal{T}$, then $\bigcup_{U \in \mathcal{T}'} U \in \mathcal{T}$. $\square$

For instance, the family of standard open subsets of $\mathbb{R}^n$ make it a topological space.

Let $f : M \to N$ be a map between topological spaces. Then, f is said to be *continuous* if the pre-image $f^{-1}(V) \subset M$ of each open subset $V \subset N$ is an open subset. Moreover, f is said to be a *homeomorphism* if it is invertible and both f and f^{-1} are continuous.

Let M be a topological space.

A *neighbourhood* of a point $x \in M$ is defined to be an open subset U which contains x.

A *base* is defined to be a subfamily $\mathcal{B} \subset \mathcal{T}$, which fulfills the following property:
- for each $x \in M$ and each neighbourhood U of x, there is an element $U' \in \mathcal{B}$, such that $x \in U' \subset U$.

For instance, the standard open n-intervals of $\mathrm{I\!R}^n$ constitutes a basis of its topology.

The topolological space is said to be *separated* if any pair of distinct points $x, y \in M$ admit disjoint neighbourhoods.

For instance, $\mathrm{I\!R}^n$ is separated.

2.1.2 Definition. A *topological manifold* of dimension m is defined to be a topological space which has a countable basis, is separate and each point has a neighbourhood homeomorphic to an open subset of $\mathrm{I\!R}^m$. □

2.1.3 Remark. Infinite dimensional manifolds are defined analogously to the finite dimensional case by requiring that each point has a neighbourhood homeomorphic to V, where V is an infinite dimensional space. But, in order to carry on analysis on M, it is required that V be a Banach space. □

Let us consider a topological manifold M.

Each local homeomorphism $x : U \to \mathrm{I\!R}^m$ is said to be a *topological chart.*

A *topological atlas* is defined to be a family $\{x_\alpha : U_\alpha \to \mathrm{I\!R}^m\}_{\alpha \in \mathcal{A}}$ of topological charts, such that $\cup_\alpha U_\alpha = M$.

2.1.4 Remark. Each chart $x : U \to \mathrm{I\!R}^n$ yields:
- the local *coordinate functions*

$$x^\alpha : M \to \mathrm{I\!R} \qquad 1 \le \alpha \le m\,;$$

- the family of local *coordinate curves*

$$x_\alpha : \mathrm{I\!R} \times M \to M \qquad 1 \le \alpha \le m\,,$$

defined by

$$x_\alpha(t, p) := x^{-1}\big(x^1(p), \dots, x^\alpha(p) + t, \dots, x^m(p)\big)\,.$$

Of course, we have the identity

$$(x^\alpha \circ x_\beta)(t, p) = x^\alpha(p) + \delta^\alpha_\beta\, t\,. \square$$

The *transition map* of two topological charts x, x' is defined to be the local map (defined where appropriate)

$$\mathrm{I\!R}^m \xrightarrow{\;x^{-1}\;} M \xrightarrow{\;x'\;} \mathrm{I\!R}^m\,.$$

A topological atlas is said to be a *smooth atlas* if its transition maps are local smooth maps $\mathrm{I\!R}^m \to \mathrm{I\!R}^m$ (in the sense of affine spaces).

A smooth atlas $\mathcal{A}$ is said to be *maximal* if it fulfills the following property:
- if $\mathcal{A}' \supset \mathcal{A}$ is a smooth atlas, then $\mathcal{A}' = \mathcal{A}$.

We can easily prove that if $\mathcal{A}$ is a smooth atlas, then there is a unique maximal smooth atlas which contains $\mathcal{A}$.

2.1.5 Definition. A *smooth manifold* is defined to be a topological manifold M together with a maximal smooth atlas. □

Of course, affine spaces are naturally equipped with a smooth structure.

Let us consider a smooth manifold M.

Let $f : M \to N$ be a map between smooth manifolds. Then, f is said to be *smooth* if, for any $x \in M$ and any topological charts

$$x : U \to \mathrm{I\!R}^m \text{at x}, \qquad y : V \to \mathrm{I\!R}^n \text{at f(x)},$$

its transition maps

$$y \circ f \circ x^{-1} : \mathrm{I\!R}^m \to \mathrm{I\!R}^n$$

are local smooth maps. Moreover, f is said to be a *diffeomorphism* if it is invertible and both f and f^{-1} are smooth.

The identity map of any smooth manifold is smooth and the composition of smooth maps is smooth.

The cartesian product of two smooth manifolds inherits naturally a smooth structure from its factor manifolds.

The coordinate functions and the coordinate curves are smooth maps.

Now on, all manifolds will be smooth and all maps between smooth manifolds will be smooth, without explicit mention.

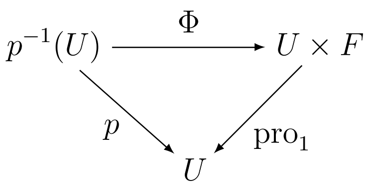
2.2 Bundles

In this section we recall a few basic facts concerning bundles, with special attention to vector bundles.

Let us consider two manifolds E, B and a surjective map $p : E \to B$. A *local bundle trivialisation* is defined to be a map

$$\Phi : p^{-1}(U) \to U \times F\,,$$

where $U \subset B$ is an open subset, F is a manifold (called *the type fibre*) and Φ is a diffeomorphism which makes the following diagram commutative



Moreover, a *bundle trivialising atlas* is defined to be a family

$$(\Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F)_{\alpha\in\mathcal{A}}$$

of local bundle trivialisations, such that $(U_\alpha)_{\alpha\in\mathcal{A}}$ be an open covering of B. [1]

2.2.1 Definition. A *bundle* is defined to be a manifold E (called the *total space*) together with a surjective map $p : E \to B$ (called the *projection*) onto a manifold B (called the *base space*), which admit a bundle trivialising atlas.

If $b \in B$, then the subset $p^{-1}(b) \subset E$ is said to be the *fibre* over b. □

2.2.2 Example. Let B and F be manifolds. Then,

$$p \equiv \mathrm{pro}_1 : E \equiv B \times F \to B$$

is a bundle. Indeed, this bundle has, by construction, a distinguished global trivialisation. □

2.2.3 Remark. We stress that a generic bundle may not admit a global trivialisation. Moreover, if a bundle admits a global trivialisation, then it admits many global trivialisations and it might be that none of them is distinguished. □

Let us consider a bundle $p : E \to B$.

[1] For the sake of simplicity, we shall consider only connected manifolds B's and the same type fibre for all bundle trivialisations of the same atlas.

2.2.4 Remark. Let $\mathcal{A}$ be a bundle trivialising atlas. A local manifold chart of the base space and a local chart of the type fibre

$$(x^\lambda) : U \subset B \to \mathbb{R}^m \qquad (y^i) : F \to \mathbb{R}^l$$

yield a local manifold chart denoted by

$$(x^\lambda, y^i) : p^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^l$$

of E, which is said to be *fibred.* □

We shall refer to a fibred manifold atlas (x^λ, y^i).

A *section* is defined to be a map $s : B \to E$, such that $p \circ s = \mathrm{id}_B$. Thus, the coordinate expression of the section s is of the type

$$(x^\lambda, y^i) \circ s = (x^\lambda, s^i),$$

where $(s^i) : B \to \mathbb{R}^l$.

2.2.5 Definition. The *fibred product* of the bundles $p : E \to B$ and $p' : E' \to B$ over the same base space B is defined to be the bundle whose total space is

$$E \underset{B}{\times} E' := \bigsqcup_{b \in B} E_b \times E'_b$$

and whose projection map is the natural projection

$$E \underset{B}{\times} E' \to B : (X, Y) \in E_b \times E'_b \mapsto b\,. \square$$

2.2.6 Definition. A *bundle morphism* between two bundles $p : E \to B$ and $p' : E' \to B'$ is defined to be a map $f : E \to E'$ which preserves thefibres, i.e. such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\;\underline{f}\;} & B' \end{array}$$

where $\underline{f} : B \to B'$ is a map which turns out to be unique.

We say that f is a bundle morphism *over* $\underline{f}$. In the particular case when $B = B'$ and $\underline{f} = \mathrm{id}_B$, we say that f is a bundle morphism *over* B. □

Let N be any manifold. We shall naturally identify the maps $f : E \to N$ with the bundle morphisms $f : E \to B \times N$ over B.

Next, we recall a few facts on vector bundles.

In the case when the fibres of the bundle are equipped with a vector structure, a local bundle trivialisation $\Phi : p^{-1}(U) \to U \times F$ is said to be *linear* if the type fibre F is a vector space and, for each $b \in U$, the map $\Phi_b : E_b \to F$ is a linear isomorphism. Moreover, a bundle trivialising atlas is said to be *linear* if it is constituted by linear local bundle trivialisations.

2.2.7 Definition. A *vector bundle* is defined to be a bundle $p : E \to B$ together with a vector structure on each fibre $E_b, b \in B$, which admits a linear bundle trivialising atlas. □

If $p : E \to B$ is a vector bundle, the bundle trivialising atlas is linear and the chart (y^i) is linear, then also the induced manifold chart is said to be *linear*.

Let $p : E \to B$ be a vector bundle and $\Phi : p^{-1}(U) \to U \times F$ a linear local bundle trivialisation. Then, an (f_i) a basis of F and the dual basis (ϕ^i) of F^* yield the local basis of sections and the linear fibred chart

$$b_i : U \to p^{-1}(U) : b \mapsto \Phi^{-1}(b, f_i) \qquad y^i : p^{-1}(U) \to \mathbb{R} : v \mapsto \phi^i\big(\operatorname{pro}_2(\phi(v))\big) .$$

If $p : E \to B$, $p' : E' \to B'$ are vector bundles, then a bundle morphism $f : E \to E'$ is said to be *linear* if it restricts to linear maps between fibres.

2.2.8 Remark. Let $p : E \to B$ be a vector bundle. Then, the set

$$E^* := \bigsqcup_{b \in B} E^*_b$$

constituted by the disjoint union of the dual spaces of the fibres of E turns out to be naturally a vector bundle over B.

Moreover, each linear fibred chart of E yields, by duality, a linear fibred chart of E^*. □

2.2.9 Remark. Let $p : E \to B$, $p' : E' \to B$ be vector bundles over the same base space. Then, the set

$$E \underset{B}{\otimes} E' := \bigsqcup_{b \in B} E_b \otimes E'_b$$

constituted by the disjoint union of tensor products of the fibres of E and E' over the same base points turns out to be naturally a vector bundle over B.

Moreover, each pair of linear fibred charts of E and E' yields, by tensor product, a linear fibred chart of $E \underset{B}{\otimes} E'$. □

2.2.10 Remark. Let $p : E \to B$ be a vector bundle. Then, there is a natural bijection

$$s : B \to E^* \qquad \mapsto \qquad f : E \to \mathbb{R}$$

between the sections of the dual bundle and the linear fibred functions of the bundle. Accordingly, we shall identify the above objects. □

2.2.11 Remark. Let $p : E \to B$, $p' : E' \to B$ be vector bundles over the same base space. Then, there is a natural bijection

$$s : B \to E^* \underset{B}{\otimes} E'^* \qquad \mapsto \qquad f : E \underset{B}{\otimes} E' \to \mathrm{I\!R}$$

between the sections of the dual tensor product bundle and the linear fibred functions of the tensor product bundle. Accordingly, we shall identify the above objects. □

Now, we introduce the concept of fibre metric of a vector bundle.

2.2.12 Definition. Let $p : E \to B$ be a vector bundle. Then, a *fibred metric* on E is defined to be a section

$$g : B \to E^* \underset{B}{\otimes} E^*$$

of the bundle $E^* \underset{B}{\otimes} E^* \to B$ such that, for any $b \in B$, the tensor

$$g_b \in E^*_b \otimes E^*_b$$

corresponds to a positive definite symmetric bilinear map

$$g_b : E_b \times E_b \to \mathrm{I\!R}\,.\;\square$$

2.3 Tangent prolongation of manifolds

In this section we recall a few basic facts concerning the tangent prolongation of manifolds.

Let us consider a manifold M and a manifold atlas $x^\lambda : U \to \mathbb{R}^m$.

Let $c, c' : \mathbb{R} \to M$ be two local curves and $t, t' \in \mathbb{R}$ two real numbers. We define the following equivalence relation

$$(t, c) \sim (t', c') \qquad \Leftrightarrow \qquad c(t) = c'(t'), \quad Dc^\lambda(t) = Dc'^\lambda(t'),$$

with respect to any manifold chart.

We denote the equivalence class of the couple (t, c) by

$$dc(t) \equiv [(t, c)].$$

Let $p \in M$ be a point. We define the *tangent space* of M *at* p to be the set

$$T_pM := \{dc(t)\}_{c(t)=p}$$

constituted by all equivalence classes $[(t, c)]$ such that $c(t) = p$.

2.3.1 Definition. The *tangent space* of M is defined to be the set

$$TM := \bigsqcup_{p \in M} T_pM . \square$$

2.3.2 Proposition. The natural projection.

$$\tau_M : TM \to M$$

is a vector bundle.

Proof. Clearly, TM turns out to be a manifold, as each manifold chart $x^\lambda : U \to \mathbb{R}^m$ yields the chart

$$(x^\lambda, \dot{x}^\lambda) : \tau^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m : dc(t) \mapsto \big(c^\lambda(t), Dc^\lambda(t)\big)$$

and the transition maps of these charts are smooth.

Moreover, the map τ turns out to be smooth.

Furthermore, τ makes TM a bundle over M.

In fact, each manifold chart $x^\lambda : U \to \mathbb{R}^m$ yields the bundle trivialisation

$$\tau^{-1}(U) \to U \times \mathbb{R}^m : dc(t) \mapsto \big(c(t), Dc^\lambda(t)\big).$$

Indeed, different charts induce the same vector structure on the fibres of the tangent bundle. Hence, $\tau : TM \to M$ turns out to be naturally a vector bundle. QED

2.3.3 Remark. We say a manifold M to be *parallelisable* if TM admits a global trivialisation $TM \simeq M \times \mathbb{R}^m$.

In general, manifolds are not parallelisable. For example, it can be proved that the unit sphere $S^2 \subset \mathbb{R}^3$ is not parallelisable.

Anyway, we have a distinguished class of parallelisable manifolds: affine spaces associated with vector spaces.

Namely, if P is an affine space associated with a vector space DP, then we obtain a natural isomorphism induced by any affine global chart of P

$$TP \simeq P \times DP\,,$$

by which we will identify the above spaces.

In particular, by regarding DP as an affine space associated with DP, then

$$TDP = DP \times DP\,.\square$$

A section $X : M \to TM$ is said to be a *vector field.*

If $c : \mathbb{R} \to M$ is a curve, then its *tangent prolongation* is defined to be the curve

$$dc : \mathbb{R} \to TM : t \to dc(t)\,.$$

By construction, we have the following coordinate expression

$$(x^\lambda, \, \dot{x}^\lambda) \circ dc = (c^\lambda, Dc^\lambda)\,.$$

2.3.4 Example. Let P be an affine space and $c : \mathbb{R} \to P$ a curve. Then, we can write

$$dc = (c, \, Dc) : \mathbb{R} \to TP = P \times DP\,.\square$$

Let us consider a family of curves parametrised by a manifold N

$$\phi : \mathbb{R} \times N \to M\,.$$

Then, we define the *variational differential* of f as

$$\partial\phi : N \to TM : q \mapsto d(\phi_q)(0)\,.$$

In particular, the tangent prolongation of the coordinate curves $x_\alpha : \mathbb{R} \times M \to M$ are the vector fields

$$\partial x_\alpha : M \to TM : p \mapsto d(x_{\alpha p})(0)\,.$$

Let $f : M \to \mathbb{R}$ be a function. Then, we define the α-th *partial derivative* of f, with respect to the coordinate curve x_α, to be the function

$$\partial_\alpha f \equiv \partial x_\alpha . f : M \to \mathbb{R} : p \mapsto \partial(f \circ x_\alpha) := D(f \circ x_{\alpha p})(0)\,.$$

The standard notation for the partial derivative is

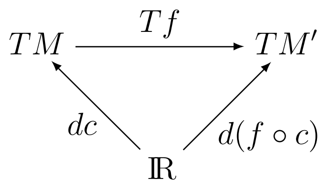
$$\frac{\partial f}{\partial x^\alpha} : M \to \mathbb{R}\,;$$

however, we prefer the above notation, because the partial derivative depends on the α-th coordinate curve x^α and not on the α-th coordinate function x_α. Moreover, we shall see soon a natural interpretation of our notation.

2.3.5 Proposition. If $f : M \to M'$ is a map, then there is a unique map

$$Tf : TM \to TM'\,,$$

such that the following diagram commutes, for each curve $c : \mathbb{R} \to M$,



Moreover, Tf is a linear fibred morphism over f, hence the following diagram commutes

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TM' \\ {\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle \tau'_M} \\ M & \xrightarrow[f]{} & M' \end{array}$$

2.3.6 Example. Let P, P' be affine spaces and $f : P \to P'$ a map. Then, we can write

$$Tf = (f \times Df) : TP = P \times DP \to TP' = P' \times DP'\,.\square$$

2.3.7 Definition. We define the *cotangent bundle* to be the dual bundle of the tangent bundle

$$\tau^M : T^*M \to M\,.\square$$

The natural fibred chart of T^*M will be denoted by

$$(x^\alpha, \dot{x}_\alpha)\,.\square$$

2.3.8 Remark. Let M be a manifold, P an affine space and $f : M \to P$ a map. Then, we define the *differential* of f to be the fibred linear map

$$\dot{f} = \mathrm{pro}_2 \circ Tf : TM \to DP\,,$$

or, equivalently, the corresponding section

$$df : M \to T^*M\,.$$

In particular, if $f : M \to \mathbb{R}$ is a function, then we obtain

$$\dot{f} : TM \to \mathbb{R} \qquad df : M \to T^*M \otimes DP \,. \square$$

This notation agrees with the notation we have already adopted for affine spaces and coordinate functions on the tangent space.

2.3.9 Example. Let P be an affine space and $f : P \to \mathbb{R}$ a function. Then, we have

$$\dot{f} : P \times DP \to \mathbb{R} : (a, h) \mapsto Df_a(h)$$
$$df : P \to P \times D^*P : a \mapsto (a,\, Df_a) \,. \square$$

2.3.10 Remark. The differentials of the coordinate functions are the fibred linear maps

$$\dot{x}^\lambda = \mathrm{pro}_2 \circ T x^\lambda : TM \to \mathbb{R} \,,$$

or equivalently the sections

$$dx^\lambda : M \to T^*M \,. \square$$

2.3.11 Lemma. For each $p \in M$, the m vectors $\partial x_\alpha(p)$ and the m forms dx^α constitute a basis of T_pM and the dual basis of T^*_pM.

Proof. In fact, by differentiating the equality

$$(x^\alpha \circ x_\beta)(t, p) = x^\alpha(p) + \delta^\alpha_\beta \, t \,,$$

we obtain

$$\partial(x^\alpha \circ x_\beta) = \delta^\alpha_\beta \,,$$

which yields the claim in virtue of a well known result of linear algebra. QED

2.3.12 Theorem. *Let $f : M \to N$ be a map. Then, the coordinate expression of Tf is*

$$(x'^\lambda,\, \dot{x}'^\lambda) \circ Tf = (f'^\lambda,\, \frac{\partial f'^\lambda}{\partial x^\mu} \dot{x}^\mu) \,.$$

In other words, by regarding Tf as a section

$$Tf : M \to T^*M \otimes TN \,,$$

we can write

$$Tf = \partial x_\mu . f'^\lambda dx^\mu \otimes \partial(x'_\lambda \circ f) \,.$$

Proof. The results follows immediately from the chain rule and the definition of partial derivative. In fact, we can write

$$(Tf')^\lambda_\mu = \langle dx'^\lambda,\, Tf(\partial x_\mu) \rangle = \partial(x'^\lambda \circ f \circ x_\mu) := \partial x_\mu . f'^\lambda \,. \text{QED}$$

2.3.13 Corollary. The coordinate expressions of the tangent prolongation of a curve $c : \mathbb{R} \to M$ and of a function $f : M \to \mathbb{R}$ are

$$dc = Dc^\lambda (\partial x_\lambda \circ c)$$
$$df = \partial_\lambda f \, dx^\lambda \,. \square$$

2.3.14 Remark. The vector subspace

$$\mathcal{F}(M) := \{ f : M \to \mathbb{R} \mid f \text{smooth} \} \subset \mathcal{M}(M, \mathbb{R})$$

is endowed by the operations in $\mathcal{M}(M, \mathbb{R})$ with the structure of a commutative associative algebra with unity. $\square$

2.3.15 Proposition. A vector field $X : M \to TM$ yields a derivation of $\mathcal{F}(M)$, i.e. a linear map

$$X. : \mathcal{F}(M) \to \mathcal{F}(M) : f \mapsto X.f := \langle df, \, X \rangle \,,$$

with the property, for each $f, f' \in \mathcal{F}(M)$,

$$X.(f f') = f X.f' + f' X.f \,. \square$$

We have the coordinate expression

$$X.f = X^\lambda \partial x_\lambda . f \,. \square$$

2.3.16 Remark. The above notation fits our notation for partial derivatives. In fact, we can write

$$\partial x_\lambda . f = \langle df, \, \partial x_\lambda \rangle \,. \square$$

Conversely, one can prove that vector fields are characterised by their action on functions.

2.3.17 Proposition. If $\delta : \mathcal{F}(M) \to \mathcal{F}(M)$ is a derivation, then there is a unique vector field $X : M \to TM$ such that, for each $f \in \mathcal{F}(M)$,

$$X.f = \delta(f) \,. \square$$

We can use the above fact in order to introduce the commutator of two vector fields.

2.3.18 Remark. If $X, Y : M \to TM$ are vector fields, then the map $X.Y. : \mathcal{F}(M) \to \mathcal{F}(M)$ is not a derivation of $\mathcal{F}(M)$, hence cannot be identified with a vector field. In fact, we have the following coordinate expression

$$X.Y.f = X^\lambda \partial_\lambda Y^\mu \partial_\mu f + X^\lambda Y^\mu \partial_{\lambda\mu} f \,. \square$$

However, we have the following result.

2.3.19 Proposition. If $X, Y : M \to TM$ are vector fields, then the map

$$X.Y. - Y.X. : \mathcal{F}(M) \to \mathcal{F}(M)$$

is a derivation of $\mathcal{F}(M)$, hence can be identified with a vector field, which will be denoted by

$$[X, Y] := X.Y. - Y.X. : M \to TM\,.$$

We have the following coordinate expression

$$[X, Y] = (X^\lambda \partial_\mu Y^\mu - Y^\lambda \partial_\mu X^\mu) \partial x_\lambda\,.$$

Proof. In fact, we have the following coordinate expression

$$X.Y.f - Y.X.f = X^\lambda \partial_\lambda Y^\mu \partial_\mu f + X^\lambda Y^\mu \partial_{\lambda\mu} f - Y^\lambda \partial_\lambda X^\mu \partial_\mu f - Y^\lambda X^\mu \partial_{\lambda\mu} f\,.\text{QED}$$

2.3.20 Remark. The set of vector fields of M

$$\mathcal{T}(M) := \{X : M \to TM\}$$

is a Lie algebra with respect to the operations

$$\mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M) : (X, X') \mapsto X + X'$$
$$\mathbb{R} \times \mathcal{T}(M) \to \mathcal{T}(M) : (k, X) \mapsto kX$$
$$\mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M) : (X, X') \mapsto [X, X']\,.$$

Moreover, we have the following property, for each $X, Y \in \mathcal{T}(M), f \in \mathcal{F}(M)$

$$[fX,\, Y] = f[X,\, Y] - Y.f\, X \qquad [X,\, fY] = f[X,\, Y] + X.f\, Y\,.\square$$

2.3.21 Definition. The operation

$$[,] : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M) : (X, X') \mapsto [X, X']$$

is said to be the *Lie bracket.* □

The Lie bracket is preserved by diffeomorphism, as we are going to prove.

Let N be a manifold, $X \in \mathcal{T}(M)$, $Y \in \mathcal{T}(N)$ and $f : M \to N$ a smooth map. Then, we say that X, Y are *f–related* if the following diagram commutes

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle Y} \\ M & \xrightarrow{f} & N \end{array}$$

2.3.22 Proposition. Let N be a manifold, $X, X' \in \mathcal{T}(M)$, $Y, Y' \in \mathcal{T}(N)$ and $f : M \to N$ a smooth map. Suppose that X, Y and X', Y' are f–related.
Then, $[X, X']$ is f–related to $[Y, Y']$. $\square$

Let $f : M \to N$ be a diffeomorphism, and $X \in \mathcal{T}(M)$.
Let us set $f_* X := Tf \circ X \circ f^{-1} \in \mathcal{T}(N)$. Then, $f_* X$ is the unique vector field on N being f–related to X.

2.3.23 Corollary. Let $f : M \to N$ be a diffeomorphism, and $X, X' \in \mathcal{T}(M)$. Then,

$$f_*[X, X'] = [f_* X, f_* X'] . \square$$

2.4 Tangent prolongation of bundles

In this section we recall a few basic facts concerning the tangent prolongation of bundles.

So, we go back to a generic bundle $p : E \to B$ and apply the constructions of the above section to it.

2.4.1 Remark. The set TE is naturally a manifold equipped with the manifold atlas

$$(x^\lambda,\, y^i;\, \dot{x}^\lambda,\, \dot{y}^i)\,.$$

Moreover, $\tau_E : TE \to E$ turns out to be naturally a vector bundle. We have the coordinate expression

$$(x^\lambda,\, y^i) \circ \tau_E = (x^\lambda,\, y^i)\,.$$

Furthermore, we have the linear bundle morphism

$$Tp : TE \to TB\,,$$

over $p : E \to B$, with coordinate expression

$$(x^\lambda,\, \dot{x}^\lambda) \circ Tp = (x^\lambda,\, \dot{x}^\lambda)\,.$$

Thus, the following diagram commutes

$$\begin{array}{ccc} TE & \xrightarrow{Tp} & TB \\ {\scriptstyle \tau_E}\downarrow & & \downarrow{\scriptstyle \tau_B} \\ E & \xrightarrow[p]{} & B \end{array}$$

2.4.2 Remark. We can see that, if $p : E \to B$ is a vector bundle with type fibre F, then also $Tp : TE \to TB$ is a vector bundle with type fibre $TF = F \times F$. □

2.4.3 Remark. We can easily see that a vector $X \in T_e E$ is tangent to the fibre $E_{p(e)}$ passing through its base point $e \in E$ if and only if $Tp(X) = 0$. Such vectors are said to be *vertical*. The subset of vertical vectors constitutes a vector subbundle

$$\iota : VE \hookrightarrow TE$$

over E, which is characterised by the system of equations

$$\dot{x}^\lambda = 0$$

and called the *vertical bundle*.

In particular, if $p : E \to B$ is a vector bundle, then we can write

$$VE = E \underset{B}{\times} E\,,$$

because we know that the tangent space of a vector space is the product of the vector space times itself. □

As a special case, we can apply the above constructions to the tangent bundle of the manifold M

$$p \equiv \tau : E \equiv TM \to B \equiv M \,.$$

In this case we obtain some interesting identifications.

2.4.4 Example. We denote the induced local chart of TTM by

$$(x^\lambda, \dot{x}^\lambda; \grave{x}^\lambda, \ddot{x}^\lambda) \,.$$

We have two vector bundle projections

$$\tau_{TM} : TTM \to TM \qquad T\tau_M : TTM \to TM \,,$$

with coordinate expressions

$$(x^\lambda, \dot{x}^\lambda) \circ \tau_{TM} = (x^\lambda, \dot{x}^\lambda) \qquad (x^\lambda, \dot{x}^\lambda) \circ T\tau_M = (x^\lambda, \grave{x}^\lambda) \,.$$

We have

$$VTM = TM \underset{M}{\times} TM \,. \square$$

2.5 Connections on bundles

In this section we introduce the concept of connection, with special attention to linear connections on vector bundles.

Let us consider a bundle $p : E \to B$ and a fibred manifold atlas (x^λ, y^i).

2.5.1 Remark. We have the following situation.

- Given a vector $\underline{X} \in T_b B$, and a point $e \in E_b$, we can prolong it to a vector $X \in T_e E$, which projects onto $\underline{X}$, in many ways; but no distinguished prolongation exists.
- Given a vector $X \in T_e E$, we can project it onto the vector $Tp(X) \in T_{p(e)} B$ of the base space B, hence we are able to say whether it is vertical or not. However, if X is not vertical, then we can project it to a vertical vector $Y \in V_e E$ in many ways; but no distinguished projection exists.
- We can easily see that the two missing operations are equivalent.
- In other words, for each $e \in E_b$, we have several splittings

$$T_e E \simeq T_b B \times V_e E,$$

which fit the natural projection $T_e E \to T_b B$ and the natural inclusion $V_e E \hookrightarrow T_e E$, but none of them is distinguished. □

So, in order to avail of such a distinguished splitting, we must postulate it by introducing a new concept, which will be discussed in next section.

2.5.2 Definition. A *connection* is defined to be a linear splitting over E

$$TE \simeq (E \underset{B}{\times} TB) \underset{E}{\times} VE,$$

provided by

- the linear bundle morphism over E

$$\gamma : E \underset{B}{\times} TB \to TE\,,$$

such that the composition

$$E \underset{B}{\times} TB \xrightarrow{\gamma} TE \xrightarrow{(\tau_E, Tp)} E \underset{B}{\times} TB$$

is the identity map of $E \underset{B}{\times} TB$,
or, equivalently, by

- the linear bundle morphism over E

$$\nu : TE \to VE\,,$$

such that the composition

$$VE \xrightarrow{\iota_E} TE \xrightarrow{\nu} VE$$

is the identity map of VE.

The map γ is called the *horizontal prolongation* and the map ν is called the *vertical projection.* □

Let us consider a connection γ or, equivalently, ν.

2.5.3 Proposition. If $\underline{X} : B \to TB$ is a vector field of the base space, then its horizontal prolongation is the vector field

$$\gamma(\underline{X}) : E \to TE\,,$$

with coordinate expressions

$$\gamma(\underline{X})^\lambda = X^\lambda \qquad \text{and} \qquad \gamma(\underline{X})^i = \gamma^i_\lambda X^\lambda;$$

if $X : E \to TE$ is a vector field of the total space, then its vertical projection is the vector field

$$\nu(X) : E \to VE\,,$$

with coordinate expressions

$$\nu(X)^\lambda = 0 \qquad \text{and} \qquad \nu(X)^i = X^i - \gamma^i_\lambda X^\lambda,$$

where

$$(\gamma^i_\lambda) : E \to \mathbb{R}^l \times \mathbb{R}^m$$

is a matrix, which characterises locally the connection. □

2.5.4 Remark. Let $s : B \to E$ be a section. The natural differential of s is its tangent prolongation

$$Ts : TB \to TE\,,$$

with coordinate expression

$$(Ts)^\lambda = \dot{x}^\lambda \qquad \text{and} \qquad (Ts)^i = \partial_\lambda s^i \dot{x}^\lambda\,.$$

The information carried by the section s is encoded just in its components s^i and analogously, the information carried by the differential Ts of the section is encoded just in its components (Ts^i). So, one can be interested in a distinguished vertical projection of Ts, which would be tangent to the fibres and would have as many significant components as the section s itself: this is provided by the connection γ. □

So, we are led to introduce the following concept.

2.5.5 Definition. Let $s : B \to E$ be a section and $X : B \to TB$ a vector field of the base space. The *covariant differential* of s is defined to be the map

$$\nabla_X s := \nu \circ Ts \circ X : B \to VE \,.\square$$

2.5.6 Remark. Of course, we have

$$\tau_E \circ \nabla_X s = s \,.\square$$

2.5.7 Remark. Let $s : B \to E$ be a section and $X, X' : B \to TB$ vector fields. The covariant derivative fulfills the property

$$\nabla_{X+X'} s = \nabla_X s + \nabla'_X s \,.$$

We have the coordinate expression

$$(\nabla_X s)^i = (\partial_\lambda s^i - \gamma^i_\lambda \circ s) X^\lambda \,.\square$$

Next, let us assume that $p : E \to B$ is a vector bundle and let us refer to a linear fibred atlas.

2.5.8 Remark. Let us recall that we can write $VE = E \underset{B}{\times} E$. Accordingly, the first component of the covariant derivative $\nabla_X s$ is just s itself. Therefore, in order to simplify the notation we shall omit, by abuse of language, the first component of the vertical projection and of the covariant derivative and keep only the second components; hence, we write

$$\nu : TE \to E \qquad \text{and} \qquad \nabla_X s : B \to E \,.\square$$

We recall that $Tp : TE \to TB$ is a vector bundle.

2.5.9 Definition. The connection γ is said to be *linear* if γ, regarded as a bundle morphism over TB, is linear. $\square$

2.5.10 Proposition. The connection γ is linear if and only if its coordinate expression is of the type

$$\gamma^i_\lambda = \gamma^i_{\lambda j} y^j \,,$$

where

$$(\gamma^i_{\lambda j}) : B \to \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^l \,.\square$$

Let us suppose that the connection γ be linear.

2.5.11 Proposition. Let $s, s' : B \to E$ be sections, $f : B \to \mathbb{R}$ a function and $X : B \to TB$ a vector field. Then, the covariant derivative fulfills the properties

$$\nabla_X (s + s') = \nabla_X s + \nabla_X s' \qquad \text{and} \qquad \nabla_X (fs) = X.f s + f \nabla_X s \,.$$

We have the coordinate expression

$$(\nabla_X s)^i = (\partial_\lambda s^i - \gamma^i_{\lambda j} s^j) X^\lambda \,. \square$$

2.5.12 Proposition. Conversely, let us assume to have defined a law ∇ which maps each local section $s : B \to E$ and local vector field $X : B \to TB$ onto a local section

$$\nabla_X s : B \to E \,,$$

with the properties

$$\nabla_{X+X'} s = \nabla_X s + \nabla'_X s \,,$$
$$\nabla_X (s + s') = \nabla_X s + \nabla_X s' \,, \qquad \nabla_X (fs) = X.fs + f \nabla_X s$$

and which commutes with local restrictions.

Then, there exist a unique linear connection γ which yields the above law. $\square$

Next, we introduce the concept of metrical connection.

2.5.13 Definition. Let us assume a fibred metric on the bundle $p : E \to B$

$$g : B \to E^* \underset{B}{\otimes} E^* \,.$$

The linear connection γ is said to be *metric* if, for any section $s, s' : B \to E$ and vector field $X : B \to TB$, we have

$$X.\big(g \circ (s, s')\big) = g \circ (\nabla_X s, s') + g \circ (s, \nabla_X s') \,. \square$$

2.6 Connections on manifolds

In this section we apply the above constructions on linear connections to the tangent bundle $\tau_M : TM \to M$ of a manifold M, with special attention to Riemannian connections on Riemannian manifolds.

2.6.1 Remark. A linear connection on the vector bundle $\tau_M : TM \to M$ can be regarded, equivalently, as the horizontal prolongation

$$\gamma : TM \underset{M}{\times} TM \to TTM \,,$$

or as the vertical projection

$$\nu : TTM \to TM \,.$$

The coordinate expression of the connection is of the type

$$\gamma^\alpha_\lambda = \gamma^\alpha_{\lambda\mu} \dot{x}^\mu \,.$$

The covariant derivative of a vector field $Y : M \to TM$, with respect to a vector field $X : M \to TM$, is the vector field

$$\nabla_X Y : M \to TM \,,$$

with coordinate expression

$$(\nabla_X Y)^\alpha = (\partial_\lambda Y^\alpha - \gamma^\alpha_{\lambda\mu} Y^\mu) X^\lambda \,.$$

Thus, we can write

$$\gamma^\alpha_{\lambda\mu} \dot{x}^\mu = -(\nabla_{\partial x_\lambda} \partial x_\mu)^\alpha \,. \square$$

2.6.2 Remark. Let $c : \mathbb{R} \to M$ be a curve and $\nu : TTM \to TM$ a linear connection. We know that the tangent prolongation of c is the curve

$$dc : \mathbb{R} \to TM \,.$$

Moreover, by iterating the tangent prolongation, we obtain the second tangent prolongation of c, which is the curve

$$d^2 c : \mathbb{R} \to TTM \,,$$

with coordinate expression

$$(x^\lambda, \dot{x}^\lambda; \mathring{x}^\lambda, \ddot{x}^\lambda) \circ d^2 c = (c^\lambda, Dc^\lambda, Dc^\lambda, D^2 c^\lambda) \,.$$

Thus, $d^2 c$ is a natural second derivative that we can perform on c by taking into account only the differentiable structure of M. But, this map has some disadvantages:

- it has values in TTM, while it might be desirable it had values in the same space of the first tangent prolongation, namely in TM;
- its coordinate expression carries a reduntant information, as the first derivatives Dc^λ are repeated twice.

These, problems can be easily overcome by taking into account the connection and introducing the map

$$\nabla dc := \nu \circ D^2 c : \mathbb{R} \to TM\,,$$

with coordinate expression

$$\nabla dc = \big(D^2 c^\lambda - (\gamma_{\mu\nu}^\lambda \circ c) Dc^\mu Dc^\nu\big)\partial x_\lambda \circ c\,.$$

We can relate the above map with the standard covariant derivative in this way. Let $X : M \to TM$ be any extension of dc, i.e. any vector field such that $X \circ c = dc$. Then, by a simple computation in coordinates we can prove that

$$(\nabla_X X) \circ c = \nabla dc\,. \square$$

2.6.3 Example. Let M be a parallelisable manifold. Then, the choice of a bundle isomorphism $TM \to M \times \mathbb{R}^m$ yields the bundle isomorphism

$$TTM \to M \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m\,,$$

which restricts to the bundle isomorphism

$$VTM \to M \times \mathbb{R}^m \times \{0\} \times \mathbb{R}^m\,.$$

Hence, we have the connection

$$\mathrm{pro} : TTM \to VTM : (x, v, v', w) \mapsto (x, v, 0, w)\,,$$

and, if $c : \mathbb{R} \to M$, then

$$\nabla dc := \mathrm{pro} \circ D^2 c$$

with $\gamma_{\mu\nu}^\lambda = 0$.

If M is an affine space associated with the vector space DM, then the affine structure induces a distinguished bundle isomorphism $TM \to M \times DM$. Hence, as before, we have a distinguished connection on M which is induced by the affine structure. $\square$

2.6.4 Proposition. Let γ be a linear connection of the vector bundle $\tau_M : TM \to M$. The map

$$\tau : (X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y]$$

is bilinear, hence can be regarded as a bilinear fibred morphism over M

$$\tau : TM \underset{M}{\times} TM \to TM\,.$$

Its coordinate expression is

$$\tau^{\alpha}_{\lambda\mu} = -\gamma^{\alpha}_{\lambda\mu} + \gamma^{\alpha}_{\mu\lambda} \,.$$

PROOF. If $f : M \to \mathbb{R}$ is a function, then we have

$$\tau(fX, Y) = f(\nabla_X Y - \nabla_Y X - [X, Y]) - (Y.f)X + (Y.f)X = \tau(fX, Y)$$
$$\tau(X, fY) = f(\nabla_X Y - \nabla_Y X - [X, Y]) + (X.f)Y - (X.f)Y = \tau(X, fY) \,.$$

Moreover, we have

$$\tau^{\alpha}_{\lambda\mu} = (\nabla_{\partial x_\lambda} x_\mu - \nabla_{\partial x_\mu} x_\lambda)^{\alpha} \,. \text{QED}$$

The map τ is said to be the *torsion* of the linear connection γ. Moreover, if τ vanishes, then γ is said to be *torsion free.*

Of course, the linear connection γ is torsion free if and only if its coordinate expression is symmetric in the two subscripts λ and μ with respect to any chart.

2.6.5 Definition. A *linear connection* on the manifold M is defined to be a linear torsion free connection on the vector bundle $\tau_M : TM \to M$. □

2.6.6 Definition. A *Riemannian metric* g on the manifold M is defined to be a fibred metric on TM, i.e.

$$g : M \to T^*M \underset{M}{\otimes} T^*M \to \mathbb{R} \,.$$

A manifold M, together with a Riemannian metric g, is said to be a *Riemannian manifold.* □

Let us consider a Riemannian manifold (M, g).

The coordinate expression of g is

$$g = g_{\lambda\mu}\, dx^{\lambda} \otimes dx^{\mu} \,.$$

2.6.7 Theorem (Levi-Civita)**.** *There is a unique metric torsion free linear connection* $\varkappa$ *on the Riemannian manifold* (M, g)*. More precisely, the coordinate expression of* $\varkappa$ *is*

$$\varkappa^{\alpha}_{\lambda\mu} = -\frac{1}{2} g^{\alpha\beta} (\partial_\lambda g_{\beta\mu} + \partial_\mu g_{\beta\lambda} - \partial_\beta g_{\lambda\mu}) \,.$$

PROOF. The metricity condition implies

$$\partial_\beta g_{\lambda\mu} = -g_{\alpha\mu} \varkappa^{\alpha}_{\beta\lambda} - g_{\alpha\lambda} \varkappa^{\alpha}_{\beta\mu}$$

i.e.

$$\partial_\beta g_{\lambda\mu} = -\varkappa_{\beta\mu\lambda} - \varkappa_{\beta\lambda\mu} \,.$$

Moreover, by circular permutation of the subscripts in the above formula, we obtain

$$\begin{aligned} \partial_\mu g_{\beta\lambda} &= -\varkappa_{\lambda\beta\mu} - \varkappa_{\mu\beta\lambda} \\ \partial_\beta g_{\lambda\mu} &= -\varkappa_{\lambda\beta\mu} - \varkappa_{\mu\beta\lambda}\,. \end{aligned}$$

Eventually, we obtain the result by subtracting the above two equalities from the previous one and by taking into account the symmetry of the symbols of the connection. QED

2.7 Lie Groups

In this section we recall some basic fact from the theory of Lie groups.

Moreover, we will deal with some distinguished examples which will play an important role in the second part of the book.

2.7.1 Lie Groups

Here, we recall the definition and the main properties of Lie groups. Some of the deepest result of the theory of Lie groups will only be stated. The interested reader can go through details of proofs in the bibliographical references.

2.7.1 Definition. A *Lie group* is defined to be a group G endowed with a manifold structure such that the maps

$$l : G \times G \to G : (g, g') \mapsto gg' , \qquad i : G \to G : g \mapsto g^{-1} ,$$

are smooth. □

2.7.2 Remark. Let G be a Lie group. Then, for each $g \in G$, the restriction of the multiplication

$$l_g : G \to G : h \mapsto gh$$

is a diffeomorphism of G with itself, which is said to be the *left translation* by g. □

Left translation provide a distinguished global trivialisation of TG, as we are going to see.

2.7.3 Definition. We say a vector field $X : G \to TG$ on a lie group G to be *left invariant* if for each $g \in G$ we have $l_{g*}X = X$. □

We denote by $\mathfrak{g}$ the set of left invariant vector fields on G.

In virtue of corollary 2.3.23, $\mathfrak{g} \subset \mathcal{T}(G)$ is a Lie subalgebra. We say $\mathfrak{g}$ to be the *Lie algebra of the Lie group G*.

2.7.4 Proposition. There is a natural isomorphism

$$\mathfrak{g} \to T_{1_G}G : X \mapsto X_{1_G} .$$

PROOF. We can extend any $v \in T_{1_G}G$ to a vector field $X_v : G \to TG$ by the formula

$$X_v : G \to TG : g \mapsto Tl_g(X) ;$$

of course, we have $X_v(1_G) = v$. This ensures both injectivity and surjectivity of the linear map of the statement. QED

Of course, $\mathfrak{g}$ is a finite dimensional vector space; namely, $\dim \mathfrak{g} = \dim T_{1_G} G = \dim G$. We have the tensor representation $[,] = V^* \otimes V^* \otimes V$, and, if (e_i) is a basis of $\mathfrak{g}$, then we have the matrix representation

$$[,] = c^k{}_{ij} \epsilon^i \otimes \epsilon^j \otimes e_k \,.$$

The real numbers $c^k{}_{ij}$ are said to be the *structure constants* of G with respect to (e_i).

2.7.5 Proposition. If G is a Lie group, then we have the natural linear bundle isomorphism over G

$$G \times \mathfrak{g} \to TG : (g, X) \mapsto X(g) \,,$$

by which we identify the above spaces. □

2.7.6 Remark. Each Lie group G admits a natural parallelisation. Hence, thefollowing consequences hold in a straightforward way.

1. A non parallelisable manifold M cannot be endowed with the structure of a Lie group.
2. G is orientable.
3. G is endowed with the natural connection induced by the distinguished parallelisation (remark 2.6.3). □

Let G, H be Lie groups. Then $f : G \to H$ is said to be a *Lie group morphism* if f is a smooth group morphism. The following straightforward property holds.

2.7.7 Proposition. Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$, and $f : G \to H$ a Lie group morphism. Then, f induces a Lie algebra morphism

$$\mathfrak{g} \to \mathfrak{h} : X \mapsto Y_X$$

where $Y_X(1_H) = T_{1_G} f(X_{1_G})$. □

2.7.8 Definition. Let G be a Lie group, and $H \subset G$. Then, H is said to be a *Lie subgroup* of G if H is a subgroup and a submanifold of G. □

2.7.9 Remark. A remarkable example of Lie subgroup of a Lie group G is provided by the connected component G_{1_G} of G containing 1_G. The other connected components are diffeomorphic to G_{1_G} by left translations. □

2.7.10 Theorem. *Let G be a Lie group. Then, there is a bijective correspondence between connected Lie subgroups of G and Lie subalgebras of the Lie algebra $\mathfrak{g}$ of G.—END*

2.7.11 Theorem. *Let G be a Lie group, and H a Lie subgroup. Then, H is an embedded submanifold of G if and only if H is closed.* □

2.7.12 Theorem. *Let G be a Lie group, and $H \subset G$. Suppose that H be a closed set and a subgroup of G. Then, H has a unique manifold structure which makes H a Lie subgroup of G.*

Moreover, as a consequence of the above theorem, with respect to this manifold structure H is an embedded submanifold of G. $\square$

Given a Lie group G, we can consider the quotient G/H with a closed subgroup $H \subset G$. If H is a normal subgroup of G, then the group structure of G passes to the quotient, endowing G/H with a natural group structure. But, even if H is not normal, we are able to give a manifold structure on G/H.

2.7.13 Theorem. *Let G be a Lie group, and H a closed subgroup. Denote by* $\mathrm{pro} : G \to G/H$ *the natural projection. Then, G/H has a unique manifold structure such that* $\mathrm{pro} : G \to G/H$ *is a bundle.*

Moreover, if H is normal, then the manifold G/H with its natural quotient group structure is a Lie group. $\square$

A manifold of the above type G/H is said to be a *homogeneous manifold*. It has to be remarked that several concrete examples of manifold are homogeneous manifolds.

Let G be a Lie group, and M a manifold. Then, a (left) action of G on M is a smooth map

$$a : G \times M \to M$$

which is an action of the group G on the set M.

Suppose that a be a transitive action, and $m \in M$. Then, the isotropy subgroup H_m (see remark 1.6.2) is a closed subgroup of G, hence a Lie subgroup.

2.7.14 Proposition. Let G be a Lie group, M a manifold and $a : G \times M \to M$ a transitive action. Let $m \in M$, and consider the map

$$[a_m] : G/H_m \to M : [g] \mapsto a(g, m) .$$

Then, $[a_m]$ is a diffeomorphism. $\square$

2.7.2 Affine spaces associated with Lie groups

Let C be an affine space associated with a Lie group DC. In this subsection, we show that C can be endowed with the structure of a manifold which is diffeomorphic to DC.

2.7.15 Proposition. There exists a unique differentiable structure on C such that, for each $o \in C$, the map $l_o : DC \to C$ is smooth.

Proof. For each $o \in C$, the bijection $l_o : DC \to C$ makes C a smooth manifold. Moreover, if $o' \in C$, then l_o and $l_{o'}$ yield the same smooth structure on C, because the transition map, given by following composition

$$\delta_{o'} \circ l_o : DC \to DC : g \mapsto (go)o'^{-1} \equiv g(oo'^{-1})$$

is the left translation $l_{oo'^{-1}} : DC \to DC$, which is smooth. QED

It turns out that for any $o \in C$ the map $\delta_o : C \to DC$ is smooth.

2.7.16 Proposition. Let $D\mathfrak{c}$ be the Lie algebra of DC. Then, we have the natural isomorphism

$$TC \simeq C \times \mathfrak{g}\,.$$

by which we identify the above spaces.

Proof. Let $o \in C$. Then, by using the natural parallelisation $TDC \simeq DC \times D\mathfrak{c}$

$$(l_o, \mathrm{id}_{D\mathfrak{c}}) \circ T\delta_o : TC \to C \times D\mathfrak{c}$$

is a parallelisation, and is independent from the choice of o. QED

2.7.17 Corollary. The affine space C associated with the Lie group DC is orientable and is endowed with a natural connection associated with the natural parallelisation (remark 2.6.3). □

2.7.3 Automorphisms of a vector space

Let V be a finite dimensional vector space, with $\dim V = n$. We study the group $\mathrm{Aut}(V)$. Namely, we prove that $\mathrm{Aut}(V)$ is a Lie group, and study its Lie algebra. Finally, we study some of its subgroup.

2.7.18 Lemma. The vector space V, regarded as an abelian group with respect to the sum of vectors, has a natural structure of Lie group. The Lie algebra of V turns out to be naturally isomorphic to V itself.

2.7.19 Proposition. The group $\mathrm{Aut}(V)$ is a Lie group with respect to the composition of maps.

The Lie algebra of $\mathrm{Aut}(V)$ is $\mathrm{End}(V)$ endowed with the commutator.

Proof. The group $\mathrm{Aut}(V)$ is an open subset of the vector space $\mathrm{End}(V)$, namely $\mathrm{Aut}(V) = \det^{-1}(\mathbb{R} \setminus \{0\})$ (see definition 1.5.5). Hence, $\mathrm{Aut}(V)$ is a manifold and

$$T_{\mathrm{id}_V} \mathrm{Aut}(V) \equiv T_{\mathrm{id}_V} \mathrm{End}(V) \simeq \mathrm{End}(V)\,,$$

where the last map is the restriction of the natural parallelisation of $\mathrm{End}(V)$.

The matrix representation of the composition of maps is smooth, hence $\mathrm{Aut}(V)$ is a Lie group.

Let (b_i) be a basis of V. Then, the matrix representation induces the vector space isomorphism $\mathrm{End}(V) \to \mathcal{M}^n{}_n$ and the group isomorphism $\mathrm{Aut}(V) \to \mathcal{I}^n{}_n$, where $\mathcal{I}^n{}_n$ are the invertible matrices.

A left invariant vector field X on $\mathcal{I}^n{}_n$ is of the type

$$X : \mathcal{I}^n{}_n \to T\mathcal{I}^n{}_n : (a^i{}_j) \mapsto a^i{}_k x^k{}_j \partial_i{}^j\,,$$

with $x^i{}_j \partial_i{}^j \in \mathcal{M}^n{}_n$. If Y is another left invariant vector field generated by $y^i{}_j \partial_i{}^j \in \mathcal{M}^n{}_n$, then we see that

$$[X, Y](a^i{}_j) = a^i{}_k (x^k{}_j y^j{}_h - y^k{}_j x^j{}_h) \partial_i{}^h\,,$$

hence $[X, Y](\delta^i_j) = (x^k{}_j y^j{}_h - y^k{}_j x^j{}_h)\partial_k{}^h$. This is the matrix representation of the commutator in $\mathrm{End}(V)$. QED

2.7.20 Proposition. The Lie group $\mathrm{Aut}(V)$ has two connected components. The group

$$\mathrm{SAut}(V) \subset \mathrm{Aut}(V)$$

is the connected component of $\mathrm{Aut}(V)$ containing id_V .

Hence, $\mathrm{SAut}(V)$ is a Lie subgroup of $\mathrm{Aut}(V)$ having the same Lie algebra. □

From now on, we assume a metric g on V .

2.7.21 Proposition. The subgroup $O(V)$ is a Lie subgroup of $\mathrm{Aut}(V)$.

The Lie algebra of $O(V)$ is the following Lie subalgebra $so(V)$ of $\mathrm{End}(V)$

$$so(V) := \{f \in \mathrm{End}(V) \mid g(f(v), w) + g(v, f(w)) = 0 \ \forall v, w \in V\}\,.$$

Proof. In fact, $O(V)$ is a closed subgroup of $\mathrm{Aut}(V)$.

Let us consider a smooth curve $c : \mathrm{I\!R} \to O(V)$ such that $c(0) = \mathrm{id}$. If $v, w \in V$, by differentiating the identity $g(c(t)(v), c(t)(w)) = g(v, w)$ in 0 we obtain the identity

$$g(Dc(0)(v), w) + g(v, Dc(0)(w)) = 0\,.\ \text{QED}$$

2.7.22 Proposition. The Lie group $O(V)$ has two connected components. The group

$$SO(V) \subset O(V)$$

is the connected component of $O(V)$ containing id_V .

Hence, $SO(V)$ is a Lie subgroup of $O(V)$ having the same Lie algebra. □

2.7.23 Remark. If $\dim V = 1$, then $O(V) \equiv \{\pm\,\mathrm{id}\}$.

If $\dim V = 2$, then an orthonormal basis yields an isomorphism of $SO(V)$ with the unit sphere $S^1 \subset \mathrm{I\!R}^2$. □

There is a natural action of $O(V)$ on V , namely

$$O(V) \times V \to V : (f, v) \mapsto f(v)\,.$$

Let us set

$$S_V := \{v \in V \mid g(v, v) = 1\}\,;$$

the choice of an orthonormal basis of V yields a diffeomorphism $S_V \to S^{n-1} \subset \mathrm{I\!R}^n$, hence S_V is a manifold.

2.7.24 Lemma. The above natural action of $O(V)$ on V restricts to a transitive action

$$O(V) \times S_V \to S_V : (f, v) \mapsto f(v)\,. \square$$

2.7.25 Proposition. Let $v \in S_V$, and let us set $v^\perp := \operatorname{span}(v)^\perp$.
Then, the isotropy group of the above transitive action of $O(V)$ on S_V at v is isomorphic to the set of orthogonal maps $O(v^\perp)$ of the Euclidean space $(v^\perp, g^\perp)$.
Hence, we have the diffeomorphism

$$O(V)/O(v^\perp) \to S_V : [f] \mapsto f(v) \,.$$

Proof. It is easily proved in coordinates adapted to the splitting $V = \operatorname{span}(v) \oplus v^\perp$. The last statement comes from proposition 2.7.14. QED

2.7.26 Remark. Analogous considerations hold for $SO(V)$:
given $v \in V$, we have the natural diffeomorphism

$$SO(V)/SO(v^\perp) \to S_V : [f] \mapsto f(v) \,. \square$$

We end this subsection by showing that the group of rigid transformations of an affine space associated with a Euclidean space can be endowed with the structure of a Lie group.

2.7.27 Proposition. Let P be an affine space associated with the Euclidean space (V, g) . Then, the group $R(P)$ can be endowed with a unique Lie group structure such that the natural isomorphism

$$R(P) \to DP \times O(DP, g) : f \mapsto (\tau_{f(p)},\ Df)$$

is a Lie group isomorphism for any $p \in P$.

Proof. In fact, the choice of $p \in P$ yields a group isomorphism which endows $R(P)$ with the structure of a Lie group. Then, it is easy to see that this structure does not depend on the choice of p . QED