Hermitian vector fields and special phase functions

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Abstract

We start by analysing the Lie algebra of Hermitian vector fields of a Hermitian line bundle.

Then, we specify the base space of the above bundle by considering a Galilei, or an Einstein spacetime. Namely, in the first case, we consider, a fibred manifold over absolute time equipped with a spacelike Riemannian metric, a spacetime connection (preserving the time fibring and the spacelike metric) and an electromagnetic field. In the second case, we consider a spacetime equipped with a Lorentzian metric and an electromagnetic field.

In both cases, we exhibit a natural Lie algebra of special phase functions and show that the Lie algebra of Hermitian vector fields turns out to be naturally isomorphic to the Lie algebra of special phase functions.

Eventually, we compare the Galilei and Einstein cases.

Key words. Hermitian vector fields, quantum bundle, special phase functions, Galilei spacetime, Lorentz spacetime.

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Contents

Introduction

A covariant formulation of classical and quantum mechanics on a curved spacetime with absolute time ("curved Galilei spacetime") based on fibred manifolds, jets, non linear connections, cosymplectic forms and Frölicher smooth spaces has been proposed by A. Jadczyk and M. Modugno some years ago [10, 11] and further developed by several authors (see, for instance, [2, 8, 12, 13, 16, 19, 20, 21, 22, 23, 24, 31, 32, 33, 34, 37, 38]). We shall briefly call this approach "Covariant Quantum Mechanics".

It presents analogies with the approach due to K. Kuchař [26], to Geometric Quantisation (see, for instance, [1, 6, 7, 25, 36, 35, 40] and references therein) and to the approach due to C. Duval, Künzle et al. (see, for instance, [4, 5]) in the Bargmann framework. But it presents several differences and novelties as well. In fact, it produces an effective procedure for quantum operators and overcomes typical difficulties of Geometric Quantisation, such as the problems of polarisations and quantum energy operator; moreover, in the flat case, it reproduces the standard quantum mechanics (hence, it allows us to recover all classical examples).

The main original features of this Covariant Quantum Mechanics are the following.

We are not concerned with a very general and ambitious quantisation programme (such as, for instance, that of Geometric Quantisation), but we deal with a rather concrete scheme of spacetime equipped only with classical fundamental fields and follow a criterion of minimal assumptions.

The main input of our scheme is the requirement of 'manifest covariance', which plays an essential role throughout the theory from the very beginning. Namely, we assume the principle of general relativity in our curved Galileian framework with absolute time and Riemannian spacelike metric, detaching it from the more usual Lorentzian viewpont of the Einstein theory. Even more, we formulate the theory in a way which is manifestly independent of the choice of units of measurement.

The first consequence of this viewpoint is the nature of time. Thus, spacetime is equipped with an absolute surjective time map, which yields the fibring into simultaneity subspaces; but, we have no absolute spacelike space. We need the choice of an observer in order to achieve (locally) a splitting of the tangent space of spacetime into timelike and spacelike components. According to the principle of general relativity, the fundamental steps of the general theory are observer independent; on the other hand, an observer is required to describe specific measurements. In general, we deal with "accelerated" observers; inertial observers can be considered only in the flat case. Thus, time is never considered as a parameter, but it is intimately linked with the other objects of the theory with strong consequences at any step. Moreover, the absolute time is assumed to be an oriented affine space and not the real line, as it is most usually done. This is not just a sophisticated mathematical approach. In fact, the usual viewpoint corresponds implicitly to the choice of a distinguished time scale; indeed, the fact that we do not assume distinguished time scales plays a key role in some steps of the theory. We observe that our "spacetime" and its structures should not be understood in a strict sense; in fact, spacetime can also play the role of a configuration space for one or several free particles, or for a constrained system, allowing more general developments of the theory.

Another of the main consequences of the manifest covariance is the choice of the 1st jet space of motions as classical phase space, instead of the more usual tangent, or cotangent, or vertical, or covertical spaces of spacetime. If we had chosen the tangent space of spacetime as classical phase space, then we should have been concerned with the constraint of time normalisation and the related difficulties. An even worst choice would have been the cotangent space of spacetime, because of the lack of a Legendre map due to the spacelike character of the metric. Moreover, the choice of the vertical or covertical tangent space of spacetime as classical phase space would conflict with the requirement of covariance for dynamical laws. Indeed, the vertical approach is suitable to describe covariantly only some geometric aspects of the theory, but not the dynamical laws which have an "horizontal" character with respect to the time fibring. Other theories assume implicitly the vertical or covertical tangent space of spacetime as classical phase space and try to bypass the above problem by additional methods. In our opinion, some of the typical difficulties of these theories arise from this non dynamically covariant choice of the phase space. Our classical phase space is odd dimensional; this technical fact has several important consequences.

Besides the time fibring, we assume as source fields just the spacelike Riemannian metric, the gravitational connection and the electromagnetic field, linked by the natural interaction equations which are allowed by the covariance requirement in the Galileian framework. In fact, these are the only classical fundamental fields and we are involved only with a "fundamental" theory. Even in the Einstein framework, the metric and the gravitational connection describe essentially distinct phenomena, hence should be considered as distinct fields; but the fact that the gravitational connection is determined by the metric induces us to consider the metric just as the potential of the gravitational connection. However, in the Galilei context, the metric and the gravitational field still describe essentially different phenomena, but the second fact is no longer true. The spacelike metric, regarded as a metric of spacetime, is degenerate, hence it determines the gravitational connection only up to a local closed 2–form. The curvature of a (pseudo-)Riemannian metric has a symmetry property which is not guaranteed in the Galilei case: it should be assumed as a postulate (this assumption has been considered also by other authors [4, 26]). Indeed, this property is equivalent to the closure of the cosymplectic form and turns out to be an esential integrability condition for the quantum connection.

It is wellknown that we cannot write in a Galilei framework the Maxwell and Einstein equations, which link the gravitational and electromagnetic fields with their charge and masses sources. This is due to the degeneracy of the spacetime metric. However, we can write in a covariant way the above equations in a reduced form, which is able to account only for the static effects of charges and masses on the corresponding fields (see also [28]. Actually, this fact is a weak aspect not specifical of our approach, but of the Galilei framework, any way. Indeed, the Einstein spacetime is the right framework for the true Maxwell and Einstein equations. On the other hand, the standard quantum mechanics and the Schrödinger equation are so important that it is worth considering the Galilei framework. Actually, in the present theory the gravitational and electromagnetic fields are considered as given external fields. Therefore, we are not explicitly involved with their sources. For this reason, in this paper, we are not interested in the above reduced equations.

In the Galilei framework, we can merge in a covariant way the electromagnetic field into the gravitational field, by exploiting the metric and the time fibring. In this way, we obtain a "joined spacetime connection", which incorporates both the gravitational and the electromagnetic fields. This "joining" effects the theory at all steps in a convenient way.

Our phase space is equipped with a cosymplectic form, instead of the more usual symplectic form. This technical fact has several strategic consequences. For instance, the classical Hamiltonian function is not an additional, absolute starting object of the theory, but it is locally extracted from the cosymplectic form by choosing an observer. Indeed, the Hamiltonian function is an "horizontal" object with respect to the time fibring and is introduced independently of the momentum, on the same footing of this. This strategic fact allows us to skip the difficulties of ordering, because energy is not derived from momentum. Actually, our cosymplectic form restricts to a fibrewise symplectic form on the vertical or covertical tangent spaces of spacetime. But this symplectic form has a purely geometric role. Again, trying to derive dynamical consequences from this symplectic form would lead us to artificial and problematic procedures. We stress that the cosymplectic 2-form cannot be even regarded as the family of those fibrewise symplectic forms. Indeed, in order to do this we should add a "horizontal" term, which cannot be expressed in a covariant way. In the Geometric Quantisation, the symplectic form is usually assumed as a postulate. In our approach, the cosymplectic form is not postulated, but is naturally generated by the starting fields, hence by the metric and the joined spacetime connection. On the other hand, the cosymplectic form encodes completely these fields. A typical aspect of the approaches based on a symplectic framework is the Darboux theorem and the symmetry between p's and q's. Actually, this viewpoint does not play a role in our approach, because it would break the time fibring; we do not believe that this is a real problem, because the above symmetry has no true physical necessity, but it is just suggested by the usual formalism.

Our approach is not concerned with the quantisation of any cosymplectic manifold. But, we deal only with the cosymplectic form which arises from the starting fundamental fields (metric, gravitational and electromagnetic fields) on a manifold equipped with a time fibring. Indeed, this cosymplectic form has some specific properties induced by the above physical structure. In particular, we stress the fact that this cosymplectic form admits "horizontal" potentials.

In the quantum theory for a scalar charged particle effected by a given gravitational and electromagnetic field we assume a Hermitian line bundle over spacetime. Moreover, on the pullback of the quantum bundle with respect to the classical phase space, we assume a "phase quantum connection", i.e. a Hermitian connection which is "universal" and whose curvature is proportional to the classical cosymplectic form. Thus, the quantum bundle lives on spacetime (and not on the classical phase space), on one hand, and the phase quantum connection is "universal", on the other hand. The existence of such a connection is strictly linked to the fact that the cosymplectic form admits horizontal potentials. We have to mention analogies with the earlier works by C. Duval, Künzle et al. [4, 5]. These are original aspects of our approach with respect to Geometric Quantisation, which have strategic and fruitful consequences. Actually, in this way we skip the problem of polarisation, by replacing a difficult search for an inclusion of a subspace with an easier and successful criterion of projectability. This criterion turns out to be the way of implementing the principle of relativity in our context, because it yields observer independent objects. By the way, we observe that a similar scheme can be applied to a spin particle with a few additional assumptions [2].

The Schrödinger equation on a curved spacetime can be achieved in a covariant way by different geometric procedures from the only starting classical and quantum objects (the time fibring, the spacelike metric, the joined spacetime connection, the quantum Hermitian metric and the phase quantum connection). In fact, we can exhibit a global, gauge free and observer independent quantum Lagrangian, which yields the Schrödinger equation by a usual procedure. Moreover, we can achieve the Schrödinger equation through a purely differential procedure induced by the quantum connection. Even more, we can show that the Schrödinger equation is determined just by a covariance requirement, which involves not only the independence of observers but also of time scales. Thus, our approach to the Schrödinger equation is detached from any Hamiltonian scheme and has nothing to do with energy at first step; the link with energy comes into only later by a comparison with the pre–quantum energy operator. This viewpoint is conceptually quite different from most usual approaches. However, we have to mention some partial analogies with the earlier works by Kuchař [26] and C. Duval, Künzle et al. [4, 5]. The explicit discussion of the Schrödinger equation is not the subject of the present paper: the reader can refer to [11, 20].

Perhaps, the most original aspect of our approach consists in the Lie algebra of special phase functions and the way the quantum operators are achieved. The special phase functions can be selected among all phase functions by taking into account just the time fibring and the metric. These functions are quadratic with respect to the velocity coordinates and the coefficient of the quadratic term is proportional to the metric. Indeed, this space of functions includes the spacetime coordinates, the components of the classical momentum and the classical Hamiltonian, treating them on the same footing. In order to achieve their Lie bracket we need the cosymplectic form. This bracket can be regarded as a modification of the Poisson bracket achieved in a covariant way by adding to it a "horizontal" term. In fact, in the Galilei framework, the phase 2–vector generating the Poisson bracket is vertical and does not encode all fields of spacetime. The special bracket apparently resembles the Jacobi bracket [30], but it is really a new bracket which makes sense only for special phase functions. We stress that this bracket depends on the 2nd jet of the functions. The special Lie bracket reduces to the Poisson bracket in the particular case of affine special phase functions. But it is essentially different from the Poisson bracket if one of the two special functions is energy. Thus, the Lie algebra of special phase functions turns out to be one of the key points, which allow us to quantise energy without the usual difficulties. In particular, we stress that in this context it would be useless to express the

Hamiltonian via the momentum, because the special bracket has not the usual behaviour with respect to the scalar multiplication of functions. One of the main features of the special phase functions is that they admit in a covariant way, besides the linear and affine Hamiltonian lift, also a tangent lift. This last allows us to achieve also a holonomic lift. It is noticeable that the special phase functions, and their bracket, arise naturally and independently in several aspects of the classicasl and quantum theory. For instance, we can prove that a distinguished Lie subalgebra of the special phase functions generates all classical and quantum infinitesimal symmetries [31]. However, in the present paper we treat explicitly only those aspects of special phase functions which are directly related to the main aim of this paper. In Geometric Quantisation [1], the wellknown Gröenewald and van Hove no go theorems show the role of quadratic functions. In the context of Covariant Quantum Mechanics, the special phase functions have a clear link with the above result, but also relevant conceptual differences.

Another basic aspects of Covariant Quantum Mechanics concerns quantum operators on quantum sections associated with special phase functions. In the original formulation of the theory, this goal was achieved by a rather intricate way. The present paper is aimed at presenting a greatly improved approach to this correspondence.

The essential idea is the following. The Lie derivatives are natural candidates as 1st order covariant operators on sections of the quantum bundle. But, we want to select Lie derivatives with respect to vector fields which reflect the geometric (hence physical) structure of spacetime and quantum bundle. For this purpose, we consider the Hermitian vector fields and classify them. Actually, by the help of an auxiliary quantum connection, we prove, in a general context, that the Lie algebra of Hermitian vector fields is isomorphic to a Lie algebra of pairs constituted by a spacetime function and a spacetime vector field. In the Galilei framework, we obtain a further result. In fact, we can prove that each observer yields an isomorphism of the Lie algebra of special phase functions with the above Lie algebra of pairs. Moreover, the phase quantum connection can be regarded as a system of observed quantum connections with a certain transition law. Hence, if we classify the Hermitian vector fields by means of any observed quantum connection of the above system, we find a natural isomorphism with the Lie algebra of special phase functions. Moreover, we can prove that this correspondence turns out to be observer independent. Summing up, we exhibit the "correspondence principle" as a consequence of the classification of Hermitian vector fields and show a covariant isomorphism between the Lie algebras of Hermitian vector fields and special phase functions. Thus, in our approach, we do not start from a postulate of quantisation of some classical Lie algebra. The principle of covariance naturally suggests a class of 1st order operators on quantum sections as candidate for quantum operators. Then, the link with a Lie algebra of classical phase functions arises from a classification theorem and not from a postulate. By the way, we stress that the Lie algebra of special phase functions appears naturally in our classical theory, but it could be recovered independently while classifying the Hermitian vector fields.

In a covariant formulation of quantum mechanics we do not deal just with one Hilbert space, but we need a Hilbert space for each time. In other words, we deal with a Hilbert bundle, which is not naturally trivial, even in the flat case. We need the choice of an observer in order to obtain a splitting of this bundle.

If we apply the above correspondence principle to the position and momentum observables we obtain the standard quantum operators, which act on the quantum sections fibrewisely with rewspect to the Hilbert bundle. But, we obtain the partial derivative with respect to time for the energy; this operator does not act fibrewisely. On the other hand, by combining this operator with the Schrödinger operator, we obtain the quantum operator for energy. Actually, in the present paper, we discuss only the operators on the sections of the quantum bundle. The further developments related to the Hilbert quantum bundle are beyond the scope of the present paper and can be found in the literature (see, for instance, [20]).

It is well known that quantum mechanics fails in an Einstein relativistic context. On the other hand, we can prove that all quantum results of Covariant Quantum Mechanics in the Galilei framework, previous the stuff related to the Hilbert quantum bundle, can be essentially rephrased in an Einstein framework. The basic ideas work on the same footing in the two cases. However, several technical differences appear, due to the different structure of spacetime in the two cases. These developments in the Einstein case seem to be interesting by themselves. Moreover, we deem that the reader can understand better the Galilei case by seeing how the results of this theory look like in the Einstein case. For these reasons and aims, this paper deals also with the Einstein case (see also [14, 15, 17, 18]).

Here, we just list a few typical features of our approach to the Einstein case, as a specific section is devoted to the comparison between the Galilei and the Einstein cases. The classical phase space is the 1st jet space of timelike one dimensional submanifolds of spacetime. In this framework we can recover the contact structure via the Lorentz metric. The time form lives on the phase space, instead of spacetime. The gravitational cosymplectic form is globally exact and has a distinguished potential. The special phase functions and their Lie bracket can be defined analogously to the Galilei case, but they are not quadratic and we do not need an observer to split them. We can split the phase quantum connection into the electromagnetic quantum connection and the gravitational correcting term.

Thus, the paper is aimed at discussing the updated approach to quantum operators via the classification of Hermitian vector fields and comparing these results achieved in the Galilei case with analogous results for the Einstein case. The paper is organised in the following way.

First, we consider a generic spacetime and quantum bundle and classify the Hermitian vector fields by an auxiliary quantum connection.

Then, we specify the geometric structures of the Galilei spacetime and quantum bundle, and analyse several classical and quantum consequences of these postulates, summarising as briefly as possible all introductory matter.

Accordingly, we achieve the classification of Hermitian vector fields in terms of special phase functions.

Next, we repeat an analogous procedure in the Einstein case.

Eventually, we devote a specific section for the discussion on the main differences between the Galilei and Einstein cases.

In order to make classical and quantum mechanics explicitly independent from scales, we introduce explicitly the "spaces of scales", treating this aspect of the theory in a rigorous mathematical way [11]. Even if several formulas appear in an unusual aspect, this method has several technical effects, which produce some strategic consequences. An example is related to the affine Hamiltonian lift of phase functions and its consequences on the energy pre–quantum operator. Another example is related to the additional scalar curvature term in the Schrödinger equation.

Roughly speaking, a space of scales $\mathbb{S}$ has the algebraic structure of $\mathbb{R}^+$ but has no distinguished 'basis'. We can define the tensor product of spaces of scales and the tensor product of spaces of scales and vector spaces. We can define rational tensor powers $\mathbb{U}^{m/n}$ of a space of scales $\mathbb{U}$. Moreover, we can make a natural identification $\mathbb{S}^* \simeq \mathbb{S}^{-1}$.

The basic objects of our theory (metric, electromagnetic field, etc.) will be valued into *scaled* vector bundles, that is into vector bundles multiplied tensorially with spaces of scales. In this way, each tensor field carries explicit information on its "scale dimension".

Actually, we assume the following basic spaces of scales: the space of *time intervals* $\mathbb{T}$, the space of *lengths* $\mathbb{L}$, the space of *masses* $\mathbb{M}$.

We assume the following "universal scales": the *Planck's constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ and the *speed of light* $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$. Moreover, we will consider a *particle* of *mass* $m \in \mathbb{M}$ and *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$.

If $\boldsymbol{M}$ and $\boldsymbol{N}$ are manifolds, then the sheaf of local smooth maps $\boldsymbol{M} \to \boldsymbol{N}$ is denoted by $\mathrm{map}(\boldsymbol{M}, \boldsymbol{N})$. If $\boldsymbol{F} \to \boldsymbol{B}$ is a fibred manifold, then the sheaf of local sections $\boldsymbol{B} \to \boldsymbol{F}$ is denoted by $\sec(\boldsymbol{B}, \boldsymbol{F})$. If $\boldsymbol{F} \to \boldsymbol{B}$ and $\boldsymbol{F}' \to \boldsymbol{B}$ are fibred manifolds, then the sheaf of local fibred morphisms $\boldsymbol{F} \to \boldsymbol{F}'$ over $\boldsymbol{B}$ is denoted by $\mathrm{fib}(\boldsymbol{F}, \boldsymbol{F}')$.

If $\boldsymbol{F} \to \boldsymbol{B}$ is a fibred manifold, then the vertical restriction of forms will be denoted by a check symbol $^\vee$.

1 Hermitian vector fields

First of all, we analyse the Lie algebra of Hermitian vector fields of a Hermitian line bundle.

Let us consider a manifold $\boldsymbol{E}$, which will be specified in the next sections as Galilei, or Einstein *spacetime*. We denote the charts of $\boldsymbol{E}$ by (x^λ) and the associated local bases of vector fields of $T\boldsymbol{E}$ and forms of $T^*\boldsymbol{E}$ by ∂_λ and d^λ, respectively.

1.1 Quantum bundle

We consider a *Hermitian line bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, called *quantum bundle*, i.e. a complex vector bundle with 1-dimensional fibres, equipped with a scaled Hermitian product $\mathtt{h} : \boldsymbol{E} \to (\mathbb{L}^{-3} \otimes \mathbb{C}) \otimes (\boldsymbol{Q}^* \otimes \boldsymbol{Q}^*)$.

We shall refer to (local) *quantum bases*, i.e. to scaled sections $\mathtt{b} \in \sec(\boldsymbol{E}, \mathbb{L}^{3/2} \otimes \boldsymbol{Q})$, such that $\mathtt{h}(\mathtt{b}, \mathtt{b}) = 1$, and to the associated (local) scaled complex linear dual functions $z \in \mathrm{map}(\boldsymbol{Q}, \mathbb{L}^{-3/2} \otimes \mathbb{C})$. We shall also refer to the associated (local) real basis $(\mathtt{b}_{\mathrm{a}}) \equiv (\mathtt{b}_1, \mathtt{b}_2) := (\mathtt{b}, \mathfrak{i}\, \mathtt{b})$ and to the associated scaled real linear dual basis $(w^{\mathrm{a}}) \equiv (w^1, w^2) = \Big(\frac{1}{2}\,(z + \bar{z}),\, \frac{1}{2}\,\mathfrak{i}\,(\bar{z} - z)\Big)$. We denote the associated vertical vector fields by $(\partial_{\mathrm{a}}) \equiv (\partial_1, \partial_2)$.

The small Latin indices $\mathrm{a}, \mathrm{b} = 1, 2$ will span the real indices of the fibres.

For each $\Phi, \Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we write

$$\Psi = \Psi^{\mathrm{a}}\, \mathtt{b}_{\mathrm{a}} = \psi\, \mathtt{b} \qquad \text{and} \qquad \mathtt{h}(\Phi, \Psi) = (\Phi^1\, \Psi^1 + \Phi^2\, \Psi^2) + \mathfrak{i}\,(\Phi^1\, \Psi^2 - \Phi^2\, \Psi^1) = \bar{\phi}\, \psi\,,$$

with $\Psi^1, \Psi^2 \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathbb{R})$ and $\psi = \Psi^1 + \mathfrak{i}\, \Psi^2 \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathbb{C})$.

Each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ can be regarded as a vertical vector field $\Psi \simeq \tilde{\Psi} \in \sec(\boldsymbol{Q}, V\boldsymbol{Q}) : q_e \mapsto \big(q_e, \Psi(e)\big)$, according to the coordinate expression $\Psi \simeq \tilde{\Psi} = \Psi^{\mathrm{a}}\, \partial_{\mathrm{a}}$. We can regard $\mathtt{h}$ as a scaled complex vertical valued form $\mathtt{h} : \boldsymbol{Q} \to (\mathbb{L}^{-3} \otimes \mathbb{C}) \otimes V^*\boldsymbol{Q}$, according to the coordinate expression $\mathtt{h} = (w^1\, \check{d}^1 + w^2\, \check{d}^2) + \mathfrak{i}\,(w^1\, \check{d}^2 - w^2\, \check{d}^1)$.

The *unity* and the *imaginary unity* tensors

$$1 = \mathrm{id}_{\boldsymbol{Q}} : \boldsymbol{E} \to \boldsymbol{Q}^* \otimes \boldsymbol{Q} \qquad \text{and} \qquad \mathfrak{i} = \mathfrak{i}\, \mathrm{id}_{\boldsymbol{Q}} : \boldsymbol{E} \to \boldsymbol{Q}^* \otimes \boldsymbol{Q}$$

will be identified, respectively, with the *Liouville* and the *imaginary Liouville* vector fields

$$\mathbb{I} : \boldsymbol{Q} \to V\boldsymbol{Q} = \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q, q) \qquad \text{and} \qquad \mathfrak{i}\,\mathbb{I} : \boldsymbol{Q} \to V\boldsymbol{Q} = \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q, \mathfrak{i}\, q)\,.$$

We have the coordinate expressions

$$\begin{aligned} 1 = \mathrm{id}_{\boldsymbol{Q}} &= w^1\, \mathtt{b}_1 + w^2\, \mathtt{b}_2 = z \otimes \mathtt{b}\,, & \mathbb{I} &= w^1\, \partial_1 + w^2\, \partial_2 = z \otimes \partial_1\,, \\ \mathfrak{i} = \mathfrak{i}\, \mathrm{id}_{\boldsymbol{Q}} &= w^1\, \mathtt{b}_2 - w^2\, \mathtt{b}_1 = \mathfrak{i}\, z \otimes \mathtt{b}\,, & \mathfrak{i}\,\mathbb{I} &= w^1\, \partial_2 - w^2\, \partial_1 = \mathfrak{i}\, z \otimes \partial_1\,. \end{aligned}$$

Each quantum basis $\mathfrak{b}$ yields (locally) the flat connection $\chi[\mathfrak{b}] : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$, with coordinate expression $\chi[\mathfrak{b}] = d^\lambda \otimes \partial_\lambda$.

Next, let us consider a Hermitian connection of the quantum bundle, i.e. a tangent valued form [9, 39] $c : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$, which is projectable on $\mathbf{1}_{\boldsymbol{E}}$, complex linear over its projection and such that $\nabla \mathrm{h} = 0$.

Then, c can be written (locally) as $c = \chi[\mathfrak{b}] + \mathfrak{i}\, A[\mathfrak{b}] \otimes \mathbb{I}$, with $A[\mathfrak{b}] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$.

Moreover, we obtain $c^1_{\lambda 1} = c^2_{\lambda 2} = 0$ and $c^2_{\lambda 1} = -c^1_{\lambda 2}$, and the coordinate expression $c = d^\lambda \otimes (\partial_\lambda + \mathfrak{i}\, A_\lambda \mathbb{I})$, with $A_\lambda = c^2_{\lambda 1} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

We have the coordinate expression $\nabla\Psi = (\partial_\lambda \psi - \mathfrak{i}\, A_\lambda\, \psi)\, d^\lambda \otimes \mathfrak{b}$, $\forall\, \Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$.

The curvature of c is $R[c] := -[c, c] = -\mathfrak{i}\, \Phi[c] \otimes \mathbb{I}$, where $[\,,]$ is the Frölicher-Nijenhuis bracket and $\Phi[c] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$ is the closed 2–form given locally by $\Phi[c] = 2\, dA[\mathfrak{b}]$ [9, 29, 39]. Thus, we have the coordinate expression $\Phi[c] = 2\, \partial_\mu A_\lambda\, d^\mu \wedge d^\lambda$.

1.2 Hermitian vector fields

1.2.1 Projectable vector fields

A vector field $Y \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ is said to be *projectable* (on $\boldsymbol{E}$) if $T\pi \circ Y \in \mathrm{fib}(\boldsymbol{Q}, T\boldsymbol{E})$ factorises through a section $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$. Thus, $Y \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ is projectable if and only if its coordinate expression is of the type $Y = X^\lambda\, \partial_\lambda + Y^{\mathrm{a}}\, \partial_{\mathrm{a}} = X^\lambda\, \partial_\lambda + Y^z\, \mathfrak{b}$, where $X^\lambda \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$, $Y^{\mathrm{a}} \in \mathrm{map}(\boldsymbol{Q}, \mathrm{I\!R})$, $Y^z = Y^1 + \mathfrak{i}\, Y^2 \in \mathrm{map}(\boldsymbol{Q}, \mathbb{C})$.

The projectable vector fields constitute a subsheaf $\mathrm{proj}(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \sec(\boldsymbol{Q}, T\boldsymbol{Q})$, which is closed with respect to the Lie bracket. Moreover, the projection $T\pi : \mathrm{proj}(\boldsymbol{Q}, T\boldsymbol{Q}) \to \sec(\boldsymbol{E}, T\boldsymbol{E})$ turns out to be a morphism of Lie algebras.

1.2.2 Linear vector fields

A vector field $Y \in \mathrm{proj}(\boldsymbol{Q}, T\boldsymbol{Q})$ is (real) linear over its projection $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$ if and only if its coordinate expression is of the type $Y = X^\lambda\, \partial_\lambda + Y^{\mathrm{a}}_{\mathrm{b}}\, w^{\mathrm{b}}\, \partial_{\mathrm{a}}$, with X^λ, $Y^{\mathrm{a}}_{\mathrm{b}} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$, i.e., of the type $Y = X^\lambda\, \partial_\lambda + Y^z_{\mathrm{b}}\, w^{\mathrm{b}}\, \mathfrak{b}$, with $X^\lambda \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$ and $Y^z_{\mathrm{b}} = Y^1_{\mathrm{b}} + \mathfrak{i}\, Y^2_{\mathrm{b}} \in \mathrm{map}(\boldsymbol{E}, \mathbb{C})$.

The linear projectable vector fields constitute a subsheaf $\mathrm{lin}_{\mathrm{I\!R}}(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \mathrm{proj}(\boldsymbol{Q}, T\boldsymbol{Q})$, which is closed with respect to the Lie bracket.

1.1 Lemma. If $Y \in \mathrm{lin}_{\mathrm{I\!R}}(\boldsymbol{Q}, T\boldsymbol{Q})$ and $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, then, by regarding Ψ as a vertical vector field $\tilde{\Psi} \in \sec(\boldsymbol{E}, V\boldsymbol{Q})$, we obtain the Lie derivative $L[Y]\, \tilde{\Psi} \in \sec(\boldsymbol{Q}, V\boldsymbol{Q})$, which can be regarded as a section $Y.\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$. We have the coordinate expression $Y.\Psi = (X^\lambda\, \partial_\lambda \Psi^{\mathrm{a}} - Y^{\mathrm{a}}_{\mathrm{b}}\, \Psi^{\mathrm{b}})\, \mathfrak{b}_{\mathrm{a}}$. □

1.2 Lemma. If $\alpha \in \sec(\boldsymbol{Q}, V^*\boldsymbol{Q})$ and $Y \in \mathrm{proj}(\boldsymbol{Q}, T\boldsymbol{Q})$, then the Lie derivative $L(Y)\alpha$ is well defined, in spite of the fact that the form α is vertical valued, and has coordinate expression $L(Y)\alpha = (Y^\mu\, \partial_\mu \alpha_{\mathrm{a}} + Y^{\mathrm{b}}\, \partial_{\mathrm{b}}\, \alpha_{\mathrm{a}} + \alpha_{\mathrm{b}}\, \partial_{\mathrm{a}}\, Y^{\mathrm{b}})\, \check{d}^{\mathrm{a}}$.

Proof. If $\tilde{\alpha} \in \sec(\boldsymbol{Q}, T^*\boldsymbol{Q})$ is any extension of α (obtained, for instance through a connection of the line bundle), then let us prove that the vertical restriction $L(Y)\alpha := (L(Y)\tilde{\alpha})^{\vee} \in \sec(\boldsymbol{Q}, V^*\boldsymbol{Q})$ does not depend on the choice of the extension $\tilde{\alpha}$. The coordinate expression of $\tilde{\alpha}$ is of the type $\tilde{\alpha} = \alpha_\mu \, d^\mu + \alpha_{\mathrm{a}} \, d^{\mathrm{a}}$.

Then, the expression $Y = Y^\lambda \, \partial_\lambda + Y^{\mathrm{a}} \, \partial_{\mathrm{a}}$, with $\partial_{\mathrm{b}} Y^\lambda = 0$, yields

$$L(Y)\,\tilde{\alpha} = (Y^\mu \, \partial_\mu \alpha_\lambda + Y^{\mathrm{b}} \, \partial_{\mathrm{b}} \, \alpha_\lambda + \alpha_\mu \, \partial_\lambda Y^\mu + \alpha_{\mathrm{b}} \, \partial_\lambda Y^{\mathrm{b}}) \, d^\lambda + (Y^\mu \, \partial_\mu \alpha_{\mathrm{a}} + Y^{\mathrm{b}} \, \partial_{\mathrm{b}} \, \alpha_{\mathrm{a}} + \alpha_{\mathrm{b}} \, \partial_{\mathrm{a}} \, Y^{\mathrm{b}}) \, d^{\mathrm{a}} \, .$$

Eventually, by considering the natural vertical projection ${}^{\vee} : T^*\boldsymbol{Q} \to V^*\boldsymbol{Q}$, we obtain the section $\big(L(Y)\,\tilde{\alpha}\big)^{\vee} = (Y^\mu \, \partial_\mu \alpha_{\mathrm{a}} + Y^{\mathrm{b}} \, \partial_{\mathrm{b}} \alpha_{\mathrm{a}} + \partial_{\mathrm{a}} Y^{\mathrm{b}} \, \alpha_{\mathrm{b}}) \, \check{d}^{\mathrm{a}}$. □

For each $Y \in \mathrm{lin}_{\mathrm{I\!R}}(\boldsymbol{Q}, \, T\boldsymbol{Q})$, we have the coordinate expression

$$\begin{aligned} L(Y)\,\mathfrak{h} = {} & \Big(2\, Y_1^1 \, w^1 + (Y_1^2 + Y_2^1) \, w^2 - \mathfrak{i} \, Y_{\mathrm{a}}^{\mathrm{a}} \, w^2\Big) \, \check{d}^1 \\ & + \Big(2\, Y_2^2 \, w^2 + (Y_1^2 + Y_2^1) \, w^1 + \mathfrak{i} \, Y_{\mathrm{a}}^{\mathrm{a}} \, w^1\Big) \, \check{d}^2 \, . \end{aligned}$$

Each $Y \in \mathrm{lin}_{\mathrm{I\!R}}(\boldsymbol{Q}, T\boldsymbol{Q})$ is complex linear over its projection X if and only if $L[Y]\,(\mathfrak{i}\,\mathbb{I}) = 0$, i.e. if and only if $L[Y]\,(\mathfrak{i}\,\Psi) = \mathfrak{i}\,Y.\Psi$, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, i.e. if and only if $Y_1^1 = Y_2^2$ and $Y_1^2 = -Y_2^1$, i.e. if and only if its coordinate expression is of the type $Y = X^\lambda \, \partial_\lambda + Y^z \, \mathbb{I}$, with $X^\lambda \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$ and $Y^z = Y_1^1 + \mathfrak{i}\, Y_1^2 = Y_2^2 - \mathfrak{i}\, Y_2^1 \in \mathrm{map}(\boldsymbol{Q}, \mathbb{C})$.

The complex linear vector fields constitute a subsheaf $\mathrm{lin}_{\mathbb{C}}(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \mathrm{lin}_{\mathrm{I\!R}}(\boldsymbol{Q}, T\boldsymbol{Q})$, which is closed with respect to the Lie bracket.

If $Y \in \mathrm{lin}_{\mathbb{C}}(\boldsymbol{Q}, T\boldsymbol{Q})$ and $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, then we obtain the coordinate expression $Y.\Psi = (X^\lambda \, \partial_\lambda \psi - Y^z \, \psi) \, \mathsf{b}$. If $\check{Y} \in \mathrm{map}(\boldsymbol{E}, \mathbb{C})$, then we obtain $(\check{Y}\,\mathbb{I}).\Psi = -\check{Y}\,\Psi$.

1.2.3 Hermitian vector fields

A vector field $Y \in \mathrm{lin}_{\mathrm{I\!R}}(\boldsymbol{Q}, T\boldsymbol{Q})$ projectable on $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$ is said to be *Hermitian* if $L[Y]\,\mathfrak{h} = 0$, where we regard $\mathfrak{h}$ as a vertical valued form.

In other words, Y is Hermitian if and only if

$$L[X]\Big(\mathfrak{h}(\Psi, \, \Phi)\Big) = \mathfrak{h}\Big(Y.\Psi, \, \Phi\Big) + \mathfrak{h}\Big(\Psi, \, Y.\Phi\Big) \, , \qquad \forall \, \Psi, \Phi \in \sec(\boldsymbol{E}, \boldsymbol{Q}) \, . \tag{1.1}$$

1.3 Proposition. Each Hermitian vector field Y turns out to be complex linear. Moreover, $Y \in \mathrm{lin}_{\mathrm{I\!R}}(\boldsymbol{Q}, T\boldsymbol{Q})$ is Hermitian if and only if $Y_1^1 = Y_2^2 = 0$ and $Y_1^2 = -Y_2^1$, i.e. if and only if its coordinate expression is of the type $Y = X^\lambda \partial_\lambda + \mathfrak{i}\, \check{Y}\, \mathbb{I}$, with $X^\lambda \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$ and $\check{Y} = Y_1^2 = -Y_2^1 \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

Proof. If Y is Hermitian, then, for each $\Phi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain

$$\begin{aligned} \mathfrak{h}\big(Y.(\mathfrak{i}\,\Psi), \Phi\big) &= L[X]\Big(\mathfrak{h}\big((\mathfrak{i}\,\Psi), \Phi\big)\Big) - \mathfrak{h}\big((\mathfrak{i}\,\Psi), Y.\Phi\big) = -\mathfrak{i}\, L[X]\,\Big(\mathfrak{h}\big(\Psi, \Phi\big)\Big) + \mathfrak{i}\,\mathfrak{h}\big(\Psi, Y.\Phi\big) \\ &= -\mathfrak{i}\,\mathfrak{h}\big(Y.\Psi, \Phi\big) = \mathfrak{h}\big((\mathfrak{i}\,Y.\Psi), \Phi\big) \, , \end{aligned}$$

which yields $Y.(\mathfrak{i}\,\Psi) = \mathfrak{i}\,Y.\Psi$, hence Y is complex linear. Hence, its coordinate expression is of the type $Y = X^\lambda \partial_\lambda + Y^z \, \mathbb{I}$, with $X^\lambda \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$ and $Y^z = Y_1^1 + \mathfrak{i}\, Y_1^2 = Y_2^2 - \mathfrak{i}\, Y_2^1 \in \mathrm{map}(\boldsymbol{E}, \mathbb{C})$.

Moreover, the equality (1.1) reads as $X^\lambda \, \partial_\lambda(\bar{\psi}\,\phi) = (\overline{X^\lambda \, \partial_\lambda \psi - Y^z \, \psi}) \, \phi + \bar{\psi} \, (X^\lambda \, \partial_\lambda \phi - Y^z \, \phi)$, which implies $\bar{Y}^z + Y^z = 0$, i.e. $Y^z = \mathfrak{i}\,\check{Y}$, with $\check{Y} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$. QED

1.4 Proposition. The Hermitian vector fields constitute a subsheaf $\mathrm{her}\,(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ of $\big(\mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})\big)$-modules, which is closed with respect to the Lie bracket.

Proof. If $Y \in \mathrm{her}\,(\boldsymbol{Q}, T\boldsymbol{Q})$ and $\alpha \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$, then

$$L[\alpha\, X](h(\Psi, \Phi)) = (\alpha\, L[X])(h(\Psi, \Phi))$$
$$(\alpha\, Y).\Psi = \alpha\,(Y.\Psi)\,, \qquad (\alpha\, Y).\Phi = \alpha\,(Y.\Phi)\,,$$

hence $\alpha\, Y \in \mathrm{her}\,(\boldsymbol{Q}, T\boldsymbol{Q})$. Clearly, if $Y_1, Y_2 \in \mathrm{her}\,(\boldsymbol{Q}, T\boldsymbol{Q})$, then $Y_1 + Y_2 \in \mathrm{her}\,(\boldsymbol{Q}, T\boldsymbol{Q})$.

The closure of $\mathrm{her}\,(\boldsymbol{Q}, T\boldsymbol{Q})$ with respect to the Lie bracket follows from the identities

$$L\big[\,[X_1, X_2]\,\big] = \big[L[X_1], L[X_2]\big]\,, \qquad L\big[\,[Y_1, Y_2]\,\big] = \big[L[Y_1], L[Y_2]\big]\,.\ \mathrm{QED}$$

1.2.4 Global classification of Hermitian vector fields

Let us consider a Hermitian connection c.

If $\xi \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, then $c(\xi) \in \mathrm{her}\,(\boldsymbol{Q},\, T\boldsymbol{Q})$.

1.5 Proposition. We have the following mutually inverse isomorphisms

$$\mathfrak{h}[c] : \mathrm{her}\,(\boldsymbol{Q},\, T\boldsymbol{Q}) \to \sec(\boldsymbol{E},\, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E},\, \mathrm{I\!R})\,,$$
$$\mathfrak{j}[c] : \sec(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E},\, \mathrm{I\!R}) \to \mathrm{her}\,(\boldsymbol{Q}, T\boldsymbol{Q})\,,$$

given by $\mathfrak{h}[c] : Y \mapsto \Big(X,\, -\mathfrak{i}\ \mathrm{tr}\,\big(\nu[c](Y)\big)\Big)$ and $\mathfrak{j}[c] : (X, \check{Y}) \mapsto c(X) + \mathfrak{i}\,\check{Y} \otimes \mathbb{I}$, i.e., in coordinates, $\mathfrak{h}[c](Y) = \big(Y^\lambda\, \partial_\lambda\,, Y^2_1 - A_\lambda\, Y^\lambda\big)$ and $\mathfrak{j}[c](X, \check{Y}) = X^\lambda\, \partial_\lambda + \mathfrak{i}\,(A_\lambda\, X^\lambda + \check{Y}) \otimes \mathbb{I}\,.$ □

1.6 Lemma. Let us consider a closed 2-form Φ of $\boldsymbol{E}$ and define the bracket of $\sec(\boldsymbol{E},\, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E},\, \mathrm{I\!R})$ by

$$\big[(X_1, \check{Y}_1)\,,\ (X_2, \check{Y}_2)\big]_\Phi := \big([X_1, X_2]\,, \quad \Phi(X_1, X_2) + X_1.\check{Y}_2 - X_2.\check{Y}_1\big)\,.$$

Then, the above bracket turns out to be a Lie bracket.

Proof. The 1st component $[X_1, X_2]$ is just the Lie bracket.

Moreover, the anticommutativity of the 2nd component is evident.

Next, let us prove the Jacobi property.

Let us consider three pairs $\Pi_i := (X_i, \check{Y}_i)$, with $X_i \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, $\check{Y}_i \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$, $i = 1, 2, 3$, and set $(X, \check{Y}) := \big[\Pi_1,\ [\Pi_2,\ \Pi_3]_\Phi\big]_\Phi + \big[\Pi_2,\ [\Pi_3,\ \Pi_1]_\Phi\big]_\Phi + \big[\Pi_3,\ [\Pi_1,\ \Pi_2]_\Phi\big]_\Phi$, where

$$[\Pi_i, \Pi_j]_\Phi := \big([X_i, X_j]\,, \quad \Phi(X_i, X_j) + X_i.\check{Y}_j - X_j.\check{Y}_i\big)\,.$$

Then, the Jacobi property of the 1st component follows from the Jacobi property of the Lie bracket

$$X := \big[X_1,\ [X_2,\ X_3]\big] + \big[X_2,\ [X_3,\ X_1]\big] + \big[X_3,\ [X_1,\ X_2]\big] = 0\,.$$

Moreover, the Jacobi property of the 2nd component follows from the following equalities

$$\begin{aligned}
\check{Y} &= \Phi\big(X_1,\,[X_2,X_3]\big) + \Phi\big(X_2,\,[X_3,X_1]\big) + \Phi\big(X_3,\,[X_1,X_2]\big)\\
&\quad + X_1.\Phi(X_2,X_3) + X_2.\Phi(X_3,X_1) + X_3.\Phi(X_1,X_2)\\
&\quad + \big(X_1.X_2. - X_2.X_1. - [X_1\,,\ X_2].\big)\,\check{Y}_3\\
&\quad + \big(X_2.X_3. - X_3.X_2 - [X_2\,,\ X_3].\big)\,\check{Y}_1\\
&\quad + \big(X_3.X_1. - X_1.X_3. - [X_3\,,\ X_1].\big)\,\check{Y}_2\\
&= \Phi(X_1,\,[X_2,X_3]) + \Phi(X_2,\,[X_3,X_1]) + \Phi(X_3,\,[X_1,X_2])\\
&\quad + X_1.\Phi(X_2,X_3) + X_2.\Phi(X_3,X_1) + X_3.\Phi(X_1,X_2)\\
&= \Phi\big(X_1,\,[X_2,X_3]\big) + \Phi\big(X_2,\,[X_3,X_1]\big) + \Phi\big(X_3,\,[X_1,X_2]\big)\\
&\quad + X_1.\Phi(X_2,X_3) + X_2.\Phi(X_3,X_1) + X_3.\Phi(X_1,X_2)\\
&= d\Phi(X_1,X_2,X_3) = 0\,.\ \mathrm{QED}
\end{aligned}$$

Now, let us refer to the 2–form $\Phi[c] := \mathfrak{i}\,\mathrm{tr}\,R[c]$ associated with the curvature of c.

1.7 Theorem. *The map $\mathfrak{j}[c]$ is a Lie algebra isomorphism with respect to the Lie bracket $[\,,]_{\Phi[c]}$ and the standard Lie bracket.*

Proof. We have

$$[c(X_1),\,c(X_2)] = c\big([X_1,X_2]\big) - R[c](X_1,X_2) = c\big([X_1,X_2]\big) + \mathfrak{i}\,\Phi[c](X_1,X_2)\,\mathbb{I}\,,$$
$$\big[c(X_1),\,\mathfrak{i}\check{Y}_2\,\mathbb{I}\big] = \mathfrak{i}\,(X_1.\check{Y}_2)\,\mathbb{I}\,,\qquad \big[c(X_2),\,\mathfrak{i}\check{Y}_1\,\mathbb{I}\big] = \mathfrak{i}\,(X_2.\check{Y}_1)\,\mathbb{I}\,,\qquad [\mathfrak{i}\check{Y}_1\,\mathbb{I},\,\mathfrak{i}\check{Y}_2\,\mathbb{I}] = 0\,,$$

which implies

$$\begin{aligned}
\big[\mathfrak{j}(X_1,\check{Y}_1)\,,\ \mathfrak{j}(X_2,\check{Y}_2)\big] &= \big[c(X_1) + \mathfrak{i}\check{Y}_1\,\mathbb{I}\,,\quad c(X_2) + \mathfrak{i}\check{Y}_2\,\mathbb{I}\big]\\
&= \big[c(X_1),\ c(X_2)\big] + \big[c(X_1),\ \mathfrak{i}\check{Y}_2\,\mathbb{I}\big] + \big[\mathfrak{i}\check{Y}_1\,\mathbb{I},\ c(X_2)\big] + \big[\mathfrak{i}\check{Y}_1\,\mathbb{I},\ \mathfrak{i}\check{Y}_2\,\mathbb{I}\big]\\
&= c([X_1,X_2]) + \mathfrak{i}\,\big(\Phi[c](X_1,X_2) + X_1.\check{Y}_2 - X_2.\check{Y}_1\big)\,\mathbb{I}\\
&= \mathfrak{j}\big([X_1,X_2]\,,\ \ \Phi[c](X_1,X_2) + X_1.\check{Y}_2 - X_2.\check{Y}_1\big)\\
&= \mathfrak{j}\big(\big[(X_1,\check{Y}_1)\,,\ (X_2,\check{Y}_2)\big]_{\Phi[c]}\big)\,.\ \mathrm{QED}
\end{aligned}$$

1.8 Corollary. The map $\mathrm{her}\,(\boldsymbol{Q},T\boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{E},\,T\boldsymbol{E}) : Y \mapsto X$ is a central extension of Lie algebras by $\mathrm{map}(\boldsymbol{E},\mathrm{I\!R})$. $\square$

So far, we have considered a generic Hermitian connection c in order to achieve a global classification of the Lie algebra of vector fields.

In the next sections, dealing with the Galilei and Einstein frameworks, we shall be involved with two more specific base manifolds $\boldsymbol{E}$ equipped with an additional structure, which yields a distinguished system of Hermitian connections.

This circunstance will provide a further isomorphism of the Lie algebra of Hermitian vector fields with a Lie algebra of functions. Indeed, this isomorphism is at the basis of the theory of quantum operators in CQM.

2 Galilei case

Now, we specify the setting of the first section, by considering the base manifold $\boldsymbol{E}$ as a Galilei spacetime equipped with a certain fundamental structure.

2.1 Classical setting

2.1.1 Spacetime

We consider the absolute *time*, consisting of an affine 1–dimensional space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes \mathrm{I\!R}$.

We assume spacetime $\boldsymbol{E}$ to be oriented and equipped with a *time fibring* $t : \boldsymbol{E} \to \boldsymbol{T}$.

We shall refer to a *time unit* $u_0 \in \mathbb{T}$, or, equivalently, to its dual $u^0 \in \mathbb{T}^*$, and to a *spacetime chart* $(x^\lambda) \equiv (x^0, x^i)$ adapted to the orientation, to the fibring, to the affine structure of $\boldsymbol{T}$ and to the time unit u_0. Greek indices will span all spacetime coordinates and Latin indices will span the fibre coordinates. The induced local bases of $V\boldsymbol{E}$ and $V^*\boldsymbol{E}$ are denoted, respectively, by (∂_i) and $(\check{d}^i)$.

In general, the vertical restriction of forms will be denoted by a "check" $^\vee$ symbol.

The differential of the time fibring is a scaled form $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$, with coordinate expression $dt = u_0 \otimes d^0$.

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$. The *1st differential* of the motion s is the map $ds : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$. We have $dt(ds) = 1$.

2.1.2 Spacelike metric

We assume spacetime to be equipped with a *scaled spacelike Riemannian metric* $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$. With reference to a mass $m \in \mathbb{M}$, it is convenient to introduce the *rescaled metric* $G := \frac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$. The associated contravariant tensors are $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})$ and $\bar{G} = \frac{\hbar}{m}\, \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})$.

We have the coordinate expressions $g = g_{ij}\, \check{d}^i \otimes \check{d}^j$ and $G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j$, with $g_{ij} \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^2 \otimes \mathrm{I\!R})$ and $G^0_{ij} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

The spacetime orientation and the metric g yield the scaled spacelike volume 3–form $\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E}$ and its dual $\bar{\eta} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^3 V\boldsymbol{E}$, with coordinate expressions $\eta = \sqrt{|g|}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3$ and $\bar{\eta} = (1/\sqrt{|g|})\, \partial_1 \wedge \partial_2 \wedge \partial_3$.

2.1.3 Phase space

We assume as classical *phase space* the 1st jet space $J_1\boldsymbol{E}$ of motions $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$.

The 1st jet space can be naturally identified with the subbundle $J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$, of scaled vectors which project on $\boldsymbol{1} : \boldsymbol{T} \to \mathbb{T}^* \otimes \mathbb{T}$. Hence, the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ turns out to be affine and associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$.

The *velocity* of a motion $s : \boldsymbol{T} \subset \boldsymbol{E}$ is defined to be its 1-jet $j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E}$.

A space time chart (x^λ) induces a chart (x^λ, x^i_0) on $J_1\boldsymbol{E}$.

The time fibring yields naturally the *contact map* $д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ and the *complementary contact map* $\theta := 1 - д \circ dt : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$, with coordinate expressions $д = u^0 \otimes (\partial_0 + x^i_0\, \partial_i)$ and $\theta = (d^i - x^i_0\, d^0) \otimes \partial_i$. The fibred morphism $д$ is injective. Indeed, it makes $J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ the fibred submanifold over $\boldsymbol{E}$ characterised by the constraint $\dot{x}^0_0 = 1$. We have $д \lrcorner dt = 1$. For each motion s, we have $д \circ j_1 s = ds$.

2.1.4 Contact splitting

The *dt–vertical* tangent space of spacetime and the *dt–horizontal* cotangent space of spacetime are defined to be, respectively, the vector subbundles over $\boldsymbol{E}$

$$V\boldsymbol{E} := \{X \in T\boldsymbol{E} \mid X \in \ker\, dt\} \quad \text{and} \quad H^*\boldsymbol{E} := \{\omega \in T^*\boldsymbol{E} \mid \omega \in \mathrm{im}\, dt\}.$$

Moreover, we define the $д$*–horizontal* tangent space of spacetime and the $д$*–vertical* cotangent space of spacetime, to be, respectively, the vector subbundles over $J_1\boldsymbol{E}$

$$\begin{aligned} H_д\boldsymbol{E} &:= \{(e_1, X) \in J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \;\mid\; X \in \mathrm{im}\; д(e_1)\} \\ V^*_д\boldsymbol{E} &:= \{(e_1, \omega) \in J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} \;\mid\; \omega \in \ker\; д(e_1)\}. \end{aligned}$$

We have the natural linear fibred splittings over $J_1\boldsymbol{E}$ and the projections

$$\begin{aligned} J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} &= H_д\boldsymbol{E} \oplus V\boldsymbol{E}, & J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} &= H^*\boldsymbol{E} \oplus V^*_д\boldsymbol{E}, \\ д \otimes \tau : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} &\to H_д\boldsymbol{E}, & \tau \otimes д : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} &= H^*\boldsymbol{E}, \\ \theta : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} &\to V\boldsymbol{E}, & \theta^* : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} &\to V^*_д\boldsymbol{E}. \end{aligned}$$

2.1.5 Vertical bundle of the phase space

Let $V_0J_1\boldsymbol{E} \subset VJ_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}$ be the vertical tangent subbundle over $\boldsymbol{E}$ and the vertical tangent subbundle over $\boldsymbol{T}$, respectively. The affine structure of the phase space yields the equality $V_0J_1\boldsymbol{E} = J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{T}^* \otimes V\boldsymbol{E})$, hence the natural map $\nu : J_1\boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V_0J_1\boldsymbol{E})$, with coordinate expression $\nu = u_0 \otimes \check{d}^i \otimes \partial^0_i$.

2.1.6 Observers

An *observer* is defined to be a section $o \in \sec(\boldsymbol{E},\, J_1\boldsymbol{E})$.

Each observer yields the scaled vector field $д[o] := д \circ o \in \sec(\boldsymbol{E},\, \mathbb{T}^* \otimes T\boldsymbol{E})$ and the tangent valued 1–form $\nu[o] \equiv \theta[o] := \theta \circ o \in \sec(\boldsymbol{E},\, T^*\boldsymbol{E} \otimes T\boldsymbol{E})$, with coordinate expressions $д[o] = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$ and $\theta[o] = (d^i - o^i_0\, d^0) \otimes \partial_i$, where $o^i_0 := x^i_0 \circ o$. Each of the above objects characterises o. Thus, an observer can be regarded as the velocity of a continuum.

A spacetime chart (x^λ) is said to be *adapted* to o if $o^i_0 = 0$, i.e. if the spacelike functions x^i are constant along the integral motions of o. Actually, infinitely many spacetime charts are adapted to an observer o; the transition maps of two such charts (x^λ) and $(\acute{x}^\lambda)$ are of the type $\partial_0 \acute{x}^i = 0$. Conversely, each spacetime chart (x^0, x^i) is adapted to the unique observer o determined by the equality $д[o] = u^0 \otimes \partial_0$.

Each observer o yields the affine fibred isomorphism $\nabla[o] := \mathrm{id} - o : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ and the linear fibred projection $\nu[o] : T\boldsymbol{E} \to V\boldsymbol{E}$, with coordinate expressions $\nabla[o] = (x^i_0 - o^i_0)\, u^0 \otimes \partial_i$ and $\nu[o] = (d^i - o^i_0\, d^0) \otimes \partial_i$.

For each observer o, we define the *kinetic energy* and the *kinetic momentum* as $\mathcal{K}[o] = \frac{1}{2}\, G\,(\nabla[o], \nabla[o]) \in \mathrm{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ and $\mathcal{Q}[o] = \nu[o] \lrcorner \left(G^\flat(\nabla[o])\right) \in \mathrm{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$.

In an adpeted chart, we have $\mathcal{K}[o] = \frac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0\, d^0$ and $\mathcal{Q}[o] = G^0_{ij}\, x^j_0\, d^j$.

We define the *kinetic Poincaré–Cartan form* $\Theta[o] := -\mathcal{K}[o] + \mathcal{Q}[o] \in \mathrm{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ and obtain $\mathcal{K}[o] = -д[o] \lrcorner \Theta[o]$ and $\mathcal{Q}[o] = \theta[o] \lrcorner \Theta[o]$.

For each motion s and observer o, we define the *observed velocity* to be the map $\vec{v} := \nabla[o] \circ j_1 s = \nu[o] \circ ds : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E}$. Then, we can write $j_1 s = o \circ s + \vec{v}$ and $д \circ j_1 s = д[o] + \vec{v}$.

2.1.7 Gravitational and electromagnetic fields

We assume spacetime to be equipped with a given torsion free linear spacetime connection, called *gravitational field*, $K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$, which fulfills the identities $\nabla^\natural dt = 0$, $\nabla^\natural g = 0$, $R^\natural{}_{\lambda i \mu j} = R^\natural{}_{\mu j \lambda i}$. The coordinate expression of $K^\natural$ is

$$\begin{aligned}
K^\natural{}_\lambda{}^0{}_\mu &= 0 \\
K^\natural{}_0{}^i{}_0 &= -G^{ij}_0\, \Phi^\natural{}_{0j} \\
K^\natural{}_h{}^i{}_0 = K^\natural{}_0{}^i{}_h &= -\tfrac{1}{2}\, G^{ij}_0\, (\partial_0 G^0_{hj} + \Phi^\natural{}_{hj}) \\
K^\natural{}_h{}^i{}_k = K^\natural{}_k{}^i{}_h &= -\tfrac{1}{2}\, G^{ij}_0\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\,,
\end{aligned}$$

where we have set $K^\natural{}_\lambda{}^\nu{}_\mu := -(\nabla^\natural{}_\lambda \partial_\mu)^\nu$, and where $\Phi^\natural = \Phi[K^\natural, o] = \Phi^\natural{}_{\lambda\mu}\, d^\lambda \wedge d^\mu$ is a closed spacetime form, which depends on the spacetime chart, through the associated observer o.

We assume spacetime to be equipped with a given *electromagnetic field*, which is a closed scaled 2–form $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$. With reference to a particle with mass m and charge q, we obtain the unscaled 2–form $\frac{q}{\hbar}\, F : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$.

We define the *magnetic field* and the *observed electric field* to be the scaled vector fields

$$\begin{aligned}
\vec{B} &:= \tfrac{1}{2}\, i(\check{F})\, \bar{\eta} : \boldsymbol{E} \to (\mathbb{L}^{-5/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E} \\
\vec{E}[o] &:= -\bar{g} \lrcorner (i(o) \lrcorner F) : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}\,,
\end{aligned}$$

where $\check{F} : \boldsymbol{E} \to \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2} \otimes \Lambda^2 V^*\boldsymbol{E}$ is the spacelike restriction of the electromagnetic field. We have the coordinate expressions

$$\vec{B} = \tfrac{1}{2}\, \frac{1}{\sqrt{|g|}}\, \epsilon^{hki}\, F_{hk}\, \partial_i \qquad \text{and} \qquad \vec{E}[o] = -g^{ij}\, F_{0j}\, u^0 \otimes \partial_i\,.$$

Then, we obtain the observed splitting $F = -2\,dt \wedge g^{\flat}(\vec{E}[o]) + 2\,\nu^*[o]\,\big(i(\vec{B})\,\eta\big)$.

The closure of F yields the Galilei version of the 1st two Maxwell equations

$$\mathrm{curl}_\eta\, \vec{E}[o] + L(o)\,\vec{B} + \vec{B}\,\mathrm{div}_\eta\, o = 0 \qquad \text{and} \qquad \mathrm{div}_\eta\, \vec{B} = 0\,.$$

In the case of a "flat spacetime" and of an "inertial observer", the above equations reduce to the standard equations $\mathrm{curl}_\eta\, \vec{E}[o] + \partial_0 \vec{B} = 0$ and $\mathrm{div}_\eta\, \vec{B} = 0$.

The fact that the metric g is spacelike does not allow us to write, in the Galilei framework, the 2nd two Maxwell equations, which are related to the source charges. Only a reduced version of these equations can be written in covariant way in this framework. On the other hand, we consider the electromagnetic field as given, hence, in the present scheme, we are not essentially involved with its source.

The electromagnetic field can be merged into the gravitational connection in a covariant way, so that we obtain the *joined connection*

$$K := K^{\natural} + K^{e} = K^{\natural} - \tfrac{q}{2m}\,(dt \otimes \widehat{F} + \widehat{F} \otimes dt)\,, \qquad \text{with} \qquad \widehat{F} = g^{\sharp 2}(F)\,,$$

which fulfills the same identities of the gravitational connection.

Thus, from now on, we shall refer to this joined connection, which incoroporates both the gravitational and the electromagnetic fields.

2.1.8 Induced objects on the phase space

We have a natural bijective map χ between time preserving linear spacetime connections K and affine phase connections $\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T J_1\boldsymbol{E}$, with coordinate expression $\Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma_{\lambda}{}^{i}_{0}\, \partial^0_i)$, where $\Gamma_{\lambda}{}^{i}_{0} = \Gamma_{\lambda}{}^{i}_{0}{}^{0}_{0} + \Gamma_{\lambda}{}^{i}_{0}{}^{0}_{j}\, x^j_0$. In coordinates, the map χ reads as $\Gamma_{\lambda}{}^{i}_{0}{}^{0}_{\mu} = K_\lambda{}^i{}_\mu$.

Then, the joined spacetime connection K yields a torsion free affine connection, called *joined phase connection*, $\Gamma := \chi(K) : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T J_1\boldsymbol{E}$, which splits as $\Gamma = \Gamma^{\natural} + \Gamma^{\mathfrak{e}}$, where $\Gamma^{\mathfrak{e}} = -\frac{q}{2m}\, g^{\sharp 2}\Big(F + 2dt \wedge (\text{д} \,\lrcorner\, F)\Big) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E})$ and $\Gamma^{\natural} = \chi(K^{\natural})$. We have $\Gamma^{\mathfrak{e}} = -\frac{q}{2\hbar}\, G^{ih}_0 \,\Big(F_{jh}\, d^j + (F_{jh}\, x^j_0 + 2\, F_{0h})\, d^0\Big) \otimes \partial^0_i$.

The joined phase connection Γ yields the 2nd order connection, called *joined dynamical phase connection*, $\gamma := \text{д} \,\lrcorner\, \Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T J_1\boldsymbol{E}$, with coordinate expression $\gamma = u_0 \otimes (\partial_0 + x^i_0\, \partial_i + \gamma_{0}{}^{i}_{0}\, \partial^0_i)$, where $\gamma_{0}{}^{i}_{0} = K_\lambda{}^i{}_\mu\, \breve{\delta}^\lambda_0\, \breve{\delta}^\mu_0$, where $\breve{\delta}^\alpha_0 := \delta^\alpha_0 + \delta^\alpha_h\, x^h_0$. Moreover, γ splits as $\gamma = \gamma^{\natural} + \gamma^{\mathfrak{e}}$, where $\gamma^{\natural} = \text{д} \,\lrcorner\, \Gamma^{\natural}$ and $\gamma^{\mathfrak{e}} = -\frac{q}{m}\, \text{д} \,\lrcorner\, \widehat{F} : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$.

Indeed, $\gamma^{\mathfrak{e}}$ turns out to be just the *Lorentz force*, whose observed expression is $\gamma^{\mathfrak{e}} = -\frac{q}{m}\,(\vec{E}[o] + \nabla[o] \times \vec{B})$ and in coordinates $\gamma^{\mathfrak{e}} = -\frac{q_0}{m}\,(F_0{}^i + F_h{}^i\, x^h_0)\, u^0 \otimes u^0 \otimes \partial_i$.

Next, let us consider the vertical projection $\nu[\Gamma] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes (T^*J_1\boldsymbol{E} \otimes V\boldsymbol{E})$ associated with Γ, whose coordinate expression is $\nu[\Gamma] = (d^i_0 - \Gamma_{\lambda}{}^{i}_{0}\, d^\lambda)\, u^0 \otimes \partial_i$.

The joined phase connection Γ and the rescaled spacelike metric G yield the 2–form, called *joined phase 2–form*, $\Omega := G \,\lrcorner\, \Big(\nu[\Gamma] \wedge \theta\Big) : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$, with coordinate expression $\Omega = G^0_{ij}\,(d^i_0 - \Gamma_{\lambda}{}^{i}_{0}\, d^\lambda) \wedge (d^j - x^j_0\, d^0)$. Moreover, Ω splits as $\Omega = \Omega^{\natural} + \Omega^{\mathfrak{e}}$, where $\Omega^{\natural} = G \,\lrcorner\, \Big(\nu[\Gamma^{\natural}] \wedge \theta\Big)$ and $\Omega^{\mathfrak{e}} = \frac{q}{2\hbar}\, F$.

The joined phase 2–form Ω is *cosymplectic*, i.e. $d\Omega = 0$ and $dt \wedge \Omega \wedge \Omega \wedge \Omega \not\equiv 0$.

Moreover, Ω admits potentials, called *horizontal*, of the type $A^{\uparrow} \in \mathrm{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$, which are defined up to a gauge of the type $\alpha \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$. Indeed, for each observer o, we have $A^{\uparrow} = \Theta[o] + A[o]$, where $A[o] = o^* A^{\uparrow}$.

We define the *Lagrangian* and the *momentum* associated with a horizontal potential $A^{\uparrow}$ to be the horizontal 1–forms $\mathcal{L} := д \lrcorner A^{\uparrow}$ and $\mathcal{P} := \theta \lrcorner A^{\uparrow}$, with coordinate expressions $\mathcal{L} = (\frac{1}{2} G^0_{ij} x^i_0 x^j_0 + A_i x^i_0 + A_0)\, d^0$ and $\mathcal{P} = (G^0_{ij} x^j_0 + A_i)\theta^i$.

Each observer o yields the closed spacetime 2–form $\Phi[o] = \Phi[\Gamma, G, o] := 2\, o^*\Omega$ and, for each potential $A^{\uparrow}$, the spacetime 1-form $A[o] = A[\Gamma, G, o] := o^* A^{\uparrow}$. Clearly, we have $\Phi[o] = 2\, dA[o]$. Moreover, we have $\Phi[\Gamma, G, o] = \Phi[K, o]$.

The joined phase connection Γ and the rescaled spacelike metric G yield the vertical 2–vector, called *joined phase 2–vector*, $\Lambda := \bar{G} \lrcorner (\Gamma \wedge \nu) : J_1\boldsymbol{E} \to \Lambda^2 V J_1\boldsymbol{E}$, with coordinate expression $\Lambda = G^{ij}_0 \left(\partial_i + \Gamma_i{}^h_0\, \partial^0_h\right) \wedge \partial^0_j$. Moreover, Λ splits as $\Lambda = \Lambda^{\natural} + \Lambda^{\mathfrak{e}}$, where $\Lambda^{\natural} = \bar{G} \lrcorner (\Gamma^{\natural} \wedge \nu)$ and $\Lambda^{\mathfrak{e}} = \frac{q}{2\hbar} G^{\sharp}(F) : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes \Lambda^2 V\boldsymbol{E}$. We have the coordinate expression $\Lambda^{\mathfrak{e}} = \frac{q}{2\hbar} G^{ih}_0 G^{jk}_0 F_{hk}\, \partial^0_i \wedge \partial^0_j$.

From now on, we shall refer to the joined objects Γ, γ, Ω, Λ.

Summing up, we have the following identities

$$i(\gamma)\, dt = 1\,, \quad i(\gamma)\,\Omega = 0\,, \quad \gamma = д \lrcorner \Gamma\,, \quad \Omega = G \lrcorner \left(\nu[\Gamma] \wedge \theta\right), \quad \Lambda = \bar{G} \lrcorner (\Gamma \wedge \nu)\,.$$

2.1.9 Hamiltonian lift of phase functions

Given a time scale $\sigma \in \mathrm{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$, we define the *$\sigma$–Hamiltonian lift* to be the map

$$X^{\uparrow}{}_{\mathrm{ham}}[\sigma] : \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\mathrm{ham}}[\sigma, f] := \gamma(\sigma) + i(df)\Lambda\,,$$

with $X^{\uparrow}{}_{\mathrm{ham}}[\sigma, f] = \sigma^0\, (\partial_0 + x^i_0\, \partial_i + \gamma_{00}{}^i\, \partial^0_i) - G^{ij}_0\, \partial^0_j f\, \partial_i + \left(G^{ij}_0\, \partial_j f + (\Gamma^{i\,j}_{00} - \Gamma^{j\,i}_{00})\, \partial^0_j f\right) \partial^0_i$, where $\Gamma^{i\,j}_{00} := G^{ih}_0\, \Gamma_h{}^j_0$.

Indeed, for each $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we obtain the distinguished time scale

$$\sigma[f] := \tfrac{1}{3}\, \bar{G} \lrcorner D^2 f \;\equiv\; f^0\, u_0 = \tfrac{1}{3}\, G^{ij}_0\, (\partial^0_i \partial^0_j f)\, u_0 \in \mathrm{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})\,.$$

2.1.10 Poisson bracket of phase functions

We define the *Poisson bracket* of $\mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$ as $\{f, g\} := i(df \wedge dg)\,\Lambda$.

Its coordinate expression is $\{f, g\} = G^{ij}_0\, (\partial_i f\, \partial^0_j g - \partial_i g\, \partial^0_j f) - (\Gamma^{i\,j}_{00} - \Gamma^{j\,i}_{00})\, \partial^0_i f\, \partial^0_j g$.

The Poisson bracket makes $\mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$ a sheaf of $(\mathrm{map}(\boldsymbol{T}, \mathrm{I\!R}))$–Lie algebras.

2.1.11 The sheaf of special phase functions

An $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is said to be a *special phase function* if $D^2 f = f'' \otimes G$, with $f'' \in \mathrm{map}(\boldsymbol{E}, \bar{\mathbb{T}})$. If f is a special phase function, then we obtain $\sigma[f] = f'' \in \mathrm{map}(\boldsymbol{E}, \bar{\mathbb{T}})$.

The special phase functions constitute a $(\mathrm{map}(\boldsymbol{E}, \mathrm{I\!R}))$–linear subsheaf $\mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

Let us consider an $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$, an observer o and a spacetime chart.

Then, $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ if and only if $f = f'' \lrcorner \mathcal{K}[o] + f'[o] \lrcorner (\mathcal{Q}[o]) + f[o]$, where $f'[o] := G^\sharp(Df) \circ o \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})$ and $f[o] := f \circ o \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

Moreover, $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ if and only if $f = f^0 \, \frac{1}{2} \, G^0_{ij} \, x^i_0 \, x^j_0 + f^i \, G^0_{ij} \, x^j_0 + \breve{f}$, with $f^0, f^i, \breve{f} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

Hence, with reference to a chart adapted to o, we obtain $f'[o] = f^i \, \partial_i$ and $f[o] = \breve{f}$.

If $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ and $o, \acute{o} = o + v \in \sec(\boldsymbol{E}, J_1\boldsymbol{E})$, then we obtain the transition formulas $f'[\acute{o}] = f'[o] + f'' \lrcorner v$ and $f[\acute{o}] = f[o] + f'[o] \lrcorner G^\flat(v) + \frac{1}{2} \, f'' \lrcorner G\,(v, v)$.

For each $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$, the map $f'' \lrcorner д - G^\sharp(Df) \in \mathrm{fib}(J_1\boldsymbol{E}, T\boldsymbol{E})$ factorises through a spacetime vector field, $X[f] \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, called the *tangent lift* of f, whose coordinate expression is $X[f] = f^0 \, \partial_0 - f^i \, \partial_i$.

For each $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ and $o \in \sec(\boldsymbol{E}, J_1\boldsymbol{E})$, we obtain $f = -X[f] \lrcorner \Theta[o] + f[o]$.

2.1 Proposition. For each observer o, we have the mutually inverse $(\mathrm{map}(\boldsymbol{E}, \mathrm{I\!R}))$–linear isomorphisms

$$\mathfrak{s}[o] : \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \sec(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R}) : f \mapsto \big(X[f], \; f \circ o\big) .$$
$$\mathfrak{r}[o] : \sec(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) : (X, \breve{f}) \mapsto X \lrcorner \Theta[o] + \breve{f} .$$

Their coordinate expressions are

$$\mathfrak{s}[o] : f^0 \, \tfrac{1}{2} \, G^0_{ij} \, x^i_0 \, x^j_0 + f^i \, G^0_{ij} \, x^j_0 + \breve{f} \mapsto \big((f^0 \, \partial_0 - f^i \, \partial_i) \, , \; \breve{f}\big)$$
$$\mathfrak{r}[o] : (X^\lambda \, \partial_\lambda, \, \breve{Y}) \mapsto X^0 \, \tfrac{1}{2} \, G^0_{ij} \, x^i_0 \, x^j_0 - X^i \, G^0_{ij} \, x^j_0 + \breve{Y} \, . \square$$

We can characterise the special phase functions via the Hamiltonian lift, as follows.

2.2 Proposition. Let $\sigma \in \mathrm{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$ and $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$. Then, the following conditions are equivalent:

1) $X^\uparrow{}_{\mathrm{ham}}[\sigma, f] \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ projects on a vector field $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$,

2) $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ and $\sigma = f''$.

Moreover, if the above conditions are fulfilled, then we obtain $X = X[f]$.

Proof. $X^\uparrow{}_{\mathrm{ham}}[\sigma, f] = \sigma^0 \, \gamma_0 - G^{ij}_0 \, \partial^0_j f \, \partial_i + \big(G^{ij}_0 \, \partial_j f + (\Gamma_{0 0}^{i j} - \Gamma_{0 0}^{j i}) \, \partial^0_j f\big) \, \partial^0_i$ is projectable if and only if $\sigma^0 \, \gamma_0 - G^{ij}_0 \, \partial^0_j f \, \partial_i$ is projectable, i.e., if and only if $\partial^0_h \sigma^0 = 0$ and $\sigma^0 \, \partial^0_h x^i_0 - G^{ij}_0 \, \partial^{00}_{hj} f = 0$, i.e. if and only if $\partial^0_h \sigma^0 = 0$ and $G^0_{ik} \, \sigma^0 \, \delta^i_h - \delta^j_k \, \partial^{00}_{hj} f = 0$, i.e. if and only if $\partial^0_h \sigma^0 = 0$ and $\partial^{00}_{hk} f = G^0_{hk} \, \sigma^0$, i.e., by integration on the affine fibres of $J_1\boldsymbol{E} \to \boldsymbol{E}$, if and only if $f = \sigma^0 \, \frac{1}{2} \, G^0_{ij} \, x^i_0 \, x^j_0 + f^i \, G^0_{ij} \, x^j_0 + \breve{f}$, with $\breve{f} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$. Moreover, if $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$, then $X = \sigma^0 \, \partial_0 + (\sigma^0 \, x^i_0 - G^{ij}_0 \, \partial^0_j f) \, \partial_i = \sigma^0 \, \partial_0 - f^i \, \partial_i$. QED

2.3 Example. Let us consider a potential $A^\uparrow$ of Ω, an observer o and an adapted chart. Then, we define the *observed Hamiltonian*, the *observed momentum* and the *square of the observed momentum* to be, respectively, $\mathcal{H}[o] := -д[o] \lrcorner A^\uparrow \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$, $\mathcal{P}[o] := \nu[o] \lrcorner A^\uparrow \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ and $\mathcal{C}[o] := \bar{G} \lrcorner \mathcal{P}[o] \otimes \mathcal{P}[o] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ with $\mathcal{H}[o] = (\frac{1}{2} \, G^0_{ij} \, x^i_0 \, x^j_0 - A_0) \, d^0$, $\mathcal{P}[o] = (G^0_{ij} \, x^j_0 + A_i) \, d^i$ and $\mathcal{C}[o] = G^{ij}_0 \, x^i_0 \, x^j_0 + 2 \, A^i_0 \, G^0_{ij} \, x^j_0 + A^i_0 \, A_i$, where $A^i_0 := G^{ij}_0 \, A_j$.

Indeed, $x^\lambda, \mathcal{H}_0, \mathcal{P}_i, \mathcal{C}_0 \in \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$. Moreover, we have $X[x^\lambda] = 0$, $X[\mathcal{H}_0] = \partial_0$, $X[\mathcal{P}_i] = -\partial_i$, $X[\mathcal{C}_0] = 2\,(\partial_0 - A^i_0\,\partial_i)$. $\square$

2.1.12 The special bracket

We define the *special bracket* of $\operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ by $[\![f, g]\!] := \{f, g\} + \gamma(f'').g - \gamma(g'').f$.

2.4 Theorem. *The sheaf* $\operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ *is closed with respect to the special bracket. For each* $f_1, f_2 \in \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ *and for each observer* o, *we obtain*

$$[\![f_1, f_2]\!] = -\Big[X[f_1], X[f_2]\Big] \lrcorner\, \Theta[o] + \Big[(X[f_1], \breve{f}_1), (X[f_2], \breve{f}_2)\Big]_{\Phi[o]},$$

i.e. in coordinates

$$\begin{aligned} [\![f, g]\!]^\lambda &= f^0\,\partial_0 g^\lambda - g^0\,\partial_0 f^\lambda - f^h\,\partial_h g^\lambda + g^h\,\partial_h f^\lambda \\ \breve{[\![f, g]\!]} &= f^0\,\partial_0 \breve{g} - g^0\,\partial_0 \breve{f} - f^h\,\partial_h \breve{g} + g^h\,\partial_h \breve{f} - (f^0\,g^h - g^0\,f^h)\,\Phi_{0h} + f^h\,g^k\,\Phi_{hk}. \end{aligned}$$

Thus, $X\Big[[\![f_1, f_2]\!]\Big] = \Big[X[f_1], X[f_2]\Big]$ *and* $[\![f_1, f_2]\!][o] = \Big[(X[f_1], \breve{f}_1), (X[f_2], \breve{f}_2)\Big]_{\Phi[o]}$.

Indeed, the special bracket makes $\operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ *a sheaf of* $\mathbb{R}$*–Lie algebras and the tangent prolongation is a morphism of* $\mathbb{R}$*–Lie algebras.* $\square$

2.5 Corollary. The map $\mathfrak{s}[o] : \operatorname{spec}(\mathcal{J}_1\boldsymbol{E}, \mathbb{R}) \to \sec(\boldsymbol{E}, T\boldsymbol{E}) \times \operatorname{map}(\boldsymbol{E}, \mathbb{R})$ turns out to be an isomorphism of Lie algebras, with respect to the brackets $[\![\ ,\ ]\!]$ and $[\,,]_{\Phi[o]}$. $\square$

For instance, we have $[\![x^\lambda, x^\mu]\!] = 0$, $[\![x^\lambda, \mathcal{H}_0]\!] = -\delta^\lambda_0$, $[\![x^\lambda, \mathcal{P}_i]\!] = \delta^\lambda_i$, $[\![x^\lambda, \mathcal{C}_0]\!] = -2\,\delta^\lambda_0 + 2\,A^h_0\,\delta^\lambda_h$, $[\![\mathcal{H}_0, \mathcal{P}_i]\!] = 0$, $[\![\mathcal{P}_i, \mathcal{P}_j]\!] = 0$, $[\![\mathcal{H}_0, \mathcal{C}_0]\!] = (\partial_0 G^{hk}_0)\,\mathcal{P}_h \mathcal{P}_k + 2\,\partial_0\mathcal{L}_0$, $[\![\mathcal{P}_i, \mathcal{C}_0]\!] = -\partial_i G^{hk}_0\,\mathcal{P}_h\,\mathcal{P}_k - 2\,\partial_i\mathcal{L}_0$.

2.2 Quantum setting

Let us consider a *quantum bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ over the Galilei spacetime.

We define the *phase quantum bundle* as $\pi^\uparrow : \boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$.

Let $\{Ч[o]\}$ be a "system" of connections of the quantum bundle parametrised by the observers $o \in \sec(\boldsymbol{E}, J_1\boldsymbol{E})$. Then, there is a unique connection $Ч^\uparrow$ of the phase quantum bundle, called *universal*, such that $Ч[o] = o^* Ч^\uparrow$, for each o. The universal connection fulfills the property $X^\uparrow \lrcorner\, Ч^\uparrow = X^\uparrow$, for each $X^\uparrow \in \sec(J_1\boldsymbol{E}, VJ_1\boldsymbol{E})$. Conversely, each connection $Ч^\uparrow$ of $\boldsymbol{Q}^\uparrow$ of the above type yields a system of connections of the quantum bundle, whose universal connection is $Ч^\uparrow$. Indeed, the curvatures of the universal connection and of the connections of the associated system fulfill the property $o^* R[Ч^\uparrow] = R[Ч[o]]$.

Moreover, the universal connection is Hermitian if and only if the connections of the associated system are Hermitian.

Let us suppose that the cohomolgy class of Ω be integer.

Then, we assume a connection $\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to T^* J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$, called *phase quantum connection*, which is Hermitian, universal and whose curvature is given by the equality $R[\text{Ч}^\uparrow] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^\uparrow$. The existence of such a universal connection and the fact that Ω admits horizontal potentials are strictly related. Moreover, the closure of Ω is an integrability condition for the above equation.

With reference to a quantum basis b and to an observer o, the expression of $\text{Ч}^\uparrow$ is of the type $\text{Ч}^\uparrow = \chi^\uparrow[\mathsf{b}] + \mathfrak{i}\,\big(\Theta[o] + A[\mathsf{b}, o]\big) \otimes \mathbb{I}^\uparrow$, where $A[\mathsf{b}, o]$ is a potential of $\Phi[o]$ selected by $\text{Ч}^\uparrow$ and b. Hence, the coordinate expression of $\text{Ч}^\uparrow$, in a chart adapted to b and o, is $\text{Ч}^\uparrow = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}\,\big((-\frac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i\big) \otimes \mathbb{I}^\uparrow$.

For each observer o, we obtain $R\big[\text{Ч}[o]\big] = -\mathfrak{i}\,\Phi[o] \otimes \mathbb{I}$.

For each observer o, the expression of $\text{Ч}[o]$, with reference to a quantum basis b, is $\text{Ч}[o] = \chi[\mathsf{b}] + \mathfrak{i}\, A[\mathsf{b}, o] \otimes \mathbb{I}$. Hence, in a chart adapted to b and o, $\text{Ч}[o] = d^\lambda \otimes \partial_\lambda + \mathfrak{i}\, A_\lambda\, d^\lambda \otimes \mathbb{I}$.

If b is a quantum basis and $o, \acute{o} = o + v$ are two observers, then we obtain the transition law $A[\mathsf{b}, \acute{o}] = A[\mathsf{b}, o] - \frac{1}{2}\, G(v, v) + \nu[o] \lrcorner\, G^\flat(v)$.

2.3 Classification of Hermitian vector fields

Eventually, we apply to the Galilei framework the classification of Hermitian vector fields achieved in Theorem 1.7. For this purpose, we choose any observed quantum connection $\text{Ч}[o]$ as auxiliary connection c, use the observed representation $\mathfrak{s}$ of special phase functions achieved in Proposition 2.1 and show an identity.

2.6 Lemma. If $f \in \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ and $o, \acute{o}$ are two observers, then we have the identity $\text{Ч}[\acute{o}]\,(X[f]) + \mathfrak{i}\, f[\acute{o}]\, \mathbb{I} = \text{Ч}[o]\,(X[f]) + \mathfrak{i}\, f[o]\, \mathbb{I}$. $\square$

2.7 Theorem. *For each observer $o \in \sec(\boldsymbol{E}, J_1\boldsymbol{E})$, we have the mutually inverse Lie algebra isomorphisms, with respect to special bracket and the Lie bracket of vector fields,*

$$\begin{aligned} \mathfrak{F} &:= \mathfrak{j}\big[\text{Ч}[o]\big] \circ \mathfrak{s}[o] \ : \operatorname{spec}(J_1\boldsymbol{E}, I\!\!R) \to \operatorname{her}(\boldsymbol{Q}, T\boldsymbol{Q}), \\ \mathfrak{H} &:= \mathfrak{r}[o] \circ \mathfrak{h}\big[\text{Ч}[o]\big] : \operatorname{her}(\boldsymbol{Q}, T\boldsymbol{Q}) \ \to \operatorname{spec}(J_1\boldsymbol{E}, I\!\!R), \end{aligned}$$

given by $\mathfrak{F}(f) = \text{Ч}[o](X[f]) + \mathfrak{i}\, f[o]\, \mathbb{I}$ and $\mathfrak{H}(Y) = -T\pi(Y) \lrcorner\, \Theta[o] - \mathfrak{i}\, \operatorname{tr}\big(\nu\big[\text{Ч}[o]\big](Y)\big)$.

We have the coordinate expressions

$$\begin{aligned} \mathfrak{F}(f^0\, \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + f^i\, G^0_{ij}\, x^j_0 + \breve{f}) &= f^0\, \partial_0 - f^i\, \partial_i + \mathfrak{i}\, (f^0\, A_0 - f^i\, A_i + \breve{f}) \otimes \mathbb{I}, \\ \mathfrak{H}(X^\lambda\, \partial_\lambda + \mathfrak{i}\, \breve{Y}\, \mathbb{I}) &= X^0\, \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - X^i\, G^0_{ij}\, x^j_0 + \breve{Y}. \end{aligned}$$

Indeed, the above maps turns out to be independent on the choice of the observer o.

Proof. The fact that the map $\mathfrak{F}$ is a Lie algebra isomorphism follows immediately from Theorem 1.7 and Theorem 2.4.

The independence of the above maps on the choice of the observer follows from Lemma 2.6. QED

For instance, we have $\mathfrak{F}(x^\lambda) = \mathfrak{i}\, x^\lambda\, \mathbb{I}$, $\mathfrak{F}(\mathcal{H}_0[o]) = \partial_0$, $\mathfrak{F}(\mathcal{P}_i[o]) = -\partial_i$ and $\mathfrak{F}(\mathcal{C}_0[o]) = 2\,\partial_0 - 2\,A_0^i\,\partial_i + \mathfrak{i}\,(2\,A_0 - A_0^i\,A_i)\,\mathbb{I}$.

These vector fields yield “quantum operators” after introducing the “sectional quantum bundle” and the Schrödinger operator (see, for instance, [8, 20]), but this further development is beyond the scope of the present paper.

3 Einstein case

Next, we specify the setting of the first section, by considering the base manifold $\boldsymbol{E}$ as an Eisntein spacetime equipped with a certain fundamental structure.

3.1 Classical setting

3.1.1 Spacetime and Lorentz metric

We assume *spacetime* to be an oriented and time oriented 4–dimensional manifold $\boldsymbol{E}$ equipped with a scaled Lorentzian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ with signature $(-+++)$. With reference to a mass $m \in \mathbb{M}$, it is convenient to introduce the *rescaled metric* $G := \frac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$. The associated contravariant tensors are $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (T\boldsymbol{E} \otimes T\boldsymbol{E})$ and $\bar{G} = \frac{\hbar}{m}\, \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes T\boldsymbol{E})$.

We shall refer to a *spacetime chart* $(x^\lambda) \equiv (x^0, x^i)$ adapted to the spacetime orientation and such that the vector ∂_0 is timelike and time oriented and the vectors $\partial_1, \partial_2, \partial_3$ are spacelike. Greek indices will span all spacetime coordinates and Latin indices will span the spacelike coordinates. We shall also refer to a time unit $u_0 \in \mathbb{T}$ and its dual $u^0 \in \mathbb{T}^*$.

We have the coordinate expressions $g = g_{\lambda\mu}\, d^\lambda \otimes d^\mu$ and $G = G^0_{\lambda\mu}\, u_0 \otimes d^\lambda \otimes d^\mu$, with $g_{\lambda\mu} \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^2 \otimes \mathrm{I\!R})$ and $G^0_{\lambda\mu} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

A *motion* is defined to be a 1–dimensional timelike submanifold $s : \boldsymbol{T} \subset \boldsymbol{E}$.

Let us consider a motion $s : \boldsymbol{T} \subset \boldsymbol{E}$. Moreover, let us consider a spacetime chart (x^λ) and the induced chart $(\breve{x}^0) \in \mathrm{map}(\boldsymbol{T}, \mathrm{I\!R})$. Let us set $\partial_0 s^\lambda := \frac{ds^\lambda}{d\breve{x}^0}$. For every arbitrary choice of a "*proper time origin*" $t_0 \in \boldsymbol{T}$, we obtain the "*proper time scaled function*" given by the equality $\sigma : \boldsymbol{T} \to \bar{\mathbb{T}} : t \mapsto \frac{1}{c} \int_{[t_0,t]} \| \frac{ds}{d\breve{x}^0} \| \, d\breve{x}^0$. This map yields, at least locally, a bijection $\boldsymbol{T} \to \bar{\mathbb{T}}$, hence a (local) affine structure of $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}}$. Indeed, this (local) affine structure does not depend on the choice of the proper time origin and of the spacetime chart.

Let us choose a time origin $t_0 \in \boldsymbol{T}$ and consider the associated proper time scaled function $\sigma : \boldsymbol{T} \to \bar{\mathbb{T}}$ and the induced linear isomorphism $T\boldsymbol{T} \to \boldsymbol{T} \times \bar{\mathbb{T}}$.

The *1st differential* of the motion s is the map $ds := \frac{ds}{d\sigma} : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$.

We have $g(ds, ds) = -c^2$ and the coordinate expression

$$ds = \frac{ds^\lambda}{d\sigma}\, (\partial_\lambda \circ s) = \frac{c_0\, u^0 \otimes \big((\partial_0 \circ s) + \partial_0 s^i\, (\partial_i \circ s)\big)}{\sqrt{|(g_{00} \circ s) + 2\, (g_{0j} \circ s)\, \partial_0 s^j + (g_{ij} \circ s)\, \partial_0 s^i\, \partial_0 s^j|}}\,.$$

3.1.2 Jets of submanifolds

In view of the definition of the phase space, let us consider a manifold $\boldsymbol{M}$ of dimension n and recall a few basic facts concerning jets of submanifolds.

Let $k \geq 0$ be an integer. A *k–jet* of 1–dimensional submanifolds of $\boldsymbol{M}$ at $x \in \boldsymbol{M}$ is defined to be an equivalence class of 1–dimensional submanifolds touching each other at x with a contact of order k. The k–jet of a 1-dimensional submanifold $s : \boldsymbol{N} \subset \boldsymbol{M}$ at

$x \in \boldsymbol{N}$ is denoted by $j_k s(x)$. The set of all k–jets of all 1-dimensional submanifolds at $x \in \boldsymbol{M}$ is denoted by $J_{k\,x}(\boldsymbol{M}, 1)$. The set $J_k(\boldsymbol{M}, 1) := \bigsqcup_{x \in \boldsymbol{M}} J_{k\,x}(\boldsymbol{M}, 1)$ is said to be the *k–jet space* of 1–dimensional submanifolds of $\boldsymbol{M}$.

For each 1–dimensional submanifold $s : \boldsymbol{N} \subset \boldsymbol{M}$ and each integer $k \geq 0$, we have the map $j_k s : \boldsymbol{N} \to J_k(\boldsymbol{M}, 1) : x \mapsto j_k s(x)$.

In particular, for $k = 0$ and for each 1 dimensional submanifold $s : \boldsymbol{N} \subset \boldsymbol{M}$, we have the natural identification $J_0(\boldsymbol{M}, 1) = \boldsymbol{M}$, given by $j_0 s(x) = x$.

For each integers $k \geq h \geq 0$, we have the natural projection $\pi^k_h : J_k(\boldsymbol{M}, 1) \to J_h(\boldsymbol{M}, 1) : j_k s(x) \mapsto j_h s(x)$.

A chart of $\boldsymbol{M}$ is said to be *divided* if the set of its coordinate functions is divided into two subsets of 1 and $n-1$ elements. Our typical notation for a divided chart will be (x^0, x^i), with $1 \leq i \leq n-1$. A divided chart and a 1–dimensional submanifold $s : \boldsymbol{N} \subset \boldsymbol{M}$ are said to be *related* if the map $\breve{x}^0 := x^0|_{\boldsymbol{N}} \in \mathrm{map}(\boldsymbol{N}, \mathbb{R})$ is a chart of $\boldsymbol{N}$. In such a case, the submanifold $\boldsymbol{N}$ is locally characterised by $s^i \circ (\breve{x}^0)^{-1} := (x^i \circ s) \circ (\breve{x}^0)^{-1} \in \mathrm{map}(\mathbb{R}, \mathbb{R})$. In particular, if the divided chart is adapted to the submanifold, then the chart and the submanifold are related.

Let us consider a divided chart (x^0, x^i) of $\boldsymbol{M}$.

Then, for each submanifold $s : \boldsymbol{N} \subset \boldsymbol{M}$ which is related to this chart, the chart yields naturally the local fibred chart $(x^0, x^i; x^i_{\underline{\alpha}})_{1 \leq |\underline{\alpha}| \leq k} \in \mathrm{map}(J_k(\boldsymbol{M}, 1),\ \mathbb{R}^n \times \mathbb{R}^{k(n-1)})$ of $J_k(\boldsymbol{M}, 1)$, where $\underline{\alpha} := (h)$ is a multi–index of "range" 1 and "length" $|\underline{\alpha}| = h$ and the functions $x^i_{\underline{\alpha}}$ are defined by $x^i_{\underline{\alpha}} \circ j_1 \boldsymbol{N} := \partial_{0 \dots 0}\, s^i$, with $1 \leq |\underline{\alpha}| \leq k$.

We can prove the following facts:

1) the above charts $(x^0, x^i; x^i_{\underline{\alpha}})$ yield a smooth structure of $J_k(\boldsymbol{M}, 1)$;

2) for each 1 dimensional submanifold $s : \boldsymbol{N} \subset \boldsymbol{M}$ and for each integer $k \geq 0$, the subset $j_k s(\boldsymbol{N}) \subset J_k(\boldsymbol{M}, 1)$ turns out to be a smooth 1–dimensional submanifold;

3) for each integers $k \geq h \geq 1$, the maps $\pi^k_h : J_k(\boldsymbol{M}, 1) \to J_h(\boldsymbol{M}, 1)$ turn out to be smooth bundles.

We shall always refer to such diveded charts (x^0, x^i) of $\boldsymbol{M}$ and to the induced fibred charts $(x^0, x^i; x^i_{\underline{\alpha}})$ of $J_k(\boldsymbol{M}, 1)$.

Let $m_1 \in J_1(\boldsymbol{M}, 1)$, with $m_0 = \pi^1_0(m_1) \in \boldsymbol{M}$. Then, the tangent spaces at m_0 of all 1–dimensional submanifolds $\boldsymbol{N}$, such that $j_1 s(m_0) = m_1$, coincide. Accordingly, we denote by $T[m_1] \subset T_{m_0}\boldsymbol{M}$ the tangent space at m_0 of the above 1–dimensional submanifolds $\boldsymbol{N}$ generating m_1. We have the natural fibred isomorphism $J_1(\boldsymbol{M}, 1) \to \mathrm{Grass}(\boldsymbol{M}, 1) : m_1 \mapsto T[m_1] \subset T_{m_0}\boldsymbol{M}$ over $\boldsymbol{M}$ of the 1st jet bundle with the Grassmannian bundle of dimension 1. If $s : \boldsymbol{N} \subset \boldsymbol{M}$ is a submanifold, then we obtain $T[j_1 s] = \mathrm{span}\langle \partial_0 + \partial_0 s^i\,,\ \partial_i \rangle$, with reference to a related chart.

3.1.3 Phase space

We assume as *phase space* the subspace of all 1st jets of motions $\mathcal{J}_1\boldsymbol{E} \subset J_1(\boldsymbol{E}, 1)$.

For each 1–dimensional submanifold $s : \boldsymbol{T} \subset \boldsymbol{E}$ and for each $x \in \boldsymbol{T}$, we have $j_1 s(x) \in \mathcal{J}_1\boldsymbol{E}$ if and only if $T[j_1 s(x)] = T_x\boldsymbol{T}$ is timelike. The *velocity* of a motion $s : \boldsymbol{T} \subset \boldsymbol{E}$ is defined to be its 1-jet $j_1 s : \boldsymbol{T} \to \mathcal{J}_1(\boldsymbol{E}, 1)$.

Any spacetime chart (x^0, x^i) is related to each motion $s : \boldsymbol{T} \to \boldsymbol{E}$. Hence, the fibred chart (x^0, x^i, x^i_0) is defined on tubelike open subsets of $\mathcal{J}_1\boldsymbol{E}$. We shall always refer to the above fibred charts.

We define the *contact map* to be the unique fibred morphism д $: \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ over $\boldsymbol{E}$ such that д $\circ\, j_1 s = ds$, for each motion s. We have the coordinate expression д $= c_0\, \alpha^0\, u^0 \otimes (\partial_0 + x^i_0\, \partial_i)$, where $\alpha^0 := 1/\sqrt{|g_{00} + 2\, g_{0j}\, x^j_0 + g_{ij}\, x^i_0\, x^j_0|}$.

The fibred morphism д is injective. Indeed, it makes $\mathcal{J}_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ the fibred submanifold over $\boldsymbol{E}$ characterised by the constraint $g_{\lambda\mu}\, \dot{x}^\lambda_0\, \dot{x}^\mu_0 = -(c_0)^2$.

It is convenient to set $b_0 := \partial_0 + x^i_0\, \partial_i$ and $\breve{g}_{0\lambda} := g(b_0, \partial_\lambda) = g_{0\lambda} + g_{i\lambda}\, x^i_0$. Then, we obtain $(\alpha^0)^2\, (\breve{g}_{00} + \breve{g}_{0i}\, x^i_0) = -1$.

We define the *time form* as the fibred morphism $\tau := -\frac{1}{c^2}\, g^\flat(\text{д}) : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$, with coordinate expression $\tau = \tau_\lambda\, d^\lambda$, where $\tau_\lambda = -\frac{\alpha^0}{c_0}\, \breve{g}_{0\lambda}\, u_0$. We have $\tau(\text{д}) = 1$ and $g\,(\text{д}, \text{д}) = -c^2$.

We define the *complementary contact map* as $\theta := 1 - \text{д} \otimes \tau : \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{E}$. We have the coordinate expressions $\theta = d^\lambda \otimes \partial_\lambda + (\alpha^0)^2\, \breve{g}_{0\lambda}\, d^\lambda \otimes (\partial_0 + x^j_0\, \partial_j)$.

For each motion s, we have $(\tau \circ j_1 s)(ds) = 1$.

With reference to a particle of mass m, we define the unscaled 1–form $\Theta := -\frac{mc^2}{\hbar}\, \tau$, with coordinate expression $\Theta = \alpha^0\, c_0\, \breve{G}^0_{0\lambda}\, d^\lambda$.

3.1.4 Contact splitting

We define the д–*horizontal* tangent space of spacetime, the τ–*vertical* tangent space of spacetime, the τ–*horizontal* cotangent space of spacetime and the д–*vertical* cotangent space of spacetime to be, respectively, the vector subbundles over $\mathcal{J}_1\boldsymbol{E}$

$$
\begin{aligned}
H_{\text{д}}\boldsymbol{E} &:= \{(e_1, X) \in \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \mid X \in T[e_1]\} && \subset \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E}\\
V_\tau\boldsymbol{E} &:= \{(e_1, X) \in \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \mid X \in T[e_1]^\perp\} && \subset \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E}\\
H^*_\tau\boldsymbol{E} &:= \{(e_1, \omega) \in \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} \mid \langle \omega\,, T[e_1]^\perp \rangle = 0\} && \subset \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E}\\
V^*_{\text{д}}\boldsymbol{E} &:= \{(e_1, \omega) \in \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} \mid \langle \omega\,, T[e_1] \rangle = 0\} && \subset \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E}\,.
\end{aligned}
$$

We have the natural orthogonal linear fibred splittings over $\mathcal{J}_1\boldsymbol{E}$ and the projections

$$
\begin{aligned}
\mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} &= H_{\text{д}}\boldsymbol{E} \oplus V_\tau\boldsymbol{E}\,, & \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} &= H^*_\tau\boldsymbol{E} \oplus V^*_{\text{д}}\boldsymbol{E}\,,\\
\text{д} \otimes \tau : \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} &\to H_{\text{д}}\boldsymbol{E}\,, & \tau \otimes \text{д} : \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} &= H^*_\tau\boldsymbol{E}\,,\\
\theta : \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} &\to V_\tau\boldsymbol{E}\,, & \theta^* : \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} &\to V^*_{\text{д}}\boldsymbol{E}\,.
\end{aligned}
$$

We have the mutually dual local bases $(b_0,\, b_i)$ and $(\beta^0,\, \beta^i)$ adapted to the above splittings, where

$$\begin{aligned} b_0 &:= \partial_0 + x^i_0\,\partial_i \quad \in \mathrm{fib}(\mathcal{J}_1\boldsymbol{E},\, H_{д}\boldsymbol{E})\,, & b_i &:= \partial_i - c\,\alpha^0\,\tau_i\,b_0 \in \mathrm{fib}(\mathcal{J}_1\boldsymbol{E},\, V_\tau\boldsymbol{E})\,,\\ \beta^0 &:= d^0 + c\,\alpha^0\,\tau_i\,\beta^i \in \mathrm{fib}(\mathcal{J}_1\boldsymbol{E},\, H^*_\tau\boldsymbol{E})\,, & \beta^i &:= d^i - x^i_0\,d^0 \quad \in \mathrm{fib}(\mathcal{J}_1\boldsymbol{E},\, V^*_{д}\boldsymbol{E})\,. \end{aligned}$$

The restriction of g to $H_{д}\boldsymbol{E}$ and $V_\tau\boldsymbol{E}$ and the restriction of $\bar{g}$ to $H^*_\tau\boldsymbol{E}$ and $V^*_{д}\boldsymbol{E}$ yield, respectively, the scaled metrics

$$\begin{aligned} g_\parallel &: \mathcal{J}_1\boldsymbol{E} \to \mathbb{L}^2 \otimes (H^*_\tau\boldsymbol{E} \otimes H^*_\tau\boldsymbol{E}) \quad \text{and} \quad g_\perp : \mathcal{J}_1\boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*_{д}\boldsymbol{E} \otimes V^*_{д}\boldsymbol{E})\\ g^\parallel &: \mathcal{J}_1\boldsymbol{E} \to \mathbb{L}^{-2} \otimes (H_{д}\boldsymbol{E} \otimes H_{д}\boldsymbol{E}) \quad \text{and} \quad g^\perp : \mathcal{J}_1\boldsymbol{E} \to \mathbb{L}^{-2} \otimes (V_\tau\boldsymbol{E} \otimes V_\tau\boldsymbol{E})\,, \end{aligned}$$

with coordinate expressions in an adapted basis

$$\begin{aligned} g_{\parallel 00} &:= g\,(b_0, b_0) &&= -\frac{1}{(\alpha^0)^2}\\ g^{\parallel 00} &:= \bar{g}\,(\beta^0, \beta^0) &&= -(\alpha^0)^2\\ g_{\perp ij} &:= g(b_i,\, b_j) &&= g_{ij} + c^2\,\tau_i\,\tau_j\\ g^{\perp ij} &:= \bar{g}(\beta^i,\, \beta^j) &&= g^{ij} - g^{i0}\,x^j_0 - g^{j0}\,x^i_0 + g^{00}\,x^i_0\,x^j_0\,. \end{aligned}$$

It is convenient to set

$$\begin{aligned} \breve{\delta}^\lambda_0 &:= \delta^\lambda_0 + \delta^\lambda_i\,x^i_0\,, & \breve{\delta}^i_\lambda &:= \delta^i_\lambda - \delta^0_\lambda\,x^i_0\,,\\ \breve{g}_{0\lambda} &:= g\,(b_0, \partial_\lambda) = g_{0\lambda} + g_{i\lambda}\,x^i_0\,, & \breve{g}^{0\lambda} &:= \bar{g}\,(\beta^0, d^\lambda) = -(\alpha^0)^2\,\breve{\delta}^\lambda_0\,,\\ \breve{g}_{i\lambda} &:= g(b_i,\, \partial_\lambda) = g_{i\lambda} + c^2\,\tau_i\,\tau_\lambda\,, & \breve{g}^{i\lambda} &:= \bar{g}(\beta^i,\, d^\lambda) = g^{i\lambda} - g^{0\lambda}\,x^i_0\,. \end{aligned}$$

Then, we obtain the following useful technical identities

$$\breve{g}_{0\lambda}\,d^\lambda = g_{\parallel 00}\,\beta^0\,, \qquad \breve{g}^{0\lambda}\,\partial_\lambda = g^{\parallel 00}\,b_0\,, \qquad \breve{g}_{i\lambda}\,d^\lambda = g_{\perp ij}\,\beta^j\,, \qquad \breve{g}^{i\lambda}\,\partial_\lambda = g^{\perp ij}\,b_j\,,$$

$$(\breve{g}_{ij})^{-1} = (g^{\perp ij}) = (\breve{g}^{ij} - \breve{g}^{i0}\,x^j_0)\,, \qquad g^{\perp jh}\,\breve{g}_{0h} = \frac{1}{(\alpha^0)^2}\,\breve{g}^{j0}\,,$$

$$\breve{g}_{\lambda\nu}\,\breve{g}^{\mu\nu} = \delta^\mu_\lambda\,, \qquad \breve{g}_{\nu\lambda}\,\breve{g}^{\nu\mu} = \delta^\mu_\lambda\,, \qquad \breve{g}_{0\lambda}\,\breve{g}^{0\mu} = -c^2\,\tau_\lambda\,\tau^\mu\,, \qquad \breve{g}_{i\lambda}\,\breve{g}^{i\mu} = \delta^\mu_\lambda + c^2\,\tau_\lambda\,\tau^\mu\,,$$

$$\breve{g}_{0i}\,\breve{g}^{i\lambda} = \frac{1}{(\alpha^0)^2}\,g^{0\lambda} + \breve{\delta}^\lambda_0\,, \qquad \breve{g}_{i0} + \breve{g}_{ij}\,x^j_0 = 0\,.$$

and

$$\partial^0_j\alpha^0 = (\alpha^0)^3\,\breve{g}_{0j}\,, \qquad \partial^0_j\frac{1}{\alpha^0} = -\alpha^0\,\breve{g}_{0j}\,, \qquad \partial^{00}_{ij}\frac{1}{\alpha^0} = -\alpha^0\,\breve{g}_{ij}\,,$$

$$\partial^0_i\tau_\mu = -\frac{\alpha^0}{c}\,\breve{g}_{i\mu}\,, \qquad \partial_\lambda\alpha^0 = \tfrac{1}{2}\,(\alpha^0)^3\,(\partial_\lambda g_{00} + 2\,\partial_\lambda g_{0h}\,x^h_0 + \partial_\lambda g_{hk}\,x^h_0\,x^k_0)\,.$$

3.1.5 Vertical bundle of the phase space

Let $V_0\mathcal{J}_1\boldsymbol{E} \subset T\mathcal{J}_1\boldsymbol{E}$ be the vertical tangent subbundle over $\boldsymbol{E}$. The vertical prolongation of the contact map yields the mutually inverse linear fibred isomorphisms $\nu_\tau : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T} \otimes V^*_\tau\boldsymbol{E} \otimes V_0\mathcal{J}_1\boldsymbol{E}$ and $\nu_\tau^{-1} : \mathcal{J}_1\boldsymbol{E} \to V^*_0\mathcal{J}_1\boldsymbol{E} \otimes \mathbb{T} \otimes V_\tau\boldsymbol{E}$, with coordinate expressions $\nu_\tau = \frac{1}{c_0\,\alpha^0}\, u_0 \otimes \beta^i \otimes \partial^0_i$ and $\nu_\tau^{-1} = c_0\,\alpha^0\, u^0 \otimes d^i_0 \otimes b_i$.

3.1.6 Observers

An *observer* is defined to be a section $o \in \sec(\boldsymbol{E}, \mathcal{J}_1\boldsymbol{E})$. Thus, an observer can be regarded as the velocity of a continuum.

Each observer yields the scaled vector field $\text{д}[o] := \text{д} \circ o \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E})$, the scaled 1–form $\tau[o] := \tau \circ o \in \sec(\boldsymbol{E}, \mathbb{T} \otimes T^*\boldsymbol{E})$ and the tangent valued 1–form $\theta[o] := \theta \circ o \in \sec(\boldsymbol{E}, T^*\boldsymbol{E} \otimes T\boldsymbol{E})$, with coordinate expressions $\text{д}[o] = c_0\,\alpha^0[o]\,u^0 \otimes (\partial_0 + o^i_0\,\partial_i)$, $\tau[o] = -\frac{1}{c_0}\,\alpha^0[o]\,(g_{0\lambda} + g_{i\lambda}\,o^i_0)\,u_0 \otimes d^\lambda$ and $\theta[o] = d^\lambda \otimes \partial_\lambda - \alpha^0[o]\,(g_{0\lambda} + g_{i\lambda}\,o^i_0)\,d^\lambda \otimes (\partial_0 + o^i_0\,\partial_i)$, where $o^i_0 := x^i_0 \circ o$ and $\alpha^0[o] = 1/\sqrt{|g_{00} + 2\,g_{0j}\,o^j_0 + g_{ij}\,o^i_0\,o^j_0|}$. Each of the above objects characterises o.

A spacetime chart (x^λ) is said to be *adapted* to an observer o if $o^i_0 = 0$, i.e. if the spacelike functions x^i are constant along the integral motions of o. Actually, infinitely many spacetime charts are adapted to an observer o; the transition maps of two such charts (x^λ) and $(\acute{x}^\lambda)$ are of the type $\partial_0\acute{x}^i = 0$. Conversely, each spacetime chart (x^0, x^i) is adapted to the unique observer o determined by the equality $\text{д}[o] := (c/\|\partial_0\|)\,\partial_0$.

An *observing frame* is defined to be a pair (o, ζ), where o is an observer and $\zeta \in \sec(\boldsymbol{E}, \mathbb{T} \otimes T^*\boldsymbol{E})$ is timelike and positively time oriented. In particular, each observer o determines the observing frame $(o, \tau[o])$. An observing frame is said to be *integrable* if ζ is closed. In this case, there exists locally a scaled function $t \in \operatorname{map}(\boldsymbol{E}, \bar{\mathbb{T}})$, called the *observed time function*, such that $\zeta = dt$.

A spacetime chart (x^λ) is said to be *adapted* to an integrable observing frame (o, ζ) if it is adapted to o and $x^0 = u^0 \lrcorner\, t$. Actually, infinitely many spacetime charts are adapted to an integrable observing frame (o, ζ); the transition maps of two such charts (x^λ) and $(\acute{x}^\lambda)$ are of the type $\partial_0\acute{x}^i = 0$, $\partial_0\acute{x}^0 \in \mathrm{I\!R}^+$. Conversely, each spacetime chart (x^0, x^i) is adapted to the observing frames (o, ζ) such that $\text{д}[o] := (c/\|\partial_0\|)\,\partial_0$ and $\zeta = u_0 \otimes d^0$ (thus, (o, ζ) is determined up to a constant positive factor for ζ).

With reference to an observing frame (o, ζ), we define the $\text{д}[o]$–*horizontal* tangent space of spacetime, the ζ–*vertical* tangent space of spacetime, the ζ–*horizontal* cotangent space of spacetime and the $\text{д}[o]$–*vertical* cotangent space of spacetime to be, respectively, the vector subbundles over $\boldsymbol{E}$

$$\begin{aligned}
H_{\text{д}[o]}\boldsymbol{E} &:= \{X \in T\boldsymbol{E} \mid X = X^0\,\text{д}[o]_0\} \subset T\boldsymbol{E} \\
V_\zeta\boldsymbol{E} &:= \{X \in T\boldsymbol{E} \mid X \lrcorner\, \zeta = 0\} \qquad \subset T\boldsymbol{E} \\
H^*_\zeta\boldsymbol{E} &:= \{\omega \in T^*\boldsymbol{E} \mid \omega = \omega_0\,\zeta^0\} \qquad \subset T^*\boldsymbol{E} \\
V^*_{\text{д}[o]}\boldsymbol{E} &:= \{\omega \in T^*\boldsymbol{E} \mid \omega \lrcorner\, \text{д}[o] = 0\} \quad \subset T^*\boldsymbol{E}\,.
\end{aligned}$$

We have the natural linear fibred splittings over $\boldsymbol{E}$ and the projections

$$T\boldsymbol{E} = H_{\text{д}[o]}\boldsymbol{E} \oplus V_\zeta \boldsymbol{E}\,, \qquad T^*\boldsymbol{E} = H^*_\zeta \boldsymbol{E} \oplus V^*_{\text{д}[o]}\boldsymbol{E}\,,$$
$$(1/\varsigma)\,\text{д}[o] \otimes \zeta : T\boldsymbol{E} \to H_{\text{д}}[o]\boldsymbol{E}\,, \qquad (1/\varsigma)\,\zeta \otimes \text{д}[o] : T^*\boldsymbol{E} = H^*_\zeta \boldsymbol{E}\,,$$
$$\theta[o,\zeta] : T\boldsymbol{E} \to V_\zeta \boldsymbol{E}\,, \qquad \theta^*[o,\zeta] : T^*\boldsymbol{E} \to V^*_{\text{д}}[o]\boldsymbol{E}\,,$$

where $\varsigma := \text{д}[o] \lrcorner \zeta \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}^+)$ and $\theta[o,\zeta] := 1 - (1/\varsigma)\,\text{д}[o] \otimes \zeta$.

With reference to an integrable observing frame and to an adapted chart (x^λ) , the coordinate expression of the above splittings are $X = X^0\,\partial_0 + X^i\,\partial_i$ and $\omega = \omega_0\,d^0 + \omega_i\,d^i$.

In the particular case when $\zeta = \tau[o]$, the above subspaces, splittings and projections turn out to be obtained from the corresponding contact subspaces, splittings and projections, by pullback with respect to o .

For each observing frame (o,ζ) , the orientation of spacetime and the metric g yield a scaled volume form $\eta[o,\zeta] : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*_{\text{д}[o]}$ and the inverse scaled volume vector $\bar\eta[o,\zeta] : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^3 V_\zeta$.

For each observing frame (o,ζ) , by splitting Θ into the horizontal and vertical components, we define the observed *kinetic energy* and *kinetic momentum* as $\mathcal{K}[o,\zeta] = -(1/\varsigma)\,\zeta(\text{д}[o] \lrcorner \Theta) \in \operatorname{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ and $\mathcal{Q}[o,\zeta] = \theta[o,\zeta] \lrcorner \Theta \in \operatorname{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$. Thus, we have $\Theta = -\mathcal{K}[o,\zeta] + \mathcal{Q}[o,\zeta] \in \operatorname{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$. In the particular case when the observing frame is integrable, with reference to an adapted chart, we obtain $\mathcal{K}[o] = -c_0\,\alpha^0\,\breve G^0_{00}\,d^0$ and $\mathcal{Q}[o] = c_0\,\alpha^0\,\breve G^0_{0i}\,d^i$.

Let us consider a motion $s : \boldsymbol{T} \subset \boldsymbol{E}$ and an observer o and refer to an adapted chart.

We have the *observed orthogonal splitting* $\text{д}[j_1 s] = \delta\,\big(\text{д}[o] + \vec v\big)$, where $\delta := \text{д}[j_1 s] \lrcorner \tau[o] \in \operatorname{map}(\boldsymbol{T}, \mathbb{R}^+)$ and $\vec v := \frac{1}{\delta}\,\text{д}[j_1 s] \circ \theta[o] \in \operatorname{map}(\boldsymbol{T}, \mathbb{T}^* \otimes V_{\tau[o]}\boldsymbol{E})$. By setting $\beta := \frac{\|\vec v\|}{c} \in \operatorname{map}(\boldsymbol{T}, \mathbb{R}^+)$, we obtain $\delta = \frac{1}{\sqrt{1-\beta^2}} > 1$, $\beta = \frac{\sqrt{\delta^2-1}}{\delta} < 1$ and

$$\delta = -\frac{g_{00} + g_{0i}\,\partial_0 s^i}{\sqrt{|g_{00}|}\sqrt{|g_{00} + 2\,g_{0j}\,\partial_0 s^j + g_{ij}\,\partial_0 s^i\,\partial_0 s^j|}}$$

$$\vec v = -\frac{c_0\,\sqrt{|g_{00}|}\,\partial_0 s^h}{g_{00} + g_{0k}\,\partial_0 s^k}\;u^0 \otimes \Big(-\frac{g_{0h}}{g_{00}}\,\partial_0 + \partial_h\Big)$$

$$\beta = \frac{\sqrt{(-g_{00}\,g_{hk} + g_{0h}\,g_{0k})\,\partial_0 s^h\,\partial_0 s^k}}{|g_{00} + g_{0i}\,\partial_0 s^i|}\,.$$

3.1.7 Gravitational and electromagnetic fields

We assume the Levi–Civita connection $K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ induced by g (or, equivalently, by G) as *gravitational connection*. The coordinate expression of $K^\natural$ is $K^\natural{}_\lambda{}^\nu{}_\mu = -\frac{1}{2}\,G^{\nu\rho}_0\,(\partial_\lambda G^0_{\rho\mu} + \partial_\mu G^0_{\rho\lambda} + \partial_\rho G^0_{\lambda\mu})$, where we have set $K^\natural{}_\lambda{}^\nu{}_\mu := -(\nabla^\natural{}_\lambda \partial_\mu)^\nu$.

We assume spacetime to be equipped with a given *electromagnetic field*, which is a closed scaled 2–form $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$. With reference to a particle with mass m and charge q, we obtain the unscaled 2–form $\frac{q}{\hbar}\, F : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$.

Given an observer o, we define the *observed magnetic* and the *observed electric* fields

$$\begin{aligned}\vec{B}[o] &:= \tfrac{c}{2}\, i(\theta[o](F))\,\bar{\eta}[o] \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V_{\tau[o]}\boldsymbol{E}\big)\\ \vec{E}[o] &:= -g^\sharp(o \lrcorner F) \qquad \in \sec\Big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V_{\tau[o]}\boldsymbol{E}\Big).\end{aligned}$$

Then, we obtain the observed splitting $F = -2\,\tau[o] \wedge g^\flat(\vec{E}[o]) + \frac{2}{c}\, i(\vec{B}[o])\,\eta[o]$.

The local potentials of F are denoted by $A^{\mathfrak{e}}$, according to $2\,dA^{\mathfrak{e}} = F$.

In the Einstein framework there is no way to merge the electromagnetic field into the gravitational connection, hence we have no joined spacetime connection.

3.1.8 Induced objects on the phase space

We have a natural injective map χ between linear spacetime connections K and phase connections $\Gamma : \mathcal{J}_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\mathcal{J}_1\boldsymbol{E}$, with coordinate expressions $\Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma_{\lambda}{}^{i}_{0}\, \partial^0_i)$. In coordinates, the map χ is expressed by $\Gamma_{\lambda}{}^{i}_{0} = \breve{\delta}^i_\nu\, K_\lambda{}^\nu{}_\rho\, \breve{\delta}^\rho_0$.

As we have no joined spacetime connection, we start with the gravitational objects induced on the phase space.

Then, the spacetime connection $K^\natural$ yields a connection, called *gravitational phase connection*, $\Gamma^\natural := \chi(K^\natural) : \mathcal{J}_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\mathcal{J}_1\boldsymbol{E}$.

The phase connection $\Gamma^\natural$ yields the 2nd order connection, called *gravitational dynamical phase connection*, $\gamma^\natural := д \lrcorner \Gamma^\natural : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\mathcal{J}_1\boldsymbol{E}$, with coordinate expression $\gamma^\natural = c_0\, \alpha^0\, u_0 \otimes (\partial_0 + x^i_0\, \partial_i + \gamma^\natural{}_{0}{}^{i}_{0}\, \partial^0_i)$, where $\gamma^\natural{}_{0}{}^{i}_{0} = \breve{\delta}^i_\nu\, K_\lambda{}^\nu{}_\mu\, \breve{\delta}^\lambda_0\, \breve{\delta}^\mu_0$.

Next, let us consider the vertical projection $\nu_\tau[\Gamma^\natural] := \nu_\tau^{-1} \circ \circ \nu[\Gamma^\natural] : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes (T^*\mathcal{J}_1\boldsymbol{E} \otimes V_\tau\boldsymbol{E})$ associated with $\Gamma^\natural$, whose coordinate expression is $\nu_\tau[\Gamma^\natural] = c_0\, \alpha^0\, (d^i_0 - \Gamma_{\lambda}{}^{i}_{0}\, d^\lambda)\, u_0 \otimes b_i$.

The phase connection $\Gamma^\natural$ and the rescaled metric G yield the 2–form, called *gravitational phase 2–form*, $\Omega^\natural := G \lrcorner \Big(\nu_\tau[\Gamma^\natural] \wedge \theta\Big) : \mathcal{J}_1\boldsymbol{E} \to \Lambda^2 T^*\mathcal{J}_1\boldsymbol{E}$, with coordinate expression $\Omega^\natural = c_0\, \alpha^0\, \breve{G}^0_{i\mu}\, \Big(d^i_0 - \breve{\delta}^i_\nu\, K^\natural{}_\lambda{}^\nu{}_\rho\, \breve{\delta}^\rho_0)\, d^\lambda\Big) \wedge d^\mu$.

The pair $(\Theta, \Omega^\natural)$ is a "contact" structure of $\mathcal{J}_1\boldsymbol{E}$, i.e. $\Omega = d\Theta$ and $\Theta \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural \not\equiv 0$.

The phase connection $\Gamma^\natural$ and the rescaled metric G yield the vertical 2–vector, called *gravitational phase 2–vector*, $\Lambda^\natural := \bar{G} \lrcorner (\Gamma^\natural \wedge \nu_\tau) : \mathcal{J}_1\boldsymbol{E} \to \Lambda^2 V\mathcal{J}_1\boldsymbol{E}$, with coordinate expression $\Lambda^\natural = \frac{1}{c_0\, \alpha^0}\, \breve{G}^{j\lambda}_0\, (\partial_\lambda + \breve{G}^{i\mu}_0\, K^\natural{}_{\lambda\mu\rho}\, \breve{\delta}^\rho_0\, \partial^0_i) \wedge \partial^0_j$.

Summing up, the above gravitational phase objects fulfill the following identities

$$i(\gamma^\natural)\,\tau = 1\,, \quad i(\gamma^\natural)\,\Omega^\natural = 0\,, \quad \gamma^\natural = д \lrcorner \Gamma^\natural\,, \quad \Omega^\natural = G \lrcorner \Big(\nu_\tau[\Gamma^\natural] \wedge \theta\Big)\,, \quad \Lambda^\natural = \bar{G} \lrcorner (\Gamma^\natural \wedge \nu^\natural)\,.$$

Now, we are looking for *joined* phase objects, obtained by merging the electromagnetic field into the above gravitational phase objects, in such a way to preserve the above relations.

By analogy with the Galilei case, we start with the phase connection.

We define the *joined phase connection* to be the phase connection $\Gamma := \Gamma^\natural + \Gamma^\mathfrak{e}$, where $\Gamma^\mathfrak{e} := -\frac{q}{2\hbar}\,\nu_\tau \circ G^{\sharp 2} \circ \Big(F + 2\tau \wedge (д \lrcorner F)\Big)$. We have the coordinate expression $\Gamma^\mathfrak{e} = -\frac{q}{2\hbar}\,\frac{1}{c_0\,\alpha^0}\,\breve{G}^{i\mu}_0\,(F_{\lambda\mu} - (\alpha^0)^2 \breve{g}_{0\lambda}\,F_{\rho\mu}\,\breve{\delta}^\rho_0)\,d^\lambda \otimes \partial^0_i$.

The joined phase connection Γ yields the 2nd order connection, called *joined dynamical phase connection*, $\gamma := д \lrcorner \Gamma : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\mathcal{J}_1\boldsymbol{E}$, which splits as $\gamma = \gamma^\natural + \gamma^\mathfrak{e}$, where $\gamma^\mathfrak{e} = -\frac{q}{m}\,\nu_\tau \circ g^\sharp \circ (д \lrcorner F)$, i.e., in coordinates, $\gamma^\mathfrak{e} = -\frac{q_0}{m}\,\breve{g}^{i\mu}\,(F_{0\mu} + F_{j\mu}\,x^j_0)\,u^0 \otimes \partial^0_i$.

The joined phase connection Γ and the rescaled metric G yield the 2–form, called *joined phase 2–form*, $\Omega := G \lrcorner \Big(\nu_\tau[\Gamma] \wedge \theta\Big)$, which splits as $\Omega = \Omega^\natural + \Omega^\mathfrak{e}$, where $\Omega^\mathfrak{e} = \frac{q}{2\hbar}\,F$, i.e., in coordinates, $\Omega^\mathfrak{e} = \frac{q}{2\hbar}\,F_{\lambda\mu}\,d^\lambda \wedge d^\mu$. The pair (Θ, Ω) is a "cosymplectic" structure of $\mathcal{J}_1\boldsymbol{E}$, i.e, $d\Omega = d\Omega^\natural + \frac{q}{2\hbar}\,dF = 0$ and $\Theta \wedge \Omega \wedge \Omega \wedge \Omega = \Theta \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural \neq 0$.

Moreover, Ω admits potentials, called *horizontal*, of the type $A^\uparrow \in \mathrm{fib}(\mathcal{J}_1\boldsymbol{E}, T^*\boldsymbol{E})$, according to $dA^\uparrow = \Omega$. They are defined up to a gauge of the type $\alpha \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$. Indeed, we have $A^\uparrow = \Theta + \frac{q}{\hbar}\,A^\mathfrak{e}$, with coordinate expression $A^\uparrow = (c_0\,\alpha^0\,\breve{G}^0_{0\lambda} + \frac{q}{\hbar}\,A^\mathfrak{e}{}_\lambda)\,d^\lambda$.

Indeed, γ is the unique 2nd order connection such that $i(\gamma)\tau = 1$ and $i(\gamma)\Omega = 0$.

We define the *Lorentz force* as $\vec{f} := -g^\sharp \circ (д \lrcorner F) : \mathcal{J}_1\boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V_\tau\boldsymbol{E}$. We have the coordinate expression $\vec{f} = -c\,\alpha^0\,(g^{\lambda j}\,F_{0j} + g^{\lambda\mu}\,F_{i\mu}\,x^i_0)\,\partial_\lambda$ and the observed expression $\vec{f} = \vec{E}[o] + \frac{1}{c}\,\vec{\nabla}[o] \underset{\eta[o]}{\times} \vec{B}[o]$. Moreover, we have $\vec{f} := := \frac{m}{q}\,\nu_\tau^{-1} \circ \gamma^\mathfrak{e}$.

We assume the *law of motion* for the unknown motion $s \subset \boldsymbol{E}$ of a particle of mass m and charge q to be the equation $\nabla[\gamma]j_1 s := j_2 s - \gamma \circ j_1 s = 0$, i.e. $m\,\nabla^\perp[\gamma^\natural]j_1 s = q\,\vec{f} \circ j_1 s$, where $\nabla^\perp := \nu_\tau^{-1} \circ \nabla$.

The joined phase connection Γ and the rescaled metric G yield the 2–vector, called *joined phase 2–vector*, $\Lambda := \bar{G} \lrcorner (\Gamma \wedge \nu^\natural)$, which splits as $\Lambda = \Lambda^\natural + \Lambda^\mathfrak{e}$, where $\Lambda^\mathfrak{e} = \frac{q}{2\hbar}\,(\nu_\tau \wedge \nu_\tau)(G^\sharp(\theta^*(F)))$, i.e., in coordinates, $\Lambda^\mathfrak{e} = \frac{q}{2\hbar}\,\frac{1}{(c_0\,\alpha^0)^2}\,\breve{G}^{i\lambda}_0\,\breve{G}^{j\mu}_0\,F_{\lambda\mu}\,\partial^0_i \wedge \partial^0_j$. From now on, we shall refer to the above joined phase objects Γ, γ, Ω, and Λ.

3.1.9 Hamiltonian lift of phase functions

For each $\phi^\uparrow \in \sec(\mathcal{J}_1\boldsymbol{E}, T^*\mathcal{J}_1\boldsymbol{E})$, we have $\Lambda^\sharp(\phi^\uparrow) := i(\phi^\uparrow)\Lambda \in \sec(\mathcal{J}_1\boldsymbol{E}, V_\tau\mathcal{J}_1\boldsymbol{E})$.

Given a time scale $\sigma \in \mathrm{map}(\mathcal{J}_1\boldsymbol{E}, \bar{\mathbb{T}})$, we define the *$\sigma$–Hamiltonian lift* to be the map

$$X^\uparrow{}_{\mathrm{ham}}[\sigma] : \mathrm{map}(\mathcal{J}_1\boldsymbol{E}, \mathrm{I\!R}) \to \sec(\mathcal{J}_1\boldsymbol{E}, T\mathcal{J}_1\boldsymbol{E}) : f \mapsto X^\uparrow{}_{\mathrm{ham}}[\sigma, f] := \gamma(\sigma) + \Lambda^\sharp{}_0(df),$$

with coordinate expression

$$X^\uparrow{}_{\mathrm{ham}}[\sigma, f] = \sigma^0 c_0 \alpha^0\,(\partial_0 + x^i_0\,\partial_i + \gamma_0{}^i_0\,\partial^0_i) - \frac{1}{c_0\alpha^0}\,\Big(\breve{G}^{j\lambda}_0\,\partial^0_j f\,\partial_\lambda - (\breve{G}^{i\lambda}_0\,\partial_\lambda f + \breve{\Xi}^{ij}_{00}\,\partial^0_j f)\,\partial^0_i\Big),$$

where $\breve{\Xi}^{ij}_{00} = \breve{G}^{ih}_0\,\Gamma_h{}^j_0 - \breve{G}^{jh}_0\,\Gamma_h{}^i_0$.

3.1.10 Poisson bracket of the phase functions

We define the *Poisson bracket* of $\mathrm{map}(\mathcal{J}_1\boldsymbol{E}, \mathrm{I\!R})$ as $\{f, g\} := i(df \wedge dg)\,\Lambda$.

Its coordinate expression is $\{f, g\} = \frac{1}{c_0\,\alpha^0}\,\Big(\breve{G}^{i\lambda}_0(\partial_\lambda f\,\partial^0_i g - \partial_\lambda g\,\partial^0_i f) - \breve{\Xi}^{ij}_{00}\,\partial^0_i f\,\partial^0_j g\Big)$.

The Poisson bracket makes $\mathrm{map}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R})$ a sheaf of $\mathbb{R}$–Lie algebras.

3.1.11 The sheaf of special phase functions

Each $X \in \mathrm{fib}(\mathcal{J}_1\boldsymbol{E},\, T\boldsymbol{E})$ yields the time scale $\sigma := \tau(X) \in \mathrm{map}(\mathcal{J}_1\boldsymbol{E},\, \bar{\mathbb{T}})$, with coordinate expression $\sigma = -\frac{\alpha^0}{c_0}\, \breve{g}_{0\lambda}\, X^\lambda\, u_0$.

If $X,\, X_1,\, X_2 \in \mathrm{sec}(\boldsymbol{E},\, T\boldsymbol{E})$, $\phi \in \mathrm{sec}(\boldsymbol{E},\, \mathbb{T}\otimes T^*\boldsymbol{E})$ and $\breve{f}_1,\, \breve{f}_2 \in \mathrm{map}(\boldsymbol{E},\, \mathbb{R})$, then

$$\begin{aligned} -G(д, X_1) + \breve{f}_1 = -G(д, X_2) + \breve{f}_2 \quad &\Leftrightarrow \quad X_1 = X_2\,, \quad \breve{f}_1 = \breve{f}_2 \\ -G(д, X) + \breve{f}_1 = -д \,\lrcorner\, \phi + \breve{f}_2 \quad &\Leftrightarrow \quad \phi = G^\flat(X)\,, \quad \breve{f}_1 = \breve{f}_2 \\ -G(д, X_1) + \breve{f}_1 = -X_2 \,\lrcorner\, \Theta + \breve{f}_2 \quad &\Leftrightarrow \quad X_1 = X_2\,, \quad \breve{f}_1 = \breve{f}_2\,. \end{aligned}$$

We define a *special phase function* to be a function $f \in \mathrm{map}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R})$ of the type $f = -G(д, X) + \breve{f}$, with $X \in \mathrm{sec}(\boldsymbol{E},\, T\boldsymbol{E})$ and $\breve{f} \in \mathrm{map}(\boldsymbol{E},\, \mathbb{R})$.

Moreover, we say that
- $X[f] := X \in \mathrm{sec}(\boldsymbol{E},\, T\boldsymbol{E})$ is the *tangent lift* of f,
- $\phi[f] := G^\flat(X) \in \mathrm{sec}(\boldsymbol{E},\, \mathbb{T}\otimes T^*\boldsymbol{E})$ is the *cotangent lift* of f,
- $\sigma[f] := \tau(X) \in \mathrm{map}(\mathcal{J}_1\boldsymbol{E},\, \bar{\mathbb{T}})$ is the *time scale* of f,
- $\breve{f} \in \mathrm{map}(\boldsymbol{E},\, \mathbb{R})$ is the *spacetime component* of f.

Thus, if f is a special phase function, then we have the following equivalent expressions

$$f = -G(д, X) + \breve{f} = -д \,\lrcorner\, \phi[f] + \breve{f} = -X[f] \,\lrcorner\, \Theta + \breve{f} = \tfrac{m\, c^2}{\hbar}\, \sigma[f] + \breve{f}$$

and, in coordinates,

$$f = -\frac{c_0\, (G^0_{\lambda 0} + G^0_{\lambda i}\, x^i_0)\, f^\lambda}{\sqrt{|g_{00} + 2\, g_{0k}\, x^k_0 + g_{hk}\, x^h_0\, x^k_0|}} + \breve{f} = -c_0\, \alpha^0\, (f^0_0 + f^0_i\, x^i_0) + \breve{f}\,,$$

with $f^\lambda := X^\lambda = G^{\lambda\mu}_0\, \phi^0_\mu$ and $f^0_\lambda := \phi^0_\lambda = G^0_{\lambda\mu}\, X^\mu$.

The special phase functions constitute a $(\mathrm{map}(\boldsymbol{E},\, \mathbb{R}))$–linear subsheaf $\mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) \subset \mathrm{map}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R})$.

Thus, we have the linear maps $X : \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) \to \mathrm{sec}(\boldsymbol{E},\, T\boldsymbol{E}) : f \mapsto X[f]$ and $\breve{} : \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) \to \mathrm{map}(\boldsymbol{E},\, \mathbb{R}) : f \mapsto \breve{f}$.

3.1 Proposition. We have the mutually inverse $(\mathrm{map}(\boldsymbol{E}, \mathbb{R}))$–linear isomorphisms

$$\begin{aligned} &\mathfrak{s} : \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) \to \mathrm{sec}(\boldsymbol{E},\, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E},\, \mathbb{R}) : f \mapsto \big(X[f],\, \breve{f}\big) \\ &\mathfrak{r} : \mathrm{sec}(\boldsymbol{E},\, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E},\, \mathbb{R}) \to \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) : (X, \breve{f}) \mapsto -X \,\lrcorner\, \Theta + \breve{f}\,, \end{aligned}$$

with $\mathfrak{s} : -c_0\, \alpha^0\, \breve{G}^0_{0\lambda}\, f^\lambda + \breve{f} \mapsto (f^\lambda\, \partial_\lambda\,,\ \breve{f})$ and $\mathfrak{r} : (X^\lambda\, \partial_\lambda,\, \breve{f}) \mapsto -c_0\, \alpha^0\, \breve{G}^0_{0\lambda}\, X^\lambda + \breve{f}$.

Hence, we have the linear splitting $\mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) = \mathrm{spec}''(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) \oplus \mathrm{map}(\boldsymbol{E},\, \mathbb{R})$, where $\mathrm{spec}''(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) := \ker(\breve{})$ and $\mathrm{map}(\boldsymbol{E},\, \mathbb{R}) = \ker(X)$.

Moreover, with reference to an observer o, we have the mutually inverse $(\mathrm{map}(\boldsymbol{E}, \mathbb{R}))$–linear isomorphisms

$$\mathfrak{s}[o] : \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) \to \sec(\boldsymbol{E},\, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E},\, \mathbb{R}) : f \mapsto \big(X[f],\, f[o]\big)$$
$$\mathfrak{r}[o] : \sec(\boldsymbol{E},\, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E},\, \mathbb{R}) \to \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) : (X, \bar{f}) \mapsto -X \lrcorner \Theta + \bar{f} + X \lrcorner \Theta[o]\,.\square$$

We can characterise the special phase functions via the Hamiltonian lift, as follows.

3.2 Proposition. Let $\sigma \in \mathrm{map}(\mathcal{J}_1\boldsymbol{E},\, \bar{\mathbb{T}})$ and $f \in \mathrm{map}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R})$. Then, the following conditions are equivalent:
1) $X^{\uparrow}{}_{\mathrm{ham}}[\sigma, f] \in \sec(\boldsymbol{E},\, \mathcal{J}_1 T\boldsymbol{E})$ is projectable on a vector field $X \in \sec(\boldsymbol{E},\, T\boldsymbol{E})$,
2) $f \in \mathrm{spec}(\mathcal{J}_1\boldsymbol{E}, \mathbb{R})$ and $\sigma = \sigma[f]$.
Moreover, if the above conditions are fulfilled, then we obtain $X = X[f]$. $\square$

3.3 Example. For any spacetime chart (x^λ), the functions x^λ are special phase functions and we obtain $X[x^\lambda] = 0$.

Moreover, with reference to a potential $A^{\uparrow}$ and to an observing frame (o, ζ), we define the observed *Hamiltonian* and *momentum* as $\mathcal{H}[o, \zeta] := -(1/\varsigma)\,(д[o] \lrcorner A^{\uparrow})\,\zeta \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ and $\mathcal{P}[o] := \theta[o, \zeta]\, A^{\uparrow} \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$.

If the observing frame is integrable, then we have the coordinate expressions, in an adapted chart, $\mathcal{H}[o, \zeta] = (-c_0\, \alpha^0\, \breve{G}^0_{00} - \frac{q}{\hbar}\, A^{\mathfrak{e}}{}_0)\, d^0$ and $\mathcal{P}[o, \zeta] = (c_0\, \alpha^0\, \breve{G}^0_{0i} + \frac{q}{\hbar}\, A^{\mathfrak{e}}{}_i)\, d^i$.

In this case, $\mathcal{H}_0$ and $\mathcal{P}_i$ are special phase functions and we obtain $X\big[\mathcal{H}_0\big] = \partial_0$ and $X\big[\mathcal{P}_i\big] = -\partial_i$. $\square$

3.1.12 The special bracket

We define the *special bracket* of $\mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R})$ by $[\![f, g]\!] := \{f, g\} + (\sigma[f])\,(\gamma.g) - (\sigma[g])\,(\gamma.f)$.

3.4 Theorem. *The sheaf* $\mathrm{spec}(J_1\boldsymbol{E},\, \mathbb{R})$ *is closed with respect to the special bracket. For each* $f_1,\, f_2 \in \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R})$, *we have*

$$[\![f_1,\, f_2]\!] = -д \lrcorner G^{\flat}\Big[X[f_1],\, X[f_2]\Big] + X[f_1].\breve{f}_2 - X[f_2].\breve{f}_1 + \tfrac{q}{\hbar}\, F\Big(X[f_1], X[f_2]\Big)\,,$$

i.e., $[\![f_1,\, f_2]\!] = -c_0\, \alpha_0\, \breve{G}^0_{0\mu}\, (f_1^\nu\, \partial_\nu f_2^\mu - f_2^\nu\, \partial_\nu f_1^\mu) + f_1^\lambda\, \partial_\lambda \breve{f}_2 - f_2^\lambda\, \partial_\lambda \breve{f}_1 + \frac{q}{\hbar}\, f_1^\lambda\, f_2^\mu\, F_{\lambda\mu}$.

Thus, $X\Big[[\![f_1, f_2]\!]\Big] = \Big[X[f_1], X[f_2]\Big]$ *and* $\breve{[\![f_1, f_2]\!]} = \Big[(X[f_1], \breve{f}_1)\,, (X[f_2], \breve{f}_2)\Big]_{\frac{q}{\hbar} F}$.

Indeed, the special bracket makes $\mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R})$ *a sheaf of* $\mathbb{R}$*–Lie algebras and the tangent prolongation is an* $\mathbb{R}$*–Lie algebra morphism.* $\square$

3.5 Corollary. The map $\mathfrak{s} : \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathbb{R}) \to \sec(\boldsymbol{E},\, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E},\, \mathbb{R})$ turns out to be an isomorphism of Lie algebras, with respect to the brackets $[\![\, ,\,]\!]$ and $[\, ,]_{\frac{q}{\hbar} F}$. $\square$

For instance, we have $[\![x^\lambda, x^\mu]\!] = 0$ and, with reference to an integrable observing frame and to an adapted chart, we have $[\![x^\lambda, \mathcal{H}_0]\!] = \delta^\lambda_0$, $[\![x^\lambda, \mathcal{P}_i]\!] = \delta^\lambda_i$, $[\![\mathcal{H}_0, \mathcal{P}_i]\!] = 0$.

3.2 Quantum setting

Let us consider a *quantum bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ over the Einstein spacetime.

We define the *phase quantum bundle* as $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := \mathcal{J}_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathcal{J}_1\boldsymbol{E}$.

We can refrase the notion of Hermitian systems of connections and associated universal connection that we have discussed in the Galilei case, by replacing $J_1\boldsymbol{E}$ with $\mathcal{J}_1\boldsymbol{E}$.

Let us assume that the cohomology class of $\frac{q}{\hbar} F$ be integer.

Then, we assume a connection $\Psi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^*\mathcal{J}_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$, called *phase quantum connection*, which is Hermitian, universal and whose curvature is given by the equality $R[\Psi^{\uparrow}] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^{\uparrow}$. The existence of such a universal connection and the fact that Ω admits horizontal potentials are strictly related. Moreover, the closure of Ω is an integrability condition for the above equation.

We have the splitting $\Psi^{\uparrow} = \Psi^{\uparrow\mathfrak{e}} + \mathfrak{i}\,\Theta \otimes \mathbb{I}^{\uparrow}$, where $\Psi^{\uparrow\mathfrak{e}} : \boldsymbol{Q}^{\uparrow} \to T^*\mathcal{J}_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$, is the pull back of a Hermitian connection $\Psi^{\mathfrak{e}} : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$, called *electromagnetic quantum connection*, whose curvature is given by the equality $R[\Psi^{\mathfrak{e}}] = -\mathfrak{i}\,\frac{q}{\hbar}\,F \otimes \mathbb{I}$.

With reference to a quantum basis b, the expression of $\Psi^{\uparrow}$ is of the type $\Psi^{\uparrow} = \chi^{\uparrow}[\mathsf{b}] + \mathfrak{i}\,\big(\Theta + \frac{q}{\hbar}\,A^{\mathfrak{e}}[\mathsf{b}]\big) \otimes \mathbb{I}^{\uparrow}$, where $A^{\mathfrak{e}}[\mathsf{b}]$ is a potential of F selected by $\Psi^{\uparrow}$ and b. Hence, in a chart adapted to b, is $\Psi^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d^i_0 \otimes \partial^0_i + \mathfrak{i}\,(c_0\,\alpha^0\,\breve{G}^0_{0\lambda} + \frac{q}{\hbar}\,A^{\mathfrak{e}}{}_{\lambda})\,d^{\lambda} \otimes \mathbb{I}^{\uparrow}$.

For each observer o, the expression of $\Psi[o]$, is $\Psi[o] = \mathfrak{i}\,\Theta[o] \otimes \mathbb{I} + \Psi^{\mathfrak{e}}$. Hence, in a chart adapted to b, $\Psi[o] = d^{\lambda} \otimes \partial_{\lambda} + \mathfrak{i}\,(\Theta[o]_{\lambda} + \frac{q}{\hbar}\,A^{\mathfrak{e}}{}_{\lambda})\,d^{\lambda} \otimes \mathbb{I}$.

3.3 Classification of Hermitian vector fields

Eventually, we apply to the Einstein framework the classification of Hermitian vector fields achieved in Theorem 1.7. For this purpose, we choose the electromagnetic quantum connection $\Psi^{\mathfrak{e}}$ as auxiliary connection c, use the classification of special phase functions achieved in Proposition 3.1 and show an identity.

3.6 Theorem. *We have the mutually inverse Lie algebra isomorphisms*

$$\mathfrak{F} := \mathfrak{j}[\Psi^{\mathfrak{e}}] \circ \mathfrak{s} : \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, I\!R) \to \mathrm{her}\,(\boldsymbol{Q},\, T\boldsymbol{Q})\,,$$
$$\mathfrak{H} := \mathfrak{r} \circ \mathfrak{h}[\Psi^{\mathfrak{e}}] : \mathrm{her}\,(\boldsymbol{Q},\, T\boldsymbol{Q}) \to \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, I\!R)\,,$$

given by $\mathfrak{F}(f) = \Psi^{\mathfrak{e}}\big(X[f]\big) + \mathfrak{i}\,\breve{f}\,\mathbb{I}$ *and* $\mathfrak{H}(Y) = -T\pi(Y) \lrcorner \Theta - \mathfrak{i}\ \mathrm{tr}\,\big(\nu[\Psi^{\mathfrak{e}}](Y)\big)$, *with respect to the Lie bracket of vector fields and the special bracket* $[\![\,,]\!]$.

We have the coordinate expressions

$$\mathfrak{F}(f) = f^{\lambda}\partial_{\lambda} + \mathfrak{i}\,(\tfrac{q}{\hbar}\,f^{\lambda}\,A^{\mathfrak{e}}{}_{\lambda} + \breve{f})\,\mathbb{I}\,,$$
$$\mathfrak{H}\Big(X^{\lambda}\,(\partial_{\lambda} + \mathfrak{i}\,\tfrac{q}{\hbar}\,A^{\mathfrak{e}}{}_{\lambda}\,\mathbb{I})\Big) + \mathfrak{i}\,\breve{Y}\,\mathbb{I} = -c_0\,\alpha^0\,\breve{G}^0_{\lambda 0}\,X^{\lambda} + \breve{Y} + \tfrac{q}{\hbar}\,A^{\mathfrak{e}}{}_{\lambda}\,X^{\lambda}\,.\square$$

3.7 Note. If $f = -X \lrcorner \Theta + \breve{f} \in \mathrm{spec}(\mathcal{J}_1\boldsymbol{E},\, \mathrm{I\!R})$ then we obtain

$$\Big(\mathfrak{j}\Big[\Psi[o]\Big] \circ \mathfrak{s}[o]\Big)(f) := \Psi[o](X) + \mathfrak{i}\,f[o]\,\mathbb{I} = \Psi^{\mathfrak{e}}(X) + \mathfrak{i}\,\breve{f}\,\mathbb{I} := \Big(\mathfrak{j}\Big[\Psi^{\mathfrak{e}}\Big] \circ \mathfrak{s}\Big)(f)\,.$$

Hence, the Hermitian vector field associated with f by the connection $Ч[o]$ does not depend on the observer o. $\square$

For instance, we have $\mathfrak{F}(x^\lambda) = \mathfrak{i}\, x^\lambda\, \mathbb{I}$ and, with reference to an integrable observing frame and to an adapted chart, $\mathfrak{F}(\mathcal{H}_0) = \partial_0$ and $\mathfrak{F}(\mathcal{P}_i) = -\partial_i$.

4 Galilei and Einstein cases: a comparison

We conclude the paper by discussing the main analogies and differences between the Galilei and the Einstein cases.

Spacetime. The essential source of all differences between the two cases is the structure of spacetime. In both cases spacetime is a 4–dimensional manifold. In the Galilei case, we have a fibring over absolute time and a spacelike (hence degenerate) Riemannian metric. In the Einstein case, we loose the time fibring, but we gain a spacetime (hence non degenerate) Lorentz metric.

Nevertheless, in both cases, the time intervals are valued in the absolute vector space $\mathbb{T}$. Indeed, this fact has no relation with simultaneity.

In the Galilei case, we have used the light velocity c just for the sake of standard normalisation of some formulas. But, the constant c has no relation with any phenomena which can be described in the framework of the Galilei theory.

Phase space. In the Galilei theory, the motions are defined as sections of the fibred manifold; in the Einstein theory, they are defined as timelike 1–dimensional submanifolds. This fact implies an important difference with respect to the phase space. In the Galilei case, it is defined as the space of 1st jets of sections; in the Einstein case it is defined as the space of 1st jets of 1–dimensional timelike submanifolds. Thus, the phase space is an affine bundle over spacetime in the Galilei case and a projective space in the Einstein case. This difference yields several technical consequences throughout the theory.

In the Galilei case, the time fibring yields the time form on spacetime, the lift of time scales to timelike spacetime forms and the contact structure of the phase space. In the Einstein case, these objects cannot be achieved through the fibring but are recovered by means of the Lorentz metric. However, in this case, the time form is based on the phase space; indeed, this is a main feature of this case. Moreover, the coordinate expressions of these objects are more complicated in the Einstein case, due to the projective structure of the phase space, instead of an affine structure.

In particular, in the Galilei case, the vertical subspace of the phase space can be easily compared with the vertical subspace of spacetime. Such a comparison requires a more complicate description in the Einstein case.

Contact splitting. Passing from the Galilei to the Einstein case, the horizontal and vertical subspaces of spacetime with respect to the time fibring are replaced by the parallel and orthogonal subspaces with respect to the metric. However, they are based on the phase space.

Observers. The observers are defined in an analogous conceptual way in the two cases. However, relevant technical differences arise due to the different structures of the phase spaces.

In the Galilei case, an observer and the time fibring - i.e. the observer independent time form (which is obsviously integrable) - yield a splitting of the tangent space of spacetime.

In the Einstein case, there are two ways in order to achieve an analogous splitting. Namely, we consider an observer and additionally either the associated observed time form (which is not integrable, in general), or an independent time form (which may be integrable, defining locally a time function). The first pair is sufficient for several purposes; however, the components of the Hamiltonian and of the momentum turn out to be special phase functions only if they are defined through an integrable observing frame.

Gravitational and electromagnetic fields. In the Einstein case, we can formulate the standard theory of the electromagnetic field, with the standard Maxwell equations $dF = 0$ and $\delta F = j$. In the Galilei case, the 1st Maxwell equation can be formulated without any change, because it involves only the differential structure of spacetime. Conversely, the 2nd Maxwell equation, which links the electromagnetic field with its charge sources, cannot be written in a full formulation, due to the degeneracy of the metric; only a static effect of the charges on the electromagnetic field can be described covariantly. On the other hand, in the present theory, we are involved just with a given electromagnetic field; hence, the dependence on its sources does not play an essential role in the present theory. In the Galilei case, the magnetic field is observer independent; this is not true in the Einstein case. Nevertheless, the observed electric and magnetic fields can be defined in a similar conceptual way in the two cases. But differences arise from the different behaviour of observers in the two cases.

Induced objects on the phase space. In both cases, a connection of the phase space yields naturally a 2nd order connection, a 2–form and a 2–vector of the phase space, which fulfill certain identities.

In the Einstein case, the metric determines the gravitational spacetime connection. In the Galilei case, the metric determines the gravitational connection up to a closed 2–form; so, the gravitational connection needs an additional postulate.

In the Galilei case, we have a natural bijection between connections of spacetime and connections of the phase space. Moreover, there is a natural way to merge the electromagnetic field into the gravitational connection, so obtaining a joined connection. Hence, this connection yields naturally a joined 2nd order connection, a joined 2–form and a joined 2–vector of the phase space, which fulfill the same identities of the gravitational objects.

In the Einstein case, we have only a natural injection between connections of spacetime and connections of the phase space. Moreover, there is no natural way to merge the electromagnetic field into the gravitational connection. Hence, we proceed in a partially different way. We define a joined phase connection, by analogy with the Galilei case. Then, we obtain the joined 2nd order connection, 2–form and 2–vector of the phase space. Indeed, the joined phase connection is not essential by itself in our theory. What is essential is that all other joined objects be generated by the same phase connection and that they fulfill certain identities.

In the Einstein case, the gravitational 2–form is globally exact and its potential is the time form. In the Galilei case, the gravitational 2–form is only closed, but admits horizontal potentials.

Thus, in the Einstein case, the time form τ plays the roles analogous both to dt and to Θ (up to a scale factor), in the Galilei case.

Hamiltonian lift of phase functions. In both cases, we have a similar formulation of the Hamiltonian lift of phase functions and of the Poisson bracket. These aspects of the theory have strict analogies with the standard literature, but are not exactly standard because of our choice of the phase space.

Lie algebra of special phase functions. In the two cases, we have several analogies in the definition of special phase functions. However, the expression of these functions is very different in the two cases, due to the different structure of the phase space. In the Galilei case, we need an observer in order to split a special function. In the Einstein case, we have a natural splitting of special functions.

The definition of the special bracket is formally identical in the two cases. However, in the Galilei case, the special bracket involves the metric and the joined 2–form, while in the Einstein case, it involves only the metric and the electromagnetic field.

Phase quantum connections. The definition of the phase quantum connection is formally identical in the two cases. However, in the Einstein case, it can be split into a natural gravitational component and an electromagnetic component, due to the exactness of the 2–form. This fact is not true in the Galilei case.

Hence, in the Einstein case we obtain an observer independent purely electromagnetic quantum connection. Conversely, in the Galilei case, we obtain a system of observed joined quantum connections, which are related by a transition law.

Classification of Hermitian vector fields. In the first part of the paper, we have shown that, given a connection of the quantum bundle, the Lie algebra of Hermitian vector fields can be represented by a Lie algebra of pairs consisting of spacetime vector fields and spacetime functions.

In the Galilei case, we implement the above result by choosing an observer and referring to the induced joined quantum connection and the induced splitting of special phase functions. Indeed, we prove that the transition laws for the above objects are such that the final result is observer independent.

In the Einstein case, we do not need to choose an observer, because the splitting of the phase functions is observer independent and we can avail of the electromagnetic quantum connection.

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