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Connections over connections and a universal calculus

C.T.J. DODSON (*), M. MODUGNO (**)

(*) Department of Mathematics, Lancaster, U.K.

(**) Istituto di Matematica Applicata «G. Sansone», Firenze, Italia

Speaker: M. Modugno

Introduction

General relativity had its origins in riemannian geometry and the more recent gauge theories and unified field theories utilized the Cartan formalism on principal bundles. Both exploited geometrical versions of lagrangian and hamiltonian calculus.

However, differential geometry is larger and has more to offer in field theories. We have in mind in particular the fibred manifolds, jet techniques and general connections of C. Ehresmann [1] (cf. also P. Libermann [2]), also the graded Lie algebra of vector valued forms of A. Froelicher and A. Nijenhuis [3] and the more recent work on universal principal connections of P.L. Garcia [4, 5]. Out of these developments a new differential calculus for connections on fibred manifolds was devised [7, 8, 9, 10, 11] and the notion of a universal connection was extended to this context [11]. A brief report on this was given at the last National Conference [12]. A significant application of the universal connection has established [13] that general relativistic singularities are stable under variations of the connection (or metric). A preliminary approach to the principal bundle geometry of gauge theories is based on the new calculus of adjoint forms and consequent interpretation of the Utiyama conditions of L. Mangiarotti [14] (cf. also P.L. Garcia-A. Perez-Rendon [6]).

The most recent developments have yielded a deeper understanding of regularity properties for systems of vector valued forms and the existence of an appropriate bracket. In application to a system of connections these allow the construction of a universal differential calculus and prolongation theorems yielding connections over connections [15, 16]. Thus, several directions of generalization and unification are achieved, for Cartan calculus on principal bundles and lagrangian formalism and hence for riemannian geometry in general.

The common thread of all these topics and approaches is the fundamental notion of connections. It ha led to natural and powerful new geometrical ideas. Inevitably they will find further applications, in particular in general relativity.

1. Preliminaries

We recall some material which is essential for the sequel; further details can be found in [11]. We work in smooth categories.

Let $p : E \to B$ denote a fibred manifold so p is a surjective submersion.

Then a *connection* on E is a section

$$\gamma : E \to J^1 E$$

of its first jet bundle. By viewing $J^1 E$ as an affine subbundle of $T^*B \otimes TE$, over E we can equivalently interpret γ as a vector-valued 1-form

$$\gamma : E \to T^*B \otimes TE,$$

which is projectable on $1 \in T^*B \otimes TB$. We shall denote by G the sheaf of local connections.

Then we are led to consider the space of projectable vector-valued forms, like

$$\phi : E \to \Lambda^r T^*B \otimes TE$$

and the Froelicher-Nijenhuis (briefly F.N.) graded Lie bracket given by [8, 11]

$$[\ ,\ ] : P^r \times P^s \to P^{r+s}$$

were P^r serves to indicate the sheaf of projectable local sections of $\Lambda^r T^*B \otimes TE$ over E. In terms of γ we have the graded Lie algebra derivation (called *covariant derivative*)

$$d_\gamma \equiv (1/2)\,[\gamma,] : P^r \to V^{r+1}$$

where $V^{r+1} \subset P^r$ is the subsheaf of vertical forms. Then the *curvature* ρ of γ is the vertical 2-form defined by

$$\rho \equiv d_\gamma \gamma : E \to \Lambda^2 T^*B \otimes VE.$$

The properties of the bracket give identities:

$$d_\gamma{}^2 \phi = (1/4)\,[\rho, \phi] \qquad d_\gamma \rho = 0 \qquad [\rho, \rho] = 0.$$

Thus, we observe an essential generalization of tensor geometry that is valid for arbitrary fibred manifolds without linearity requirements. This can be seen, for instance, from the following coordinate expressions

$$\gamma = d^\lambda \otimes \partial_\lambda + \gamma_\lambda{}^i d^\lambda \otimes \partial_i$$

$$\rho = (1/2)\,(\partial_{[\lambda}\gamma_{\mu]}{}^i + \gamma_{[\lambda}{}^j \partial_j\, \gamma_{\mu]}{}^i)\, d^\lambda \wedge d^\mu \otimes \partial_i$$

where the connection components $\gamma_\lambda{}^i$ correspond to the classical Christoffel symbols, with Greek indices running over the base coordinates on B and the Latin indices running over the coordinates of the fibres of E.

2. Systems of vector valued forms

For our present purposes we wish to consider a certain finite-dimensional subspace of the space of all projectable vector-valued forms, in particular of the space of all connections. Our choice of subspace will reduce in the linear or principal bundle context to the corresponding bundles of linear or principal forms, in particular connections. The easiest way forward is the following (further details can be found in [15]).

DEFINITION. A *regular system of projectable vector fields* of E consists of a vector bundle

$$q : H \to B$$

and a linear fibred morphism

$$\eta : H \times_B E \to TE$$

over E which is projectable on the linear epimorphism $\underline{\eta} : H \to TB$, such that

(i) for each $x \in B$, $\eta_x : H_x \to P^0_x : h_x \to \mathbf{h}_x \equiv \eta \circ (h_x{}^{\sim})$ is injective,

(ii) there exists a fibred atlas of $H \times_B E$ (called canonical) which provides the following expression for the induced sheaf morphism

$$\boldsymbol{\eta} : H \to P^0 : h \equiv h^\lambda \epsilon_\lambda h^a e_a \mapsto \mathbf{h} \equiv \eta \circ h^{\sim} = h^\lambda \partial_\lambda + \eta^i{}_a h^a \partial_i$$

(here H denotes the sheaf of local sections of H, $h^{\sim} : E \mapsto H \times_B E$ is the natural extension of $h : B \to H$ and (ϵ_λ, e_a) is the base on H induced by the coordinates) and with the condition

$$\partial_\lambda \eta^i{}_a = 0$$

(iii) the system is closed under the Lie bracket, i.e. involutive. ■

In summary our system is to be projectable, linear, monic, involutive and with a canonical atlas. Such systems actually arise in many pratical situations.

Note also that there will be always be the distinguished *vertical subsystem* given by $A \equiv \ker \underline{\eta} \subset H$.

PROPOSITION 1. Such a system admits a Lie bracket

$$[\,,\,] : H \times H \to H : (h^\lambda \epsilon_\lambda + h^a e_a, k^\mu \epsilon_\mu + k^b e_b) \mapsto$$
$$\mapsto (h^\mu \partial_\mu k^\lambda - k^\mu \partial_\mu h^\lambda)\, \epsilon_\lambda + (h^\mu \partial_\mu k^a - k^\mu \partial_\mu h^a + c^a_{bc} h^b k^c)\, e_a$$

where the structure constants are defined by the integrability conditions

$$c^a{}_{bc} \eta^i{}_a = \eta^i{}_{[b} \partial_j \eta^i{}_{c]} .$$ ∎

COROLLARY. The bracket restricts to a bilinear form

$$c : A \times_B A \to A : (h^a e_a, k^b e_b) \mapsto c^a{}_{bc} h^b k^c e_a$$

which endows the vector bundle A with a Lie algebra structure. ∎

It is natural to extend the idea of a *system* to the case *of projectable vector valued forms* and indeed just such a system (F, θ) is induced by (H, η) as follows.

Put

$$F^r \equiv \Lambda^r T^*B \otimes H \qquad \text{and} \qquad F \equiv \oplus_r F^r ;$$

then

$$\theta^r : F^r \times_B E \to \Lambda^r T^*B \otimes TE \quad \text{and} \quad \theta = \oplus_r \theta^r$$

are well defined by tensorializing η, so we obtain a system (F, θ) with $(F^\circ, \theta^\circ) = (H, \eta)$ as a natural subsystem.

As before there is an induced sheaf morphism

$$\boldsymbol{\theta}^r : \mathcal{F}^r \to \mathcal{P}^r : \phi \mapsto \boldsymbol{\phi} \equiv \theta^r \circ \phi^\sim$$

where $\mathcal{F}^r$ is the sheaf of local sections of F^r.

PROPOSITION 2. The F.N. bracket and the involutivity of H induce a graded Lie bracket on $\mathcal{F}$

$$[\,,\,] : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$$

and it is a canonical extension of the one on $\mathcal{H}$. ∎

3. Systems of connections

A connection is a particular kind of vector-valued 1-form, so it is natural for us to consider *systems* entirely composed *of connections.*

Indeed, such a system (C, α) is induced by (F, θ) and hence by (H, η) as follows.

Denote by

$$\chi : C \to F^1$$

the affine subbundle over B of 1-forms which project onto $1 \in T^*B \otimes TB$. Define

$$\alpha : C \times_B E \to T^*B \otimes TE$$

as the restriction of θ^1.

Observe that the associated vector bundle of C is $T^*B \otimes A$.

Again there is an induced sheaf morphism

$$\boldsymbol{\alpha} : C \to G : \gamma \mapsto \boldsymbol{\gamma} \equiv \alpha \circ \gamma^{\sim},$$

where C is the sheaf of local sections of C.

We are now led to consider the sheaves $F^{(k,r)}$ of local *vector-valued r-forms parametrized by the k-jets of the connections* in our system, namely the local fibred morphisms over B like

$$\phi : J^k C \to F^r.$$

Of course we have a family of sheaf inclusions

$$F^r \subset F^{(0,r)} \subset F^{(1,r)} \subset \ldots \subset F^{(k,r)} \subset \ldots$$

Now, any local

$$\phi : J^k C \to F^r$$

determines, by composition with $\boldsymbol{\theta}^r$, a local

$$\boldsymbol{\phi} : J^k C \times_B E \to \Lambda^r T^*B \otimes TE,$$

hence a sheaf inclusion $F^{(k,r)} \to P^{(k,r)}$, where $P^{(k,r)}$ is the sheaf of local fibred morphisms

$$J^k C \times_B E \to \Lambda^r T^*B \otimes TE$$

over E.

In particular

$$\chi \in F^{(0,1)} \quad \text{and} \quad \boldsymbol{\chi} = \alpha \in P^{(0,1)}.$$

Our bracket on F depend on the first jet prolongation of sections and this yields the following.

PROPOSITION 3. There is a canonical extension of the graded Lie bracket on F to a bracket

$$[,] : F^{(k,r)} \times F^{(k,s)} \to F^{(k+1,r+s)}$$

characterized by

$$[\phi, \psi]_0 (j^{k+1}\gamma)^{\sim} \equiv [\phi \circ (j^k\gamma)^{\sim}, \psi \circ (j^k\gamma)^{\sim}]$$

for all $\gamma \in G$. ∎

As before we can lift by means of $\boldsymbol{\theta}$ to obtain a bracket

$$\boldsymbol{[}\,,\boldsymbol{]} : P^{(k,r)} \times P^{(k,s)} \to P^{(k+1, r+s)}.$$

COROLLARY. There is a derivation

$$d = (1/2)\,[\chi,] : F^{(k,r)} \to F^{(k+1, r+1)} : \phi \mapsto (1/2)\,[\chi, \phi].$$

Moreover the form χ distinguishes another form

$$\omega \equiv d\chi : J^1C \to \Lambda^2 T^*B \otimes A$$

with the properties

$$d\omega = 0 \qquad [\omega, \omega] = 0 \qquad d^2\phi = (1/2)\,[\omega, \phi].$$ ∎

Once more we can lift by means of $\boldsymbol{\theta}$ to obtain

$$\boldsymbol{d} \equiv (1/2)\,\boldsymbol{[}\alpha, \boldsymbol{]} : P^{(k,r)} \to P^{(k+1, r+1)} \quad \text{and} \qquad \boldsymbol{\omega} \equiv d\alpha$$

satisfying analogous identities.

For computations we have the following coordinate expressions:

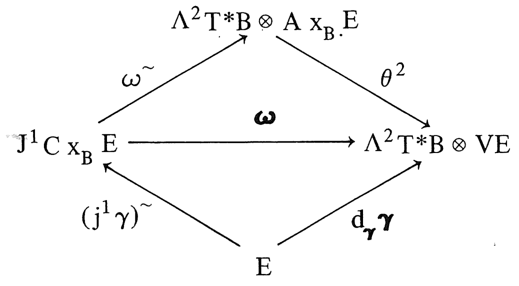
$$\boldsymbol{d}\phi = (1/2)(1/r+1!)\,(J_{[\lambda 1}\phi^{a}{}_{\lambda 2 \ldots \lambda r+1]} - z_{\mu}{}^{a}{}_{[\lambda 1}\phi^{\mu}{}_{\lambda 2 \ldots \lambda r+1]} -$$

$$- \partial_{[\lambda 1}\phi^{\mu}{}_{\lambda 2 \ldots \lambda r+1]}\, z^{a}{}_{\mu} + c^{a}{}_{bc} z^{b}{}_{[\lambda 1}\phi^{c}{}_{\lambda 2 \ldots \lambda r+1]})\, d^{\lambda 1} \wedge \ldots \wedge d^{\lambda r+1} \otimes e_a$$

$$\omega = (1/2)\,(z_{[\lambda}{}^{a}{}_{\mu]} + c^{a}{}_{bc} z^{b}{}_{\lambda} z^{c}{}_{\mu})\, d^{\lambda} \wedge d^{\mu} \otimes e_a$$

where $(z^a{}_\mu)$ and $(z_\lambda{}^a{}_\mu)$ denote the canonical fibre coordinates of C and J^1C and J_λ denotes the λ-jet-derivative.

By way of illustrating the many interrelating identities among these objects we offer one commutative diagram for the curvature:



We can summarise the position as follows:

(i) we used a system of projectable vector-valued forms to select a system of connections,

(ii) we identified the important F^k of local vector-valued forms parametrized by the k-jets of connections,

(iii) there appears a canonical calculus induced by the universal bracket and differential; in particular, in the presence of a metric on B, we obtain a universal codifferential and laplacian.

Of course we recover the usual results and more in application to principal bundles and their associated bundles, indeed we gain direct access to the latter.

By means of the universal differential and codifferential operators we can write gauge equations for any system directly and canonically.

On the other hand, in studying lagrangians which depend on the jets of the connection, the gauge-invariant lagrangians are just those whose Euler-Lagrange equations admit a representation through the universal differential and codifferential.

In the particular case that the system comes from a principal bundle then we obtain a new view of the standard theory.

4. Connections over connections

Suppose that we have the system (C, α) arising from a particular choice (H, η). Since C is an affine bundle we have automatically the system of all affine connections over C. However, there is a distinguished subsystem (K, β) of «*over-connections*» which arises in the following way.

PROPOSITION 4. There is a canonical fibred morphism

$$\zeta : J^1 H \times_B C \to TC,$$

induced by the bracket of H. It gives rise to a (non-involutive) system (J^1H, ζ) of projectable vector fields over C. The system is involutive with respect to integrable sections of J^1H and we have there

$$[j^1 h, j^1 k] = j^1 [h, k].$$

Then we obtain a Lie algebra morphism

$$H \to P^0{}_C : h^\lambda \epsilon_\lambda + h^a e_a \mapsto h^\lambda \partial_\lambda + (\partial_\lambda h^a - \partial_\lambda h^\mu z^a{}_\mu + c^a{}_{bc} z^b{}_\lambda h^c)\, \partial^\lambda{}_a . \qquad \blacksquare$$

This morphism generalizes a result of P.L. Garcia [5] for principal bundles, by utilizing the bracket and jet techniques. This way has enabled A. Perez-Rendon and A. Lopez-Almorox to achieve an extension to graded manifolds.

Following the procedure as before, we replace E by C and (H, η) by (J^1H, ζ) to obtain the system of projectable vector valued forms ($\Lambda^r T^*B \otimes J^1 H$, τ) of C.

PROPOSITION 5. The induced system (K, β) of connections of C is a subsystem of the system of affine connections. Moreover, K is an affine bundle over C by virtue of the projection $J^1H \to H$, hence any section $\Gamma : B \to K$ is projected onto a section $\gamma : B \to C$. ∎

Although we have only partial involutivity, our constructions are nevertheless sufficient to pose novel problems in riemannian geometry and its applications. For example torsion free linear connections of a manifold can be interpreted in terms of systems and their overconnections give information that is inaccessible by the usual tensor geometry.

In conclusion we suggest that two important lines of investigation for general relativity arising from our approach are:

(i) a study of operators on the space of connections to allow the spacetime connections to be handled as a free field,

(ii) a study of the universal calculus on the space of Lorentz metrics.

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