Quantum Potential in Covariant Quantum Mechanics

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Devoted to Ivan Kolář on occasion of his 80th birthday

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Abstract

We discuss several features of the classical quantum potential appearing in Covariant Quantum Mechanics.

In particular, we compare the "observed potential" $A[K,G,o]$ of the joined spacetime connection K with the potential $A^{\uparrow}$ of the cosymplectic phase 2–form $\Omega[K,G]$ and with the potential $A^{\uparrow}$ of the upper quantum connection $\text{Ч}^{\uparrow}$.

Moreover, we discuss the distinguished observer $o[\Psi]$ and the distinguished timelike potential $A[\Psi]$ associated with a non vanishing quantum section Ψ.

We show that the above objects play a natural role in the context of the kinetic quantum momentum $\mathsf{Q}[\Psi]$, of the quantum probability current $\mathsf{J}[\Psi]$, of the Schrödinger operator $\mathsf{S}[\Psi]$ and of the classical fluid associated with a non vanishing quantum section Ψ.

Key words: covariant classical mechanics, covariant quantum mechanics, galileian metric, phase space, quantum connection, quantum potential.

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Contents

Introduction

Starting from E. Cartan [8], there have been proposed several formulations of Quantum Mechanics in a curved spacetime with absolute time (see, for instance, [2, 3, 4, 12, 13, 19, 20, 22, 21, 23, 24, 25, 26, 32, 42, 43, 44, 45, 46, 47, 58, 59, 61, 68, 69] and citations therein).

Covariant Quantum Mechanics is an approach to Quantum Mechanics in a curved spacetime fibred over time and equipped with absolute time and a riemannian metric on its fibres, aimed at implementing several features of General Relativity in this riemannian framework. This formulation started some years ago [29] and has been further developed by several papers (see, for instance, [6, 7, 28, 31, 32, 36, 50, 52, 53, 55, 56, 65, 66] and citations therein).

Several ideas and methods are typical features of Covariant Quantum Mechanics. For instance, we consider as phase space the 1st jet space $J_1\boldsymbol{E}$, we couple the gravitational field $K^{\natural}$ and the electromagnetic field F into a *joined* spacetime connection, which yields several joined objects of the phase space, such as the *cosymplectic* 2–form Ω, which plays a fundamental role in classical and quantum mechanics. Moreover, we introduce the special phase functions and their Lie bracket. In the quantum theory, we introduce a complex line bundle $\boldsymbol{Q}$ over spacetime and an *upper quantum connection* $\text{Ч}^{\uparrow}$, which is hermitian and "reducible" and whose curvature is proportional to Ω. All main further quantum objects are derived in a natural way from this connection, by means of a "criterion of projectability", which allows us to get rid of observers, in view of the covariance of the theory. The quantum operators are achieved via the classification of hermitian quantum vector fields and their Lie algebra isomorphism with the special phase functions.

Scales

We deal with units of measurement on the same footing of gauges, observers and coordinates. So, in order to make our theory explicitly independent of "*units of measurement*", we use the notion of "*spaces of scales*" [37].

We define a *positive space* to be a semi-vector space $\mathbb{U}$ on the semi–field $\mathbb{R}^+$, such that the scalar product $\cdot : \mathbb{R}^+ \times \mathbb{U} \to \mathbb{U}$ is a left free and transitive action of the group $(\mathbb{R}^+, \cdot)$ on $\mathbb{U}$. We can define in a natural way the tensor product $\mathbb{U} \otimes \mathbb{U}'$ of two positive spaces, the rational powers $\mathbb{U}^{m/n}$ of a positive space and the dual $\mathbb{U}^*$ of a positive space. We make a natural identification $\mathbb{U}^* \simeq \mathbb{U}^{-1}$. Moreover, we can define in a natural way the tensor product $\mathbb{U} \otimes \boldsymbol{V}$ of a positive space with a vector space; indeed, it turns out to be a vector space.

We consider the following *basic positive spaces*: 1) the space $\mathbb{T}$ of *time intervals*, 2) the space $\mathbb{L}$ of *lengths*, 3) the space $\mathbb{M}$ of *masses*. Then, we define a *space of scales* to be any tensor product of rational powers of the above positive spaces.

We consider the *Planck constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ as a "*universal scale*". Moreover, we will consider a *mass* $m \in \mathbb{M}$ and *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$. We denote a *time unit of measurement* and its dual, respectively, by $u_0 \in \mathbb{T}$ and $u^0 \in \mathbb{T}^* \simeq \mathbb{T}^{-1}$.

1 Setting of the classical theory

We start by summarising some achievements of Covariant Classical Mechanics.

1.1 Spacetime

We consider *time* to be an oriented 1–dimenesionl affine space $\boldsymbol{T}$, associated with the vector space $\mathbb{T} \otimes \mathrm{I\!R}$, and *spacetime* to be an oriented 4–dimensional manifold $\boldsymbol{E}$ equipped with a *time fibring*

$$t : \boldsymbol{E} \to \boldsymbol{T} .$$

The time fibring yields the distinguished *time form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$.

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$.

We shall refer to *spacetime charts* $(x^\lambda) \equiv (x^0, x^i)$, defined as charts of the manifold $\boldsymbol{E}$, which are adapted to the time fibring, the affine structure of $\boldsymbol{T}$ and the orientation of $\boldsymbol{E}$ and $\boldsymbol{T}$. Every spacetime chart (x^λ) yields a *time scale* $u_0 \in \mathbb{T}$. We shall denote the associated bases of vector fields and forms by $(\partial_\lambda) \equiv (\partial_0, \partial_i)$ and $(d^\lambda) \equiv (d^0, d^i)$. Accordingly, we shall denote the linear fibred charts of the tangent bundle $T\boldsymbol{E} \to \boldsymbol{E}$ by $(x^\lambda, \dot{x}^\lambda)$.

We denote by $V\boldsymbol{E} \subset T\boldsymbol{E}$ the 3–dimensional *vertical subbundle* annihilated by dt and by $H^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ the 1–dimensional *horizontal subbundle* generated by dt. The vertical projection $T^*\boldsymbol{E} \to V^*\boldsymbol{E}$ is denoted by the restriction symbol $\vee$.

1.2 Phase space

We choose, as *phase space*, the 1st jet space [57] of motions $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$.

It turns out to be the 7–dimensional affine subbundle $J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ over $\boldsymbol{E}$ characterised by the constraint $u^0 \otimes \partial_0 = \boldsymbol{1}$. The associated vector space is $\mathbb{T}^* \otimes V\boldsymbol{E}$. We shall denote the affine fibred charts of the phase space by (x^λ, x^i_0).

The phase space is naturally equipped with the *contact map* and the *complementary contact map*

$$\text{д} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \qquad \text{and} \qquad \theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E} ,$$

with coordinate expressions

$$\text{д} = u^0 \otimes (\partial_0 + x^i_0\, \partial_i) \qquad \text{and} \qquad \theta = (d^i - x^i_0\, d^0) \otimes \partial_i .$$

The contact maps yield a splitting, over $J_1\boldsymbol{E}$, of the tangent bundle $T\boldsymbol{E} \to \boldsymbol{E}$ and of the cotangent bundle $T^*\boldsymbol{E} \to \boldsymbol{E}$.

1.3 Observers

An *observer* is defined to be a section $o : \boldsymbol{E} \to J_1\boldsymbol{E}$. Thus, an observer o is the velocity field of a classical continuum motion $c : (\mathbb{T} \otimes \mathrm{I\!R}) \times \boldsymbol{E} \to \boldsymbol{E}$, which plays the role of reference system.

An observer o is characterised by the associated *observed contact map* and the *complementary observed contact map*

$$\text{д}[o] := \text{д} \circ o : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \qquad \text{and} \qquad \theta[o] := \theta \circ o : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E} \,,$$

with coordinate expressions

$$\text{д}[o] = u^0 \otimes (\partial_0 + o^i_0 \, \partial_i) \qquad \text{and} \qquad \theta[o] = (d^i - o^i_0 \, d^0) \otimes \partial_i \,.$$

The observed contact maps yield a splitting, over $\boldsymbol{E}$, of the tangent bundle $T\boldsymbol{E} \to \boldsymbol{E}$ and of the cotangent bundle $T^*\boldsymbol{E} \to \boldsymbol{E}$.

A spacetime chart (x^λ) is said to be *adapted* to an observer o if $o^i_0 = 0$. Many spacetime charts (x^λ) are adapted to an observer o. Conversely, each spacetime chart (x^λ) is associated with a unique observer o, which is characterised by the condition $\text{д}[o] = u^0 \otimes \partial_0$.

Given an observer $o : \boldsymbol{E} \to J_1\boldsymbol{E}$, the other observers $\acute{o} : \boldsymbol{E} \to J_1\boldsymbol{E}$ are of the type $\acute{o} = o + v$, where $v : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$.

In comparison with the einsteinian general relativity, in both cases, the observers can be defined as normalised spacetime scaled vector fields [35]. But, in the einsteinian case, the normalisation is achieved via the lorentzian metric, while, in the galileian case, the normalisation is achieved via the time fibring.

1.4 Galileian metric

Next, we consider spacetime to be equipped with a *scaled spacelike riemannian metric*

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}) \,.$$

With reference to a particle of mass $m \in \mathbb{M}$, and by taking into account the Planck constant $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$, we define the rescaled spacelike metric

$$G := \tfrac{m}{\hbar} \, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}) \,.$$

Actually, in the classical theory, any value of $\hbar$ with the above scale dimension would do, while in the quantum theory the value of $\hbar$ has an essential role. We have the coordinate expressions

$$g = g_{ij} \, \overset{\vee}{d}{}^i \otimes \overset{\vee}{d}{}^j \qquad \text{and} \qquad G = G^0_{ij} \, u_0 \otimes \overset{\vee}{d}{}^i \otimes \overset{\vee}{d}{}^j \,,$$

with $g_{ij} \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^2 \otimes \mathrm{I\!R})$ and $G^0_{ij} \in \mathrm{map}(\boldsymbol{E}, \mathbb{T} \otimes \mathrm{I\!R})$.

In comparison with the einsteinian general relativity, we have replaced the lorentzian spacetime metric with the time fibring and a riemannian spacelike metric [35]. Indeed, this is the main difference between the two theories; all other differences arise from this one. In particular, in the galileian case, the speed of the light c has no meaning.

The spacelike metric g and the time form dt, along with the time and spacetime orientations yield naturally the scaled *spacelike volume form* and the scaled *spacetime volume form*

$$\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 T^* \boldsymbol{E} \qquad \text{and} \qquad \upsilon := dt \wedge \eta : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^* \boldsymbol{E} \,,$$

with coordinate expressions

$$\eta = \sqrt{|g|}\, \overset{\vee}{d}{}^1 \wedge \overset{\vee}{d}{}^2 \wedge \overset{\vee}{d}{}^3 \qquad \text{and} \qquad \upsilon = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 \,.$$

Given an observer o, we define the *observed kinetic energy*, the *observed kinetic momentum* and the *observed Poincaré–Cartan form* to be, respectively, the sections

$$\begin{aligned}
\mathcal{K}[G,o] &:= \tfrac{1}{2}\, G\,(\nabla[o], \nabla[o]) && \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})\,, \\
\mathcal{Q}[G,o] &:= \theta[o] \,\lrcorner\, (G^\flat\, \nabla[o]) && \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})\,, \\
\Theta[G,o] &:= -\,\mathcal{K}[G,o] + \mathcal{Q}[G,o] && \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})\,,
\end{aligned}$$

with coordinate expressions

$$\begin{aligned}
\nabla[o] &= (x^i_0 - o^i_0)\, u^0 \otimes \partial_i \,, \\
\mathcal{K}[G,o] &= \tfrac{1}{2}\, G^0_{ij} (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0 \,, \\
\mathcal{Q}[G,o] &= G^0_{ij} (x^j_0 - o^j_0)\, (d^i - o^i_0\, d^0)\,, \\
\Theta[G,o] &= (-\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + \tfrac{1}{2}\, G^0_{ij}\, o^i_0\, o^j_0)\, d^0 + G^0_{ij}\, (x^j_0 - o^j_0)\, d^i \,.
\end{aligned}$$

1.5 Galileian spacetime connection

We define a *metric preserving spacetime connection* to be a connection of $\boldsymbol{E}$

$$K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}\,,$$

which is linear, torsion free and which fulfills the conditions

$$\nabla dt = 0 \qquad \text{and} \qquad \nabla g = 0\,.$$

Moreover, such a spacetime connection K is said to be *galileian* if its curvature fulfills the additional condition $R_{i\mu j\nu} = R_{j\nu i\mu}$ (see, also, for instance, [42, 46]).

The coordinate expression of a metric preserving spacetime connection K is of the type

$$\begin{aligned}
K &= d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i{}_\mu\, \dot{x}^\mu\, \dot{\partial}_i) \\
&= d^\lambda \otimes \partial_\lambda - \tfrac{1}{2}\, G^{ij}_0 \left((\partial_0 G^0_{hj}\, \dot{x}^h\, d^0 + \dot{x}^0\, d^h) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, \dot{x}^k\, d^h\right) \otimes \dot{\partial}_i \\
&- G^{ij}_0 \left(\Phi_{0j}\, \dot{x}^0\, d^0 + \tfrac{1}{2}\, \Phi_{hj}\, (\dot{x}^h\, d^0 + \dot{x}^0\, d^h)\right) \otimes \dot{\partial}_i \,,
\end{aligned}$$

where

$$\Phi \;\equiv\; \Phi[K,G,o] = \Phi_{\lambda\mu}\, d^\lambda \wedge d^\mu : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$$

is a spacetime 2–form, which depends on K, on G and on the observer o associated with the chosen spacetime chart (x^λ).

Thus, a metric preserving spacetime connection K is not fully determined by the metric, as in the riemannian case. Actually, it is defined up to a gauge, which is represented by the observed spacetime 2–form $\Phi[K, G, o]$.

Moreover, we can prove that a metric preserving connection K is galileian if and only if, for any observer o, the spacetime 2–form $\Phi[K, G, o]$ is closed. In such a case, $\Phi[K, G, o]$ can be written (locally) as

$$\Phi[K, G, o] = 2\, dA[K, G, o]\,,$$

where the (local) *potential* $A[K, G, o] : \boldsymbol{E} \to T^*\boldsymbol{E}$ is locally defined up to a gauge of the type $df : \boldsymbol{E} \to T^*\boldsymbol{E}$, with $f \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

In the classical theory, there is no way to parametrise this gauge; on the other hand, it will be possible in the quantum theory, by means of the quantum basis $\mathtt{b}$.

1.6 Gravitational and electromagnetic fields

Further, we consider the *gravitational field* as a galileian spacetime connection $K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$.

In comparison with the einsteinian general relativity, in the galileian case we cannot say that the spacelike metric represents the gravitational field, because g does not determine $K^\natural$. Indeed, this fact turns out to be an opportunity to help us distinguishing the great different role of g and $K^\natural$ in the description of physical phenomena.

Moreover, we consider the *electromagnetic field* as a scaled spacetime 2–form $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$, which fulfills the 1st Maxwell equation $dF = 0$.

We define the *observed electric vector field* and the *magnetic vector field* as the scaled spacelike vector fields

$$\begin{aligned} \vec{E}[o] &:= -\, g^\sharp\big(д[o] \lrcorner F\big) : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}\,, \\ \vec{B} &:= \tfrac{1}{2}\, i_{\underset{F}{\vee}}\, \bar{\eta} : \boldsymbol{E} \to (\mathbb{L}^{-5/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}\,, \end{aligned}$$

where $\bar{\eta} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^3 V\boldsymbol{E}$ is the spacelike volume vector.

In comparison with the einsteinian general relativity, the electromagnetic field F and the 1st Maxwell equation $dF = 0$ are formally the same. On the other hand, a difference arises when we consider the observed electric field $\vec{E}[o]$ and the magnetic field $\vec{B}$. In fact, formally, the observed electric field is defined in the same way in both cases, by means a contraction of the electromagnetic field with the observer; but a difference is due to the difference of the concept of observer. Moreover, the magnetic field is defined, in the galileian case, via the vertical restriction of F and, in the einsteinian case, via the orthogonal projection of F. Eventually, in the galileian case, the magnetic field turns out to be observer independent. For a comparison with the literature, see, for instance, [18, 48].

In the present paper, we consider the gravitational and electromagnetic fields as given. On the other hand, if we would like to relate these fields to their mass and charge sources, then we could not avail of the true Einstein and Maxwell equations, because they require the lorentzian metric, hence are not consistent with the galileian framework. Indeed, we should consider just a reduced feeble version of these equations, where the effects of movement of masses and charges are lost. This is the price that we pay to couple, in a consistent way, the electromagnetic field with quantum mechanics based on a classical background with absolute time.

1.7 Joined spacetime connection

With reference to a particle of mass m and charge q, we can couple the gravitational field $K^\natural$ and the electromagnetic field F into the *joined galileian spacetime connection*

$$K \equiv K^\natural + K^\mathfrak{e} := K^\natural - \tfrac{1}{2}\tfrac{q}{\hbar}\,(dt \otimes \widehat{F} + \widehat{F} \otimes dt)\,,$$

where the scaled tensor

$$\widehat{F} := G^{\sharp 2}(F) : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E})$$

turns out to be given by

$$\widehat{F} = \tfrac{\hbar}{m}\,(-dt \otimes \vec{E}[o] + \overline{B}[o])\,, \qquad \text{where} \qquad \overline{B}[o] := 2\,g^{\sharp 2}\big(i_{\vec{B}}\theta^*[o]\eta\big)\,.$$

From now on, we shall refer to the joined spacetime connection K.

The joined observed spacetime 2–form $\Phi \equiv \Phi[K, G, o]$ splits as

$$\Phi = \Phi^\natural + \tfrac{1}{2}\tfrac{q}{\hbar}\,F\,.$$

Accordingly, the observed potential $A \equiv A[K, G, o]$ splits as $A = A^\natural + A^\mathfrak{e}$, but we stress that there is no distinguished way to assign the arbitrary gauge to the two components of A.

We consider as *law of motion* for a particle, with mass m and charge q, effected by the gravitational and electromagnetic fields $K^\natural$ and F, to be the equation

$$\nabla[K]ds = 0\,.$$

This equation splits as

$$\nabla ds = \nabla^\natural ds - \vec{f} \circ ds = 0\,,$$

where $\vec{f}$ turns out to be the *Lorentz force*

$$\vec{\mathsf{f}} = \tfrac{q}{m}\,(\vec{E}[o] + \theta[o] \times \vec{B})\,.$$

1.8 Analytical Mechanics

Further, in order to achieve a consistent formulation of the classical Analytic Mechanics and the consequent formulation of Quantum Mechanics in our framework, we define the following objects.

We define a *phase connection* to be a connection of the bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$

$$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}\,.$$

We can prove that there is a bijection between time preserving, linear spacetime connections K and affine phase connections Γ. Therefore, the joined spacetime connection K yields a distinguished affine phase connection $\Gamma \equiv \Gamma[K]$.

Each affine phase connection Γ yields the "quadratic" *dynamical phase connection*

$$\gamma \equiv \gamma[\Gamma] := д \lrcorner \Gamma : \boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}\,,$$

which can be regarded as a connection of the fibred manifold $t^1 : J_1\boldsymbol{E} \to \boldsymbol{T}$.

Therefore, the joined spacetime connection K yields the distinguished dynamical phase connection $\gamma \equiv \gamma[K]$.

Each phase connection Γ yields the *dynamical phase 2–form*

$$\Omega \equiv \Omega[\Gamma, G] := G \lrcorner \big(\nu[\Gamma] \wedge \theta\big) : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}\,,$$

where $\nu[\Gamma]$ is the vertical projection $\nu[\Gamma] : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes (\mathbb{T}^* \otimes V\boldsymbol{E})$ associated with Γ.

Therefore, the joined spacetime connection K yields the distinguished dynamical phase 2–form $\Omega \equiv \Omega[K, G]$.

Each phase connection Γ yields the *dynamical phase two vector*

$$\Lambda \equiv \Lambda[G, \Gamma] := \bar{G} \lrcorner (\check{\Gamma} \wedge \nu) : J_1\boldsymbol{E} \to \Lambda^2 V J_1\boldsymbol{E}\,,$$

where ν is the natural scaled vertical form $\nu : J_1\boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V_{\boldsymbol{E}} J_1\boldsymbol{E})$ and $\check{\Gamma} : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \otimes (\mathbb{T}^* \otimes V\boldsymbol{E})$ is the vertical restriction of Γ.

Therefore, the joined spacetime connection K yields the distinguished dynamical phase 2–vector $\Lambda \equiv \Lambda[K, G]$.

We have the coordinate expressions

$$\Gamma[K] = d^\lambda \otimes \big(\partial_\lambda + (K_\lambda{}^i{}_0 + K_\lambda{}^i{}_h\, x^h_0)\, \partial^0_i\big)\,,$$

$$\begin{aligned}
\gamma[K] &= u^0 \otimes \big(\partial_0 + x^i_0\, \partial_i + (K_0{}^i{}_0 + 2\, K_0{}^i{}_h\, x^h_0 + K_h{}^i{}_k\, x^h_0\, x^k_0)\, \dot{\partial}^i_0\big)\,,\\
\Omega[K, G] &= G^0_{ij}\, \big(d^i_0 - (K_\lambda{}^i{}_0 + K_\lambda{}^i{}_h\, x^h_0)\, d^\lambda\big) \wedge \theta^j\,,\\
\Lambda[K, G] &= G^{ij}_0\, (\partial_i + K_i{}^h{}_k\, x^k_0\, \partial^0_h) \wedge \partial^0_j\,,
\end{aligned}$$

i.e., more explicitly,

$$\Gamma[K] = d^\lambda \otimes \partial_\lambda - G_0^{ij} \big(\Phi_{0j} + \tfrac{1}{2} (\partial_0 G^0_{hj} + \Phi_{hj})\, x^h_0)\big)\, d^0 \otimes \partial^0_i$$
$$- G_0^{ij} \tfrac{1}{2} \big((\partial_0 G^0_{kj} + \Phi_{kj}) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, x^h_0)\big)\, d^k \otimes \partial^0_i \,,$$

$$\gamma[K] = u^0 \otimes \Big(\partial_0 + x^i_0\, \partial_i$$
$$- G_0^{ij} \big(\Phi_{0j} + (\partial_0 G^0_{hj} + \Phi_{hj})\, x^h_0 + (\partial_h G^0_{jk} - \tfrac{1}{2}\, \partial_j G^0_{hk})\, x^h_0\, x^k_0\big)\, \partial^0_i\Big)\,,$$

$$\Omega[K,G] = (\partial_0 G^0_{hj}\, x^h_0 + \tfrac{1}{2}\, \partial_j G^0_{hk}\, x^h_0\, x^k_0)\, d^0 \wedge d^j + (\partial_i G^0_{jh}\, x^h_0)\, d^i \wedge d^j$$
$$+ G^0_{hj}\, x^h_0\, d^0 \wedge d^j_0 - G^0_{ij}\, d^i \wedge d^j_0 + \tfrac{1}{2}\, \Phi_{\lambda\mu}\, d^\lambda \wedge d^\mu \,,$$

$$\Lambda[K,G] = G_0^{ij}\, \partial_i \wedge \partial^0_j + G_0^{ih}\, G_0^{jk} \big(\partial_h G^0_{kr}\, x^r_0 + \tfrac{1}{2}\, \Phi_{hk}\big)\, \partial^0_i \wedge \partial^0_j \,.$$

We have the splittings

$$\begin{aligned}
\Gamma &= \Gamma^\natural + \tfrac{q}{m} \big(dt \otimes (\vec{E}[o] + \tfrac{1}{2}\, \theta[o] \times \vec{B}) - \tfrac{1}{2}\, \vec{B}[o]\big)\,,\\
\gamma &= \gamma^\natural - \tfrac{q}{\hbar}\, G^\sharp\, (д \lrcorner F)\,,\\
\Omega &= \Omega^\natural + \tfrac{1}{2}\, \tfrac{q}{\hbar}\, F\,,\\
\Lambda &= \Lambda^\natural + \tfrac{1}{2}\, \tfrac{q}{m}\, \overline{B}\,,
\end{aligned}$$

where $\overline{B} := 2\, G^{\sharp 2} \big(i_{\vec{B}}\, \eta\big)$.

The map $\Gamma \mapsto \gamma[\Gamma]$ turns out to be a bijection. Moreover, we can prove that γ is the unique dynamical phase connection which fulfills the condition $i_\gamma \Omega = 0$. Therefore, we obtain a sequence of bijections $K \mapsto \Gamma \mapsto \Omega \mapsto \gamma$, which is circularly closed.

We stress that $\Omega[K,G]$ encodes full information on the metric and the joined galileian connection K, while $\Lambda[K,G]$ turns out to be determined only by the metric G and the magnetic field $\vec{B}$.

We can prove the equality

$$\Omega[K,G] = d\Theta[G,o] + \tfrac{1}{2}\, \Phi[K,G,o]\,.$$

So, $\Omega[K,G]$ turns out to be closed if and only if K is galileian.

Hence, the pair (dt, Ω) turns out to be a scaled *cosymplectic structure* of the phase space [17, 34]. In other words, $dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^7 T^* J_1 \boldsymbol{E}$ is a scaled volume form of the phase space and $d\Omega = 0$.

The dynamical phase 2–form Ω turns out to play a fundamental role both in the classical and the quantum theory.

The law of motion $\nabla[K] ds = 0$ turns out to be equivalent to the equation $\nabla[\gamma] \circ j_1 s = 0$.

In comparison with Geometric Quantisation [1, 15, 16, 62, 63, 71], in our context, such a main role is played by a cosymplectic 2–form and not by a symplectic 2–form. This fact is related to the fundamental role of time (which, in the present theory, is not just a parameter, but a fundamental object) and to our strategic choice of the odd dimensional phase space $J_1\boldsymbol{E}$.

1.9 Horizontal potential

One of the main features of the cosymplectic 2–form Ω is to admit (locally) an *"upper" horizontal potential* of the type

$$A^{\uparrow} : J_1\boldsymbol{E} \to T^*\boldsymbol{E},$$

according to the equation $\Omega = dA^{\uparrow}$. Clearly, the horizontal potential $A^{\uparrow}$ is locally defined up to a gauge of the type $df : \boldsymbol{E} \to T^*\boldsymbol{E}$, with $f \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

Given an observer o, we can prove the equality

$$\Phi[K, G, o] = 2\, o^*\Omega[K, G].$$

Therefore, the observed potential $A[K, G, o]$ of $\Phi[K, G, o]$ turns out to be given (up to a gauge) by the equality

$$A[K, G, o] = o^* A^{\uparrow}.$$

We can split, in a natural way, the potential $A^{\uparrow}$ into its horizontal and vertical components $A^{\uparrow} = \mathcal{L}[A^{\uparrow}] + \mathcal{P}[A^{\uparrow}]$, where

$$\begin{aligned} \mathcal{L} &\equiv \mathcal{L}[A^{\uparrow}] := д \lrcorner A^{\uparrow} \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E}) \\ \mathcal{P} &\equiv \mathcal{P}[A^{\uparrow}] := \theta \lrcorner A^{\uparrow} \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E}) \end{aligned}$$

are the *classical lagrangian* and the *classical momentum*. Clearly, the above objects are local, observer independent and defined up to the gauge of $A^{\uparrow}$.

The *Euler–Lagrange equation* associated with the lagrangian $\mathcal{L}$

$$\mathcal{E}[\mathcal{L}] \circ j_2 s = 0,$$

where

$$\mathcal{E}[\mathcal{L}] = \big(\partial_i \mathcal{L}_0 - (\partial_0 + x_0^j\, \partial_j + x_{00}^j \partial_j^0)\, \partial_i^0 \mathcal{L}_0\big)\, u^0 \otimes (d^i - x_0^i\, d^0),$$

turns out to be equivalent to the equations $\nabla[K]ds = 0$ and $\nabla[\gamma] j_1 s = 0$.

Indeed, the *Euler–Lagrange operator* $\mathcal{E}[\mathcal{L}]$ turns out to be global, gauge independent and observer independent.

The triplet $(\Omega, \mathcal{L}, \mathcal{E})$ turns out to be an excerpt of a *variational sequence* [40, 67], whose starting source, in our context, is Ω.

The pair (Ω, γ) accounts for the fact that the equation of motion $\nabla[\gamma] j_1 s = 0$ can be derived from a lagrangian [9, 10, 38, 39, 41, 54].

Moreover, given an observer o, the potential $A^{\uparrow}$ splits into its horizontal and vertical components

$$A^{\uparrow} = -\mathcal{H}[A^{\uparrow}, o] + \mathcal{P}[A^{\uparrow}, o],$$

where

$$\mathcal{H}[A^{\uparrow}, o] := -\text{д}[o] \lrcorner A^{\uparrow} = \mathcal{K}[G, o] - A[G, o] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})$$
$$\mathcal{P}[A^{\uparrow}, o] := \quad \theta[o] \lrcorner A^{\uparrow} = \mathcal{Q}[G, o] + A[G, o] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$$

are the *observed classical hamiltonian* and the *observed classical momentum.*

We have the coordinate expressions

$$\mathcal{L}[A^{\uparrow}] = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_j\, x^j_0 + A_0)\, d^0\,, \qquad \mathcal{P}[A^{\uparrow}] = (G^0_{ij}\, x^j_0 + A_i)\,(d^i - x^i_0\, d^0)\,,$$
$$\mathcal{H}[A^{\uparrow}, o] = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0\,, \qquad \mathcal{P}[A^{\uparrow}, o] = (G^0_{ij}\, x^j_0 + A_i)\,(d^i - o^i_0\, d^0)\,,$$

and the splittings

$$\mathcal{L}[A^{\uparrow}] = \mathcal{L}[A^{\uparrow\natural}] + \text{д} \lrcorner A^{\mathfrak{e}}\,, \qquad \mathcal{P}[A^{\uparrow}] = \mathcal{P}[A^{\uparrow\natural}] + \theta \lrcorner \overset{\vee}{A}{}^{\mathfrak{e}}\,,$$
$$\mathcal{H}[A^{\uparrow}, o] = \mathcal{H}[A^{\uparrow\natural}, o] - \text{д}[o] \lrcorner A^{\mathfrak{e}}\,, \qquad \mathcal{P}[A^{\uparrow}, o] = \mathcal{P}[A^{\uparrow\natural}, o] + \theta[o] \lrcorner \overset{\vee}{A}{}^{\mathfrak{e}}\,.$$

Clearly, the above objects are local, observer dependent and defined up to the gauge of $A^{\uparrow}$.

The equation of motion can be equivalently written, in terms of the phase functions $f : J_1\boldsymbol{E} \to \mathrm{I\!R}$, via the Poisson bracket, as

$$(\gamma.f)_0 = \partial_0 f - \{\mathcal{H}_0,\, f\} - G^{ij}_0\, \partial_0 \mathcal{P}_i\, \partial^0_j f\,.$$

However, while the equations $\nabla[K]ds = 0$, $\nabla[\gamma]j_1 s = 0$ and $\mathcal{E}[\mathcal{L}] \circ j_2 s = 0$ are global, gauge independent and observer independent, the above equation has not so, hence it is not explicitly covariant.

In comparison with Geometric Quantisation [1, 15, 16, 52, 62, 63, 64, 71], we are not dealing with a symplectic phase 2–form Ω, but with a cosymplectic phase 2–form Ω. Moreover, our observed hamiltonian $\mathcal{H}$ is not provided by an assumption, but is derived from the cosymplectic 2–form Ω. Furthermore, the standard Hamilton equation $i_X\Omega = -d\mathcal{H}$ is replaced by the identity $i_\gamma\Omega = 0$.

2 Setting of the quantum theory

Next, we summarise some achievements of Covariant Quantum Mechanics.

2.1 Quantum bundle

We consider the *quantum bundle* to be a 1–dimensional complex vector bundle over spacetime

$$\pi : \boldsymbol{Q} \to \boldsymbol{E} \,.$$

The quantum states are represented by the *quantum sections* $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$.

In comparison with Geometric Quantisation, our quantum bundle is over spacertime, not over the phase space. This fact will agree, later, with our choice of the upper quantum connection $\text{Ч}^{\uparrow}$ and the horizontal "upper quantum potential" $A^{\uparrow}$.

We consider the quantum bundle to be equipped with a scaled *hermitian quantum metric*

$$\mathsf{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{C} \,.$$

By taking into account the spacelike volume form η , we define the vertical valued *hermitian quantum metric* $\mathsf{h}_\eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \Lambda^3 V^* \boldsymbol{E} \otimes \mathbb{C}$.

We shall refer to normalised scaled *quantum bases*

$$\mathsf{b} : \boldsymbol{E} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q} \,, \qquad \text{which fulfill the condition} \qquad \mathsf{h}_\eta(\mathsf{b}, \mathsf{b}) = \eta \,.$$

Accordingly, we shall refer to scaled linear fibred charts (x^λ, z) , where the scaled complex function $z : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{C}$, fulfills the condition $z(\mathsf{b}) = 1$.

We shall write

$$\Psi = \psi \, \mathsf{b} \,, \qquad \text{with} \qquad \psi \equiv |\psi| \exp(\mathfrak{i}\, \varphi) \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathbb{C}) \,.$$

In view of the forthcoming upper quantum connection $\text{Ч}^{\uparrow}$, we define the *upper quantum bundle* to be the 1–dimensional complex vector bundle $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$ over the phase space, where $\boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}$. Thus, this bundle is obtained, by pullback, via an enlargement the base space, leaving the fibres untouched. In the present context, the base space $J_1\boldsymbol{E}$ plays the role of the space of all possible observers o .

2.2 Upper quantum connection

We say that a complex linear connection $\text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} TJ_1\boldsymbol{E} \to T\boldsymbol{Q}^{\uparrow}$ is *reducible* if it factorises through a system of quantum connections $\text{Ч}[o] : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{Q}$. We can prove that $\text{Ч}^{\uparrow}$ is reducible if and only if, in coordinates, $\text{Ч}^{\uparrow}{}^{0}_{i} = 0$.

We define a *galileian upper quantum connection* to be a connection of $\boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$

$$\text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^* J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow} \,,$$

such that it is hermitian and reducible and with a curvature fulfilling the condition

$$R[Ч^\uparrow] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^\uparrow\,,$$

where $\mathbb{I}^\uparrow : \boldsymbol{Q}^\uparrow \to \boldsymbol{Q}^\uparrow$ is the Liouville vector field of $\boldsymbol{Q}^\uparrow$ (see also [46]).

The closure of Ω turns out to be a necessary integrability condition for the local existence of $Ч^\uparrow$, because of the Bianchi identity.

The integer cohomology class of Ω turns out to be a necessary integrability condition for the global existence of $Ч^\uparrow$ [66].

The upper quantum connections $Ч^\uparrow$ are defined locally up to a gauge of the type $\mathfrak{i}\,df \otimes \mathbb{I}^\uparrow$, where $f : \boldsymbol{E} \to \mathbb{R}$.

With reference to a quantum basis $\mathfrak{b}$, the coordinate expression of an upper quantum connection $Ч^\uparrow$ is locally of the type

$$\begin{aligned} Ч^\uparrow &= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\,A^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow \\ &= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\,\big(\Theta[o] + A[\mathfrak{b}, o]\big) \otimes \mathbb{I}^\uparrow \\ &= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\,\big(-\mathcal{K}[o] + \mathcal{Q}[o] + A[\mathfrak{b}, o]\big) \otimes \mathbb{I}^\uparrow \\ &= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\,\big(-\mathcal{H}[\mathfrak{b}, o] + \mathcal{P}[\mathfrak{b}, o]\big) \otimes \mathbb{I}^\uparrow \\ &= d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}\,\big(-(\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 - A_0)\,d^0 + (G^0_{ij}\,x^j_0 + A_i)\,d^i\big) \otimes \mathbb{I}^\uparrow\,, \end{aligned}$$

where $\chi^\uparrow[\mathfrak{b}] : \boldsymbol{Q}^\uparrow \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$ is the trivial hermitian upper quantum connection induced by the quantum basis $\mathfrak{b}$.

Thus, the *upper quantum potential* $A^\uparrow[\mathfrak{b}]$ appearing in the above expression of $Ч^\uparrow$ is just a potential of Ω and a potential of K, that have been discussed previously.

Then, we suppose the cohomology class of Ω to be integer and consider, as source of all further quantum developments, a *galileian upper quantum connection* $Ч^\uparrow$.

In comparison with Geometric Quantisation [1, 15, 16, 52, 62, 63, 64, 71], we have a natural polarisation, our upper quantum connection is reducible and the upper quantum potential is horizontal.

On the other hand, the upper quantum connection $Ч^\uparrow$ lives on the pullback quantum bundle $\boldsymbol{Q}^\uparrow$, which includes all observers on the base space $J_1\boldsymbol{E}$, and, in the subsequent developments of our theory, all further quantum objects will be derived from $Ч^\uparrow$. Then, in order to get rid of observers, according to the principle of general relativity, we shall follow a *criterion of projectability*, namely we shall look for objects defined on the phase space, which factorise through spacetime. This method of projectability replaces the search of polarisations typical of Geometric Quantisation.

2.3 Observed quantum potential

We observe that the quantum bases $\mathfrak{b}$ allow us to parametrise the upper quantum potentials $A^\uparrow$, hence the *observed quantum potentials* $A[\mathfrak{b}, o]$.

If $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \exp(\mathfrak{i}\,\vartheta)\,\mathfrak{b}$ are two quantum bases, then we have

$$A^\uparrow[\acute{\mathfrak{b}}] = A^\uparrow[\mathfrak{b}] - \mathfrak{i}\,d\vartheta\,.$$

Moreover, now we can find the transition rule for the observed quantum potential. In fact, with reference to two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \exp(\mathfrak{i}\,\vartheta)\,\mathfrak{b}$ and two observers o and $\acute{o} = o + v$, we have the following transition rule

$$A[\acute{\mathfrak{b}}, \acute{o}] = A[\mathfrak{b}, o] - d\vartheta + \theta[o] \lrcorner G^{\flat}(v) - \tfrac{1}{2}\, G(v, v)\,,$$

i.e., in a chart adapted to $\mathfrak{b}$ and o,

$$A[\acute{\mathfrak{b}}, \acute{o}] = A_{\lambda}[\mathfrak{b}, o]\, d^{\lambda} - \partial_{\lambda}\vartheta\, d^{\lambda} + G^{0}_{ij}\, v^{j}_{0}\, d^{i} - \tfrac{1}{2}\, G^{0}_{ij}\, v^{i}_{0}\, v^{j}_{0}\, d^{0}\,.$$

2.4 Distinguished observer and observed potential

The upper quantum connection $\text{Ч}^{\uparrow}$ yields the following distinguished objects.

Given a quantum basis $\mathfrak{b}$, there exists a unique *distinguished observer* $o[\mathfrak{b}]$, associated with $\mathfrak{b}$, such that the vertical restriction of the induced observed potential vanishes, i.e., such that $\check{A}\big[\mathfrak{b}, o[\mathfrak{b}]\big] = 0$.

Thus, for every quantum basis $\mathfrak{b}$, we obtain the *distinguished timelike observed potential* $A[\mathfrak{b}] := A\big[\mathfrak{b}, o[\mathfrak{b}]\big] \in \sec(\boldsymbol{E}, H^{*}\boldsymbol{E})$ determined by $\mathfrak{b}$.

Indeed, if o is any observer, then we obtain the equalities (where $\vec{A} := G^{\sharp}(\check{A})$)

$$o[\mathfrak{b}] = o - \vec{A}[\mathfrak{b}, o] \qquad \text{and} \qquad A[\mathfrak{b}] = д[o] \lrcorner A[\mathfrak{b}, o] - \tfrac{1}{2}\, G\big(\vec{A}[\mathfrak{b}, o],\ \vec{A}[\mathfrak{b}, o]\big)\,,$$

i.e., in a chart adapted to o,

$$o^{i}_{0}[b] = -A^{i}_{0}[\mathfrak{b}, o]\,, \qquad \text{and} \qquad A[\mathfrak{b}] = \big(A_{0}[\mathfrak{b}, o] - \tfrac{1}{2}\, A_{i}[\mathfrak{b}, o]\, A^{i}_{0}[\mathfrak{b}, o]\big)\, d^{0}\,.$$

Given two quantum bases $\acute{\mathfrak{b}}$ and $\acute{\mathfrak{b}} = \exp(\mathfrak{i}\,\vartheta)\,\mathfrak{b}$, we have the transition rules (where $\vec{d} := G^{\sharp} \circ \check{d}$)

$$o[\acute{\mathfrak{b}}] = o[\mathfrak{b}] - \vec{d}\,\vartheta \qquad \text{and} \qquad A[\acute{\mathfrak{b}}] = A[\mathfrak{b}] - д[o].\vartheta - \tfrac{1}{2}\, G\,(\vec{d}\,\vartheta,\ \vec{d}\,\vartheta)\,.$$

Now, we consider the *proper quantum subbundle* $\boldsymbol{Q}_{/0} := \boldsymbol{Q}/\{0\} \subset \boldsymbol{Q}$.

Let us consider a *proper quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ and the associated distinguished quantum basis $\mathfrak{b}[\Psi] := \Psi/\|\Psi\|$.

Then, we obtain the *distinguished observer* and the *distinguished potential*, which are determined only by Ψ,

$$\begin{aligned} o[\Psi] &:= o\big[\mathfrak{b}[\Psi]\big] \ : \boldsymbol{E} \to J_{1}\boldsymbol{E}\,,\\ A[\Psi] &:= A\big[\mathfrak{b}[\Psi]\big] : \boldsymbol{E} \to H^{*}\boldsymbol{E}\,. \end{aligned}$$

Thus, $o[\Psi]$ is, by definition, the unique observer, associated with Ψ, such that the observed potential $A\big[\mathfrak{b}[\Psi], o[\Psi]\big]$ be timelike, i.e. such that

$$\check{A}\big[\mathfrak{b}[\Psi], o[\Psi]\big] = 0\,.$$

Indeed, if $\mathfrak{b}$ is any quantum basis and o any observer, then we obtain the equalities

$$\begin{aligned} o[\Psi] &= o - \overrightarrow{A}[\mathfrak{b}, o] + \overrightarrow{d}\,\varphi\,, \\ A[\Psi] &= д[o] \lrcorner A[\mathfrak{b}, o] - д\big[o[\mathfrak{b}]\big].\varphi - \tfrac{1}{2}\, G\,\big(\overrightarrow{A}[\mathfrak{b}, o],\, \overrightarrow{A}[\mathfrak{b}, o]\big) - \tfrac{1}{2}\, G\,(\overrightarrow{d}\,\varphi,\, \overrightarrow{d}\,\varphi) \\ &= -д[o] \lrcorner \big(d\varphi - A[\mathfrak{b}, o]\big) - \tfrac{1}{2}\, G\,\Big(\big(\overrightarrow{d}\,\varphi - \overrightarrow{A}[\mathfrak{b}, o]\big),\, \big(\overrightarrow{d}\,\varphi - \overrightarrow{A}[\mathfrak{b}, o]\big)\Big)\,, \end{aligned}$$

with coordinates expressions, in a chart adapted to o and $\mathfrak{b}$,

$$\begin{aligned} o^i_0[\Psi] &= -A^i_0[\mathfrak{b}, o] + G^{ij}_0\, \partial_j\varphi \\ &= G^{ij}_0\, (\partial_j\varphi - A_j[\mathfrak{b}, o])\,, \\ A[\Psi] &= \big(A_0[\mathfrak{b}, o] - \partial_0\varphi + A^i_0[\mathfrak{b}, o]\, \partial_i\varphi - \tfrac{1}{2}\, G^0_{ij}\, A^i_0[\mathfrak{b}, o]\, A^j_0[\mathfrak{b}, o] - \tfrac{1}{2}\, G^{ij}_0\, \partial_i\varphi\, \partial_j\varphi\big)\, d^0 \\ &= -\Big(\big(\partial_0\varphi - A_0[\mathfrak{b}, o]\big) + \tfrac{1}{2}\, G^0_{ij}\, \big(\partial_i\varphi - A_i[\mathfrak{b}, o]\big)\, \big(\partial_i\varphi - A_i[\mathfrak{b}, o]\big)\Big)\, d^0\,, \end{aligned}$$

where we have set

$$\varphi := \varphi[\Psi, \mathfrak{b}]\,, \qquad \overrightarrow{d}\,\varphi := G^\sharp(\overset{\vee}{d}\phi)\,, \qquad \overrightarrow{A} := G^\sharp(\overset{\vee}{A})\,.$$

In particular, if o is any observer and we refer to the distinguished quantum basis $\mathfrak{b}[\Psi]$, then we obtain the equality

$$o[\Psi] = o - \overrightarrow{A}\big[\mathfrak{b}[\Psi], o\big]\,,$$

with coordinate expression, in a chart adapted to o and $\mathfrak{b}[\Psi]$,

$$o^i_0[\Psi] = -A^i_0\big[\mathfrak{b}[\Psi], o\big]\,.$$

Given a proper quantum section Ψ , we are led to regard $o[\Psi]$ as the distinguished *observer at rest* with respect to Ψ and $A[\Psi]$ as the distinguished *observed potential* “seen” by Ψ , regardless of any further gauge.

2.5 Kinetic quantum momentum

Given a quantum section Ψ , we have the following distinguished scaled fibred morphisms defined on the phase space $J_1\boldsymbol{E}$

$$д \otimes \Psi : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q}) \qquad \text{and} \qquad \overrightarrow{\nabla}^{\uparrow}\Psi : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})\,,$$

where $\overrightarrow{\nabla}^{\uparrow} := G^\sharp \circ \overset{\vee}{\nabla}^{\uparrow}$.

Indeed, according to the criterion of projectability, there is a unique complex combination of these fibred morphisms, which factorises through a section of spacetime (so getting rid of observers)

$$\mathrm{Q}[\Psi] := д \otimes \Psi - \mathfrak{i}\, \overrightarrow{\nabla}^{\uparrow}\Psi : \boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})\,,$$

Thus, the above section $\mathsf{Q}[\Psi]$ of spacetime, called *kinetic quantum momentum*, is global, gauge independent and observer independent. We have the identity

$$dt \lrcorner \mathsf{Q}[\Psi] = д\big[o[\Psi]\big] \otimes \Psi\,.$$

We have the coordinate expression

$$\mathsf{Q}[\Psi] = \big(\psi\,\partial_0 - \mathfrak{i}\,G^{ij}_0\,(\partial_j\psi - \mathfrak{i}\,A_j[\mathsf{b},o]\,\psi)\,\partial_i\big) \otimes u^0 \otimes \mathsf{b}\,,$$

i.e., in the proper domain of Ψ,

$$\mathsf{Q}[\Psi] = (\partial_0 - A^i_0\,\partial_i) \otimes u^0 \otimes \Psi - \mathfrak{i}\,G^{ij}_0(\partial_j \log|\psi| + \mathfrak{i}\,\partial_j\varphi)\,\partial_i \otimes u^0 \otimes \Psi\,.$$

In the proper domain of Ψ, with reference to the distinguished quantum basis $\mathsf{b}[\Psi]$ and the distinguished observer $o[\Psi]$, the coordinate expression of the kinetic quantum momentum can be written as

$$\mathsf{Q}[\Psi] = \partial_0 \otimes u^0 \otimes \Psi - \mathfrak{i}\,G^{ij}_0\partial_j \log|\psi|\,\partial_i \otimes u^0 \otimes \Psi\,.$$

In other words, we obtain the equality

$$\mathsf{Q}[\Psi] = \Big(д\big[o[\Psi]\big] - \mathfrak{i}\,\vec{d}\,(\log\|\Psi\|)\Big) \otimes \Psi\,, \qquad \text{where} \qquad \vec{d} := G^\sharp \circ \check{d}\,.$$

Hence, we obtain the scaled complexified spacetime vector field

$$\mathsf{V}[\Psi] := \mathsf{Q}[\Psi]/\Psi = д\big[o[\Psi]\big] - \mathfrak{i}\,\vec{d}\,(\log\|\Psi\|) : \boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \mathbb{C})\,,$$

which splits into its real and imaginary components

$$\operatorname{re}\mathsf{V}[\Psi] = д\big[o[\Psi]\big] : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \quad \text{and} \quad \operatorname{im}\mathsf{V}[\Psi] = -\,\vec{d}\,(\log\|\Psi\|) : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}\,.$$

Thus, the above real component turns out provide again the distinguished observer $o[\Psi]$.

2.6 Probability current

Given a quantum section Ψ, we have the following distinguished scaled fibred morphisms defined on the phase space

$$д \otimes \|\Psi\|^2 : J_1\boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E}) \quad \text{and} \quad \operatorname{re}\mathsf{h}(\Psi, \mathfrak{i}\,\Psi) : J_1\boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\,.$$

Indeed, according to the criterion of projectability, there is a unique real combination of these fibred morphisms, which factorises through a section of spacetime (so getting rid of observers)

$$\mathsf{J}[\Psi] := д \otimes \|\Psi\|^2 - \operatorname{re}\mathsf{h}(\Psi,\, \mathfrak{i}\,\vec{\nabla}^\uparrow\Psi) : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\,.$$

Thus, the above section $\mathsf{J}[\Psi]$ of spacetime, called *quantum probability current*, is global, gauge independent and observer independent.

We have the identity

$$dt \lrcorner \mathsf{J}[\Psi] = \|\Psi\|^2 .$$

We have the coordinate expression

$$\mathsf{J}[\Psi] = \big(|\psi|^2 \, \partial_0 + (\mathfrak{i}\, \tfrac{1}{2}\, G_0^{ij} \, (\psi \, \partial_j \bar{\psi} - \bar{\psi} \, \partial_j \psi) - A_0^i[\mathfrak{b}, o] \, |\psi|^2) \, \partial_i\big) \otimes u^0 ,$$

i.e., in the proper domain of Ψ ,

$$\mathsf{J}[\Psi] = |\psi|^2 \, u^0 \otimes \big(\partial_0 + (G_0^{ij} \, \partial_j \varphi[\Psi, \mathfrak{b}] - A_0^i[\mathfrak{b}, o]) \, \partial_i\big) .$$

In the particular case of a flat spacetime with vanishing magnetic field and an inertial observer, this operator coincides with the standard quantum probability current.

In the proper domain of Ψ , with reference to the distinguished quantum basis $\mathfrak{b}[\Psi]$ and the distinguished observer $o[\Psi]$, the coordinate expression of the quantum probability current can be written as

$$\mathsf{J}[\Psi] = |\psi|^2 \, u^0 \otimes \partial_0 .$$

In other words, we obtain the equality

$$\mathsf{J}[\Psi] = \|\Psi\|^2 \, д\big[o[\Psi]\big] .$$

Hence, once more, we obtain the distinguished observer

$$д\big[o[\Psi]\big] = \mathsf{J}[\Psi] / \|\Psi\|^2 .$$

2.7 Schrödinger operator

Given a quantum section Ψ , we have the following distinguished scaled fibred morphisms defined on the phase space

$$д \lrcorner \nabla^{\uparrow} \Psi : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q} \qquad \text{and} \qquad \delta^{\uparrow} \mathrm{Q}[\Psi] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q} ,$$

where $\delta^{\uparrow}$ is the codifferential operator induced by the upper quantum covariant differential $\nabla^{\uparrow}$ and the rescaled metric G .

Indeed, according to the criterion of projectability, there is a unique complex combination of these fibred morphisms, which factorises through a section of spacetime (so getting rid of observers)

$$\mathsf{S}[\Psi] := \tfrac{1}{2} \big(д \lrcorner \nabla^{\uparrow} \Psi + \delta^{\uparrow}(\mathrm{Q}.\Psi)\big) : \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q} .$$

Thus, the above section $\mathsf{S}[\Psi]$ of spacetime, called *Schrödinger operators*, is global, gauge independent and observer independent.

With reference to any observer o , we have the expression

$$\mathsf{S}[\Psi] = \big(\nabla_{д[o]}[o] + \tfrac{1}{2} \, \mathrm{div}_\eta \, д[o] - \mathfrak{i} \, \tfrac{1}{2} \, \Delta[G, o]\big) \Psi ,$$

where $\Delta[G, o]$ is the quantum laplacian operator associated with the rescaled metric G and the observer o.

In other words, we have the coordinate expression

$$\begin{aligned}\mathsf{S}_0[\Psi] &= (\partial_0\psi - \tfrac{1}{2}\,\mathrm{i}\,G_0^{ij}\,\partial_{ij}\psi)\,\mathsf{b} - \mathrm{i}\,(A_0 - \tfrac{1}{2}\,A_i\,A_0^i)\,\psi\,\mathsf{b} \\ &\quad - \big((A_0^j + \tfrac{1}{2}\,\mathrm{i}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}})\,\partial_j\psi\big)\,\mathsf{b} + \tfrac{1}{2}\,\big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}}\big)\,\psi\,\mathsf{b}\,.\end{aligned}$$

In the particular case of a flat spacetime with vanishing magnetic field and an inertial observer, this operator coincides with the standard Schrödinger operator [49].

We stress that our approach to Schrödinger operator does not involve hamiltonian methods and the energy.

In the proper domain of Ψ, the Schrödinger equation $\mathsf{S}[\Psi] = 0$ splits into the system

$$\begin{aligned}0 &= \partial_0|\psi| + \tfrac{1}{2}\,|\psi|\,G_0^{ij}\,\partial_{ij}\varphi + G_0^{ij}\,\partial_i|\psi|\,(\partial_j\varphi - A_j) \\ &\quad - \tfrac{1}{2}\,G_0^{ij}\,|\psi|\,\partial_i A_j + \tfrac{1}{2}\,|\psi|\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \tfrac{1}{2}\,|\psi|\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_j\varphi - A_j)\,,\end{aligned}$$

$$\begin{aligned}0 &= -\tfrac{1}{2}\,G_0^{ij}\,\partial_{ij}|\psi| - \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j|\psi| \\ &\quad + |\psi|\,\big(\partial_0\varphi + \tfrac{1}{2}\,G_0^{ij}\,\partial_i\varphi\,\partial_j\varphi - A_0^i\,\partial_i\varphi - A_0 + \tfrac{1}{2}\,A_i\,A_0^i\big)\,,\end{aligned}$$

i.e., with reference to the distinguished quantum basis $\mathsf{b}[\Psi]$ and the distinguished observer $o[\Psi]$,

$$0 = \partial_0|\psi| + \tfrac{1}{2}\,|\psi|\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,,$$

$$0 = -\tfrac{1}{2}\,G_0^{ij}\,\partial_{ij}|\psi| - \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j|\psi| - A_0\,|\psi|\,.$$

Hence, with reference to the distinguished quantum basis $\mathsf{b}[\Psi]$ and the distinguished observer $o[\Psi]$, the above system can be written as

$$\begin{aligned}0 &= д[o_\Psi].\|\Psi\| + \tfrac{1}{2}\,\|\Psi\|\,\mathrm{div}_\eta\,д[o_\Psi]\,, \\ 0 &= \Delta[G]\|\Psi\| + 2\,\|\Psi\|\,A[\Psi]\,,\end{aligned}$$

We can regard the 1st equation as the continuity equation for the quantum probability current $\mathsf{J}[\Psi]$ and the 2nd equation as an equation for the distinguished timelike quantum potential $A[\Psi]$ "seen" by Ψ.

2.8 Associated classical fluid

Regardless the Schrödinger equation, given a proper quantum section Ψ, we can naturally associate with it a classical fluid whose mass density and velocity are given by the equalities [5, 11, 14, 27, 30, 60, 70]

$$\mu := m\,\|\Psi\|^2 \qquad \text{and} \qquad \mathscr{V}[\Psi] := д\big[o[\Psi]\big] = \operatorname{re} \mathsf{V}[\Psi]\,,$$

with coordinate expressions

$$\mu[\Psi] = m\,|\psi|^2 \qquad \text{and} \qquad \mathscr{V}[\Psi] = u^0 \otimes \big(\partial_0 + G^{ij}_0\,(\partial_j\varphi - A_j)\,\partial_i\big)\,.$$

Just in virtue of the definition of $\mathscr{V}[\Psi]$, this fluid turns out to fulfill the equation of motion

$$\mu[\Psi]\,\nabla^\natural{}_{\mathscr{V}[\Psi]}\mathscr{V}[\Psi] = -\mu[\Psi]\,q\,g^\sharp(\mathscr{V}[\Psi]\,\lrcorner\,F) - \mu[\Psi]\,\tfrac{\hbar}{m}\,\vec{d}A[\Psi]\,,$$

where

$$\vec{d}\mathfrak{p}[\Psi] := -\tfrac{\hbar}{m}\,g^\sharp\big(\overset{\vee}{d}A[\Psi]\big)$$

can be interpreted as the gradient of a *quantum pressure*. Once more, the distinguished timelike potential $A[\Psi]$ "seen" by Ψ appears in the above equation.

This equation can be written, in eulerian form, as

$$\mu[\Psi]\,\mathscr{E}^\natural\big[o[\Psi]\big] = \rho[\Psi]\,\vec{E}\big[o[\Psi]\big] + \mu[\Psi]\,\vec{d}\,\mathfrak{p}[\Psi] - \mu[\Psi]\,G^\sharp\Big(д\big[o[\Psi]\big]\,\lrcorner\,\Phi^\natural\big[o[\Psi]\big]\Big)\,,$$

where $\mathscr{E}^\natural[o_\Psi]$ is the eulerian acceleration observed by the distinguished observer $o[\Psi]$.

Next, by assuming that Ψ fulfills the Schrödinger equation, the classical fluid turns out to fulfill the system of continuity equation and equation of motion

$$\begin{aligned}
&\operatorname{div}_\upsilon\big(\mu[\Psi]\,\mathscr{V}[\Psi]\big) = \mathscr{V}[\Psi].\mu[\Psi] + \mu[\Psi]\,\operatorname{div}_\eta\mathscr{V}[\Psi] = 0\,,\\
&\mu[\Psi]\,\nabla^\natural{}_{\mathscr{V}[\Psi]}\mathscr{V}[\Psi] = -\rho[\Psi]\,g^\sharp(\mathscr{V}[\Psi]\,\lrcorner\,F) + \mu[\Psi]\,\vec{d}\,\mathfrak{p}[\Psi]\,,
\end{aligned}$$

where the quantum pressure fulfills the constitutive equation

$$\mathfrak{p}[\Psi] = \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\,.$$

This result extends the well known hydrodynamical formulation of quantum mechanics [11] to our more general framework.

3 Further developments

Further developments of Covariant Classical and Quantum Mechanics are out of the scope of the present paper.

Nevertheless, here we shortly mention a few achievements just to indicate the perspectives of the theory.

The phase 2–vector Λ yields the Poisson bracket $\{f, \acute{f}\} := \Lambda(df, d\acute{f})$.

However, this bracket cannot be taken as the key issue of Analytical Mechanics, because Λ does not carry full information on the structure of the phase space, but it accounts only for its spacelike structure.

On the other hand, the *special phase functions* of the type

$$f = f^0 \, \tfrac{1}{2} \, G^0_{ij} \, x^i_0 \, x^j_0 + f^i \, G^0_{ij} \, x^j_0 + \breve{f} \,, \qquad \text{with} \qquad f^0 \,, f^i \,, \breve{f} \in \sec(\boldsymbol{E}, \mathrm{I\!R}) \,,$$

are equipped with the *special Lie bracket*

$$[\![f, \acute{f}]\!] := \Lambda(df, d\acute{f}) + \gamma_0(f^0).\acute{f} - \gamma_0(\acute{f}^0).f \,.$$

This special Lie algebra turns out to be the source of classical and quantum symmetries [50, 55, 56].

Moreover, the projectable hermitian quantum vector fields $Y_\eta : \boldsymbol{Q} \to \boldsymbol{Q}$ turn out to be of the type [33]

$$Y_\eta[f] = f^0 \, \partial_0 - f^i \, \partial_i + \Big(\mathfrak{i} \, (\breve{f} + f^0 \, A_0 - f^i \, A_i) - \tfrac{1}{2} \, \big(f^0 \, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (f^i \, \sqrt{|g|})}{\sqrt{|g|}} \big) \Big) \, \mathbb{I} \,,$$

and constitute a Lie algebra naturally isomorphic to the Lie algebra of special phase functions.

Then, the Schrödinger operator $\mathsf{S}[\Psi]$ and the above projectable hermitian vector fields $Y_\eta[f]$ yield in a natural way our *quantum operators* associated with the special phase functions f [29, 33]

$$\begin{aligned} \mathsf{O}[f].\Psi &= \mathfrak{i} \, \big(Y_\eta[f].\Psi - f^0 \, \mathsf{S}_0[\Psi] \big) \\ &= \Big(\big(\breve{f} - A_i \, f^i - \mathfrak{i} \, (f^i \, \partial_i + \tfrac{1}{2} \, \frac{\partial_i (f^i \, \sqrt{|g|})}{\sqrt{|g|}}) - \tfrac{1}{2} \, f^0 \, \Delta_0 \big) \, \psi \Big) \, \mathsf{b} \,. \end{aligned}$$

Indeed, the above combination makes the partial derivative ∂_0 to disappear. In other words, these operators act on the fibres of spacetime.

So, in comparison with Geometric Quantisation [1, 15, 16, 52, 62, 63, 64, 71], we replace the Poisson Lie algebra of phase functions with the special phase Lie algebra and we obtain the quantum operators associated with special phase functions via the Schrödinger operator and the classification of projectable hermitian quantum vector fields. Indeed, energy and momentum are treated on the same footing.

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