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Fibered Spaces, Jet Spaces and Connections for Field Theories

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Abstract. *The geometrical framework for field theories, constituted by fibered spaces, jet spaces and connections, is analysed together with some new results. The fibered spaces constitute a weak structure, but sufficient enough to describe many facts. Further structures can be introduced in different ways such as bundles, bundles with symmetries and S-fibered spaces. However they might not always play a real role in physical applications.*
The natural structures of the tangent and jet spaces are the theoretical basis for all the differential operations. In particular a bracket on the projectable vector valued forms and a canonical differential operator d are presented.
A theory of connections is developed in detail, in the weak framework of fibered spaces. In particular the curvature is introduced both via the braket and the operator d. The usual theory of differential forms results as a particular case. The concept of structure of connections is introduced and a universal connection and curvature are found. The contact and symplectic forms turn out to be a particular case.
The previous results lead naturally to writing Maxwell-type equations on a fibered space, by requiring only a metric on the base space.

Introduction

The geometrical theory of principal bundles, associated bundles and connections is well established and its ideas and language can be considered as standard in the current theoretical physics, mainly in Lagrangian and gauge theories.

However we expect that also weaker geometrical structures, such as fibered spaces and related jet spaces and connections, could play an interesting role in Physics. In fact it is possible to develop rich geometrical theories also at this basic level. A further enrichment of the starting structure will lead to a subsequent specialization of the results. For instance, we can formulate connections, curvatures and related objects on any fibered space; moreover, if we deal with a linear bundle and linear connections, then we find linear curvatures and so on. So we are led to an analysis of successive levels of structures and of their possibilities of geometrical developments. This search for generality is not an end in itself, but is an essential contribution to understanding which geometrical structures in models of Physics play a real role. In this way, surprising results could occur. Let us also remark that many general results can be stated very easily.

A «field» is a fibered space $p : E \to M$, where possibly M is the «space-time» and E the «configuration space», while a «state» is a section $s : M \to E$. The jet spaces $J_k E \to M$ will be the natural spaces to describe geometrically the «field equation», which is a sub-fibered space $\mathcal{E} \hookrightarrow J_k E$. A «solution» is a section $s : M \to E$ such that its prolongation $j_k s : M \to J_k E$ takes its values in $\mathcal{E}$. In particular, when $\dim M = 1$ (for instance in Analytical Mechanics), the jet spaces reduce to the tangent spaces. In order to incorporate further physical information, we need to assume further geometrical objects. A leading role is played by connections. In any case a connection can be viewed as the choice of an origin in a linear affine space, which then is made into a vector space. Connections lead to differential operators, equations and new fibered spaces.

There is an extensive literature on the geometry of fibered spaces and related jet spaces and connections. In this paper we shall be concerned with a survey of some classical ideas, together with some new results which could possibly have interest in Physics. Very influential in the development of our theme has been the pionering work of C. Eheresmann and P. Libermann [4].

A fibered space is a surjective submersion $p : E \to M$. It is characterized by the existence of local trivializations $\Phi : V \subset E \to p(V) \times F$. This structure, although weak, allows us to develop rich theories of jets and connections. Bundles are particular fibered spaces which admit local trivializations $\Phi : p^{-1}(U) \to U \times F$. There are different ways in order to give further information on a fibered space. Besides the bundles associated with a principal bundle, which are endowed with a distinguished group of symmetries, we consider the $\mathcal{S}$-fibered spaces, by introducing the concept of category of structure. It leads to an axiomatic unified treatment of fibered spaces whose fibres are endowed with algebraic structures. The theories of associated bundles and $\mathcal{S}$-fibered spaces are partially equivalent.

There is a natural way to prolong the $\mathcal{S}$-fibered spaces to tangent spaces.

We present the new algebra [6] $\mathcal{F}$ of vector valued forms $F : E \to \wedge T^*M \otimes TE$ projectable on $\bar{F} : M \to \wedge T^*M \otimes TM$. It is a conveniently small graded Lie subalgebra of the Frolicher-Nijenhuis algebra (whose bracket is denoted by [,]), extends the Lie algebra of vector fields of E and depends only on the differential structure.

F is not locally finitely generated on the functions of M. However we are concerned with interesting locally finitely generated reductions, such as S-forms on S-fibered spaces (for instance, if S is the linear affine or the linear structure).

We analyse in detail several features of the jet fibered spaces $J_K E \to M$. Besides the well known basic linear affine structure $J_k E \to J_{k-1} E$ and contact form $\lambda_k : J_{k+1} E \to T^*M \otimes TJ_k E$ or $\theta_k : TJ_{k+1} E \to VJ_k E$, we present the new map [5] $r_k : J_k TE \to TJ_k E$, which allows us to prolong canonically any vector field of E to $J_k E$.

We consider also the sesquiholonomic jet space $\hat{J}_{k+1} E$, which is the sub-space of $JJ_k E$ characterized by certain symmetries and plays an important role with respect to connections. Then we find a fundamental splitting $\hat{J}_{k+1} E = J_{k+1} E \oplus$ $\oplus \underset{k-1}{\overset{2}{\diamond}} T^*M \otimes VE$ (where $\underset{k-1}{\overset{2}{\diamond}} T^*M \otimes VE \hookrightarrow \overset{2}{\wedge} T^*M \otimes \underset{k-1}{\vee} T^*M \otimes VE$) and the associated projection $\mathrm{d} : \hat{J}_{k+1} E \to \underset{k-1}{\overset{2}{\diamond}} T^*M \otimes VE$.

In the particular case when $E \equiv R \times V \to R$ or $E \equiv M \times R \to M$, the jet spaces turn out to be the tangent and cotangent spaces and we obtain interesting specifications of the previous objects. In particular, the 1-forms $\alpha : M \to T^*M$ can be viewed as sections $E \to JE$ and $M \xrightarrow{j\alpha} JT^*M \xrightarrow{\mathrm{d}} \overset{2}{\wedge} T^*M$ is nothing but the usual differential $\mathrm{d}\alpha$.

There is a natural way to prolong the S-fibered spaces to jet spaces. In this context we show interesting properties of the jet spaces of principal bundles.

We analyse in detail several equivalent ways in order to introduce the concept of connection on a fibered space. The most direct approach is to choose an origin on each affine fibre of $JE \to E$, by means of a section $\Gamma : E \to JE$. This leads, among other things, to a projection $\nabla : JE \to T^*M \otimes VE$ and to a projectable vector valued form $\Gamma : E \to T^*M \otimes TE$, hence to the covariant derivative $\nabla s \equiv \nabla \circ js : M \to T^*M \otimes s_* VE$ of sections $s : M \to E$ and to the covariant differential $\mathrm{d}_\Gamma F \equiv [\Gamma, F]$ of projectable vector valued forms $F : E \to \overset{r}{\wedge} T^*M \otimes TE$.

We have two different but equivalent definitions of the curvature $R \equiv -$ $-\mathrm{d} \circ J\Gamma \circ \Gamma = -1/2\, [\Gamma, \Gamma] : E \to \overset{2}{\wedge} T^*M \otimes VE$, which involve two fundamental structures of E and generalize the Cartan equations of structure. Each of the two definitions shows some interesting features, such as the generalized Bianchi identities $\mathrm{d}_\Gamma R = 0$ and the integrability conditions. In the particular case when $E \equiv M \times R \to M$, the usual forms $\alpha : M \to T^*M$ and their differential $\mathrm{d}\alpha : M \to$ $\to \overset{2}{\wedge} T^*M$ turn out to be nothing but connections and their curvature. Moreover $\mathrm{d}^2 \alpha = 0$ is a particular case of the Bianchi identities.

The previous definitions and results work suitably on higher order jet spaces. We are led to the reductions of connections $\Gamma : J_{k-1} E \to J_k E \hookrightarrow JJ_{k-1} E$ and their curvature $R : J_{k-1} E \to \underset{k-1}{\overset{2}{\diamond}} T^*M \otimes VE \hookrightarrow \overset{2}{\wedge} T^*M \otimes VJ_{k-1} E$. Such connections have been used in [7] in order to obtain a scheme for fields theories, where the field equation E is given directly, by means of a second order connection Γ, by the operator $J_2E \xrightarrow{\nabla} \underset{2}{\vee} T^*M \otimes VE \xrightarrow{g} VE$, where g is a metric. This scheme fits several fields such as the relativistic one-body dynamics, the Klein-Gordon, Maxwell, Dirac, Einstein, Yang-Mills fields and their possible interactions.

The space of connections is not locally finite dimensional on M. However we are concerned with interesting locally finitely generated reductions, given by a fibered space $C \to M$ and a fibered morphism $\gamma : C \underset{M}{\times} E \to JE$, such as S--connections on S-fibered spaces (for instance if S is the linear affine or the linear structure). Moreover, if such a reduction come from a structure of projectable vector valued forms, then we obtain also a reduction of the space of curvatures. In the particular case when γ is invertible (for instance, if we deal with principal connections or symmetric linear connections on $\underset{2}{\vee} T^*M$), we can introduce the concept of potential of a connection, which generalizes the Riemannian connections.

On the bundle $C \underset{M}{\times} E \to C$ we can show a canonical connection Λ and its curvature Ω, which are characterized by a universal property with respect to all the connections of C. Then a suitable non degeneracy of Ω is ensured by its universal property. In the particular case when $E \equiv M \times R \to M$ and $C \equiv T^*M$ is the space of usual forms, then Λ and Ω turn out to be the Liouville and the symplectic forms. Moreover, when C is the space of principal connections, then we recover the canonical connection and curvature introduced by Garcia [1]. It is reasonable to expect that the previous results could lead to a Hamilton calculus on fibered spaces.

We define the sesquiholonomic connections as sections $\hat{\Gamma} : J_kE \to \hat{J}_{k+1}E$. Then the projection $d : \hat{J}_{k+1}E \to \underset{k+1}{\overset{2}{\diamond}} T^*M \otimes VE$ leads immediately to the concept of torsion $d \circ \hat{\Gamma} : J_kE \to \underset{k-1}{\overset{2}{\diamond}} T^*M \otimes VE$. In the particular case when $E \equiv M \times R \to M$, we recover the usual torsion of connections $T^*M \to JT^*M$.

So we are in the position to write in a natural way Maxwell-type equations on a fibered space, which coincide with the usual Maxwell equations in the particular case when $E \equiv M \times R \to M$ (or $E \equiv M \times U_1 \to M$). We might expect that this weak approach could help in by-passing some difficulties due to hard problems on groups.

I - FIBERED SPACES

I - 1 Fibered Spaces

Henceforth all manifolds will be Hausdorff, paracompact, C^∞ and with constant and finite dimension. We shall denote the category of manifolds by $\mathcal{D}$ and the C^∞ maps and diffeomorphisms by Diff and Diffeo.

a) *Fibered spaces*

Let us recall that a *fibered space* is a C^∞ surjective submersion (i.e. of maximum rank) $p : E \to M$. E is the *space*, M is the *base space* and $E_x \equiv p^{-1}(x)$, with $x \in M$, are the *fibres*. We assume $m \equiv \dim M$, $1 \equiv \dim E - \dim M$.

Then, for each $y \in E$, there is an open neighbourhood $V \subset E$ of y, a manifold F and a diffeomorphism, called *fibered chart*, $\Phi : V \to p(V) \times F$, such that $p = \Pi^1 \circ \Phi$. A *fibered atlas* is a family $\{\Phi_\alpha : V_\alpha \to p(V_\alpha) \times F_\alpha\}$ of fibered charts, where $\{V_\alpha\}$ is a covering of E.

An *adapted chart* is a manifold chart of E $\{x^\lambda, y^i\} : V \to R^m \times R^1$, where $\{x^\lambda\}$ factors through $p(V)$.

A *section* on an open sub-set $U \subset M$ is a map $s : U \to E$, such that $p \circ s = \mathrm{id}_U$. Its coordinate expression is $\{x^\lambda, y^i\} \circ s \equiv \{x^\lambda, s^i\}$. Fibered spaces are characterized by the existence of local sections.

A *fibered morphism* between the fibered spaces $p : E \to M$ and $q : F \to N$ is a C^∞ map $H : E \to F$, which preserves the fibres; i.e. such that there is a map $h : M \to N$ (which turns out to be unique and C^∞) such that $q \circ H = h \circ p$. Its coordinate expression is $\{x'^\lambda, y'^i\} \circ H \equiv \{h^\lambda, H^i\}$.

The fibered spaces constitute a category.

Let us remark that the notion of *pull-back* works at the level of fibered spaces.

A way to introduce a further structure on a fibered space $E \to M$ is to select a group of fibered automorphisms $\mathrm{Sym}(E)$, called the group of *symmetries*.

b) *Bundles*

Let us recall that a *bundle* is a fibered space $p : E \to M$, which admits a fibered atlas, called *bundle atlas*, $\{\Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F_\alpha\}$, where $\{U_\alpha\}$ is an open covering of M and F_α are diffeomorphic manifolds. The equivalence class $\mathcal{F} \equiv [F_\alpha]$ of diffeomorphic manifolds is called the *type-fibre*.

Let $E \to M$ be a bundle. If $x \in M$ and $F' \subset E_x$ is a closed subset such that $F' \neq E_x$, then $E - F' \to M$ is a fibered space, but not a bundle. If $U \subset M$ is a closed subset such that $U \neq M$ and $s : M \to E$ is a local section, then $E - s(U) \to M$ is a fibered space, but not a bundle.

The bundle-morphisms are just the fibered morphisms between bundles.

Let us remark that the construction of fibered morphism from transition family, the construction of bundle from bundle-atlas and the construction of bundle from cocycle work at the level of bundles (even without further structures).

c) *Bundles with symmetries*

A way to endow a bundle $p : E \to M$ with a further structure is to select a (maximal) bundle-atlas $\{\Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F\}$, whose cocycle is $\{\Phi_{\beta\alpha} : U_\alpha \cap \cap U_\beta \to G \subset \mathrm{Diffeo}(F)\}$, where G is a group. A bundle together with such an atlas is called a *G-bundle*.

In particular, we recall the bundle $E \to M$ *associated* with the principal bundle $P \to M$, whose structure Lie group G acts on the left on the manifold F. We have a distinguished bundle-atlas and $E \to M$ turns out to be a G-bundle. Moreover, any principal automorphism $P \to P$ induces a fibered automorphism $E \to E$, called *symmetry*. The symmetries constitute a group $\mathrm{Sym}(E)$ and we obtain a group-epimorphism $\mathrm{Aut}(P) \to \mathrm{Sym}(E)$, which is an isomorphism if G acts effectively on F. If C is the center of G, then we have a natural group-monomorphism $C \to \mathrm{Aut}(P)$, hence a group-morphism $C \to \mathrm{Sym}(E)$. Moreover, if $P \to M$ is trivial, then we have a group-monomorphism $G \to \mathrm{Aut}(P)$, hence a group morphism $G \to \mathrm{Sym}(E)$.

Let us remark that when we are concerned with a G-cocycle, where G is a Lie group acting on the left on a manifold F, then the construction theorems (b) can be re-interpreted in terms of bundles associated with a principal bundle. In fact, the G-cocycle leads, by the construction procedure, to a principal bundle $P \to M$, whose structure group is G, and to a bundle $E \to M$, which is nothing but the bundle associated with $P \to M$ and F.

I - 2 S-Fibered spaces

We introduce the concept of «category of structure» S in order to obtain a unifying framework for a large class of important algebraic structures which are candidate to endow the fibres of fibered spaces. So we can define the S-fibered morphisms, between the S-symmetries. Most of the groups and the associated bundles relevant to Physics could be obtained in this context. We are concerned with the example of affine spaces over any group, which generalize the usual linear-affine spaces. By the way we obtain a unifying review of principal bundles.

a) *Categories of structure*

Roughly speaking a category of structure S is a category whose objects are manifolds with some algebraic C^∞ operations and whose morphisms are the algebraic morphisms. So we obtain a forgetful functor $\mathcal{S} \to \mathcal{D}$.

DEFINITION. A *category of structure* is a category $\mathcal{S}$ together with a covariant functor $\mathcal{S} \to \mathcal{D}$ denoted by $\overline{F} \in \mathrm{Ob}\, \mathcal{S} \longmapsto F \in \mathrm{Ob}\, \mathcal{D}$ and $\overline{f} \in \mathrm{Mor}\, \mathcal{S} \longmapsto f \in \mathrm{Mor}\, \mathcal{D}$, where $\overline{F}$ is called an $\mathcal{S}$*-structure* on F, such that, $\forall \overline{F}, \overline{F}' \in \mathrm{Ob}\, \mathcal{S}$,

a) $\mathrm{Mor}\,(\overline{F}, \overline{F}') \hookrightarrow \mathrm{Diff}(F, F')$ is injective;

b) $\overline{f} \in \mathrm{Iso}\,(\overline{F}, \overline{F}')$ iff $\overline{f} \in \mathrm{Mor}\,(\overline{F}, \overline{F}')$ and $f \in \mathrm{Diffeo}(F, F')$;

and such that

~~c) if $F, F' \in \mathrm{Ob}\, \mathcal{D}$, $\mathrm{Diffeo}(F, F') \neq \phi$, then there are $\overline{F}, \overline{F}'$ such that $\mathrm{Iso}(\overline{F}, \overline{F}') \neq \phi$;~~

d) if F is a manifold, $\overline{G} \in \mathrm{Ob}\, \mathcal{S}$ and $f : F \to G$ is a diffeomorphism, then there is an $\mathcal{S}$-structure $\overline{F}$ on F such that $f \in \mathrm{Iso}(\overline{F}, \overline{G})$.

Moreover $\mathcal{S}$ is said to be *transitive* if the previous structure $\overline{F}$ in d) is unique and for any further diffeomorphism $f' : F \to G'$, such that $f' \circ f^{-1} \in \mathrm{Iso}\,(\overline{G}, \overline{G}')$, the induced $\mathcal{S}$-structure $\overline{F}'$ on F coincides with $\overline{F}$.

Let us remark that, by taking into account a), we can replace $\overline{f} \in \mathrm{Mor}(\overline{F}, \overline{F}')$ with $f \in \mathrm{Diff}(F, F')$.

Let $\mathcal{S}$ be a category of structure. If $\overline{F} \in \mathrm{Ob}\, \mathcal{S}$, then $\mathrm{Aut}(\overline{F})$ is a group. Moreover if $\overline{F}'$ is $\mathcal{S}$-isomorphic to $\overline{F}$, then $\mathrm{Aut}(\overline{F}')$ is isomorphic to $\mathrm{Aut}(\overline{F})$. Hence $\mathcal{S}$ associates (up to an isomorphism) a group with each class of diffeomorphic manifolds which admit structure in $\mathcal{S}$.

We can naturally define the *sub-categories of structure*.

Now let us consider some important examples.

b) *Affine spaces*

DEFINITION. A *right affine space* is a set A together with a free and transitive right action $A \times DA \to A$ of a group DA, which is called the *derived group*. (We will omit the prefix «right» when it is understood). Let A and A' be affine spaces; an *affine morphism* is a map $f : A \to A'$, such that there is a group morphism (which turns out to be unique) $Df : DA \to DA'$, called the *derivative*, which gives $f(ag) = f(a)\, Df(g)$, $\forall (a, g) \in A \times DA$. In particular, if $DA \subseteq DA'$, a *translation* is an affine morphism such that $Df = \mathrm{id}_{DA}$.

We have the «chain rule» $\mathrm{Did}_A = \mathrm{id}_{DA}$ and $D(f \circ f') = Df \circ Df'$. Moreover, an affine morphism f is a mono-, epi-, isomorphism iff Df is such.

Let G be a group and put $A \equiv G \equiv DA$. Then the right action $R : A \times DA \to A$ induces an affine structure, the left translations $L_g : A \to A$ are translations in the previous sense and the map $g \longmapsto L_g$ is a group-isomorphism of G onto the group of translations of A. A group-morphism $f : A \to A$ is an affine morphism and $Df = f$.

Let A be an affine space and $o, a \in A$. Then there is a unique element of Da, denoted by $o^{-1}a$, such that $o(o^{-1}a) = a$. Moreover, if $o \in A$ is fixed, then the map $A \to DA, a \longmapsto o^{-1}a$, is a translation.

Moreover, if DA is a Lie group, then the previous translations induce the same C^∞ structure on A and the action turns out to be C^∞. Such affine spaces are said *differentiable*.

Analogous concepts and results hold by replacing «right» by «left».

If Da is a vector space, then A is called a *linear affine space*.

Henceforth we will refer to differentiable affine spaces.

The affine spaces constitute a category of structure A. We have also the sub--categories of structure G of Lie groups, A_ℓ of linear affine space and V of vector spaces. Moreover D turns out to be a covariant functor $\mathsf{A} \to G$.

Let G be a given Lie group. The affine spaces P, such that $DP = G$, together with the translations, constitute a category of structure P_G, called the category of *principal spaces* with structure group G.

Further examples of categories of structure are given by the algebras and pseudo-euclidean and symplectic spaces. But we could also find very different algebraic examples.

c) *S-fibered spaces*

DEFINITION. An *S-fibered space* is a fibered space $E \to M$ together with an S-structure $i : M \to \text{Ob } S$ on its fibres, such that there is a fibered atlas $\{\Phi_\alpha : V_\alpha \to \to p(V_\alpha) \times F'_\alpha\}$ and a family $\{\overline{F}_\alpha\} \subset \text{Ob } S$, such that $F'_\alpha \subset F_\alpha$ are open subsets and $\forall x \in p(V_\alpha)$, $\Phi_{\alpha x} : V_\alpha \cap E_x \to F'_\alpha$ are restrictions of S-isomorphisms $\overline{E}_x \to \overline{F}_\alpha$. Let $(E \to M, i)$ and $(E' \to M', i')$ be S-fibered spaces; an *S-fibered morphism* is a fibered morphism $H : E \to E'$ over $h : M \to M'$, such that, $\forall x \in M$, $H_x : E_x \to \to E'_{h(x)}$ is an S-morphism.

In the following, when there is a natural way to give the S-structure i, we will omit «i» and say that $E \to M$ is an S-fibered space.

The S-fibered spaces constitute a category denoted also by S.

Let us remark that the notion of pullback fits naturally the S-fibered spaces.

We can define naturally the *reductions* of an S-fibered space.

The previously considered examples of categories of structure lead immediately to the corresponding examples of S-fibered spaces.

d) *Affine fibered spaces*

In particular, the previous definitions lead immediately to that of *affine* and *group-fibered space.*

Let $(E \to M, i)$ be an affine fibered space. Then, by deriving its fibres, we obtain a group-fibered space $(DE \to M, Di)$, called the *derived fibered space*, whose group-fibered atlas is obtained by deriving the affine fibered atlas. Moreover, if $H : E \to F$ is an affine fibered morphism, then by deriving H on the fibres, we obtain a group-fibered morphism $DH : DE \to DF$, called the *derivative*. So D turns out to be a covariant functor $A \to G$.

If $(E \to M, i)$ is an affine fibered space, then we have the *action* morphism $a : E \times_M DE \to E$, which characterizes the affine structure i itself.

If $(E \to M, i)$ is a group-fibered space, then we have the *multiplication* morphism $m : E \times_M E \to E$, the *inversion* morphism $in : E \to E$ and the *unity* section $e : M \to E$, which characterize the group-structure i itself.

Analogous results hold with respect to the other examples previously considered.

In particular, we can define the *principal fibered spaces* according to the previous definitions of the category of principal spaces and of S-fibered spaces.

e) *S-bundles*

DEFINITION. An *S-bundle* is a bundle $E \to M$ together with an S-structure $i : M \to \to \text{Ob } S$ on its fibres, such that there is a bundle-atlas $\{\Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F_\alpha\}$ with an S-structure $\{F_\alpha\} \to \text{Ob } S$, such that, $\forall x \in U_\alpha$, $\Phi_{\alpha x} : E_x \to F_\alpha$ is an S-isomorphism.

The S-bundles constitute a subcategory of the S-fibered spaces.

The construction theorems $(1, b)$ can be specified in the case of S-bundles in a natural way.

If $(E \to M, i)$ is an S-bundle and S is transitive, then the S-structure is characterized by the (maximal) S-bundle-atlas.

If S is a transitive category of structure, $\bar{F} \in \text{Ob } S$, $G \equiv \text{Aut }(\bar{F})$ is a Lie group and $P \to M$ is a principal bundle with structure group G, then we have a natural S-structure on the associated bundle $E \to M$ and $\text{Sym}(E) = \text{Aut }(E)$.

However, let us remark that, even if S is not transitive, the construction theorems could provide a canonical way in order to recover an S-bundle-structure.

f) *Affine bundles*

The previous definitions lead immediately to that of affine bundles, group-bundles, and so on.

We can give a simple example of a group-fibered space, which is not a group-bundle. Let $(E \to M, i)$ be a group-bundle with type-fibre $[\bar{F}]$, where $\bar{F}$ is a group with two connected components. Then we obtain the example by dropping, for a certain $x \in M$, the component of E_x which does not contain the unity.

Conversely, under suitable hypotesis on S or on E, we can prove that the S-fibered spaces are S-bundles.

For instance, by taking into account the local translations, we see that any principal fibered space is a principal bundle, according to the previous definitions. So the usual definition of principal bundle and our one, via the structure of principal spaces, coincide. Moreover we have the following reduction result. Let $(E \to M, i)$ be an affine bundle and G a Lie group. Then the following conditions are equivalent: a) there is P_G-atlas $\{\Psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G\}$, i.e. such that $D\overline{\Psi}_{\beta\alpha} = \mathrm{id}_G$; b) there is an affine fibered isomorphism $DE \to M \times G$ over M. If such conditions hold, then the atlas or the isomorphism determine a reduction of the affine bundle to a principal bundle.

II - JET SPACES

II - 1 The tangent space of a fibered space

a) *The tangent functor*

Let us recall some notations. Let M be a manifold and $\{x^\lambda\}$ a chart. Then TM is the *tangent space* and $\{x^\lambda, \dot{x}^\lambda\}$ the induced chart. The linear *tangent bundle* is denoted by $\pi_M : TM \to M$. If $f : M \to N$ is a C^∞ map, then $Tf : TM \to TN$ is the *tangent map*, which is a linear fibered morphism over f. Its coordinate expression with respect to a chart $\{y^\mu\}$ of N is $\{y^\mu, \dot{y}^\mu\} \circ Tf = \{f^\mu, \partial_\lambda f^\mu \dot{x}^\lambda\}$, where $\partial_\lambda f^\mu \equiv \partial x_\lambda \cdot (y^\mu \circ f)$. Hence T is a covariant functor $\mathcal{D} \to \mathcal{D}$.

$\mathcal{T}M$ will denote the space of C^∞ sections $M \to TM$.

Let $p : E \to M$ be a fibered space (a bundle with type-fibre $[F]$), $\Phi : V \to p(V) \times \times F$ a fibered chart ($\Phi : p^{-1}(U) \to U \times F$ its bundle-chart) and $\{x^\lambda, y^i\}$ an adapted chart. We can consider the tangent space TE and the induced chart $\{x^\lambda, \dot{x}^\lambda; y^i, \dot{y}^i\}$. We have three fibrations on TE. First we have the linear bundle $\pi_E : TE \to E$. So $Tp : TE \to TM$ turns out to be a linear fibered morphism over $p : E \to M$. Second, $Tp : TE \to TM$ is a fibered space (a bundle with type-fibre $[TF]$). The *vertical bundle* is the linear subbundle $\pi_E : VE \equiv \ker_E Tp \to E$ of $\pi_E : TE \to E$ and the *transverse bundle* is the pullback bundle $HE \equiv E \underset{M}{\times} TM \to E$ of $\pi_M : TM \to M$ with respect to $p : E \to M$. Third, we have the linear affine bundle $\mu_E \equiv (\pi_E, Tp) : TE \to HE$, whose derived bundle is (up to an obvious pullback) $\pi_E : VE \to E$.

If $s : M \to E$ is a section, then $Ts : TM \to TE$ is a section of $Tp : TE \to TM$.

If $p : E \equiv h^*(F) \to M$ is the pullback fibered space of $q : F \to N$ with respect to $h : M \to N$, then we have a canonical fibered isomorphism $(Th)^*(TF) \to T(h^*F)$ over TM.

$\mathcal{P}E$ will denote the space of C^∞ sections $E \to TE$ projectable on $M \to TM$.

b) *The tangent prolongation of categories of structure*

Let $\mathcal{S}$ be a category of structure. A *tangent prolongation* of $\mathcal{S}$ is a covariant functor $\mathcal{S} \to \mathcal{S}$ such that $\overline{F} \in \mathrm{Ob}\, \mathcal{S} \longmapsto \overline{TF} \in \mathrm{Ob}\, \mathcal{S}$, $\overline{f} \in \mathrm{Mor}\, \mathcal{S} \longmapsto \overline{Tf} \in \mathrm{Mor}\, \mathcal{S}$ and $TF \to F$ is an $\mathcal{S}$-morphism. Such a functor will be also denoted by T.

If $\mathcal{S}$ is given by some algebraic operations, then their tangent map could provide such a functor. In this way we obtain a canonical tangent prolongation of groups and affine spaces. Moreover, if A is an affine space and $f : A \to A'$ an affine morphism, then we have $DTA = TDA$ and $DTf = TDf$. Let us remark that $T : \mathcal{A} \to \mathcal{A}$ induces a covariant functor $T : \mathcal{P}_G \to \mathcal{P}_{TG}$.

c) *The tangent prolongation of $\mathcal{S}$-fibered spaces*

Let $\mathcal{S}$ be a category of structure, $T : \mathcal{S} \to \mathcal{S}$ a tangent prolongation and $(E \to$

$\to M, i)$ an S-fibered space (S-bundle). Then $(TE \to TM, Ti)$ is an S-fibered space (S-bundle). Moreover $TE \to E$ is an S-fibered morphism over $TM \to M$.

In particular, this result applies to affine and group-fibered spaces and, up to obvious modifications, to principal bundles. Let us remark that if $(E \to M, i)$ is an affine fibered space, then $VE = E \underset{M}{\times} DE$.

The tangent functor works naturally also with respect to the bundles associated with a principal bundle.

d) *The graded Lie algebra of projectable vector valued forms*

If E is a manifold, then we have the fundamental Frolicher-Nijenhuis [6] graded Lie algebra of vector valued forms $E \to \wedge T^*E \otimes TE$, which extends the Lie algebra of vector fields $E \to TE$. Now, if $E \to M$ is a fibered space, then we will be concerned with the more manageable subalgebra [6] $\mathcal{F} \equiv \underset{o \leqslant r \leqslant m}{\oplus} \mathcal{F}^r$ of vector valued forms $F: E \to \wedge T^*M \otimes TE$ projectable on $\overline{F}: M \to \wedge T^*M \otimes TM$. This algebra plays an essential role in the treatment of connections.

The bracket in $\mathcal{F}$ is defined as follows. If $F \in \mathcal{F}^r$ and $G \in \mathcal{F}^s$, then $[F, G] \in \mathcal{F}^{r+s}$ is given by $[F, G](u_1, \ldots, u_{r+s}) \equiv$

$$\equiv \frac{1}{r!s!} \sum_{\sigma} \in(\sigma) \Big\{ [F(u_{\sigma_1}, \ldots, u_{\sigma_r}), G(u_{\sigma_{r+1}}, \ldots, u_{\sigma_{r+s}})] -$$
$$- r F(u_{\sigma_1}, \ldots, u_{\sigma_{r-1}}, [u_{\sigma_r}, \overline{G}(u_{\sigma_{r+1}}, \ldots, u_{\sigma_{r+s}})]) -$$
$$- s G([\overline{F}(u_{\sigma_1}, \ldots, u_{\sigma_r}), u_{\sigma_{r+1}}], u_{\sigma_{r+2}}, \ldots, u_{\sigma_{r+s}}) +$$
$$+ \frac{rs}{2} F(u_{\sigma_1}, \ldots, u_{\sigma_{r-1}}, \overline{G}([u_{\sigma_r}, u_{\sigma_{r+1}}], u_{\sigma_{r+2}}, \ldots, u_{\sigma_{r+s}})) +$$
$$+ \frac{rs}{2} G(\overline{F}(u_{\sigma_1}, \ldots, u_{\sigma_{r-1}}, [u_{\sigma_r}, u_{\sigma_{r+1}}]), u_{\sigma_{r+2}}, \ldots, u_{\sigma_{r+s}}) \Big\}.$$

Its coordinate expression is

$$[F, G] = \{(F^{\rho}_{\lambda_1 \ldots \lambda_r} \partial_{\rho} G^{\mu}_{\lambda_{r+1} \ldots \lambda_{r+s}} - G^{\rho}_{\lambda_{r+1} \ldots \lambda_{r+s}} \partial_{\rho} F^{\mu}_{\lambda_1 \ldots \lambda_r} -$$
$$- r F^{\mu}_{\lambda_1 \ldots \lambda_{r-1} \rho} \partial_{\lambda_r} G^{\rho}_{\lambda_{r+1} \ldots \lambda_{r+s}} + s G^{\mu}_{\rho \lambda_{r+2} \ldots \lambda_{r+s}} \partial_{\lambda_{r+1}} F^{\rho}_{\lambda_1 \ldots \lambda_r}) \partial_{\mu} +$$
$$+ (F^{\rho}_{\lambda_1 \ldots \lambda_r} \partial_{\rho} G^{i}_{\lambda_{r+1} \ldots \lambda_{r+s}} - G^{\rho}_{\lambda_{r+1} \ldots \lambda_{r+s}} \partial_{\rho} F^{i}_{\lambda_1 \ldots \lambda_r} -$$
$$- r F^{i}_{\lambda_1 \ldots \lambda_{r-1} \rho} \partial_{\lambda_r} G^{\rho}_{\lambda_{r+1} \ldots \lambda_{r+s}} + s G^{i}_{\rho \lambda_{r+2} \ldots \lambda_{r+s}} \partial_{\lambda_{r+1}} F^{\rho}_{\lambda_1 \ldots \lambda_r} +$$
$$+ F^{j}_{\lambda_1 \ldots \lambda_r} \partial_j G^{i}_{\lambda_{r+1} \ldots \lambda_{r+s}} - G^{j}_{\lambda_{r+1} \ldots \lambda_{r+s}} \partial_j F^{i}_{\lambda_1 \ldots \lambda_r}) \partial_i\} \otimes dx^{\lambda_1} \wedge \ldots \wedge dx^{\lambda_{r+s}}$$

We have the following properties

$[F, G] = (-1)^{1+rs}[G, F]$

$(-1)^{rt}[F, [G, H]] + (-1)^{ts}[H, [F, G]] + (-1)^{rs}[G, [H, F]]$.

In particular the vertical forms constitute a subalgebra.

Let us remark that $\mathcal{F}$ is not locally finitely generated on the ring $\mathcal{D}M$ of C^∞ functions of M. However we can introduce the following concept.

DEFINITION. A *structure of projectable vector-valued forms* $\mathcal{B}$ is a vector fibered space $B \to M$ together with a linear fibered morphism $\beta : B \underset{M}{\times} E \to \wedge T^*M \otimes TE$ over E, projectable on the fibered morphism $\bar{\beta} : B \to \wedge T^*M \otimes TM$ over M, such that its sections constitute a graded subalgebra of $\mathcal{F}$, called the subalgebra of *$\mathcal{B}$-forms*.

Let $\mathcal{S}$ be a category of structure together with a tangent prolongation and $(E \to M, i)$ an $\mathcal{S}$-fibered space. An *$\mathcal{S}$-form* is a form $F \in \mathcal{F}^r$ such that, $\forall u_1, \ldots, u_r \in TM$, $F(u_1, \ldots, u_r) : E \to TE$ is an $\mathcal{S}$-fibered morphism over $\bar{F}(u_1, \ldots, u_r) : M \to TM$. Under suitable conditions the $\mathcal{S}$-forms constitute a $\mathcal{B}$-structure. For instance we have the structures of linear affine and linear forms.

Analogously we can consider $\mathcal{B}$-structures determined by the symmetries of a bundle associated with a principal bundle.

However we can find simple examples of $\mathcal{B}$-structures which do not originate in these ways.

The $\mathcal{B}$-structures could be viewed as an other means to endow a fibered space with a structure. Let $(E \to M, \beta)$ and $(E' \to M', \beta')$ be fibered spaces together with $\mathcal{B}$-structures. Then a *$\mathcal{B}$-fibered morphism* is a fibered isomorphism $H : E \to E'$ which induces a fibered morphism of the $\mathcal{B}$-structures.

II - 2 Symmetrization and antisymmetrization

We wish to make a few points by way of introduction to jet notation.

a) *Notation on indices and multi-indices*

Let us recall some useful notations.

A *family of indices* $\{1, \ldots, k\} \to \{1, \ldots, m\}$ of degree k and range m will be denoted by $\{\lambda_1, \ldots, \lambda_k\}$, with $1 \leqslant \lambda_1, \ldots, \lambda_k \leqslant m$. A *multi-index* $\Lambda : \{1, \ldots, m\} \to \{0, \ldots, l\}$ of length l and range m will be denoted by $\Lambda \equiv (\Lambda_1, \ldots, \Lambda_m)$, with $0 \leqslant \Lambda_1, \ldots, \Lambda_m \leqslant l$ and $|\Lambda| \equiv \Lambda_1 + \ldots + \Lambda_m = l$. We set $\Lambda + \Phi \equiv (\Lambda_1 + \Phi_1, \ldots, \Lambda_m + \Phi_m)$ and $\lambda \equiv (\Lambda_1, \ldots, \Lambda_\lambda, \ldots, \Lambda_m) \equiv (0, \ldots, 1, \ldots, 0)$; then we get, by definition, $\lambda + \Lambda \equiv (\Lambda_1, \ldots, \Lambda_\lambda + 1, \ldots, \Lambda_m)$. We have the symmetrization $s : (\lambda_1, \ldots, \lambda_k) \longmapsto (\sum_{1 \leqslant h \leqslant k} \delta_1^{\lambda_h}, \ldots, \sum_{1 \leqslant h \leqslant k} \delta_m^{\lambda_h})$ which maps the families of indices into the multi-indices and is such that $s(\lambda_1, \ldots, \lambda_k) = s(\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_k})$ for any permutation σ.

b) *The symmetrization and antisymmetrization operators*

Let F be a vector space. We recall that the *symmetrization* and *antisymmetrization* operators are the linear maps

$$s : \overset{k}{\otimes} F \to \underset{k}{\vee} F \equiv s(\overset{k}{\otimes} F) \subset \overset{k}{\otimes} F, \; u_1 \otimes \ldots \otimes u_k \longmapsto u_1 \vee \ldots \vee u_k \equiv \frac{1}{k!} \sum_\sigma u_{\sigma_1} \otimes \ldots \otimes u_{\sigma_k}$$

$$a : \overset{k}{\otimes} F \to \overset{k}{\wedge} F \equiv a(\overset{k}{\otimes} F) \subset \overset{k}{\otimes} F, \; u_1 \otimes \ldots \otimes u_k \longmapsto u_1 \wedge \ldots \wedge u_k \equiv \frac{1}{k!} \sum_\sigma \in(\sigma) u_{\sigma_1} \otimes \ldots \otimes u_{\sigma_k}$$

We have $s^2 = s$ and $a^2 = a$. Then we set $A \equiv \mathrm{id} - s : \overset{k}{\otimes} F \to \overset{k}{A} F \equiv A(\overset{k}{\otimes} F) \subset \overset{k}{\otimes} F$. We have $A^2 = A$, $\overset{k}{\otimes} F = \underset{k}{\vee} F \oplus \overset{k}{A} F$, $\overset{k}{A} F = \ker s$, $\overset{k}{\wedge} F \subset \overset{k}{A} F$ and, in particular, $\overset{2}{\wedge} F = \overset{2}{A} F$.

In particular, we shall be concerned with the subspace $F \otimes \underset{k}{\vee} F \hookrightarrow \overset{k+1}{\otimes} F$. So we set $\underset{k-1}{\overset{2}{\Diamond}} F \equiv \ker (s \,|\, F \otimes \underset{k}{\vee} F) = A(F \otimes \underset{k}{\vee} F) \hookrightarrow F \otimes \underset{k}{\vee} F$. Then we have $F \otimes \otimes \underset{k}{\vee} F = \underset{k}{\vee} F \oplus \underset{k-1}{\overset{2}{\Diamond}} F$, given by $u_{\lambda\Lambda} \equiv u_\lambda \otimes u_\Lambda \longmapsto u_{\lambda+\Lambda} + (u_{\lambda\Lambda} - u_{\lambda+\Lambda})$.

We recall the *Spencer operator* $\delta : \overset{r}{\wedge} F \otimes \underset{s}{\vee} F \to \overset{r+1}{\wedge} F \otimes \underset{s-1}{\vee} F, u_1 \wedge \ldots \wedge u_r \otimes w_1 \vee \ldots \vee w_s \longmapsto \Sigma_i \, u_1 \wedge \ldots \wedge u_r \wedge w_i \otimes w_1 \vee \ldots \vee \hat{w}_i \vee \ldots \vee w_s$. We have $\delta^2 = 0$ and the sequence $0 \to \underset{k}{\vee} F \to \ldots \to \overset{r}{\wedge} F \otimes \underset{k-r}{\vee} F \to \ldots \to \overset{k}{\wedge} F \to 0$. We have a linear monomorphism $\underset{k-1}{\overset{2}{\Diamond}} F \hookrightarrow F \otimes \underset{k}{\vee} F \overset{\delta}{\longrightarrow} \overset{2}{\wedge} F \otimes \underset{k-1}{\vee} F$ which allows us to identify $\underset{k-1}{\overset{2}{\Diamond}} F$ with its image on $\overset{2}{\wedge} F \otimes \underset{k-1}{\vee} F$.

II - 3 The k-jet space of a fibered space

a) *The k-jet space of a fibered space*

Let $p : E \to M$ be a fibered space, with $\dim M \equiv m$, $\dim E \equiv m + l$, $\Phi : V \to \to p(V) \times F$ a fibered chart and $\{x^\lambda, y^i\}$ an adapted chart. The indices and multi-indices relative to the basis and the fibres will be denoted by Greek and Latin letters, respectively. If $f : E \to R$ is a local C^∞ function, then we set $\partial_\lambda \equiv \partial x_\lambda \cdot f$, $\partial_\Lambda f \equiv \partial_{\Lambda_1} \ldots \partial_{\Lambda_m} f$, where $\partial_{\Lambda_\lambda} \equiv \partial x_\lambda \cdot \ldots \cdot \partial x_\lambda \cdot f \Lambda_\lambda$ times and $\partial_{\Lambda_\lambda} f \equiv f$ if $\Lambda_\lambda = 0$; and analogously with respect to the fibre coordinates.

Let us recall that the *k-jet space* of $p : E \to M$, with $0 \leqslant k$, is the set $J_k E \equiv \bigsqcup_{x \in M} [s]_{kx}$ of equivalence classes of local sections $s : M \to E$ given by

$s \sim s'$ iff $(\partial_\Lambda s^i)(x) = (\partial_\Lambda s'^i)(x)$, with $0 \leqslant |\Lambda| \leqslant k$, with respect to any adapted chart (the equivalence does not depend on the choice of such a chart). In particular, we have $J_o E \equiv E$.

By definition, there are the natural projections $p_k : J_k E \to M$ and $p_k^h : J_k E \to J_h E$ for $0 \leqslant h \leqslant k$. We have $p_h^l \circ p_k^h = p_k^l$ and $p_h \circ p_k^h = p_k$.

The adapted charts $\{x^\lambda, y^i\} : V \to R^m \times R^l$ induce naturally the maps $\{x^\lambda, y^i_\Lambda\} : (p_k^o)^{-1}(V) \to R^m \times R^{l_k}$, which make $J_k E$ a manifold. Moreover $p_k : J_k E \to M$ turns out to be a fibered space and $p_k^h : J_k E \to J_h E$ a bundle with type-fibre $[R^{l_k^h}]$. Henceforth we will always refer to the canonical manifold-, fibered and bundle-atlases of $J_k E$.

The canonical atlas induces a canonical linear affine structure on the bundle $J_k E \to J_{k-1} E$, with $1 \leqslant k$, whose derived bundle is (up to an obvious pullback) $DJ_k E = \bigvee_k T^*M \otimes VE$.

If $s : M \to E$ is a local section, then the map $x \longmapsto [s]_{kx}$ is a C^∞ local section $j_k s : M \to J_k E$, called the *k-prolongation* of s. Of course, we have $p_k^h \circ j_k s = j_h s$. The coordinate expression is $\{x^\lambda, y^i_\Lambda\} \circ j_k s = \{x^\lambda, \partial_\Lambda s^i\}$, with $0 \leqslant |\Lambda| \leqslant k$.

b) *The k-jet functor. (See* [5]*)*

Let $H : E \to F$ be a fibered morphism over the diffeomorphism $h : M \to N$. For any local section $s : M \to E$, we have the local section $H_* s \equiv H \circ s \circ h^{-1} : N \to F$. Then there is a unique fibered morphism $J_k H : J_k E \to J_k F$ over $h : M \to N$ such that, for each local section $s : M \to E$, we have $J_k H \circ j_k s = j_k(H_* s) \circ h$. In particular, we have $J_o H \equiv H$. Of course, we have $q_k^h \circ J_k H = J_h H \circ p_k^h$ and $q_h \circ J_h H = h \circ p_h$. The coordinate expression of $J_k H$, when $h \equiv \mathrm{id}_M$, is $z^j_\Phi \circ J_k H = \Sigma\, \partial_\Lambda \partial_I H^j\, y^1_{\psi(1,1)} \cdots y^1_{\psi(1,I_1)} \cdots y^l_{\psi(l,1)} \cdots y^l_{\psi(l,I_l)}$, where the sum is extended to all the multi-indices with the conditions $\Phi = \Lambda + \psi^{(1,1)} + \ldots + \psi^{(l,I_l)}$ and $I_h > 0 \Rightarrow \psi^{(h,1)}, \ldots, \psi^{(h,I_h)} \neq 0$, with $1 \leqslant h \leqslant l$.

So J_k turns out to be a covariant functor. Moreover, if H is injective (surjective), then $J_k H$ is such.

The fibered morphism $E \underset{M}{\times} F \to E$ and $E \underset{M}{\times} F \to F$ over M induce a fibered isomorphism $J_k(E \underset{M}{\times} F) \to J_k E \underset{M}{\times} J_k F$.

II - 4 Relations between tangent and k-jet spaces

a) *The spaces $TJ_k E$ and $J_k TE$. (See* [5]*)*

We will be concerned with the space $TJ_k E$. Its natural charts are $\{x^\lambda, y^i_\Lambda; \dot{x}^\lambda, \dot{y}^i_\Lambda\}$. We have the linear bundle $TJ_k E \to J_k E$, the fibered space $TJ_k E \to TM$ and the bundles $TJ_k E \to TJ_h E$, with $0 \leqslant h < k$.

We have the vertical sub-bundles $VJ_kE \equiv \ker Tp_k \to J_kE$ and $V^hJ_kE \equiv \ker Tp_k^h \to \to J_kE$, with $0 \leqslant h < k$, which are locally characterized by $\dot{x}^\lambda = 0$ and $\dot{x}^\lambda = \dot{y}^i_\Phi = = 0$, with $0 \leqslant |\Phi| \leqslant h$, respectively. We have the following sequence of natural inclusions $V^{k-1}J_kE \hookrightarrow \ldots \hookrightarrow V^oJ_kE \hookrightarrow TJ_kE$.

We have the transverse bundles $HJ_kE \equiv J_kE \underset{M}{\times} TM \to J_kE$ and $H^hJ_kE \equiv \equiv J_kE \,{}_{J_hE}{\times}\, TJ_hE \to J_kE$. We have the natural sequence of projections $TJ_kE \to \to H^{k-1}J_kE \to \ldots \to H^oJ_kE \to HJ_kE$.

Moreover we will be concerned with the exact sequence over J_kE $\quad 0 \to VJ_kE \to \to TJ_kE \to HJ_kE \to 0$.

We have the linear affine bundles $TJ_kE \to HJ_kE$ and $TJ_kE \to H^h_kJ_kE$, whose derived bundles are (up to an obvious pull-back) the vertical bundles.

We shall be concerned also with the spaces J_kTE and J_kTM. Their natural charts are $\{x^\lambda, y^i_\Lambda; \dot{x}^\lambda, \dot{y}^i_\Lambda\}$ and $\{x^\lambda, \dot{x}^\lambda_\Lambda\}$. We have the linear bundle $J_kTE \to J_kE$, the linear sub-bundle $J_kVE \to J_kE$ and the linear affine bundles $J_kTE \to J_kE \underset{E}{\times} TE$, $J_kTE \to J_kE \,{}_{J_{k-1}E}{\times}\, J_{k-1}TE$, $\quad J_kTE \to J_kE \underset{M}{\times} J_kTM$.

b) *The k-jet contact form. (See* [5]*)*

The exact sequence $0 \to VJ_kE \to TJ_kE \to HJ_kE \to 0$ does not have in general a canonical splitting, but its pull back over $J_{k+1}E$ splits canonically.

If $0 \leqslant k$, then there is a unique fibered morphism $\lambda_k : J_{k+1}E \underset{M}{\times} TM \to TJ_kE$ over $J_{k+1}E \to J_kE$, such that, for each local section $s : M \to E$, we have $Tj_ks = = \lambda_k \circ (j_{k+1}s \circ \pi_M, \mathrm{id}_{TM}) : TM \to TJ_kE$. Moreover λ_k is linear and induces a splitting of the previous sequence. We can also view λ_k as a linear affine fibered monomorphism $\lambda_k : J_{k+1}E \to T^*M \otimes TJ_kE$ over J_kE. Its coordinate expression is $\lambda_k = \mathrm{d}x^\lambda \otimes \partial_\lambda + y^i_{\lambda+\Lambda}\,\mathrm{d}x^\lambda \otimes \partial_i^\Lambda$, with $0 \leqslant |\Lambda| \leqslant k$.

The complementary linear epimorphism of λ_k will be denoted by $\theta_k : J_{k+1}E \,{}_{J_kE}{\times}\, TJ_kE \to VJ_kE$. We have also the equivalent vector valued form $\theta_k : J_{k+1}E \to T^*J_{k+1}E \otimes VJ_kE$. Its coordinate expression is $\theta_k = (\mathrm{d}y^i_\Lambda - y^i_{\lambda+\Lambda}\mathrm{d}x^\lambda) \otimes \otimes \partial_i^\Lambda$, with $0 \leqslant |\Lambda| \leqslant k$. The form θ_k is called the *contact k-jet form*. We set $\Delta_k \equiv \ker \theta_k \subset TJ_{k+1}E$.

If $S : M \to J_kE$ is a local section, then the following conditions are equivalent: a) $S = j_ks$, b) $Ts : TM \to \Delta_k$, where $s \equiv p^o_k \circ s : M \to E$.

An *infinitesimal contact transformation* of order $0 \leqslant k$ is a vector field $u : J_kE \to TJ_kE$ such that $L_u\Delta_k \subset \Delta_k$. Such u is characterized by the coordinate expression $u^i_{\lambda+\Phi} = (\partial_\lambda u^i_\Phi + \partial_j^\psi u^i_\Phi y^j_{\lambda+\psi}) - y^i_{\mu+\Phi}(\partial_\lambda u^\mu + \partial_j^\psi u^\mu y^j_{\lambda+\psi})$, where $\partial_j^\Theta u^i_\Phi = y^i_{\lambda+\Phi}\,\partial_j^\Theta u^\lambda$, with $0 \leqslant |\Phi| \leqslant k-1$, $|\Theta| = k$.

c) *The k-jet prolongation map. (See [5])*

Let $0 \leqslant k$. There is a unique fibered morphism $i_k : J_k VE \to VJ_k E$ over $J_k E$, such that, for each C^∞ 1-parameter family of local sections $\sigma : R \times M \to E$, we have $j_k \sigma = i_k \circ j_k \partial\sigma$, where ∂ is the derivative with respect to the parameter evaluated at $0 \in R$. Moreover i_k is a linear isomorphism. Its coordinate expression is $\{x^\lambda, y^i_\Lambda; \dot{y}^i_\Lambda\} \circ i_k = \{x^\lambda, y^i_\Lambda; \dot{y}^i_\Lambda\}$.

There is a unique affine morphism $r_k : J_k TE \to TJ_k E$ over $J_k E \underset{M}{\times} J_k TM \to$ $\to J_k E \underset{M}{\times} TM$, such that, for each local section $s : M \to E$, $Tj_k s \circ (\pi_M)^o_k = r_k \circ J_k TS$: $J_k TM \to TJ_k E$ and $Dr_k = i_k$. Moreover r_k is a surjective map. Its coordinate expression is $\{x^\lambda, y^i_\Lambda; \dot{x}^\lambda, \dot{y}^i_\Lambda\} \circ r_k = \{x^\lambda, y^i_\Lambda; \dot{x}^\lambda, \dot{y}^i_\Lambda - y^i_{\mu+\Phi}\, \dot{x}^\mu_\Psi\}$, with $0 \leqslant |\Lambda| \leqslant k$ and where the sum is extended to all the multi-indices Φ and Ψ such that $\Phi + \Psi = \Lambda$ and $0 < |\Psi|$. We call r_k the *k-jet prolongation map*.

In fact r_k allows us to prolong the vector fields of E. Let $u_o : E \to TE$ be a local vector field. Then there is a unique infinitesimal contact transformation $u_k : J_k E \to TJ_k E$ which is projectable on u_o. Namely, we have $u_k = r_k \circ J_k u_o$. The map $u_o \longmapsto u_k$ is a Lie algebra morphism. Moreover, in the particular case when u_o is projectable on $u : M \to TM$, then we have $u_k = \partial J_k H$, where $H : R \times$ $\times E \to E$ is the fibered flow of u_o over the flow $h : R \times M \to M$ of u.

II - 5 Double jet spaces. (See [2] and [5])

a) *Double jet spaces*

Let $0 \leqslant h$ and $0 \leqslant k$. As $J_h E \to M$ is a fibered space, then, by iterating the procedure, we can consider the k-jet space $p_{hk} : J_k J_h E \to M$. The induced charts are $\{x^\lambda, y^i_{\Lambda\Phi}\}$, with $0 \leqslant |\Phi| \leqslant h$, $0 \leqslant |\Lambda| \leqslant k$. We have the two bundles

$$p^o_{hk} : J_k J_h E \to J_h E \qquad \text{and} \qquad J_k p^o_h : J_k J_h E \to J_k E.$$

Their coordinate expressions are

$$\{x^\lambda, y^i_\Phi\} \circ p^o_{hk} = \{x^\lambda, y^i_{0\Phi}\} \qquad \text{and} \qquad \{x^\lambda, y^i_\Lambda\} \circ J_k p^o_h = \{x^\lambda, y^i_{\Lambda 0}\}.$$

There is a unique fibered morphism $\sigma_{kh} : J_{h+k} E \to J_k J_h E$ over $J_h E$, such that, for each local section $s : M \to E$, we have $j_k j_h s = \sigma_{kh} \circ j_{h+k} s$. It is a monomorphism. Its coordinate expression is $\{x^\lambda, y^i_{\Lambda\Phi}\} \circ \sigma_{kh} = \{x^\lambda, y^i_{\Lambda+\Phi}\}$. We will identify $J_{h+k} E$ with its image and will write $j_{h+k} s = j_k\, j_h s$.

b) *The k-jet Spencer operator*

Let $k \geqslant 1$. We have two fibered morphisms $\sigma_{1\,k-1} \circ p^o_{k1} : JJ_k E \to JJ_{k-1} E$ and $Jp^{k-1}_k : JJ_k E \to JJ_{k-1} E$ over M. Then, by taking their difference over $J_{k-1} E$, we obtain the fibered morphism $\mathcal{D}_k : JJ_kE \to T^*M \otimes VJ_{k-1}E$ over $J_{k-1}E$, called the *k-jet Spencer operator*. Its coordinate expression is $\{x^\lambda, y^i_\Phi; \dot{x}_\mu \otimes y^i_\Phi\} \circ \mathcal{D}_k = \{x^\lambda, y^i_\Phi; y^i_{\mu\Phi} - y^i_{\mu+\Phi}\}$, with $0 \leqslant |\Phi| \leqslant k-1$.

c) *The sesquiholonomic k-jet spaces. (See also* [4]*)*

The kernel of $\mathcal{D}_k$ is a fibered sub-space $\hat{p}_{k+1} : \hat{J}_{k+1}E \to E$ of $JJ_kE \to M$, called the *sesquiholonomic* k-jet fibered space. The induced charts are $\{x^\lambda, y^i_\Lambda, y^i_{\mu\Theta}\}$, with $0 \leqslant |\Lambda| \leqslant k$, $|\Theta| = k$. The coordinate expression of the fibered monomorphism $i_k : \hat{J}_{k+1}E \to JJ_kE$ over J_kE is $\{x^\lambda, y^i_\Lambda; y^i_{\mu\Phi}, y^i_{\mu\Theta}\} \circ i_k = \{x^\lambda, y^i_\Lambda; y^i_{\mu+\Phi}, y^i_{\mu\Theta}\}$, with $0 \leqslant |\Lambda| \leqslant k$, $0 \leqslant |\Phi| \leqslant k-1$, $|\Theta| = k$. Moreover $\hat{J}_{k+1}E \to J_kE$ is a linear affine sub-bundle of $JJ_kE \to J_kE$, whose derived space is (up to an obvious pullback) $T^*M \otimes \bigvee_k T^*M \otimes VE$.

The fibered monomorphism $\sigma_{1k} : J_{k+1}E \to JJ_kE$ over J_kE factorizes through $\hat{\sigma}_{k+1} : J_{k+1}E \hookrightarrow \hat{J}_{k+1}E$. Its coordinate expression is $\{x^\lambda, y^i_\Lambda, y^i_{\mu\Theta}\} \circ \hat{\sigma}_{k+1} = \{x^\lambda, y^i_\Lambda, y^i_{\mu+\Theta}\}$, with $0 \leqslant |\Lambda| \leqslant k$, $|\Theta| = k$. Moreover $J_{k+1}E \to J_kE$ is a linear affine subbundle of $\hat{J}_{k+1}E \to J_kE$, whose derived space is (up to an obvious pullback) $\bigvee_{k+1} T^*M \otimes VE$. We have natural charts of $\hat{J}_{k+1}E$ adapted to $J_{k+1}E$, namely $\{x^\lambda, y^i_\Lambda; y^i_{(\mu\Theta)}, y^i_{[\mu\Theta]}\}$, where $y^i_{(\mu\Theta)} \equiv \frac{1}{n} \sum_{1 \leqslant \lambda \leqslant m} y^i_{\lambda(\Theta-\lambda+\mu)}$ and $y^i_{[\mu\Theta]} \equiv y^i_{\mu\Theta} - y^i_{(\mu\Theta)}$, with $0 \leqslant |\Lambda| \leqslant k$, $|\Theta| = k$, $1 \leqslant \mu < \inf \nu$ such that $\Theta_\nu \neq 0$, $n \equiv$ number of terms in Σ.

The coordinate expression of $\hat{\sigma}_{k+1}$ then becomes $\{x^\lambda, y^i_\Lambda; y^i_{(\mu\Theta)}, y^i_{[\mu\Theta]}\} \circ \hat{\sigma}_{k+1} = \{x^\lambda, y^i_\Lambda; y^i_{\mu+\Theta}, 0\}$.

We have the following integrability condition. Let $S : M \to J_kE$ be a local section. Then the following conditions are equivalent: a) $S = j_ks$, b) $jS : M \to \hat{J}_{k+1}E$, c) $jS : M \to J_{k+1}E$, where $s \equiv p^o_k \circ S : M \to E$.

The following fundamental splitting will lead to the curvature of connections and to the usual differential as a particular case.

THEOREM. *The linear affine bundle $\hat{J}_{k+1}E \to J_kE$ is the «direct sum» of the linear affine sub-space $J_{k+1}E$ and the linear sub-space $\overset{2}{\underset{k-1}{\Diamond}} T^*M \otimes VE$, over J_kE, namely $\hat{J}_{k+1}E = J_{k+1}E \oplus \overset{2}{\underset{k-1}{\Diamond}} T^*M \otimes VE$. Its coordinate expression is $\{x^\lambda, y^i_\Lambda; y^i_{\mu\Theta}\} = \{x^\lambda, y^i_\Lambda; y^i_{(\mu\Theta)} + y^i_{[\mu\Theta]}\}$. So we have two linear affine projections $\hat{J}_{k+1}E \to J_{k+1}E$ and $d : \hat{J}_{k+1}E \to \overset{2}{\underset{k-1}{\Diamond}} T^*M \otimes VE$. Moreover we have $\delta \circ d = 0$.*

Proof. $J_{k+1}E \to J_kE$ is a linear sub-bundle of $\hat{J}_{k+1}E \to J_kE$, whose derived space has the splitting $T^*M \otimes \bigvee_k T^*M \otimes VE = \bigvee_{k+1} T^*M \otimes VE \oplus \overset{2}{\underset{k-1}{\Diamond}} T^*M \otimes VE$, where the first term is the derived space of $J_{k+1}E \to J_kE$.

II - 6 The k-tangent spaces

a) *The k-jet bundle of a bundle*

Let N be an affine space, F a manifold and $N \times F \to N$ the product bundle. Let $o \in N$, $g \in DN$ and $H_g : N \times F \to N \times F$, $(x, f) \longmapsto (xg, f)$. Then the map $N \times J_{ko}(N \times F) \to J_k(N \times F)$; $(x, \sigma) \longmapsto J_k H_{o^{-1}x}(\sigma)$, is a fibered isomorphism over N, which makes $J_k(N \times F) \to N$ a trivial bundle.

In particular, we have the canonical fibered isomorphism $J_k(R^m \times F) \to R^m \times \times T_k^m F$, where we have set $T_k^m F \equiv J_{ko}(R^m \times F)$, with $0 \in R^m$.

Hence, if $E \to M$ is a bundle with type-fibre $[F]$, then $J_k E \to M$ is a bundle with type-fibre $[T_k^m F]$.

Moreover, if $E \equiv M \times F \to M$ is a product bundle, then we have the fibered monomorphism $j_k : E \hookrightarrow J_k E$, $(x, y) \longmapsto (j_k y)(x)$, over M, which satisfies $p_k^h \circ j_k = j_h$.

b) *The k-tangent and cotangent spaces of a manifold*

Let V be a manifold and let us consider the bundle $E \equiv R \times V \to M \equiv R$. Let $\{x, y^i\}$ be an adapted chart of E. The *k-tangent space* of V is the manifold $T_k V \equiv \equiv J_{ko} E$. By definition, we have $V \equiv T_o V$, $TV \equiv T_1 V$ and $J_k E = R \times T_k V$. The induced chart of $T_k V$ is $\{y^i_\Lambda\}$, with $0 \leqslant |\Lambda| \leqslant k$.

By iterating the procedure, we find $JJ_k E = R \times TT_k V$. Then we obtain the fibered morphism $\lambda_k : T_{k+1} V \to TT_k V$ over $T_k V$ and $r_k : T_k TV \to TT_k V$ over $T_k V \underset{V}{\times} TV$. In particular, $r_1 : TTV \to TTV$ turns out to be the canonical involution and $\lambda_1 : T_2 V \to TTV$ the symmetric subbundle over V. Since $\dim R = 1$, we have $\hat{J}_{k+1} E = J_{k+1} E$ and d vanishes.

Let M be a manifold and let us consider the bundle $E \equiv M \times R \to M$. Let $\{x^\lambda, y\}$ be an adapted chart of E. The *k-cotangent space* of M is the manifold $T_k^* M \equiv J_{ko} E$. By definition, we have $M \equiv T_o^* M$, $T^* M \equiv T_1^* M$ and $J_k E = R \times \times T_k^* M$. The induced chart of $T_k^* M$ is $\{x^\lambda, y_\Lambda\}$, with $1 \leqslant |\Lambda| \leqslant k$. Then we obtain a 1-form $\theta_k : T_{k+1}^* M \to T^* T_{k+1}^* M$ and a fibered morphism $r_k : T_k^* TM \to TT_k^* M$ over $T_k^* M \underset{M}{\times} TM$. In particular, $\theta_o : T^* M \to T^* T^* M$ turns out to be (up to the sign) the Liouville form and $r_1 : T^* TM \to TT^* M$ the fibered isomorphism over M.

The *k-sesquiholonomic cotangent space* of M is the manifold $\hat{T}_{k+1}^* M \equiv \hat{J}_{k+1\,o} E$. By definition, we have $T^* M \equiv \hat{T}_1^* M$ and $\hat{J}_{k+1} E = R \times T_{k+1}^* M$. The induced chart of $\hat{T}_{k+1}^* M$ is $\{x^\lambda, y_\Lambda, y_{\lambda\theta}\}$, with $1 \leqslant |\Lambda| \leqslant k$, $|\theta| = k$. Then we obtain the linear affine splitting $\hat{T}_{k+1}^* M = T_{k+1}^* M \oplus \underset{k-1}{\overset{2}{\diamond}} T^* M$ over $T_k^* M$ and the projection d : $\hat{T}_{k+1}^* M \to \underset{k-1}{\overset{2}{\diamond}} T^* M$, which satisfies $\delta \circ d = 0$.

In particular, we have $\hat{T}_2^* M = JT^* M$. Hence we obtain the linear affine splitting

$JT^*M = T_2^*M \oplus \overset{2}{\wedge} T^*M$ and the projection $\mathrm{d} : JT^*M \to \overset{2}{\wedge} T^*M$.

Now, let $\alpha : M \to T^*M$ be a local section, i.e. a 1-form. Then we obtain $\mathrm{d}\alpha \equiv \equiv \mathrm{d} \circ j\alpha : M \to \overset{2}{\wedge} T^*M$, where d is the usual differential. Moreover T_2^*M turns out to be the space of jets of locally integrable forms and $\mathrm{d}\alpha$ the obstruction to the local integrability of α. So we obtain an unusual approach to the usual forms as a very particular case of a theory which deals with intrinsical properties of the jet spaces. These facts will be re-interpreted also in terms of connections and curvatures (III, 2, e).

II - 7 The k-jet space of an $\mathcal{S}$-fibered space

a) *The k-jet functor and $\mathcal{S}$-fibered spaces*

Let $\mathcal{S}$ be category of fibered spaces with structure. A *k-jet prolongation* of $\mathcal{S}$ is a covariant functor $\mathcal{S} \to \mathcal{S}$ such that $(E \to M, i) \in \mathrm{Ob}\, \mathcal{S} \to (J_k E \to M, i_k) \in \in \mathrm{Ob}\, \mathcal{S}$, $f \in \mathrm{Mor}\, \mathcal{S} \longmapsto J_k F \in \mathrm{Mor}\, \mathcal{S}$ and $J_k E \to J_h E$, with $0 \leqslant h < k$, are $\mathcal{S}$--fibered morphisms over M. Such a functor will be also denoted by J_k.

b) *The k-jet space of affine fibered spaces*

The natural prolongation of the algebraic fibered morphisms and the unity section lead to the k-jet prolongation of affine and group-fibered spaces. Moreover, if $(E \to M, i)$ is an affine fibered space and $H : E \to F$ an affine fibered morphism over the diffeomorphism $h : M \to N$, then $DJ_k E = J_k DE$ and $DJ_k f = J_k Df$.

Let us analyse the trivial case, in order to study the k-jet of a principal bundle. Let $E \equiv M \times G \to M$, where G is a Lie group. We have the group-fibered monomorphism $j_k : E \hookrightarrow J_k E$ over M and the fibered epimorphism (if G is abelian, a group-fibered epimorphism) $h_k : J_k E \to J_{ke} E$, $\sigma \longmapsto (j_k \circ p_k^o)(\sigma^{-1}) \cdot \sigma$, which satisfies $h_{k \mid J_{ke} E} = \mathrm{id}$. So we get the fibered isomorphism $J_k E \to E \underset{M}{\times} J_{ke} E$ over M, whose inverse is $(g, \sigma) \longmapsto j_k g \sigma$. Let us remark that, even if $E \to M$ is trivial, $J_k E \to M$ is not trivial, in general. Hence we have not in general a k-jet prolongation of principal fibered spaces. However we have the following result.

THEOREM. *Let $(P \to M, i)$ be a principal bundle, with structure group G. By taking into account the group-fibered monomorphism $M \times G \to DJ_k P$, we have the quotient space $J_k P/G$ and the bundles $[\pi_k] : J_k P \to J_k P/G$, $\tilde{\pi}_k : J_k P/G \to M$ and $\tilde{\pi}_k^h : J_k P/G \to J_h P/G$, which satisfy $[\pi_h] \circ \pi_k^h = \tilde{\pi}_k^h \circ [\pi_k]$, with $0 \leqslant h < k$. Then the map $J_k P \times G \to J_k P$, $(\sigma, g) \longmapsto \sigma g$, induce a principal structure with structure group G on the bundle $[\pi_k] : J_k P \to J_k P/G$.*

Moreover $J_k P \to J_h P$ is a translation fibered morphism over $J_k P/G \to J_h P/G$ and if $H : P \to Q$ is a principal fibered morphism over the diffeomorphism $h : M$

$\to N$, then $J_k H$ is a principal fibered morphism over $\widetilde{J_k H} : J_k P/G \to J_k Q/G$.

COROLLARY. *The fibered morphism $J_k P \times G \to J_k P$ induces a principal structure with structure group G on the bundle $J_k P/G \underset{J_h P/G}{\times} J_h P \to J_k P/G$. Moreover $J_k P \to$ $\to J_k P \underset{J_h P/G}{\times} J_h P$ is a principal fibered isomorphism over $J_k P/G$, whose inverse is $\gamma_k^h : ([\sigma_k], \sigma_h) \longmapsto \sigma_k \; \pi_k^h(\sigma_k^{-1})\sigma_h$ (it does not depend on the representative $\sigma_k \in [\sigma_k]$).*

The map γ_k^h will play an interesting role in the context of principal connections (III, 2, d).

c) *The k-jet space of a tensor product bundle*

Let $(E \to M, i)$ and $(F \to M, j)$ be linear bundles, $E \otimes F \to M$ their tensor product bundle and $\{x^\lambda, y^i\}$, $\{x^\lambda, z^j\}$, $\{x^\lambda, y^i \otimes z^j\}$ some adapted charts. We have the bilinear fibered morphisms $t : E \underset{M}{\times} F \to E \otimes F$ and $Jt : JE \underset{M}{\times} JF \to J(E \otimes F)$ over M. Their coordintate expressions are $y^i \otimes z^j \circ t = y^i z^j$ and $(y^i \otimes z^j)_\mu \circ Jt = y^i_\mu z^j +$ $+ y^i z^j_\mu$. Now, let $H : E \underset{M}{\times} F \to G$ be a bilinear fibered morphism over M; then there is a unique linear fibered morphism $\tilde{H} : E \otimes F \to G$ over M such that $\tilde{H} \circ t =$ $= H$. In particular, we obtain the linear fibered morphism $\widetilde{Jt} : JE \otimes JF \to J(E \otimes F)$. Its coordinate expression is $(y^i \otimes z^j)_\mu \circ Jt = y^i_\mu \otimes z^j + y^i \otimes z^j_\mu$.

III - CONNECTIONS

III - 1 Connections on a fibered space

a) *Connections. (See also* [4] *and* [6]*)*

Let $E \to M$ be a fibered space. We can introduce the notion of connection in several ways. The most direct approach is the following.

DEFINITION. *A connection on E* is a section $\Gamma : E \to JE$.

Hence it is a choice of an «origin» for each affine fibre of $JE \to E$.

A connection Γ determines the following objects:

– the linear affine translation $\nabla : JE \to T^*M \otimes VE$ over E, given by the composition $JE \to JE \times_M E \xrightarrow{\mathrm{id} \times \Gamma} JE \times_E JE \xrightarrow{\mathrm{diff}} T^*M \otimes VE$;

– the vertical vector field $X \equiv \nabla : JE \to T^*M \otimes VE = V^oJE \hookrightarrow TJE$;

– the splitting of the exact sequence $0 \to VE \to TE \to TM \to 0$ over E, given by each one of the following three objects;

– the section $H : E \times_M TM \to TE$ of the affine bundle $TE \to HE$ (which is a linear fibered morphism over E), given by the composition $E \times_M TM \xrightarrow{\Gamma \times \mathrm{id}} JE \times_M TM \xrightarrow{\lambda} TE$;

– the vector valued form $F : E \to T^*M \otimes TE$ (which is projectable on $\overline{F} = \mathrm{id}_{TM} : M \to T^*M \otimes TM$) associated with H;

– the linear projection fibered morphism $V : TE \to VE$ over E, given by the composition $TE \to TE \times_E (E \times_M TM) \xrightarrow{\mathrm{id} \times H} TE \times_E TE \xrightarrow{\mathrm{diff}} VE$;

– the prolongation of vector fields $P : TM \to PE$, given by $u \longmapsto F(u)$;

– the distribution $D \equiv P(TM) \hookrightarrow TE$ ($Tp : D \to TM$ is an isomorphism).

The previous objects characterize Γ itself, by means of the following main relations $\Gamma(e) = \nabla^{-1}(0_e)$ and $\Gamma = F : E \to JE \overset{\lambda}{\hookrightarrow} T^*M \otimes TE$.

When no confusion may arise, we will denote the previous objects by the same notation.

Their coordinate expressions are $\{x^\lambda, y^i; y^i_\lambda\} \circ \Gamma = \{x^\lambda, y^i; -\Gamma^i_\lambda\}$, $\Gamma^i_\lambda : E \to R$;

$$X = \nabla = (y^i_\lambda + \Gamma^i_\lambda) \mathrm{d}x^\lambda \otimes \partial_i \quad ; \quad \{x^\lambda, y^i; \dot{x}^\lambda, \dot{y}^i\} \circ H = \{x^\lambda, y^i; \dot{x}^\lambda, -\Gamma^i_\mu \dot{x}^\mu\};$$

$$P = F = (\partial_\mu - \Gamma^i_\mu \partial_i) \otimes \mathrm{d}x^\mu \quad ; \quad \{x^\lambda, y^i; \dot{y}^i\} \circ V = \{x^\lambda, y^i; \dot{y}^i + \Gamma^i_\mu \dot{x}^\mu\};$$

$$D = \langle \partial_\mu - \Gamma^i_\mu \partial_i \rangle.$$

A given connection Γ leads to the *covariant derivative* of local sections $s : M \to E$, given by $\nabla s \equiv \nabla \circ js : M \to T^*M \otimes s^*VE$. Its coordinate expression is $\nabla s = (\partial_\mu s^i + \Gamma^i_\mu \circ s)\, \mathrm{d}x^\mu \otimes (\partial_i \circ s)$.

A given connection Γ leads also to the *covariant derivative* of fibered morphisms $H : E \to E$ over M, given by the morphism $\nabla H \equiv \nabla \circ JH \circ \Gamma : E \to T^*M \otimes VE$. Its coordinate expression is $\{x^\lambda, y^i, \dot{x}_\lambda \otimes \dot{y}^i\} \circ \nabla H = \{x^\lambda, H^i, \partial_\lambda H^i - \partial_j H^i \Gamma^j_\lambda \circ H + + \Gamma^i_\lambda \circ H\}$.

Moreover a given connection Γ leads to the *covariant differential* of local projectable vector valued forms $F \in \mathcal{F}^r$, given by $d_\Gamma F \equiv [\Gamma, F] \in \mathcal{V}^{r+1}$. More explicitely, we have

$$d_\Gamma F(u_1, \dots, u_{r+1}) = \frac{(-1)^{r+1}}{r!} \sum_\sigma \in(\sigma) \; [F(u_{\sigma_1}, \dots, u_{\sigma_r}), \Gamma(u_{\sigma_{r+1}})] - \Gamma[\bar{F}(u_{\sigma_1}, \dots, u_{\sigma_r}), u_{\sigma_{r+1}}] - $$

$$- \frac{r}{2} F(u_{\sigma_1}, \dots, u_{\sigma_{r-1}}, [u_{\sigma_r}, u_{\sigma_{r+1}}]) + \frac{r}{2} \bar{F}(u_{\sigma_1}, \dots, u_{\sigma_{r-1}}, [u_{\sigma_r}, u_{\sigma_{r+1}}]).$$

Its coordinate expression is

$$d_\Gamma F = (\partial_\lambda F^i_{\lambda_1 \dots \lambda_r} + \partial_\lambda F^\mu_{\lambda_1 \dots \lambda_r} \Gamma^i_\mu - \partial_j F^i_{\lambda_1 \dots \lambda_r} \Gamma^j_\lambda +$$
$$+ \partial_j \Gamma^i_\lambda F^j_{\lambda_1 \dots \lambda_r} + \partial_\mu \Gamma^i_\lambda F^\mu_{\lambda_1 \dots \lambda_r}) \, \partial_i \otimes dx^\lambda \wedge dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r}.$$

The covariant differential is a derivation of degree 1, i.e. we have

$$d_\Gamma [F, G] = [d_\Gamma F, G] + (-1)^r [F, d_\Gamma G].$$

Let us remark that the prolongation of vector fields from M to E requires a connection, while the further prolongations to $J_k E$ are provided by the canonical maps r_k.

b) *Curvature. (See also* [4] *and* [6]*)*

We have two independent, but equivalent, ways to introduce the curvature of a connection. The equivalence is proved by their coordinate expressions.

Let us remark that $J\Gamma \circ \Gamma : E \to \hat{J}_2 E \hookrightarrow JJE$.

DEFINITION. Let $\Gamma : E \to JE$ be a connection.

a) The *curvature* is the section $R \equiv -\frac{1}{2} d_\Gamma \Gamma \equiv -\frac{1}{2_2} [\Gamma, \Gamma] : E \to \overset{2}{\wedge} T^*M \otimes VE$.

b) The *curvature* is the section $R \equiv - d \circ J\Gamma \circ \Gamma : E \to \wedge T^*M \otimes VE$.

We have the explicit formula $R(u, v) \equiv -[\Gamma(u), \Gamma(v)] + \Gamma[u, v]$ and the coordinate expression $R \equiv R^i_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes \partial_i = (\partial_{[\lambda} \Gamma^i_{\mu]} + \partial_j \Gamma^i_{[\lambda} \Gamma^j_{\mu]}) \, dx^\lambda \wedge dx^\mu \otimes \partial_i$.

Let us remark that, if $\partial_j \Gamma^i_\lambda = 0$, then $R = \partial_\lambda \Gamma^i_\mu \, dx^\lambda \wedge dx^\mu \otimes \partial_i$.

The two definitions give important integrability conditions.

THEOREM. a) *The prolongation P is a Lie algebra morphism iff the distribution D is involutive iff $R = 0$.* b) *(see* [4]*) Γ is locally integrable (i.e. for each $\sigma \in$*

$\in \Gamma(E) \hookrightarrow J_x E$, *with* $x \in M$, *there is a local section* $s : M \to E$ *in a neighbourhood of* x, *such that* $js(x) = \sigma$ *and* $js = \Gamma \circ s$, *i.e.* $\nabla s = 0$) *iff* $R = 0$.

Moreover we have the following results.

THEOREM. Generalized Bianchi identities. *We have* $d_\Gamma R = [\Gamma, R] = 0$ *and* $[R, R] = 0$.

PROPOSITION. *If* $F \in \mathcal{F}$ *then* $d_\Gamma^2 F = [F, R]$ *(which involves the first jets of* R *and* F*).*

We remark that, if $u : M \to TM$ is a vector field, then $d_\Gamma \Gamma(u) = R(u)$.

c) *k-order connections. (See also* [4]*)*

We can generalize the concept of connection to k-jet space in several ways. First we can apply the previous definitions and results by replacing $E \to M$ with $J_k E \to M$. However we assume the more particular definition, which involves the important linear affine bundle $J_k E \to J_{k-1} E$.

DEFINITION. *A connection of order* $k \geqslant 1$ on E is a section $\Gamma : J_{k-1} E \to J_k E$.

By taking into account the fibered monomorphism $J_k E \hookrightarrow J J_{k-1} E$ over M, a connection of order k on E can be viewed as a particular connection of order 1 on $J_{k-1} E$. Then we can apply the previous definitions and results to this case. In particular we obtain the following reductions.

PROPOSITION. *Let* $\Gamma : J_{k-1} E \to J_k E$ *be a connection. Then*

$$X \equiv \nabla : J_k E \to \underset{k}{\vee} T^*M \otimes VE \hookrightarrow T^*M \underset{k-1}{\vee} T^*M \otimes VE \hookrightarrow T^*M \otimes VJ_{k-1} E;$$

$$R \equiv -\frac{1}{2}[\Gamma, \Gamma] = -d \circ J\Gamma \circ \Gamma : J_{k-1} E \to {}_{k-1}\overset{2}{\diamond} T^*M \otimes VE \hookrightarrow \overset{2}{\wedge} T^*M \otimes \underset{k-1}{\vee} T^*M \otimes VE \hookrightarrow \overset{2}{\wedge} T^*M \otimes VJ_{k-1} E.$$

We have the following coordinate expressions.

$$\{x^\lambda, y^i_\Phi; y^i_\Theta\} \circ \Gamma = \{x^\lambda, y^i_\Phi; -\Gamma^i_\Theta\}, \qquad \Gamma^i_\Theta : J_{k-1} E \to R; \qquad X = \nabla = (y^i_\Theta + \Gamma^i_\Theta)\, dx^\Theta \otimes \partial_i;$$

$$\{x^\lambda, y^i_\Phi; \dot{x}^\lambda, \dot{y}^i_\Psi, \dot{y}^i_\Sigma\} \circ H = \{x^\lambda, y^i_\Phi; \dot{x}^\lambda, y^i_{\Psi+\mu} \dot{x}^\mu, -\Gamma^i_{\Sigma+\mu} \dot{x}^\mu\};$$

$$P = F = (\partial_\mu + y^i_{\Psi+\mu} \partial_i^\Psi - \Gamma^i_{\Sigma+\mu} \partial_i^\Sigma) \otimes dx^\mu{}_i; \qquad D = (\partial_\mu + y^i_{\Psi+\mu} \partial_i^\Psi - \Gamma^i_{\Sigma+\mu} \partial_i^\Sigma);$$

$$R = (\partial_{[\lambda} \Gamma^i_{\Theta]} + y^i_{\Psi+[\lambda} \partial_j^\Psi \Gamma^i_{\Theta]} - \Gamma^j_{\Psi+[\lambda} \partial_j^\Sigma \Gamma^i_{\Theta]})\, dx^\lambda \wedge dx^\Theta \otimes \partial_i,$$

with $0 \leqslant |\Phi| \leqslant k-1$, $|\Theta| = k$, $0 \leqslant |\Psi| \leqslant k-2$, $|\Sigma| = k-1$.

d) *Pullback of a connection*

We will be concerned with the following facts (III, 2, b). Let $F \to N$ be a fibered space, $h : M \to N$ a C^∞ map, $E \equiv h^*F \to M$ the pullback fibered space and $H : E \to F$ the canonical fibered morphism. Let $\Lambda : TF \to VF$ be a connection on $F \to N$. Then the map $\Gamma : TE \to VE$ given by the composition $T(h^*F) = (Th)^*TF \to h^*TF \to M \underset{M}{\times} TF \xrightarrow{\mathrm{id} \times \Lambda} M \underset{M}{\times} VF = V(h^*F)$ is a connection on $E \to M$, called the *pullback connection* of Λ with respect to h. Moreover, the curvature $R : E \to \overset{2}{\wedge} T^*M \otimes VE$ of Γ is the pullback of the curvature $\Omega : F \to \overset{2}{\wedge} T^*N \otimes VF$ of Λ, with respect to h. The coordinate expressions are $\Gamma^i_\lambda = (\Lambda^i_\alpha \circ H)\, \partial_\lambda h^\alpha$ and $R^i_{\lambda\mu} = (\Omega^i_{\alpha\beta} \circ H)\, \partial_\lambda h^\alpha \partial_\mu h^\beta$.

e) *Sesquiholonomic connections. (See also* [4]*)*

By taking into account the linear affine bundle $\hat{J}_{k+1}E \to \hat{J}_k E$, we have another interesting kind of connections.

DEFINITION. *A sesquiholonomic connection of order* $1 \leqslant k$ on $E \to M$ is a section $\hat{\Gamma} : J_k E \to \hat{J}_{k+1}E$. The *torsion* of Γ is the linear affine fibered morphism $T \equiv \mathrm{d} \circ \hat{\Gamma} : J_k E \to {}_{k-1}\overset{2}{\diamond}\, T^*M \otimes VE$ over $J_{k-1}E \to E$.

The coordinate expressions are $\{x^\lambda, y^i_\Lambda ; y^i_{\mu\Theta}\} \circ \hat{\Gamma} = \{x^\lambda, y^i_\Lambda ; -\Gamma^i_{\mu\Theta}\}$, with $\Gamma^i_{\mu\Theta} : J_k E \to R$, and $T = \Gamma^i_{[\mu\Theta]}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\Theta \otimes \partial_i$.

Hence, any sesquiholonomic connection $\hat{\Gamma}$ of order k is the linear affine direct sum of a connection Γ of order $k+1$ and its torsion T.

We will find (III, 2, e) the torsion of the usual connections on a manifold as a particular case.

III - 2 Structures of connections

a) *Structures of connections*

Let us remark that the connections on E are not locally finite dimensional on M. However we can introduce the following concept.

DEFINITION. *A structure of connections* $\mathcal{C}$ *of order* $1 \leqslant k$ on E is a fibered space $C \to M$ together with a fibered morphism $\gamma : K \equiv C \underset{M}{\times} J_{k-1}E \to J_k E$ over $J_{k-1}E$, such that the map $\tilde{\Gamma} \longmapsto \Gamma$, which associates the local connection $\Gamma : J_{k-1}E \to J_k E$, given by the composition $J_{k-1}E \to M \underset{M}{\times} J_{k-1}E \xrightarrow{\tilde{\Gamma} \times \mathrm{id}} C \underset{M}{\times} J_{k-1}E \xrightarrow{\gamma} J_k E$, with each local section $\tilde{\Gamma} : M \to C$, is injective. $\tilde{\Gamma}$ or Γ is said a $\mathcal{C}$-*connection.*

Let us consider the bundle $K \equiv C \underset{M}{\times} J_{k-1}E \to C$. If $\tilde{\Gamma} : M \to C$ is a $\mathcal{C}$-connection, then we have a natural fibered isomorphism $\tilde{\Gamma}^* K \to J_{k-1}E$ over M.

We will denote an adapted chart of C by $\{x^\lambda, z^a\}$. Then we have the coordinate expressions $\{x^\lambda, y^i_\Phi; y^i_\Theta\} \circ \gamma = \{x^\lambda, y^i_\Phi; \gamma^i_\Theta\}$, with $\gamma^i_\Theta : K \to R$, $0 \leqslant |\Phi| \leqslant k-1$, $|\Theta| = k$, and $\{x^\lambda, z^a\} \circ \Gamma = \{x^\lambda, \Gamma^a\}$, with $\Gamma^a : M \to R$. By abuse of notation, we write $\Gamma^i_\Theta = -\gamma^i_\Theta \circ \tilde{\Gamma}$.

Let $\mathcal{S}$ be a category of structure and J_k its prolongation. Then an *$\mathcal{S}$-connection* of order $1 \leqslant k$ on the $\mathcal{S}$-fibered space $E \to M$ is a connection $\Gamma : J_{k-1}E \to J_kE$, which is an $\mathcal{S}$-fibered morphism over M. Then the space $C = \bigsqcup_{x \in M} \mathrm{Mor}_x (J_{k-1\,x}E, J_{k\,x}E) \cap \mathrm{Sections}\,(J_{k-1\,x}E, J_{k\,x}E) \to M$ and the natural map $\gamma : C \underset{M}{\times} J_{k-1}E \to J_kE$ constitute a structure of connections, under suitable conditions on the differentiability.

Moreover, under the previous hypothesis, if we drop a closed subset $E'_x \hookrightarrow E_x$, such that $E'_x \neq E_x$, with $x \in M$, then $\mathcal{C}$ is still a structure of connections on $E - E'_x \to M$.

Similar considerations hold with respect to the bundles with symmetries.

But, conversely, we can give simple examples of structures of connections, which do not come from bundles with symmetries.

A very interesting case is when the structure of connections comes from a structure of projectable vector valued forms.

DEFINITION. *A* structure of connections $\mathcal{C}$ is *complete* if $C \hookrightarrow B$ is a fibered sub-space of a structure $\mathcal{B}$ of projectable vector valued forms.

In such a case if $\tilde{\Gamma}$ is a $\mathcal{C}$-connection, i.e. a $\mathcal{B}$-form, then also R (and $\mathrm{d}R = 0$) is a $\mathcal{B}$-form.

PROPOSITION. *We have the fibered morphism $\rho : JC \underset{M}{\times} J_{k-1}E \to \overset{2}{\underset{k-1}{\Diamond}} T^*M \otimes VE$ over E, given by the composition of the following maps $JC \underset{M}{\times} J_{k-1}E \to JC \underset{M}{\times} (C \underset{M}{\times} J_{k-1}E) \xrightarrow{\mathrm{id}\times\gamma} JC \underset{M}{\times} J_kE \to J(C \underset{M}{\times} J_{k-1}E) \xrightarrow{J\gamma} JJ_kE$ and $\mathrm{d} : \hat{J}_{k+1}E \to \overset{2}{\underset{k-1}{\Diamond}} T^*M \otimes VE$, by taking into account that the first map takes its values on $\hat{J}_{k+1}E \to JJ_kE$.*

Hence, if $\tilde{\Gamma} : M \to C$ is a $\mathcal{C}$-connection, then we have $R = \rho \circ j\tilde{\Gamma}$ (where we write $j\tilde{\Gamma} : J_{k-1}E \to JC \underset{M}{\times} J_{k-1}E$, by abuse of notation).

The coordinate expression is $\rho = (\partial_\lambda \gamma^i_\Theta + y^i_{\Phi+[\lambda}\gamma^i_{\Theta]} + \gamma^j_{\Sigma+[\lambda}\partial^\Sigma_j \gamma^i_{\Theta]} + z^a_{[\lambda}\,\partial_a \gamma^i_{\Theta]})\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\Theta \otimes \partial_i$.

The $\mathcal{C}$-structures could be a further way in order to give a «structure» on a fibered space. Let $(E \to M, \gamma)$ and $(E' \to M', \gamma')$ be fibered spaces together with

$\mathcal{C}$-structures. Then, a *$\mathcal{C}$-morphism* is a fibered isomorphism $H : E \to E'$, which induces a fibered morphism of the $\mathcal{C}$-structures.

b) *The universal connection and curvature*

We have an interesting result, which produces important examples as particular cases (d and e). It extends a result by Garcia on principal bundles [1].

THEOREM. *Let $\mathcal{C}$ be a structure of connections. Then there is a unique connection $\Lambda : K \to JK$ on the bundle $K \to \mathcal{C}$, such that, for each local section $\tilde{\Gamma} : M \to \mathcal{C}$, we have $\Gamma = \tilde{\Gamma}_* \Lambda$ (hence $R = \tilde{\Gamma}_* \Omega$, where R and Ω are the curvatures of Γ and Λ). More precisely, Λ is given by the composition $T(\mathcal{C} \times_M J_{k-1}E) \xrightarrow{(\pi, T\gamma)} \mathcal{C} \times_M TJ_k E \xrightarrow{\mathrm{id} \times \theta_k} \mathcal{C} \times_M VJ_{k-1}E = VK$. Moreover, Λ is the unique section $\Lambda : K \to JK$ which makes the following diagram commutative*

$$\begin{array}{ccccc}
K & \xrightarrow{\Lambda} & JK & \xhookrightarrow{\lambda} & T^*\mathcal{C} \otimes TK \\
\gamma \big\downarrow & & & & \big\downarrow \\
J_k E & \xhookrightarrow{\lambda_k} & T^*M \otimes TJ_{k-1}E & \hookrightarrow & T^*\mathcal{C} \otimes TJ_{k-1}E
\end{array}$$

The coordinate expressions are $\Lambda^i_{\mu\Psi} = -y^i_{\mu+\Psi}$, $\Lambda^i_{\mu\Sigma} = -\gamma^i_{\mu+\Sigma}$, $\Lambda^i_{a\Phi} = 0$ and $\Omega =$ $= [-(\partial_{[\lambda}\gamma^i_{\Sigma+\mu]} + y^j_{\Psi+[\lambda}\partial^\Psi_j \gamma^i_{\Sigma+\mu]} + \gamma^j_{\Theta+[\lambda}\partial^\Theta_j \gamma^i_{\Sigma+\mu]})\, dx^\lambda \wedge dx^\mu + \partial_a \gamma^i_{\lambda+\Sigma}\, dx^\lambda \wedge$ $\wedge\, dz^a] \otimes \partial^\Sigma_i$, with $0 \leqslant |\Phi| \leqslant k-1$, $0 \leqslant |\Psi| \leqslant k-2$, $|\Sigma| = |\Theta| = k-1$.

Proof. Existence. The coordinate expression shows that Λ is a connection and that, for each $\tilde{\Gamma} : M \to \mathcal{C}$, we have $\Gamma = \tilde{\Gamma}_* \Lambda$. Uniqueness. The equality $-\gamma^i_{\lambda+\Phi} \circ \tilde{\Gamma} = \Gamma^i_{\lambda\Phi} = \Gamma^i_{\lambda+\Phi} = \Lambda^i_{\lambda\Phi} \circ \tilde{\Gamma} + \Lambda^i_{a\Phi} \circ \tilde{\Gamma}\, \partial_\lambda \tilde{\Gamma}^a$ proves the result.

We call Λ and Ω the *universal* connection and curvature.

Let us remark that the universal property ensures a suitable non degeneracy of the form Ω.

If the structure of connections is determined by a category of structure S, then the universal connection is an S-connection (with respect to a natural suitable definition, which takes into account the appropriate pullbacks).

c) *Invertible structures of connections*

An interesting case is the following one.

DEFINITION. A structure of connections is *invertible* if γ is a fibered isomorphism over $J_{k-1}E$.

The following conditions are equivalent: a) C is invertible; b) $J_k E \to J_{k-1} E$ is the bullback of $C \to M$, with respect to $J_{k-1} E \to M$.

If C is invertible, then we can define the potential of a connection. In fact, if $\tilde{\Gamma} : M \to C$ and $g : M \to E$ are sections, then the following conditions are equivalent: a) $\tilde{\Gamma} = \pi^1 \circ \gamma^{-1} \circ j_k g$; b) $\nabla g = 0$. In such a case, Γ is said *exact* and g is called its *potential*.

We shall find two types of examples (d and e).

d) *Affine connections*

Let $(E \to M, i)$ be an affine fibered space. Then a connection $\Gamma : J_{k-1} E \to J_k E$ is affine if it is an affine fibered morphism over M, i.e. if $\Gamma(\sigma\tau) = \Gamma(\sigma) D\Gamma(\tau)$, where $D : DJ_{k-1} E \to DJ_k E$ is a group-fibered morphism over M. The affine connections constitute a structure of connections, under suitable hypothesis on the dimensions. Similar considerations hold with respect to group-connections.

In particular the linear affine connections constitute a complete structure of connections. The coordinate expressions of a linear affine connection and its curvature are $\Gamma_\Theta^i = \Gamma_{\Theta j}^{i\Phi} y_\Phi^j + \overline{\Gamma}_\Theta^i$, with $\Gamma_{\Theta j}^{i\Phi}$, $\overline{\Gamma}_\Theta^i : M \to R$, and $R_{\lambda\Theta}^i = \partial_{[\lambda} \Gamma_{\Theta| j}^i + \partial_{[\lambda} \overline{\Gamma}_{\Theta]}^i + y_{[\lambda+\Psi}^j \Gamma_{\Theta]j}^i - \Gamma_{[\lambda+\Sigma}^j \Gamma_{\Theta]j}^{i\,\Sigma}$, with $0 \leqslant |\Phi| \leqslant k-1$, $0 \leqslant |\Psi| \leqslant k-2$, $|\Sigma| = K-1$, $|\Theta| = k$. The coordinate expression of γ is $\gamma_\Theta^i = -z_{\Theta j}^{i\Phi} y_\Phi^j - z_\Theta^i$. The fibered space $K \to C$ is linear affine and the universal connection $\Lambda : K \to JK$ and its curvature $\Omega : K \to \overset{2}{\Lambda} T^*C \otimes \underset{k-1}{\vee} T^*C \otimes DE$ are linear affine. Their coordinate expressions are $\Lambda_{\mu\Psi}^i = -y_{\mu+\Psi}^i$, $\Lambda_{\mu\Sigma}^i = z_{\mu+\Sigma\, j}^{i\ \ \Phi} y_\Phi^j + z_{\mu+\Sigma}^i$, $\Lambda_{a\Phi}^i = 0$,

$$\Omega = [(y_{[\lambda+\Psi}^j z_{\mu]+\Sigma\, j}^{i\ \ \Psi} + z_{[\lambda+\Psi\, k}^{j\ \ \Phi} z_{\mu]+\Sigma\, j}^{i\ \ \Psi} y_\Phi^k + z_{[\lambda+\Psi}^j z_{\mu]+\Sigma\, j}^{i\ \ \Psi}) \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu - y_\Phi^j \mathrm{d}x^\lambda \wedge \mathrm{d}z_{\lambda+\Sigma\, j}^{i\ \ \Phi} - \mathrm{d}x^\lambda \wedge \mathrm{d}z_{\lambda+\Sigma}^i] \otimes \partial_i^\Sigma.$$

Similar considerations hold with respect to the linear connections. The coordinate expressions can be obtained from the previous ones, by dropping $\overline{\Gamma}_\Theta^i$, $\partial_\lambda \overline{\Gamma}_\Theta^i$, z_Θ^i and $\mathrm{d}z_{\lambda+\Sigma}^i$.

Now, let $\Gamma : E \to JE$ and $H : F \to JF$ be linear connections. Then the *tensor product connection* is the unique linear connection $\Gamma \otimes H : E \otimes F \to JE \otimes JF$ such that $\tilde{Jt} \circ (\Gamma \times H) = (\Gamma \otimes H) \circ t$. The coordinate expression is $(\Gamma \otimes H)_\mu^{ij} = \Gamma_\mu^i z^j + y^i H_\mu^j$. The tensor product connections constitute a sub-structure of connections of that of linear connections on $E \otimes F$. In particular, if $E \equiv F$, then we can consider the further sub-structures of the symmetric and antisymmetric tensor product connections.

The usual principal connections can be easily recovered as a particular case of our general approach. In fact a *principal connection* is defined as an affine connection $\Gamma : P \to JP$ on a principal bundle, such that $D\Gamma = j : DP \to DJP$, i.e. $\Gamma(yg) = \Gamma(y) g$. The principal connections constitute an invertible and complete structure of connections, where $C \equiv JP/G$ and $\gamma \equiv \gamma_1^o : C \underset{M}{\times} P \to JP$ is given in (II, 7, b). In this way we can recover the canonical connection introduced by Garcia [1].

In particular, the coordinate expressions of a linear affine principal connection and its curvature are $\Gamma^i_\Theta : M \to R$ and $R^i_{\lambda\Theta} = \partial_{[\lambda}\Gamma^i_{\Theta]}$, with $|\Theta| = k$.

e) *Connections on the line-bundle. (See also* [3]*)*

Let us consider the principal bundle $E \equiv M \times R \to M$. We can recover well known facts as a very particular case of our approach and so we can obtain a deeper understanding of their meaning.

THEOREM. *The principal connections $\Gamma : E \equiv M \times R \to JE \equiv R \times T^*M$ are the sections projectable on the usual forms $\alpha : M \to T^*M$, which characterize them. Their curvature $R \equiv \mathrm{d} \circ J\Gamma \circ \Gamma : E \equiv M \times R \to \overset{2}{\wedge} T^*M \otimes VE \equiv R \otimes \overset{2}{\wedge} T^*M$ is projectable on the usual differential $d\alpha \equiv \mathrm{d} \circ j\alpha : M \to \overset{2}{\wedge} T^*M$, which characterizes it.*

Moreover, the principal connections constitute a complete and invertible structure of connections given by $C = T^*M$ and $\gamma = \mathrm{id} : C \underset{M}{\times} E \equiv T^*M \times R \to \to JE \equiv R \times T^*M$. The universal connection $\wedge$ and its curvature Ω are projectable (up to the sign) on the Liouville form λ and the symplectic form ω which characterize them. The universal properties can be re-interpreted in terms of forms as follows: $\alpha^*\lambda = \alpha$, $\alpha^*\omega = \mathrm{d}\alpha$.

Now, let us consider the sesquiholonomic connections $\hat{\Gamma} : JE \equiv R \times T^*M \to \to \hat{J}_2 E \equiv R \times \hat{T}_2^* M = R \times JT^*M$, which are projectable on $\Gamma : T^*M \to JT^*M$, which characterize $\hat{\Gamma}$. These connections can be identified with the connections on T^*M. Hence we can define the *torsion* of such a Γ as the projection $t : T^*M \to \to \overset{2}{\wedge} T^*M$ of the torsion $T \equiv \mathrm{d} \circ \hat{\Gamma} : R \times T^*M \to R \times \overset{2}{\wedge} T^*M$ of $\hat{\Gamma}$, which characterizes T.

Finally, we have the following interesting result.

THEOREM. *Let $L \to \underset{2}{\vee} T^*M$ be a fibered sub-space of non degenerate bilinear symmetric forms. Let $\underset{2}{\vee} B \to M$ be the bundle of symmetric tensor product linear connections $T^*M \to JT^*M$ with vanishing torsion. Then the fibered morphism $\gamma : \underset{2}{\vee} B \times L \to JL$ is an invertible structure of connections.*

It is just for this reason that we can introduce the pseudo-Riemannian connections (see also [7]).

III – 3 The Maxwell-type equations on a fibered space.

Let $E \to M$ be a fibered space, $g : M \to \underset{2}{\vee} TM$ a non degenerate symmetric bilinear form and $* : \overset{r}{\Lambda} T^*M \to \overset{m-r}{\Lambda} T^*M$ the Hodge isomorphism.

Let $A : E \to JE \hookrightarrow T^*M \otimes TE$ be a connection. Then we can define its curvature $F \equiv -\mathrm{d} \circ JA \circ A = -\frac{1}{2} \mathrm{d}_A A : E \to \overset{2}{\Lambda} T^*M \otimes VE$. The generalized Bianchi identities give $\mathrm{d}_A F = 0$.

Moreover, for any projectable vector valued form $G : E \to \overset{r}{\Lambda} T^*M \otimes TE$, we set $\delta_A G \equiv (-1)^{r-1} * \mathrm{d}_A * G : E \to \overset{r-1}{\Lambda} T^*M \otimes VE$. Then we have $\delta_A^2 G = \pm * [* G, F]$. In particular, we have $\delta_A A : E \to VE$ and $\delta_A F : E \to T^*M \otimes VE$. Then, by taking into account $[* G, H] = -[* H, G]$ for any vertical projectable vector valued forms G and F, we get $\delta_A^2 F = 0$.

Further, for any projectable vector valued form $G : E \to \overset{r}{\Lambda} T^*M \otimes TE$, we set $\Delta_A G \equiv (\mathrm{d}_A \delta_A + \delta_A \mathrm{d}_A) G : E \to \overset{r}{\Lambda} T^*M \otimes VE$.

Hence we can write the Maxwell-type equation $\delta_A F = 0$, the Lorentz-type equation $\delta_A A = 0$ and the Laplace-type equation $\Delta_A A = 0$ in the unknown $A : E \to T^*M \otimes TE$. We have the two identities $\mathrm{d}_A^2 A = -2\, \mathrm{d}_A F = 0$ and $\delta_A^2 F = 0$.

Next, let us assume that we have a complete structure $\mathcal{C}$ of connections, compatible with the Hodge isomorphism (for instance, the linear affine connections or the linear connections), given by the structure of forms $B \to M$. Then, by choosing $\tilde{A} : M \to C \hookrightarrow B^1$, we obtain $\tilde{F} : M \to B^2$, $\delta_A \tilde{A} : M \to B^o$, $\delta_A \tilde{F} : M \to B^1$, $\Delta_A \tilde{A} : M \to B^1$. Hence, in such a case, the previous objects are reduced on a fibered space over M.

In the particular case when $E \equiv M \times R \to M$ and $\mathcal{C}$ is the structure of principal connections, then we recover the Maxwell equations.

Conclusions

Our starting point is that bundles and connections have already proved useful for physical field theories and also jets have been used for particular purposes.

We have provided a very general framework that can support arbitrary physical field theories by viewing them through the appropriate geometric fibered structures they demand over the underlying space.

This is achieved first by exploiting a view of symmetries, which hiterto has been rather informally handled. Second, we use the jet spaces as the general framework suitable for providing the natural unification of the geometric differential operations. Finally we present a broad theory of connections on jet spaces, which extends the usual one.

Several incidental results turns out to be interesting, for example: the $\mathcal{S}$-category structure, the generalized affine spaces, a new view-point on principal bundles, the graded Lie algebra of projectable vector valued forms, the structures of forms, the functorial techniques on jets (which provide compact calculations), the prolongation map, the splitting of sesquiholonomic jet spaces, the generalized connections and curvatures, the structures of connections, the universal connection and curvature (detached from principal bundles), the potential of connections, the analysis of tangent and cotangent spaces.

In the context of gauge theories, this new framework shows that connections do not need to be attached to principal bundles, contrary to normal belief practise. Also in application to Maxwell type theories, the approprieate equations can always be written on the configuration space and, in the case of a complete connection structure being available, the equation projects fuithfully to the base space. We expect further physically useful results for many field theories; work on such application is in progress.

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Further references on the large bibliography concerning this subject can be found in the above papers.

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