Infinitesimal Symmetries in Covariant Quantum Mechanics

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Abstract

We discuss the Lie algebras of infinitesimal symmetries of the main covariant geometric objects of covariant quantum mechanics: the time form, the hermitian metric, the upper quantum connection, the quantum lagrangian. Indeed, these infinitesimal symmetries are generated, in a covariant way, by the Lie algebra of time preserving conserved special phase functions. Actually, this Lie algebra of special phase functions generates also the Lie algebra of infinitesimal symmetries of the main classical objects: the time form and the cosymplectic 2–form.

A natural output of the classification of the quantum symmetries is a covariant approach to quantum operators and to quantum currents associated with special phase functions.

Key words: covariant classical mechanics, covariant quantum mechanics, quantum symmetries.

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Introduction

Several covariant formulations of Classical and Quantum Mechanics in a curved spacetime with absolute time have been proposed by different authors (see, for instance, [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 28, 29, 30, 31, 32, 37, 38, 39, 40] and citations therein).

In particular, "Covariant Quantum Mechanics" is an approach to Quantum Mechanics in a curved spacetime fibred over absolute time, equipped with a riemannian metric on its fibres, and aimed at implementing several features of General Relativity in this riemannian framework. This formulation started some years ago [20] and has been further developed by several papers (see, for instance, [19, 21, 22, 23, 25, 26, 34, 35, 36, 41] and citations therein).

The infinitesimal symmetries of Covariant Classical Mechanics have been discussed in [26, 34, 35, 36]. In the present paper, we discuss the infinitesimal symmetries of the fundamental objects of Covariant Quantum Mechanics: the time form dt, the η–hermitian metric h_η and the upper quantum connection $\mathrm{Ч}^\uparrow$, which is the source of all other quantum objects. We find that the Lie algebra of infinitesimal symmetries of these objects is isomorphic, in a covariant way, to the Lie algebra of time preserving conserved special phase functions [35]. Moreover, we find that the Lie algebra of infinitesimal symmetries of the quantum lagrangian L and of the time form dt coincides with the Lie algebra of the above fundamental quantum objects and also with the Lie algebra of the fundamental classical objects: the time form dt and the cosymplectic 2–form Ω. Hence, the results of this paper underline the meaning of the Lie algebra of special phase functions and its distinguished subalgebras within this approach to Classical and Quantum Mechanics. This again confirms the covariant approach, which was crucial for the discovery of special phase functions.

We deal with units of measurement on the same footing of coordinates, gauges and observers. So, in order to make our theory explicitly independent of "*units of measurement*", we use the notion of "*spaces of scales*" [25, 27].

We consider the following *basic spaces of scales*: 1) the space $\mathbb{T}$ of *time intervals*, 2) the space $\mathbb{L}$ of *lengths*, 3) the space $\mathbb{M}$ of *masses*. Then, other *space of scales* are obtained by tensor products of rational powers of the above basic spaces.

We consider the *Planck constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ as a "*universal scale*". Moreover, we will consider a *mass* $m \in \mathbb{M}$ and *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$. We denote a *time unit of measurement* and its dual, respectively, by $u_0 \in \mathbb{T}$ and $u^0 \in \mathbb{T}^* \simeq \mathbb{T}^{-1}$.

1 Sketch of the classical background

The classical background of Covariant Quantum Mechanics is provided by a suitable formulation of Classical Mechanics (for a short account of it, see, for instance, [25], where the reader can find further details).

In the present model, we postulate *time* as an oriented 1–dimensional affine space $\boldsymbol{T}$, associated with the vector space $\mathbb{T} \otimes \mathrm{I\!R}$, and *spacetime* as an oriented 4–dimensional manifold $\boldsymbol{E}$ equipped with a *time fibring* $t : \boldsymbol{E} \to \boldsymbol{T}$.

The time fibring yields the distinguished *time form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$.

We shall refer to *spacetime charts* $(x^\lambda) \equiv (x^0, x^i)$, defined as charts of the manifold $\boldsymbol{E}$, which are adapted to the time fibring, the affine structure of $\boldsymbol{T}$ and the orientation of $\boldsymbol{E}$ and $\boldsymbol{T}$. Every spacetime chart (x^λ) yields a *time scale* $u_0 \in \mathbb{T}$. The associated bases of vector fields and forms are denoted by $(\partial_\lambda) \equiv (\partial_0, \partial_i)$ and $(d^\lambda) \equiv (d^0, d^i)$. Accordingly, we obtain the linear fibred charts of the tangent bundle $T\boldsymbol{E} \to \boldsymbol{E}$ by $(x^\lambda, \dot{x}^\lambda)$.

We denote by $V\boldsymbol{E} \subset T\boldsymbol{E}$ the 3–dimensional *vertical subbundle* annihilated by dt and by $H^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ the 1–dimensional *horizontal subbundle* generated by dt. The vertical projection $T^*\boldsymbol{E} \to V^*\boldsymbol{E}$ is denoted by the restriction symbol $\vee$.

The *classical motions* are the sections $s : \boldsymbol{T} \to \boldsymbol{E}$.

The *classical phase space* is the 7-dimensional 1st jet space of motions $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$, equipped with the fibred charts (x^λ, x^i_0).

The phase space is naturally equipped with the *contact map* and the *complementary contact map* $д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ and $\theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$, with coordinate expressions $д = u^0 \otimes (\partial_0 + x^i_0\, \partial_i)$ and $\theta = (d^i - x^i_0\, d^0) \otimes \partial_i$.

The *classical observers* are the sections $o : \boldsymbol{E} \to J_1\boldsymbol{E}$.

An observer o is characterised by the “observed” *contact map* and *complementary contact map* $д[o] := д \circ o : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ and $\theta[o] := \theta \circ o : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$, with coordinate expressions $д[o] = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$ and $\theta[o] = (d^i - o^i_0\, d^0) \otimes \partial_i$.

Then, we postulate the *galileian metric* to be a spacelike riemannian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$. With reference to a particle of mass $m \in \mathbb{M}$, and to the Planck constant $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$, the *rescaled galileian metric* is $G := \frac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$.

We have the coordinate expressions $g = g_{ij}\, \check{d}^i \otimes \check{d}^j$ and $G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j$, with $g_{ij} \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^2 \otimes \mathrm{I\!R})$ and $G^0_{ij} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

The spacelike metric g and the spacetime orientation yield the scaled *spacelike volume form* $\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \wedge^3 V^*\boldsymbol{E}$, with coordinate expression $\eta = \sqrt{|g|}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3$.

Then, we obtain the scaled *spacetime volume form* $\upsilon := dt \wedge \eta : \boldsymbol{E} \to \mathbb{T} \otimes \wedge^4 T^*\boldsymbol{E}$, with coordinate expression $\upsilon = \upsilon^0\, u_0 = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3$.

Given an observer o, we define the *observed kinetic energy*, the *observed kinetic momentum* and the *observed Poincaré–Cartan form* to be, respectively, the sections

$$\begin{aligned}
\mathcal{K}[G, o] &:= \tfrac{1}{2}\, G\, (\nabla[o], \nabla[o]) && \in \mathrm{sec}(J_1\boldsymbol{E}, H^*\boldsymbol{E}), \\
\mathcal{Q}[G, o] &:= \theta[o] \lrcorner\, (G^\flat\, \nabla[o]) && \in \mathrm{sec}(J_1\boldsymbol{E}, T^*\boldsymbol{E}), \\
\Theta[G, o] &:= -\mathcal{K}[G, o] + \mathcal{Q}[G, o] && \in \mathrm{sec}(J_1\boldsymbol{E}, T^*\boldsymbol{E}),
\end{aligned}$$

with coordinate expressions

$$\begin{aligned}
\mathcal{K}[G, o] &= \tfrac{1}{2}\, G^0_{ij} (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0, \\
\mathcal{Q}[G, o] &= G^0_{ij} (x^j_0 - o^j_0)\, (d^i - o^i_0\, d^0), \\
\Theta[G, o] &= (-\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + \tfrac{1}{2}\, G^0_{ij}\, o^i_0\, o^j_0)\, d^0 + G^0_{ij}\, (x^j_0 - o^j_0)\, d^i.
\end{aligned}$$

We define a *galileian spacetime connection* to be a spacetime connection K, which is linear, torsion free and which fulfills the conditions $\nabla dt = 0$, $\nabla g = 0$ and $R_{i\mu j\nu} = R_{j\nu i\mu}$, where R is the curvature tensor of K. Its coordinate expression is of the type

$$\begin{aligned} K &= d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i{}_\mu \, \dot{x}^\mu \, \dot\partial_i) \\ &= d^\lambda \otimes \partial_\lambda - \tfrac{1}{2} G^{ij}_0 \big(\partial_0 G^0_{hj} \, (\dot{x}^h \, d^0 + \dot{x}^0 \, d^h) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) \, \dot{x}^k \, d^h\big) \otimes \dot\partial_i \\ &\quad - G^{ij}_0 \big(\Phi_{0j} \, \dot{x}^0 \, d^0 + \tfrac{1}{2} \Phi_{hj} \, (\dot{x}^h \, d^0 + \dot{x}^0 \, d^h)\big) \otimes \dot\partial_i \,, \end{aligned}$$

where $\Phi \equiv \Phi[K, G, o] = \Phi_{\lambda\mu} \, d^\lambda \wedge d^\mu : \boldsymbol{E} \to \wedge^2 T^*\boldsymbol{E}$ is a closed spacetime 2–form, which depends on K, on G and on the observer o associated with the chosen spacetime chart (x^λ) by the condition $o^i_0 = 0$.

Further, we postulate, as *gravitational and electromagnetic fields*, a galileian spacetime connection and a closed scaled spacetime 2–form [33]

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E} \qquad \text{and} \qquad F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \wedge^2 T^*\boldsymbol{E} \,.$$

With reference to a particle of mass m and charge q, we couple $K^\natural$ and F into the *joined galileian spacetime connection* $K \equiv K^\natural + K^\mathfrak{e} := K^\natural - \frac{1}{2} \frac{q}{\hbar} \, (dt \otimes \widehat{F} + \widehat{F} \otimes dt)$, where $\widehat{F} := G^{\sharp 2}(F) : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E})$.

From now on, we shall refer to the joined spacetime connection K.

The joined observed spacetime 2–form $\Phi \equiv \Phi[K, G, o]$ splits as $\Phi = \Phi^\natural + \frac{1}{2} \frac{q}{\hbar} F$.

We consider as *law of motion* for a particle, with mass m and charge q, effected by the gravitational and electromagnetic fields, the equation $\nabla[K] ds = 0$.

We define a *phase connection* to be a connection $\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$ of the affine bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$.

There is a bijection between time preserving, linear spacetime connections K and affine phase connections Γ [21].

Each affine phase connection Γ yields the "quadratic" *dynamical phase connection*, the *dynamical phase 2–form*, the *dynamical phase 2–vector*

$$\begin{aligned} \gamma &\equiv \gamma[\Gamma] &&:= \text{д} \lrcorner \Gamma : \boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E} \,, \\ \Omega &\equiv \Omega[\Gamma, G] &&:= G \lrcorner \big(\nu[\Gamma] \wedge \theta\big) : J_1\boldsymbol{E} \to \wedge^2 T^* J_1\boldsymbol{E} \,, \\ \Lambda &\equiv \Lambda[\Gamma, G] &&:= \bar{G} \lrcorner (\check{\Gamma} \wedge \nu) : J_1\boldsymbol{E} \to \wedge^2 V J_1\boldsymbol{E} \,. \end{aligned}$$

Therefore, the joined spacetime connection K yields the distinguished affine phase connection, dynamical phase connection, dynamical phase 2–form, dynamical phase 2–vector $\Gamma \equiv \Gamma[K]$, $\gamma \equiv \gamma[K]$, $\Omega \equiv \Omega[K]$, $\Lambda \equiv \Lambda[K, G]$.

We have the coordinate expressions

$$\begin{aligned} \Gamma[K] &= d^\lambda \otimes \partial_\lambda - G^{ij}_0 \big(\Phi_{0j} + \tfrac{1}{2} (\partial_0 G^0_{hj} + \Phi_{hj}) \, x^h_0)\big) \, d^0 \otimes \partial^0_i \\ &\quad - G^{ij}_0 \, \tfrac{1}{2} \big((\partial_0 G^0_{kj} + \Phi_{kj}) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) \, x^h_0)\big) \, d^k \otimes \partial^0_i \,, \end{aligned}$$

$$\begin{aligned} \Omega[K, G] &= (\partial_0 G^0_{hj} \, x^h_0 + \tfrac{1}{2} \, \partial_j G^0_{hk} \, x^h_0 \, x^k_0) \, d^0 \wedge d^j + (\partial_i G^0_{jh} \, x^h_0) \, d^i \wedge d^j \\ &\quad + G^0_{hj} \, x^h_0 \, d^0 \wedge d^j_0 - G^0_{ij} \, d^i \wedge d^j_0 + \tfrac{1}{2} \, \Phi_{\lambda\mu} \, d^\lambda \wedge d^\mu \,, \end{aligned}$$

$$\Lambda[K, G] = G_0^{ij} \, \partial_i \wedge \partial^0_j + G_0^{ih} \, G_0^{jk} \left(\partial_h G^0_{kr} \, x^r_0 + \tfrac{1}{2} \, \Phi_{hk} \right) \partial^0_i \wedge \partial^0_j \, .$$

We can prove that $\Omega[K, G]$ turns out to be closed if and only if K is galileian.

Hence, the pair (dt, Ω) turns out to be a scaled *cosymplectic structure* of the phase space [24]. In other words, $dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to \mathbb{T} \otimes \wedge^7 T^* J_1\boldsymbol{E}$ is a scaled volume form of the phase space and $d\Omega = 0$.

The cosymplectic 2–form Ω admits an *"upper" horizontal potential* of the type $A^\uparrow : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, according to the equation $\Omega = dA^\uparrow$. Clearly, the horizontal potential $A^\uparrow$ is locally defined up to a gauge of the type $df : \boldsymbol{E} \to T^*\boldsymbol{E}$, with $f \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

For each observer o, we have $\Phi[K, G, o] = 2\, o^*\Omega[K, G]$. Hence, the observed potential $A[K, G, o]$ of $\Phi[K, G, o]$ turns out to be given (up to a gauge) by the equality $A[K, G, o] = o^* A^\uparrow$.

The classical law of motion for a motion s effected by the gravitational and electromagnetic fields is expressed equivalently by the equations $\nabla[K] ds = 0$, or $dj_1 s = \gamma[K] \circ j_1 s$.

The *classical lagrangian*, the *classical momentum*, the *observed classical hamiltonian* and the *observed classical momentum* are, respectively, the horizontal and vertical components and the observed horizontal and vertical components of $A^\uparrow$

$$\begin{aligned}
\mathcal{L} \;&\equiv\; \mathcal{L}[A^\uparrow] := д \lrcorner A^\uparrow \in \mathrm{sec}(J_1\boldsymbol{E}, H^*\boldsymbol{E}) \, , \\
\mathcal{P} \;&\equiv\; \mathcal{P}[A^\uparrow] := \theta \lrcorner A^\uparrow \in \mathrm{sec}(J_1\boldsymbol{E}, T^*\boldsymbol{E}) \, , \\
\mathcal{H}[A^\uparrow, o] &:= -д[o] \lrcorner A^\uparrow = \mathcal{K}[G, o] - A[G, o] \in \mathrm{sec}(J_1\boldsymbol{E}, H^*\boldsymbol{E}) \, , \\
\mathcal{P}[A^\uparrow, o] &:= \quad \theta[o] \lrcorner A^\uparrow = \mathcal{Q}[G, o] + A[G, o] \in \mathrm{sec}(J_1\boldsymbol{E}, T^*\boldsymbol{E}) \, .
\end{aligned}$$

We have the coordinate expressions

$$\begin{aligned}
\mathcal{L}[A^\uparrow] &= (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_j\, x^j_0 + A_0)\, d^0 \, , &\quad \mathcal{P}[A^\uparrow] &= (G^0_{ij}\, x^j_0 + A_i)\,(d^i - x^i_0\, d^0) \, , \\
\mathcal{H}[A^\uparrow, o] &= (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 \, , &\quad \mathcal{P}[A^\uparrow, o] &= (G^0_{ij}\, x^j_0 + A_i)\,(d^i - o^i_0\, d^0) \, .
\end{aligned}$$

2 Setting of the quantum theory

Next, we sketch the starting setting of Covariant Quantum Mechanics (for a short account of it, see, for instance, [25], where the reader can find further details).

We postulate the *quantum bundle* to be a 1–dimensional complex vector bundle over spacetime $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, equipped with a scaled *η–hermitian quantum metric* $\mathfrak{h}_\eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \wedge^3 V^*\boldsymbol{E} \otimes \mathbb{C}$.

We shall refer to normalised scaled *quantum bases* $\mathfrak{b} : \boldsymbol{E} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q}$, which fulfill the condition $\mathfrak{h}_\eta(\mathfrak{b}, \mathfrak{b}) = \eta$. Accordingly, we shall refer to scaled linear fibred charts (x^λ, z), where the scaled complex function $z : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{C}$, fulfills the condition $z(\mathfrak{b}) = 1$, and to the associated real fibred charts (x^λ, w^1, w^2), given by $z = w^1 + \mathfrak{i}\, w^2$.

The quantum states are represented by the *quantum sections* $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$. We shall write $\Psi = \psi\, \mathfrak{b}$, with $\psi \equiv |\psi| \exp(\mathfrak{i}\, \varphi) \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathbb{C})$.

We define the *upper quantum bundle* to be the 1–dimensional complex vector bundle $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$ over the phase space, given by the pullback $\boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}$.

The η–hermitian quantum metric $\mathfrak{h}$ yields, by pullback, the η–hermitian upper quantum metric $\mathfrak{h}^{\uparrow}$.

We say that a complex linear connection $\text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} TJ_1\boldsymbol{E} \to T\boldsymbol{Q}^{\uparrow}$ is *reducible* if it factorises through a system of quantum connections $\text{Ч}[o] : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{Q}$. Indeed, $\text{Ч}^{\uparrow}$ turns out to be reducible if and only if, in coordinates, $\text{Ч}^{\uparrow}{}^0_i = 0$.

We postulate the *galileian upper quantum connection* $\text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$ to be a connection of the upper quantum bundle, which is hermitian and reducible and whose curvature fulfills the condition $R[\text{Ч}^{\uparrow}] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^{\uparrow}$, where $\mathbb{I}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to \boldsymbol{Q}^{\uparrow}$ is the Liouville vector field of $\boldsymbol{Q}^{\uparrow}$ (see also [32]). The closure of Ω turns out to be a necessary integrability condition for the local existence of $\text{Ч}^{\uparrow}$, because of the Bianchi identity. The integer cohomology class of Ω turns out to be a necessary integrability condition for the global existence of $\text{Ч}^{\uparrow}$ [41]. The upper quantum connections $\text{Ч}^{\uparrow}$ are defined locally up to a gauge of the type $\mathfrak{i}\,df \otimes \mathbb{I}^{\uparrow}$, where $f : \boldsymbol{E} \to \mathbb{R}$.

With reference to a quantum basis $\mathfrak{b}$, the coordinate expression of an upper quantum connection $\text{Ч}^{\uparrow}$ is locally of the type

$$
\begin{aligned}
\text{Ч}^{\uparrow} &= \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\,A^{\uparrow}[\mathfrak{b}] \otimes \mathbb{I}^{\uparrow} \\
&= \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\,\big(\Theta[o] + A[\mathfrak{b},o]\big) \otimes \mathbb{I}^{\uparrow} \\
&= \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\,\big(-\mathcal{K}[o] + \mathcal{Q}[o] + A[\mathfrak{b},o]\big) \otimes \mathbb{I}^{\uparrow} \\
&= \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\,\big(-\mathcal{H}[\mathfrak{b},o] + \mathcal{P}[\mathfrak{b},o]\big) \otimes \mathbb{I}^{\uparrow} \\
&= d^{\lambda} \otimes \partial_{\lambda} + d^i_0 \otimes \partial^0_i + \mathfrak{i}\,\big(-(\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 - A_0)\,d^0 + (G^0_{ij}\,x^j_0 + A_i)\,d^i\big) \otimes \mathbb{I}^{\uparrow},
\end{aligned}
$$

where $\chi^{\uparrow}[\mathfrak{b}] : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$ is the flat hermitian upper quantum connection induced by the quantum basis $\mathfrak{b}$.

Thus, the *upper quantum potential* $A^{\uparrow}[\mathfrak{b}]$ appearing in the above expression of $\text{Ч}^{\uparrow}$ is just a potential of Ω and a potential of K, that have been discussed previously.

We suppose the cohomology class of Ω to be integer and postulate a *galileian upper quantum connection* $\text{Ч}^{\uparrow}$, as source of all further quantum developments.

We observe that the quantum bases $\mathfrak{b}$ allow us to parametrise the upper quantum potentials $A^{\uparrow}$, hence the *observed quantum potentials* $A[\mathfrak{b},o]$.

With reference to two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \exp(\mathfrak{i}\,\vartheta)\,\mathfrak{b}$ and two observers o and $\acute{o} = o + v$, with $v \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})$, we have the transition rules $A^{\uparrow}[\acute{\mathfrak{b}}] = A^{\uparrow}[\mathfrak{b}] - \mathfrak{i}\,d\vartheta$ and $A[\acute{\mathfrak{b}},\acute{o}] = A[\mathfrak{b},o] - d\vartheta + \theta[o] \lrcorner G^{\flat}(v) - \frac{1}{2}\,G(v,v)$.

From the quantum connection $\text{Ч}^{\uparrow}$ we derive, by a covariant procedure, the *kinetic quantum momentum*, the *probability current*, the *Schrödinger operator*, the *quantum lagrangian* and the *quantum Poincaré–Cartan form*

$$
\begin{aligned}
\mathsf{Q}(\Psi) &:= \text{д} \otimes \Psi - \mathfrak{i}\,G^{\sharp}\nabla^{\uparrow}\Psi : \boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q}), \\
\mathsf{J}(\Psi) &:= \text{д} \otimes \|\Psi\|^2 - \mathrm{re}\,\mathfrak{h}(\Psi,\, \mathfrak{i}\,G^{\sharp}\nabla^{\uparrow}\Psi) : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E}), \\
\mathsf{S}(\Psi) &:= \tfrac{1}{2}\Big(\text{д} \lrcorner \nabla^{\uparrow}\Psi + \delta^{\uparrow}\big(\mathsf{Q}(\Psi)\big)\Big) : \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q},
\end{aligned}
$$

$$\mathsf{L}(\Psi) := -\,dt \wedge \big(\operatorname{im} \mathrm{h}_\eta(\Psi,\, д \lrcorner \nabla^{\uparrow}\Psi) + \tfrac{1}{2}\,(\bar{G} \otimes \mathrm{h}_\eta)(\check{\nabla}^{\uparrow}\Psi,\, \check{\nabla}^{\uparrow}\Psi) : \boldsymbol{E} \to \wedge^4 T^*\boldsymbol{E}\,,$$
$$\Theta[\mathsf{L}] := \mathsf{L} + \vartheta \mathbin{\bar{\wedge}} V_{\boldsymbol{Q}}\mathsf{L} : J_1\boldsymbol{Q} \to \wedge^4 T^*\boldsymbol{Q}\,,$$

with coordinate expressions

$$\begin{aligned}
\mathsf{Q}[\Psi] &= \big(\psi\,\partial_0 - \mathfrak{i}\,G^{ij}_0\,(\partial_j\psi - \mathfrak{i}\,A_j[\mathfrak{b},o]\,\psi)\,\partial_i\big) \otimes u^0 \otimes \mathfrak{b}\,,\\
\mathsf{J}(\Psi) &= \big(|\psi|^2\,\partial_0 + (\mathfrak{i}\,\tfrac{1}{2}\,G^{ij}_0\,(\psi\,\partial_j\bar{\psi} - \bar{\psi}\,\partial_j\psi) - A^i_0[\mathfrak{b},o]\,|\psi|^2)\,\partial_i\big) \otimes u^0\,,\\
\mathsf{S}(\Psi) &= \Big((\partial_0\psi - \tfrac{1}{2}\,\mathfrak{i}\,G^{ij}_0\,\partial_{ij}\psi) - \mathfrak{i}\,(A_0 - \tfrac{1}{2}\,A_i\,A^i_0)\,\psi\\
&\quad - \big((A^j_0 + \tfrac{1}{2}\,\mathfrak{i}\,\frac{\partial_i(G^{ij}_0\sqrt{|g|})}{\sqrt{|g|}})\,\partial_j\psi\big) + \tfrac{1}{2}\,\big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A^i_0\sqrt{|g|})}{\sqrt{|g|}}\big)\,\psi\Big)\,u_0 \otimes \mathfrak{b}\,,\\
\mathsf{L}(\Psi) &= \tfrac{1}{2}\,\big(-G^{ij}_0\,\partial_i\bar{\psi}\,\partial_j\psi + \mathfrak{i}\,A^\lambda_0\,(\bar{\psi}\,\partial_\lambda\psi - \psi\,\partial_\lambda\bar{\psi}) + 2\,(A_0 - \tfrac{1}{2}\,A_i\,A^i_0)\big)\,\upsilon^0\,,\\
\Theta[\mathsf{L}] &= \tfrac{1}{2}\,\mathfrak{i}\,(\bar{z}\,dz - z\,d\bar{z}) \wedge \upsilon^0_0 - \tfrac{1}{2}\,\big(G^{ij}_0\,(\bar{z}_i\,dz + z_i\,d\bar{z}) + \mathfrak{i}\,A^i_0\,(\bar{z}\,dz - z\,d\bar{z})\big) \otimes \upsilon^0_j\\
&\quad + \big(\tfrac{1}{2}\,G^{ij}_0\,\bar{z}_i\,z_j + (A_0 - \tfrac{1}{2}\,A_i\,A^i_0)\bar{z}\,z\big)\,\upsilon^0\,,
\end{aligned}$$

where $\upsilon_\lambda := i_{\partial_\lambda}\upsilon$ and $A^i_0 := G^{ij}_0\,A_j$.

In the particular case of a flat spacetime and an inertial observer, S turns out to be the standard Schrödinger operator.

3 Lie algebra of special phase functions

3.1 Definition. [20, 23] A *special phase function* (s.p.f.) is defined to be a phase function $f \in \operatorname{map}(J_1\boldsymbol{E}, \mathbb{R})$, such that its 2nd fibre derivative with respect to the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ is of the type $D^2 f = f'' \otimes G$, with $f'' \in \operatorname{map}(\boldsymbol{E}, \mathbb{T} \otimes \mathbb{R})$.

In coordinates, a special phase function is characterised by an expression of the type $f = f^0\,\frac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 + f^i\,G^0_{ij}\,x^j_0 + \breve{f}$, with $f^0,\ f^i,\ \breve{f} \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$. Accordingly, we have $f'' = f^0\,u_0$.

We denote the subsheaf of s.p.f. by $\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{map}(J_1\boldsymbol{E}, \mathbb{R})$. □

We have the following distinguished subsheaves of $\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$

subsheaf of projectable s.p.f.	$:= \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$	$:= \big\{f \mid \partial_j f^0 = 0\big\}$.
subsheaf of time preserving s.p.f.	$:= \operatorname{tim}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$	$:= \big\{f \mid \partial_\lambda f^0 = 0\big\}$,
subsheaf of affine s.p.f.	$:= \operatorname{aff}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$	$:= \big\{f \mid f^0 = 0\big\}$,
subsheaf of spacetime s.p.f.	$:= \operatorname{map}(\boldsymbol{E}, \mathbb{R})$	$:= \big\{f \mid f^\lambda = 0\big\}$.

3.2 Example. We have the distinguished special phase functions

$$x^\lambda\,,\quad A^{\uparrow}{}_i[\mathfrak{b},o] = \mathcal{P}_i[\mathfrak{b},o] = G^0_{ij}\,x^j_0 + A_i\,,\quad -A^{\uparrow}{}_0[\mathfrak{b},o] = \mathcal{H}_0[\mathfrak{b},o] = \tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 - A_0\,. \;\square$$

3.3 Proposition. For each $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, we obtain, in a covariant way, the spacetime vector field, called its *tangent lift*, $X[f] = f'' \lrcorner д - G^\sharp(Df) \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, with coordinate expression $X[f] = f^0\,\partial_0 - f^i\,\partial_i$.

For instance, we have: $X[\mathcal{P}_i] = -\partial_i$, $X[\mathcal{H}_0] = \partial_0$, $X[\mathcal{L}_0] = \partial_0 - A^i_0\,\partial_i$. □

3.4 Proposition. With reference to an observer o and to a quantum basis $\mathfrak{b}$, we can split each $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, respectively, as

$$f = -X[f] \lrcorner \Theta[o] + \breve{f}[o] \quad = (f^0\,\mathcal{K}_0 + f^i\,\mathcal{Q}_i) + \breve{f}\,,$$
$$f = -X[f] \lrcorner A^{\uparrow}[\mathfrak{b}] + \hat{f}[\mathfrak{b}] = (f^0\,\mathcal{H}_0 + f^i\,\mathcal{P}_i) + \hat{f}\,,$$

where

$$\breve{f}[o] = \breve{f} \qquad \text{and} \qquad \hat{f}[\mathfrak{b}] = \breve{f} + A_0\,f^0 - A_i\,f^i\,.\square$$

Thus, each $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ is characterised:

- with reference to an observer o, by its observer and gauge independent tangent lift $X[f]$ and observer dependent and gauge independent spacetime function $\breve{f}[o]$,

- with reference to a quantum basis $\mathfrak{b}$, by its observer and gauge independent tangent lift $X[f]$ and gauge dependent and observer independent spacetime function $\hat{f}[\mathfrak{b}]$.

3.5 Proposition. We have two distinguished phase lifts of a special phase function f :

- the *holonomic phase lift*, which involves only the time fibring of spacetime,

- the *hamiltonian phase lift*, which involves the cosymplectic structure of the phase space (here r_1 is the natural fibred morphism $r_1 : J_1T\boldsymbol{E} \to TJ_1\boldsymbol{E}$)

$$X^{\uparrow}{}_{\mathrm{hol}}[f] := r_1 \circ J_1X[f]\,, \qquad X^{\uparrow}{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df)$$

with coordinate expressions

$$X^{\uparrow}{}_{\mathrm{hol}}[f] = f^0\,\partial_0 - f^i\,\partial_i - (\partial_0 f^i + \partial_j f^i\,x^j_0 + \partial_0 f^0\,x^i_0 + \partial_j f^0\,x^j_0\,x^i_0)\,\partial^0_i\,,$$
$$\begin{aligned} X^{\uparrow}{}_{\mathrm{ham}}[f] = f^0\,\partial_0 - f^i\,\partial_i + G^{ij}_0\,\big(&- f^0\,(\partial_0\mathcal{P}_j - \partial_j A_0) + f^h\,(\partial_h\mathcal{P}_j - \partial_j A_h) \\ &+ \partial_j f^0\,\mathcal{K}_0 + \partial_j f^h\,\mathcal{Q}_h + \partial_j \breve{f}\big)\,\partial^0_i\,.\square \end{aligned}$$

3.6 Theorem. *The equality* $[\![f, \acute{f}]\!] := \Lambda(df, d\acute{f}) + \gamma(f'').\acute{f} - \gamma(\acute{f}'').f$ *equips the sheaf of special phase functions with an* $\mathbb{R}$*–lie bracket, called* special phase Lie bracket*.*

This bracket can also be expressed by the following equalities

$$[\![f, \acute{f}]\!] = -\big[X[f], X[\acute{f}]\big] \lrcorner \Theta[o] + X[f].\breve{\acute{f}} - X[\acute{f}].\breve{f} + \Phi[o]\big(X[f]\,, X[\acute{f}]\big)\,,$$
$$[\![f, \acute{f}]\!] = -\big[X[f], X[\acute{f}]\big] \lrcorner A^{\uparrow}[\mathfrak{b}] + X[f].\hat{\acute{f}} - X[\acute{f}].\hat{f}\,,$$
$$[\![f, \acute{f}]\!] = X^{\uparrow}[f].\acute{f} - X^{\uparrow}[\acute{f}].f + 2\,\Omega\big(X^{\uparrow}[f]\,, X^{\uparrow}[\acute{f}]\big)\,,$$

where $X^{\uparrow}[f] \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ *is any phase prolongation (in particular, the holonomic lift and the hamiltonian lift) of the tangent lift* $X[f] \in \sec(\boldsymbol{E}, T\boldsymbol{E})$.

In coordinates, we have the following expression

$$[\![f, \acute{f}]\!]^{\lambda} = X[f]^{\mu}\,\partial_\mu X[\acute{f}]^{\lambda} - X[\acute{f}]^{\mu}\,\partial_\mu X[f]^{\lambda}\,,$$
$$[\![f, \breve{\ } \acute{f}]\!] = X[f]^{\mu}\,\partial_\mu \breve{\acute{f}} - X[\acute{f}]^{\mu}\,\partial_\mu \breve{f} + X[f]^{\lambda}\,X[\acute{f}]^{\mu}\,(\partial_\lambda A_\mu - \partial_\mu A_\lambda)\,.$$

The projectable, time preserving and affine subsheaves of special phase functions turn out to be closed with respect to the special phase Lie bracket.

The holonomic lift and the hamiltonian lift of special phase functions turn out to be Lie algebra homomorphisms. $\square$

For each $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we set $\operatorname{div}_\eta f := \operatorname{div}_\eta X[f]$.

Indeed, we have $\operatorname{div}_\eta \llbracket f, \acute{f} \rrbracket = X[f].\operatorname{div}_\eta \acute{f} - X[\acute{f}].\operatorname{div}_\eta f$.

The subsheaves $\operatorname{uni}_\eta \operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{duni}_\eta \operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$, of projectable special phase functions with vanishing divergence and with constant divergence, respectively, are closed with respect to the special Lie bracket.

3.7 Definition. A special phase function f is said to be *holonomic* if its holonomic phase lift and hamiltonian phase lift coincide: $X^{\uparrow}{}_{\mathrm{ham}}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f]$. $\square$

3.8 Proposition. A special phase function f turns out to be holonomic if and only if it fulfills the following conditions, in coordinates,

$$\begin{aligned} \partial_i f^0 &= 0\,, \\ \partial_0 f^0\, G^0_{ij} - (f^0\, \partial_0 - f^h\, \partial_h)\, G^0_{ij} + \partial_j f^h\, G^0_{ih} + \partial_i f^h\, G^0_{jh} &= 0\,, \\ \partial_i \breve{f} + \partial_0 f^h\, G^0_{ih} - f^0\, (\partial_0 A_i - \partial_i A_0) + f^h\, (\partial_h A_i - \partial_i A_h) &= 0\,. \end{aligned}$$

The subsheaf of *holonomic* special phase functions $\operatorname{hol}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is closed with respect to the special phase Lie bracket. $\square$

A special phase function f is said to be *conserved* if it is constant along the classical motions solutions of the law of motion, i.e. if $\gamma.f = 0$. We denote the subsheaf of conserved special phase functions by $\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

3.9 Lemma. For each $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and $f \in \operatorname{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$, the following implication holds

$$i_{X^{\uparrow}}\, \Omega = df \qquad \Rightarrow \qquad X^{\uparrow} = \gamma\big(dt(X^{\uparrow})\big) + \Lambda^{\sharp}(df) \quad \text{and} \quad \gamma.f = 0\,.$$

Proof. The proof can be achieved from the identities $\Lambda\big(i_{X^{\uparrow}}\Omega\big) = X^{\uparrow} - \gamma(X^{\uparrow})$ and $i_\gamma \Omega = 0$. QED

3.10 Theorem. *For each* $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$, *the following conditions are equivalent:*

$$1)\quad 0 = \gamma.f\,, \qquad 2)\quad df = i_{X^{\uparrow}{}_{\mathrm{ham}}[f]}\, \Omega\,, \qquad 3)\quad df = i_{X^{\uparrow}{}_{\mathrm{hol}}[f]}\, \Omega\,,$$

$$4)\quad \begin{cases} 0 = \partial_i f^0\,, \\ 0 = \partial_0 f^0\, G^0_{hk} - f^0\, \partial_0 G^0_{hk} + f^i\, \partial_i G^0_{hk} + \partial_h f^i\, G^0_{ik} + \partial_k f^i\, G^0_{ih}\,, \\ 0 = \partial_h \breve{f} - f^0\, (\partial_0 A_h - \partial_h A_0) + f^i\, (\partial_i A_h - \partial_h A_i) + \partial_0 f^i\, G^0_{ih}\,, \\ 0 = \partial_0 \breve{f} - f^i\, (\partial_0 A_i - \partial_i A_0)\,. \end{cases}$$

Indeed, if the above equivalent conditions are fulfilled, then $X^{\uparrow}{}_{\mathrm{ham}}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f]$, *i.e.,* $\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{hol}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

PROOF. The proof can be achieved from the above Lemma 3.9 and from the coordinate expression of the condition for a special phase function to be conserved. QED

The time preserving conserved special phase functions constitute a further Lie subalgebra $\operatorname{tim}\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$.

For each $f \in \operatorname{tim}\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, we have $L_{X[f]}G = 0$, hence $\operatorname{div}_\eta f = 0$.

4 Quantum symmetries

A vector field $Y \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ is said to be *real linear* if it is projectable on $\boldsymbol{E}$ and a real linear morphism over its spacetime projection $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, i.e. if it is of the type $Y = X^\lambda\,\partial_\lambda + (Y^1_1\,w^1 + Y^1_2\,w^2)\,\partial w_1 + (Y^2_1\,w^1 + Y^2_2\,w^2)\,\partial w_2$, with $X^\lambda,\ Y^1_1,\ Y^1_2,\ Y^2_1,\ Y^2_2 \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$.

A vector field $Y \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ is said to be *complex linear* if it is real linear and a complex linear morphism over its spacetime projection $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, i.e. if it is of the type $Y = X^\lambda\,\partial_\lambda + Y^1_1\,(w^1\,\partial w_1 + w^2\,\partial w_2) + Y^1_2\,(w^2\,\partial w_1 - w^1\,\partial w_2)$, with $X^\lambda,\ Y^1_1,\ Y^1_2 \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$.

The sheaves $\operatorname{lin}_{\mathbb{R}}\operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$ and $\operatorname{lin}_{\mathbb{C}}\operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$ of $\mathbb{R}$–linear and $\mathbb{C}$–linear quantum vector fields turn out to be closed with respect to the Lie bracket of vector fields.

4.1 Lemma. If $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, then:

- for each observer o, the vector field $Y[f,o] := X[f] \lrcorner \text{Ч}[o] + \mathfrak{i}\,\check{f}[o]\,\mathbb{I} \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ turns out to be gauge independent;
- for each basis $\mathfrak{b}$, the vector field $Y[f,\mathfrak{b}] := X[f] \lrcorner \chi[\mathfrak{b}] + \mathfrak{i}\,\hat{f}[\mathfrak{b}]\,\mathbb{I} \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ turns out to be observer independent.

Moreover, we have $X[f] \lrcorner \text{Ч}[o] + \mathfrak{i}\,\check{f}[o]\,\mathbb{I} = X[f] \lrcorner \chi[\mathfrak{b}] + \mathfrak{i}\,\hat{f}[\mathfrak{b}]\,\mathbb{I}$.

PROOF. The proof follows from the transition rules of the quantum potential [25] and of the components of the special phase function, which fit very well. QED

4.2 Definition. We define the *η–hermitian quantum vector fields* to be the infinitesimal symmetries of the η–quantum metric $\mathfrak{h}_\eta$, i.e. the vector fields $Y_\eta \in \operatorname{lin}_{\mathbb{R}}\operatorname{pro}_{\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}, T\boldsymbol{Q})$, such that $L_{Y_\eta}\mathfrak{h}_\eta = 0$. We denote the Lie algebra subsheaf of η–hermitian quantum vector fields by $\operatorname{her}_\eta(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \sec(\boldsymbol{Q}, T\boldsymbol{Q})$. □

4.3 Theorem. *[23] The η–hermitian quantum vector fields are of the type*

$$\begin{aligned} Y_\eta = Y_\eta[f] &= X[f] \lrcorner \chi[\mathfrak{b}] + \big(\mathfrak{i}\,\hat{f}[\mathfrak{b}] - \tfrac{1}{2}\operatorname{div}_\eta X[f]\big)\,\mathbb{I} \\ &= X[f] \lrcorner \text{Ч}[o] + \big(\mathfrak{i}\,\check{f}[o] - \tfrac{1}{2}\operatorname{div}_\eta X[f]\big)\,\mathbb{I} \\ &= f^0\,\partial_0 - f^i\,\partial_i + \big(\mathfrak{i}\,(\check{f} + A_0\,f^0 - A_i\,f^i) - \tfrac{1}{2}\operatorname{div}_\eta f\big)\,\mathbb{I} \\ &= f^0\,\partial_0 - f^i\,\partial_i + \big(\mathfrak{i}\,\hat{f} - \tfrac{1}{2}\operatorname{div}_\eta f\big)\,\mathbb{I}\,, \end{aligned}$$

with $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$. Indeed, the map $Y_\eta : \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \operatorname{her}_\eta(\boldsymbol{Q}, T\boldsymbol{Q})$ turns out to be an $\mathbb{R}$–Lie algebra isomorphisms with respect to the special phase Lie bracket and the Lie bracket of vector fields.

PROOF. The proof can be achieved by comparing the splitting of Y_η into its horizontal and vertical components with respect to the observed quantum connection $\text{Ч}[o]$ (or with respect to the flat quantum connection $\chi[\mathfrak{b}]$) and the splittings of a special phase function f into its spacetime lift $X[f]$ and its observed spacetime component $f[o]$ (or its gauge components $f[\mathfrak{b}]$) (Proposition 3.4). QED

4.4 Example. We have the following distinguished η–hermitian quantum vector fields

$$Y_\eta[x^\lambda] = \mathfrak{i}\, x^\lambda\, \mathbb{I}\,, \qquad Y_\eta[A^\uparrow{}_\lambda] = -\partial_\lambda + \tfrac{1}{2}\, \frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}}\, \mathbb{I}\,.\,\square$$

4.5 Definition. We define the *η–hermitian upper quantum vector fields* to be the infinitesimal symmetries of the η–hermitian upper quantum metric $\mathfrak{h}^\uparrow{}_\eta$, i.e. the vector fields $Y^\uparrow{}_\eta \in \operatorname{lin}_{\mathrm{I\!R}} \operatorname{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$, such that $L_{Y^\uparrow{}_\eta} \mathfrak{h}^\uparrow{}_\eta = 0$.

We denote the Lie algebra subsheaf of η–hermitian upper quantum vector fields by $\operatorname{her}^\uparrow{}_\eta(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) \subset \sec(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$. $\square$

4.6 Proposition. The η–hermitian upper quantum vector fields are of the type

$$Y^\uparrow{}_\eta = Y^\uparrow{}_\eta[X^\uparrow, f] := \text{Ч}^\uparrow(X^\uparrow) + (\mathfrak{i}\, f - \tfrac{1}{2}\, \operatorname{div}_\eta X)\, \mathbb{I}^\uparrow\,,$$

with $(X^\uparrow,\, f) \in \operatorname{pro}_{\boldsymbol{E},\boldsymbol{T}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \times \operatorname{map}(J_1\boldsymbol{E})$, where $X \in \operatorname{pro}_{\boldsymbol{T}}(\boldsymbol{E}, T\boldsymbol{E})$ is the spacetime projection of $X^\uparrow$, i.e., in coordinates, of the type

$$Y^\uparrow{}_\eta = X^\lambda\, \partial_\lambda + X^i_0\, \partial^0_i + (f + A^\uparrow{}_\lambda\, X^\lambda)\, (w^1\, \partial w_2 - w^2\, \partial w_1) - \tfrac{1}{2}\, \operatorname{div}_\eta f\, (w^1\, \partial w_1 + w^2\, \partial w_2)\,,$$

where $X^0 \in \operatorname{map}(\boldsymbol{T}, \mathrm{I\!R})\,, \quad X^i \in \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R})\,, \quad X^i_0,\, f \in \operatorname{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

PROOF. The proof can be achieved by splitting $Y^\uparrow{}_\eta$ into its horizontal and vertical components with respect to the upper quantum connection $\text{Ч}^\uparrow$. QED

4.7 Proposition. The subsheaf of η–hermitian upper quantum vector fields $\operatorname{her}^\uparrow{}_\eta(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) \subset \operatorname{lin}_{\mathrm{I\!R}} \operatorname{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ turns out to be closed with respect to the Lie bracket of vector fields. Indeed, the map

$$Y^\uparrow{}_\eta : \operatorname{pro}_{\boldsymbol{E},\boldsymbol{T}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \times \operatorname{map}(J_1\boldsymbol{E}) \to \operatorname{her}^\uparrow{}_\eta(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) : (X^\uparrow, f) \mapsto Y^\uparrow{}_\eta[X^\uparrow, f]$$

turns out to be an $\mathrm{I\!R}$–Lie algebra isomorphism with respect to the Lie bracket of phase pairs $\Big[(X^\uparrow, f)\,,\, (\acute{X}^\uparrow, \acute{f})\Big]_{2\Omega} = \Big([X^\uparrow, \acute{X}^\uparrow]\ ,\ X^\uparrow.\acute{f} - \acute{X}^\uparrow.f + 2\,\Omega(X^\uparrow, \acute{X}^\uparrow)\Big)$ and the Lie bracket of vector fields. $\square$

4.8 Theorem. *An η–hermitian upper quantum vector field $Y^\uparrow{}_\eta[X^\uparrow, f]$ is projectable on $\boldsymbol{Q}$ if and only if $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$ and $X^\uparrow$ is any phase prolongation of the tangent lift $X[f] \in \operatorname{pro}\sec(\boldsymbol{E}, T\boldsymbol{E})$.*

PROOF. The proof can be achieved from the coordinate expression of $L_{Y^\uparrow{}_\eta} \mathfrak{h}^\uparrow{}_\eta$ and the splittings of the special phase functions (Proposition 3.4). QED

4.9 Corollary. For each $f \in \text{pro}\,\text{spe}(J_1\boldsymbol{E}, {\rm I\!R})$, we have two distinguished ${\rm I\!R}$–Lie algebra isomorphisms (Proposition 3.5)

$$Y^\uparrow{}_{\eta\,\text{hol}} : \text{pro}\,\text{spe}(J_1\boldsymbol{E}, {\rm I\!R}) \to \text{her}^\uparrow{}_\eta(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) : f \mapsto Y^\uparrow{}_\eta[X^\uparrow{}_{\text{hol}}, f]\,,$$
$$Y^\uparrow{}_{\eta\,\text{ham}} : \text{pro}\,\text{spe}(J_1\boldsymbol{E}, {\rm I\!R}) \to \text{her}^\uparrow{}_\eta(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) : f \mapsto Y^\uparrow{}_\eta[X^\uparrow{}_{\text{ham}}, f]\,.\square$$

4.10 Example. We have the following distinguished infinitesimal symmetries of the η–hermitian upper quantum metric

$$Y^\uparrow{}_{\eta\,\text{hol}}[x^\lambda] = \mathfrak{i}\,x^\lambda\,\mathbb{I}^\uparrow\,, \qquad Y^\uparrow{}_{\eta\,\text{hol}}[A^\uparrow{}_\lambda] = -\partial_\lambda + \tfrac{1}{2}\,\frac{\partial_\lambda\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}^\uparrow\,,$$

$$Y^\uparrow{}_{\eta\,\text{ham}}[x^\lambda] = \delta^\lambda_i\,G^0_{ij}\,\partial^i_0 + \mathfrak{i}\,x^\lambda\,\mathbb{I}^\uparrow\,, \quad Y^\uparrow{}_{\eta\,\text{ham}}[A^\uparrow{}_\lambda] = -\partial_\lambda + G^{ih}_0\,\partial_\lambda\mathcal{P}_h\,\partial^0_i + \tfrac{1}{2}\,\frac{\partial_\lambda\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}^\uparrow\,.\square$$

4.11 Definition. We define the *infinitesimal symmetries* of the upper quantum connection $\text{Ч}^\uparrow$ to be the upper quantum vector fields $Y^\uparrow \in \text{lin}_{\rm I\!R}\,\text{pro}_{J_1\boldsymbol{E}}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$, such that $L_{Y^\uparrow}\text{Ч}^\uparrow = 0\,.\square$

4.12 Proposition. The infinitesimal symmetries of $\text{Ч}^\uparrow$ are of the type

$$Y^\uparrow = \text{Ч}^\uparrow(X^\uparrow) + \check{Y}^\uparrow\,,$$

where $X^\uparrow \in \sec(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ and $\check{Y}^\uparrow \in \text{lin}_{\rm I\!R}\,\text{pro}_{J_1\boldsymbol{E}}(\boldsymbol{Q}^\uparrow,\, V_{J_1\boldsymbol{E}}\boldsymbol{Q}^\uparrow)$ fulfill the following three equivalent conditions

$$1)\quad L_{\check{Y}^\uparrow}\text{Ч}^\uparrow = -\mathfrak{i}\,(i_{X^\uparrow}\Omega)\otimes\mathbb{I}^\uparrow\,, \qquad 2)\quad \nabla^\uparrow\check{Y}^\uparrow = \mathfrak{i}\,(i_{X^\uparrow}\Omega)\otimes\mathbb{I}^\uparrow\,.$$

Indeed, the sheaf $\text{cnc}^\uparrow(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ of infinitesimal symmetries of $\text{Ч}^\uparrow$ turns out to be closed with respect to the Lie bracket of vector fields.

Proof. The proof can be achieved by means of our postulate $R[\text{Ч}^\uparrow] = -2\,\mathfrak{i}\,\Omega\otimes\mathbb{I}^\uparrow$. QED

4.13 Proposition. The infinitesimal symmetries $Y^\uparrow{}_\eta \in \text{lin}_{\rm I\!R}\,\text{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ of $\mathrm{h}^\uparrow{}_\eta$ and $\text{Ч}^\uparrow$ are of the type $Y^\uparrow{}_\eta = Y^\uparrow{}_\eta[f] := \text{Ч}^\uparrow\big(X^\uparrow[f]\big) + (\mathfrak{i}\,f - \tfrac{1}{2}\,\text{div}_\eta f)\,\mathbb{I}^\uparrow$, with $f \in \text{duni}_\eta\,\text{cns}\,\text{spe}(J_1\boldsymbol{E},\, {\rm I\!R})$ and $X^\uparrow[f] = X^\uparrow{}_{\text{hol}}[f] = X^\uparrow{}_{\text{ham}}[f]$, i.e. of the type

$$\begin{aligned}Y^\uparrow{}_\eta &= f^0\,\partial_0 - f^i\,\partial_i + X^j_0\,\partial^0_j\\ &\quad + (\check{f} + A_0\,f^0 - A_i\,f^i)\,(w^1\,\partial w_2 - w^2\,\partial w_1) - \tfrac{1}{2}\,\text{div}_\eta f\,(w^1\,\partial w_1 + w^2\,\partial w_2)\,,\end{aligned}$$

where the spacetime functions $f^0 \in \text{map}(\boldsymbol{T}, {\rm I\!R})$, $f^i,\, \check{f} \in \text{map}(\boldsymbol{E}, {\rm I\!R})$ fulfill the conditions

$$\begin{aligned}0 &= \partial_i f^0\,,\\ 0 &= \partial_0 f^0\,G^0_{hk} - f^0\,\partial_0 G^0_{hk} + f^i\,\partial_i G^0_{hk} + \partial_h f^i\,G^0_{ik} + \partial_k f^i\,G^0_{ih}\,,\\ 0 &= \partial_h\check{f} - f^0\,(\partial_0 A_h - \partial_h A_0) + f^i\,(\partial_i A_h - \partial_h A_i) + \partial_0 f^i\,G^0_{ih}\,,\end{aligned}$$

$$
\begin{aligned}
0 &= \partial_0 \breve{f} - f^i \, (\partial_0 A_i - \partial_i A_0) \,, \\
0 &= d\big(f^0 \, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (f^i \, \sqrt{|g|})}{\sqrt{|g|}}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
X^{\uparrow}[f] &= f^0 \, \partial_0 - f^i \, \partial_i - (\partial_0 f^i + \partial_j f^i \, x^j_0 + \partial_0 f^0 \, x^i_0 + \partial_j f^0 \, x^j_0 \, x^i_0) \, \partial^0_i \,, \\
&= f^0 \, \partial_0 - f^i \, \partial_i + G^{ij}_0 \, \Big(\partial_j \breve{f} + \partial_j f^0 \, \tfrac{1}{2} \, G^0_{hk} \, x^h_0 \, x^k_0 + \partial_j f^h \, G^0_{hk} \, x^k_0 \\
&\quad - f^0 \, \big(\partial_0 G^0_{hj} \, x^h_0 + (\partial_0 A_j - \partial_j A_0)\big) + f^h \, \big(\partial_h G^0_{jk} \, x^k_0 - (\partial_j A_h - \partial_h A_j)\Big) \, \partial^0_i \,,
\end{aligned}
$$

$$
\operatorname{div}_\eta f = f^0 \, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (f^i \, \sqrt{|g|})}{\sqrt{|g|}} \,.
$$

Indeed, the upper quantum vector field ${Y^{\uparrow}}_\eta[f]$ turns out to be projectable on the η–hermitian quantum vector field $Y_\eta[f] \in \operatorname{her}_\eta(\boldsymbol{Q}, T\boldsymbol{Q})$.

Moreover, the map ${Y^{\uparrow}}_\eta : \operatorname{duni}_\eta \operatorname{cns} \operatorname{spe}(J_1\boldsymbol{E}, {\rm I\!R}) \to \operatorname{cns} {\operatorname{her}^{\uparrow}}_\eta(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : f \mapsto {Y^{\uparrow}}_\eta[f]$ turns out to be an ${\rm I\!R}$–Lie algebra isomorphism with respect to the special phase Lie bracket and the Lie bracket of vector fields.

Furthermore, the map $\operatorname{pro}_{\boldsymbol{Q}} : \operatorname{cnc} {\operatorname{her}^{\uparrow}}_\eta(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : \operatorname{her}_\eta(\boldsymbol{Q}, T\boldsymbol{Q}) : {Y^{\uparrow}}_\eta[f] \mapsto Y_\eta[f]$ turns out to be an ${\rm I\!R}$–Lie algebra morphism with respect to the Lie bracket of vector fields.

Proof. The proof can be achieved from the coordinate expression of $L_{Y^{\uparrow}} Ч^{\uparrow}$ and Theorem 3.10. QED

4.14 Theorem. *The infinitesimal symmetries ${Y^{\uparrow}}_\eta \in \operatorname{lin}_{I\!R} \operatorname{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$ of dt, ${\mathrm{h}^{\uparrow}}_\eta$ and $Ч^{\uparrow}$ are of the type $Y^{\uparrow} = Y^{\uparrow}[f] := Ч^{\uparrow}\big(X^{\uparrow}[f]\big) + \mathfrak{i} \, f \, \mathbb{I}^{\uparrow}$, with $f \in$* cns tim spe$(J_1\boldsymbol{E}, I\!R)$ *and $X^{\uparrow}[f] = {X^{\uparrow}}_{\mathrm{hol}}[f] = {X^{\uparrow}}_{\mathrm{ham}}[f]$, i.e. of the type*

$$
{Y^{\uparrow}}_\eta = f^0 \, \partial_0 - f^i \, \partial_i + X^j_0 \, \partial^0_j + (\breve{f} + A_0 \, f^0 - A_i \, f^i) \, (w^1 \, \partial w_2 - w^2 \, \partial w_1) \,,
$$

where the spacetime functions $f^0 \in I\!R$, $f^i, \breve{f} \in \operatorname{map}(\boldsymbol{E}, I\!R)$ fulfill the conditions

$$
\begin{aligned}
0 &= \partial_0 f^0 \, G^0_{hk} - f^0 \, \partial_0 G^0_{hk} + f^i \, \partial_i G^0_{hk} + \partial_h f^i \, G^0_{ik} + \partial_k f^i \, G^0_{ih} \,, \\
0 &= \partial_h \breve{f} - f^0 \, (\partial_0 A_h - \partial_h A_0) + f^i \, (\partial_i A_h - \partial_h A_i) + \partial_0 f^i \, G^0_{ih} \,, \\
0 &= \partial_0 \breve{f} - f^i \, (\partial_0 A_i - \partial_i A_0) \,,
\end{aligned}
$$

$$
\begin{aligned}
X^{\uparrow}[f] &= f^0 \, \partial_0 - f^i \, \partial_i - (\partial_0 f^i + \partial_j f^i \, x^j_0 + \partial_0 f^0 \, x^i_0 + \partial_j f^0 \, x^j_0 \, x^i_0) \, \partial^0_i \,, \\
&= f^0 \, \partial_0 - f^i \, \partial_i + G^{ij}_0 \, \Big(\partial_j \breve{f} + \partial_j f^0 \, \tfrac{1}{2} \, G^0_{hk} \, x^h_0 \, x^k_0 + \partial_j f^h \, G^0_{hk} \, x^k_0 \\
&\quad - f^0 \, \big(\partial_0 G^0_{hj} \, x^h_0 + (\partial_0 A_j - \partial_j A_0)\big) + f^h \, \big(\partial_h G^0_{jk} \, x^k_0 - (\partial_j A_h - \partial_h A_j)\Big) \, \partial^0_i \,.
\end{aligned}
$$

Indeed, the upper quantum vector field $Y^{\uparrow}{}_{\eta}[f]$ turns out to be projectable on the hermitian quantum vector field $Y_{\eta}[f] \in \mathrm{her}_{\eta}(\boldsymbol{Q}, T\boldsymbol{Q})$.

Moreover, the map $Y^{\uparrow}{}_{\eta} : \mathrm{cns}\,\mathrm{tim}\,\mathrm{spe}(J_1\boldsymbol{E}, I\!R) \to \mathrm{cns}\,\mathrm{her}^{\uparrow}{}_{\eta}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}{}_{\eta}[f]$ turns out to be an $I\!R$–Lie algebra isomorphism with respect to the special phase Lie bracket and the Lie bracket of vector fields.

Furthermore, the map $\mathrm{pro}_{\boldsymbol{Q}} : \mathrm{cnc}\,\mathrm{her}^{\uparrow}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \to \mathrm{her}_{\eta}(\boldsymbol{Q}, T\boldsymbol{Q}) : Y^{\uparrow}{}_{\eta}[f] \mapsto Y_{\eta}[f]$ turns out to be an $I\!R$–Lie algebra morphism with respect to the Lie bracket of vector fields.

Proof. The proof follows from Proposition 4.13. QED

4.15 Definition. We define the *infinitesimal symmetries of the quantum lagrangian* to be the $\mathrm{I\!R}$–linear quantum vector fields $Y \in \mathrm{lin}_{\mathrm{I\!R}}\,\mathrm{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$, such that $L_{Y_1}\mathsf{L} = 0$, where $Y_1 := r_1 \circ J_1 Y \in \mathrm{lin}_{\mathrm{I\!R}}\,\mathrm{pro}_{\boldsymbol{E},\boldsymbol{Q}}(J_1\boldsymbol{Q}, TJ_1\boldsymbol{Q})$, is the 1st holonomic prolongation of Y, with coordinate expression

$$Y_1 = X^{\lambda}\,\partial_{\lambda} + Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}}\,\partial w_{\mathrm{a}} + (\partial_{\mu}Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}} + Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}}_{\mu} - \partial_{\mu}X^{\nu}\,w^{\mathrm{a}}_{\nu})\,\partial w^{\mu}_{\mathrm{a}}\,.\square$$

4.16 Proposition. The infinitesimal symmetries Y of L are characterised, in coordinates, by the following conditions

$$Y^1_1 = Y^2_2\,, \qquad Y^1_2 = -Y^2_1\,, \qquad \partial_j Y^1_1 = 0\,,$$
$$0 = X^{\lambda}\,\partial_{\lambda}(A_0 - A_j\,A^j_0) - (\partial_0 - A^j_0\,\partial_j)\,Y^2_1 + (A_0 - A_i\,A^i_0)\,(2\,Y^1_1 + \mathrm{div}_{\upsilon}\,X)\,,$$
$$0 = -(\partial_0 X^0 - A^j_0\,\partial_j)\,X^0 + (2\,Y^1_1 + \mathrm{div}_{\upsilon}\,X)\,,$$
$$0 = (\partial_0 - A^j_0\,\partial_j)\,X^i + X^{\lambda}\,\partial_{\lambda}A^i_0 - G^{ij}_0\,\partial_j Y^2_1 + A^i_0\,(2\,Y^1_1 + \mathrm{div}_{\upsilon}\,X)\,,$$
$$0 = X^{\lambda}\,\partial_{\lambda}G^{ij}_0 - G^{hj}_0\,\partial_h X^i - G^{ih}_0\,\partial_h X^j + G^{ij}_0\,(2\,Y^1_1 + \mathrm{div}_{\upsilon}\,X)\,.\square$$

4.17 Theorem. *The infinitesimal symmetries of L and dt are the η–hermitian quantum vector fields generated by time preserving conserved special phase functions*

$$Y_{\eta} = Y_{\eta}[f]\,, \qquad \text{with} \qquad f \in \mathrm{tim}\,\mathrm{cns}\,\mathrm{spe}(J_1\boldsymbol{E}, I\!R)\,.$$

Thus, they are of the type $Y_{\eta} = f^0\,\partial_0 - f^i\,\partial_i + \mathfrak{i}\,(\breve{f} + A_0\,f^0 - A_i\,f^i)\,\mathbb{I}$, where the functions $f^0 \in I\!R$, $f^i, \breve{f} \in \mathrm{map}(\boldsymbol{E}, I\!R)$ fulfill the conditions

$$0 = f^0\,\partial_0 G^0_{hk} - f^i\,\partial_i G^0_{hk} + \partial_h f^i\,G^0_{ik} + \partial_k f^i\,G^0_{ih}\,,$$
$$0 = \partial_h\breve{f} - f^0\,(\partial_0 A_h - \partial_h A_0) + f^i\,(\partial_i A_h - \partial_h A_i) + \partial_0 f^i\,G^0_{ih}\,,$$
$$0 = \partial_0\breve{f} - f^i\,(\partial_0 A_i - \partial_i A_0)\,.$$

Proof. The proof follows from the coordinate expressions of $L_{Y_1}\mathsf{L}$ and $Y_{\eta}[f]$. QED

4.18 Corollary. For each $f \in \mathrm{tim}\,\mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we have the equivalences: $L_{Y_{\eta\,1}[f]}\,\mathsf{L} = 0 \quad \Leftrightarrow \quad L_{Y_{\eta\,1}[f]}\Theta[\mathsf{L}] = 0 \quad \Leftrightarrow \quad f \in \mathrm{tim}\,\mathrm{cns}\,\mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

Proof. The 1st equivalence follows from a general result of variational calculus [42]. QED

It is remarkable that the $\mathrm{I\!R}$–Lie algebra of infinitesimal symmetries of $(\Omega\,,\ dt)$ (see [34, 36]) of $(\mathfrak{h}^\uparrow\,,\ \mathsf{Y}^\uparrow\,,\ dt)$ and of $(\mathsf{L}\,,\ dt)$ (see [35]) be generated by the same Lie subsheaf of special phase functions $\operatorname{tim}\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

The above classifications of quantum infinitesimal symmetries can be used as the source of further developments.

In particular, the classification of η–hermitian quantum vector fields yields, in a covariant way, the *quantum operators* associated with projectable special phase functions $\mathsf{O}[f] = \mathfrak{i}\,\big(Y_\eta[f] - f'' \lrcorner\, \mathsf{S}\big) : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$, with coordinate expression $\mathsf{O}[f](\Psi) = \Big(\big(\breve{f} - A_i\, f^i - \mathfrak{i}\,(f^i\,\partial_i + \frac{1}{2}\,\frac{\partial_i(f^i\,\sqrt{|g|})}{\sqrt{|g|}}) - \frac{1}{2}\,f^0\,\Delta_0\big)\,\psi\Big)\,\mathsf{b}$.

For instance, $\mathsf{O}[x^\lambda](\Psi) = x^\lambda\,\psi\,\mathsf{b}$, $\mathsf{O}[\mathcal{P}_j](\Psi) = -\mathfrak{i}\,\big(\partial_j\psi + \frac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\,\psi\big)\,\mathsf{b}$, $\mathsf{O}[\mathcal{H}_0](\Psi) = -\big(\frac{1}{2}\,\Delta_0\,\psi + A_0\,\psi\big)\,\mathsf{b}$.

Moreover, we obtain, in a covariant way, for each $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$ and $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the *quantum current* $\mathfrak{j}_\eta[f](\Psi) := -(j_1\Psi)^*(i_{Y_{\eta\,1}[f]}\,\Theta[\mathsf{L}]) \in \sec(\boldsymbol{E}, \wedge^3 T^*\boldsymbol{E})$.

For instance, the quantum current associated to the special phase function $f = 1$ turns out to be just the probability current.

These objects will be the subject of another paper.

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