Graded Lie algebra of Hermitian tangent valued forms

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Abstract

We define the Hermitian tangent valued forms of a complex 1–dimensional line bundle equipped with a Hermitian metric. We provide a local characterisation of these forms in terms of a local basis and of a local fibred chart. We show that these forms constitute a graded Lie algebra through the Frölicher–Nijenhuis bracket.

Moreover, we provide a global characterisation of this graded Lie algebra, via a given Hermitian connection, in terms of the tangent valued forms and forms of the base space. The bracket involves the curvature of the given Hermitian connection.

Resumé

On definit les formes hermitiennes à valeurs tangentes d'un fibré de ligne complexe doté d'une métrique hermitienne. On donne une caractérisation locale de ces formes à travers une base locale et une carte locale fibrée. On montre que ces formes constituent une algèbre de Lie graduée par rapport au crochet de Frölicher-Nijenhuis. En autre, on donne une caractérisation globale de cette algèbre de Lie graduée, au moyen d'une connexion hermitienne donnée, en termes des formes à valeurs tangentes et des formes de l'espace de base. Le crochet implique la courbure de la connexion hermitienne donnée.[1]

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Contents

Introduction

In the theory of so called “Covariant Quantum Mechanics” (see, for instance, [1, 3, 5]) a basic role is played by Hermitian vector fields on a complex line bundle in the frameworks of Galilei and Einstein spacetimes. In fact, it has been proved that the Lie algebra of Hermitian vector fields is naturally isomorphic to a Lie algebra of “special functions” of the phase space. Indeed, this is the source of the covariant quantisation of the above special functions. In the original version of the theory, this result was formulated and proved in a rather involved way; now, we have achieved a more direct and simple approach to the classification of Hermitian vector fields and to their representation via special phase functions.

In view of a possible covariant quantisation of a larger class of “observables” [4], it is natural to consider the Hermitian tangent valued forms. This paper is devoted to a purely geometric self–contained analysis of the graded Lie algebra of Hermitian tangent valued forms of a complex line bundle and to their classification.

We observe that tangent valued forms constitute the framework for a covariant differential calculus on fibred manifolds equipped with a non linear connection, which naturally generalises both the classical covariant differential calculus (for sections or for vector valued forms) on a vector bundle equipped with a linear connection and the classical covariant differential calculus on a pricipal bundle equipped with a principal connection [8, 7].

Here, we first classify locally the Hermitian tangent valued forms in coordinates. Then, for the global classification we need a Hermitian connection: indeed, this is just the connection required in gauge theories. By splitting each tangent valued form into its horizontal and vertical components, we show that the Hermitian tangent valued forms of the line bundle are in bijection with the pairs consisting of a tangent valued form and a form of the base space. Moreover, we introduce a natural graded Lie bracket for these pairs and show that the graded Lie algebras of Hermitian tangent valued forms and of the above pairs are naturally isomorphic.

For a quantisation procedure we need a further geometric structure on the base space, but this is beyond the scope of the present paper.

All manifolds and maps between manifolds are supposed to be smooth.

If $\boldsymbol{M}$ and $\boldsymbol{N}$ are manifolds, and $\boldsymbol{F} \to \boldsymbol{B}$ is a fibred manifold, then the sheaf of local smooth maps $\boldsymbol{M} \to \boldsymbol{N}$ is denoted by $\mathrm{map}(\boldsymbol{M}, \boldsymbol{N})$, the sheaf of local sections $\boldsymbol{B} \to \boldsymbol{F}$ is denoted by $\mathrm{sec}(\boldsymbol{B}, \boldsymbol{F})$ and the vertical restriction of forms will be denoted by the check symbol ${}^{\vee}$.

1 Hermitian line bundle

We start with some basic properties of a Hermitian line bundle.

Let us consider a manifold $\boldsymbol{E}$. The charts of $\boldsymbol{E}$ are denoted by (x^λ) and the associated local bases of vector fields and forms by ∂_λ and d^λ, respectively.

Then, we consider a *Hermitian line bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, i.e. a complex vector bundle with 1-dimensional fibres, equipped with a Hermitian product [2] $\mathfrak{h} : \boldsymbol{E} \to \mathbb{C} \otimes (\boldsymbol{Q}^* \otimes \boldsymbol{Q}^*)$.

The tensor product symbol $\otimes$ always indicates a real tensor product.

We shall refer to *quantum bases*, i.e. to (local) sections $\mathfrak{b} \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, such that $\mathfrak{h}(\mathfrak{b}, \mathfrak{b}) = 1$ and to the associated complex dual functions $z \in \mathrm{map}(\boldsymbol{Q}, \mathbb{C})$.

We shall also refer to the associated (local) real basis $(\mathfrak{b}_{\mathrm{a}}) \equiv (\mathfrak{b}_1, \mathfrak{b}_2) := (\mathfrak{b}, \mathfrak{i}\,\mathfrak{b})$ and to the associated scaled real dual basis $(w^{\mathrm{a}}) \equiv (w^1, w^2) = \big(\frac{1}{2}(z + \bar{z}), \frac{1}{2}\,\mathfrak{i}\,(\bar{z} - z)\big)$. We denote the associated vertical vector fields by $(\partial_{\mathrm{a}}) \equiv (\partial_1, \partial_2)$.

The small Latin indices $\mathrm{a}, \mathrm{b} = 1, 2$ will span the real indices of the fibres.

Thus, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we write

$$\Psi = \Psi^{\mathrm{a}}\,\mathfrak{b}_{\mathrm{a}} = \psi\,\mathfrak{b}\,, \qquad \text{with} \qquad \Psi^1, \Psi^2 \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})\,, \quad \psi = \Psi^1 + \mathfrak{i}\,\Psi^2 \in \mathrm{map}(\boldsymbol{E}, \mathbb{C})$$

and, for each $\Phi, \Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$,

$$\mathfrak{h}(\Phi, \Psi) = (\Phi^1\,\Psi^1 + \Phi^2\,\Psi^2) + \mathfrak{i}\,(\Phi^1\,\Psi^2 - \Phi^2\,\Psi^1) = \bar{\phi}\,\psi\,.$$

Each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ can be naturally regarded as the vertical vector field

$$\Psi \;\simeq\; \tilde{\Psi} \in \sec(\boldsymbol{Q}, V\boldsymbol{Q}) : q_e \mapsto \big(q_e, \Psi(e)\big)\,,$$

with coordinate expression

$$\Psi \;\simeq\; \tilde{\Psi} = \Psi^{\mathrm{a}}\,\partial_{\mathrm{a}}\,.$$

We can regard $\mathfrak{h}$ also as a complex vertical valued form $\mathfrak{h} : \boldsymbol{Q} \to \mathbb{C} \otimes V^*\boldsymbol{Q}$, with coordinate expression $\mathfrak{h} = (w^1\,\check{d}^1 + w^2\,\check{d}^2) + \mathfrak{i}\,(w^1\,\check{d}^2 - w^2\,\check{d}^1)$.

The *unity* and the *imaginary unity* tensors

$$1 = \mathrm{id}_{\boldsymbol{Q}} : \boldsymbol{E} \to \boldsymbol{Q}^* \otimes \boldsymbol{Q} \qquad \text{and} \qquad \mathfrak{i} = \mathfrak{i}\;\mathrm{id}_{\boldsymbol{Q}} : \boldsymbol{E} \to \boldsymbol{Q}^* \otimes \boldsymbol{Q}$$

will be naturally identified, respectively, with the *Liouville* and the *imaginary Liouville* vector fields

$$\mathbb{I} : \boldsymbol{Q} \to V\boldsymbol{Q} = \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q, q) \qquad \text{and} \qquad \mathfrak{i}\,\mathbb{I} : \boldsymbol{Q} \to V\boldsymbol{Q} = \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q, \mathfrak{i}\,q)\,.$$

We have the coordinate expressions

$$\begin{aligned} 1 = \mathrm{id}_{\boldsymbol{Q}} &= w^1\,\mathfrak{b}_1 + w^2\,\mathfrak{b}_2 = z \otimes \mathfrak{b}\,, & \mathbb{I} &= w^1\,\partial_1 + w^2\,\partial_2 = z \otimes \partial_1 \\ \mathfrak{i} = \mathfrak{i}\;\mathrm{id}_{\boldsymbol{Q}} &= w^1\,\mathfrak{b}_2 - w^2\,\mathfrak{b}_1 = \mathfrak{i}\,z \otimes \mathfrak{b}\,, & \mathfrak{i}\,\mathbb{I} &= w^1\,\partial_2 - w^2\,\partial_1 = \mathfrak{i}\,z \otimes \partial_1\,. \end{aligned}$$

2 Tangent valued forms

2.1 Tangent valued forms of a manifold

First of all, we summarise a few essential recalls on tangent valued forms of a manifold.

Let us consider a manifold $\boldsymbol{M}$ and denote a generic chart by (x^λ).

For each integer $0 \leq r$, let us consider the sheaf $\sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M} \otimes T\boldsymbol{M})$ of *tangent valued forms* of degree r. In particular, for $r = 0$, we have the sheaf $\sec(\boldsymbol{M}, T\boldsymbol{M})$ of vector fields.

Let us consider a $\Xi \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M} \otimes T\boldsymbol{M})$. Then, we obtain the derivations [7]

$$\begin{aligned} i(\Xi) &: \sec(\boldsymbol{M}, \Lambda^s T^*\boldsymbol{M}) \to \sec(\boldsymbol{M}, \Lambda^{r+s-1} T^*\boldsymbol{M}) \\ L(\Xi) &: \sec(\boldsymbol{M}, \Lambda^s T^*\boldsymbol{M}) \to \sec(\boldsymbol{M}, \Lambda^{r+s} T^*\boldsymbol{M}) \,, \end{aligned}$$

which are characterised, via decomposable tangent valued forms, by the equalities

$$\begin{aligned} i(\xi \otimes X)\, \alpha &= \xi \wedge i(X)\, \alpha \\ L(\xi \otimes X)\, \alpha &= \xi \wedge L(X)\, \alpha - (-1)^{r-1} d\xi \wedge i(X)\, \alpha \,. \end{aligned}$$

We have the natural (real) linear injective morphisms

$$\begin{aligned} \sec(\boldsymbol{E}, \Lambda^r T^*\boldsymbol{E}) &\to \sec(\boldsymbol{Q}, \Lambda^r T^*\boldsymbol{E} \otimes V\boldsymbol{Q}) : \xi \mapsto \xi \otimes \mathbb{I} \\ \sec(\boldsymbol{E}, \Lambda^r T^*\boldsymbol{E}) &\to \sec(\boldsymbol{Q}, \Lambda^r T^*\boldsymbol{E} \otimes V\boldsymbol{Q}) : \xi \mapsto \mathfrak{i}\, \xi \otimes \mathbb{I} \,, \end{aligned}$$

whose inverse are, respectively, the maps

$$\mathrm{tr}_{\mathbb{C}} : \xi \otimes \mathbb{I} \mapsto \xi \qquad \text{and} \qquad -\mathfrak{i}\, \mathrm{tr}_{\mathbb{C}} : \mathfrak{i}\, \xi \otimes \mathbb{I} \mapsto \xi \,.$$

The sheaf of tangent valued forms turns out to be a graded Lie algebra with respect to the Frölicher-Nijenhuis bracket (FN bracket) [7]

$$\sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M} \otimes T\boldsymbol{M}) \times \sec(\boldsymbol{M}, \Lambda^s T^*\boldsymbol{M} \otimes T\boldsymbol{M}) \to \sec(\boldsymbol{M}, \Lambda^{r+s} T^*\boldsymbol{M} \otimes T\boldsymbol{M}) \,,$$

which is characterised, via decomposable tangent valued forms, by the equality

$$\begin{aligned} [\xi \otimes X,\, \sigma \otimes Y] = \xi \wedge \sigma \otimes [X, Y] &+ \xi \wedge L(X)\, \sigma \otimes Y - (-1)^{rs} \sigma \wedge L(Y)\, \xi \otimes X \\ &+ (-1)^r d\xi \wedge i(X)\, \sigma \otimes Y - (-1)^{s+rs} d\sigma \wedge i(Y)\, \xi \otimes X \,. \end{aligned}$$

We have the coordinate expression

$$\begin{aligned} [\Xi,\, \Sigma] = (&\Xi^\rho_{\lambda_1 \dots \lambda_r}\, \partial_\rho \Sigma^\mu_{\lambda_{r+1} \dots \lambda_{r+s}} - (-1)^{rs}\, \Sigma^\rho_{\lambda_1 \dots \lambda_s}\, \partial_\rho \Xi^\mu_{\lambda_{s+1} \dots \lambda_{r+s}} \\ &- r\, \Xi^\mu_{\lambda_1 \dots \lambda_{r-1} \rho}\, \partial_{\lambda_r} \Sigma^\rho_{\lambda_{r+1} \dots \lambda_{r+s}} + (-1)^{rs}\, s\, \Sigma^\mu_{\lambda_1 \dots \lambda_{s-1} \rho}\, \partial_{\lambda_s} \Xi^\rho_{\lambda_{s+1} \dots \lambda_{r+s}})\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+s}} \otimes \partial_\mu \,. \end{aligned}$$

We have the identity

$$[L(\Xi),\, L(\Sigma)] := L(\Xi) \circ L(\Sigma) - (-1)^{rs} L(\Sigma) \circ L(\Xi) = L\Big([\Xi, \Sigma]\Big)\,.$$

2.2 Projectable tangent valued forms

Now, we analyse a distinguished subsheaf of the tangent valued forms of the line bundle.

Let us devote our attention to the sheaf $\sec(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})$.
The coordinate expression of $\Xi \in \sec(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})$ is of the type

$$\begin{aligned}\Xi &= d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes (\Xi^{\mu}_{\lambda_1\ldots\lambda_r}\, \partial_\mu + \Xi^{\mathrm{a}}_{\lambda_1\ldots\lambda_r}\, \partial_{\mathrm{a}}) \\ &= d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes (\Xi^{\mu}_{\lambda_1\ldots\lambda_r}\, \partial_\mu + \Xi^{z}_{\lambda_1\ldots\lambda_r}\, \partial_1)\,,\end{aligned}$$

with $\Xi^{\mu}_{\lambda_1\ldots\lambda_r}, \Xi^{\mathrm{a}}_{\lambda_1\ldots\lambda_r} \in \mathrm{map}(\boldsymbol{Q}, \mathrm{I\!R})$ and $\Xi^{z}_{\lambda_1\ldots\lambda_r} = \Xi^{1}_{\lambda_1\ldots\lambda_r} + \mathfrak{i}\, \Xi^{2}_{\lambda_1\ldots\lambda_r}$.

Ξ is said to be *projectable* if $T\pi \circ \Xi \in \sec(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{E})$ factorises through a section $\underline{\Xi} \in \sec(\boldsymbol{E},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{E})$.

Thus, Ξ is projectable if and only if $\Xi^{\mu}_{\lambda_1\ldots\lambda_r} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

We denote the subsheaf of projectable tangent valued forms of degree r by

$$\mathrm{proj}(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q}) \subset \sec(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})\,.$$

In particular, we have the subsheaf of *vertical valued forms*

$$\sec(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes V\boldsymbol{Q}) \subset \mathrm{proj}(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})\,.$$

The sheaf of projectable tangent valued forms is closed with respect to the FN bracket.
For projectable tangent valued forms, we have the identity

$$[\underline{\Xi}, \underline{\Sigma}] = \underline{[\Xi, \Sigma]}\,.$$

For projectable tangent valued forms, we obtain the coordinate expression

$$\begin{aligned}[\Xi, \Sigma] = (&\Xi^{\rho}_{\lambda_1\ldots\lambda_r}\, \partial_\rho \Sigma^{\mu}_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\, \Sigma^{\rho}_{\lambda_1\ldots\lambda_s}\, \partial_\rho \Xi^{\mu}_{\lambda_{s+1}\ldots\lambda_{r+s}} \\ &- r\, \Xi^{\mu}_{\lambda_1\ldots\lambda_{r-1}\rho}\, \partial_{\lambda_r} \Sigma^{\rho}_{\lambda_{r+1}\ldots\lambda_{r+s}} + (-1)^{rs}\, s\, \Sigma^{\mu}_{\lambda_1\ldots\lambda_{s-1}\rho}\, \partial_{\lambda_s} \Xi^{\rho}_{\lambda_{s+1}\ldots\lambda_{r+s}}) \cdot \\ &\qquad \cdot d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_\mu \\ &+ (\Xi^{\rho}_{\lambda_1\ldots\lambda_r}\, \partial_\rho \Sigma^{\mathrm{a}}_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\, \Sigma^{\rho}_{\lambda_1\ldots\lambda_s}\, \partial_\rho \Xi^{\mathrm{a}}_{\lambda_{s+1}\ldots\lambda_{r+s}} \\ &+ \Xi^{\mathrm{b}}_{\lambda_1\ldots\lambda_r}\, \partial_{\mathrm{b}} \Sigma^{\mathrm{a}}_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\, \Sigma^{\mathrm{b}}_{\lambda_1\ldots\lambda_s}\, \partial_{\mathrm{b}} \Xi^{\mathrm{a}}_{\lambda_{s+1}\ldots\lambda_{r+s}} \\ &- r\, \Xi^{\mathrm{a}}_{\lambda_1\ldots\lambda_{r-1}\rho}\, \partial_{\lambda_r} \Sigma^{\rho}_{\lambda_{r+1}\ldots\lambda_{r+s}} + (-1)^{rs}\, s\, \Sigma^{\mathrm{a}}_{\lambda_1\ldots\lambda_{s-1}\rho}\, \partial_{\lambda_s} \Xi^{\rho}_{\lambda_{s+1}\ldots\lambda_{r+s}}) \cdot \\ &\qquad \cdot d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_{\mathrm{a}}\,.\end{aligned}$$

Moreover, for decomposable projectable tangent valued forms, we obtain

$$[\xi \otimes X,\ \sigma \otimes Y] = \xi \wedge \sigma \otimes [X, Y] + \xi \wedge L_{\underline{X}}\sigma \otimes Y - (-1)^{rs}\, \sigma \wedge L_{\underline{Y}}\xi \otimes X \\ + (-1)^{r}\, d\xi \wedge i_{\underline{X}}\sigma \otimes Y - (-1)^{rs+s}\, d\sigma \wedge i_{\underline{Y}}\xi \otimes X \,.$$

2.3 Linear tangent valued forms

Next, we analyse the subsheaf of linear tangent valued forms of the line bundle.

A projectable tangent valued form Ξ is said to be (real) *linear* if it is a (real) linear fibred morphism over its projection $\underline{\Xi}$.

Thus, a projectable tangent valued form Ξ is (real) linear if and only if

$$\Xi^{\mathrm{a}}_{\lambda_1\dots\lambda_r} = \Xi^{\mathrm{a}}_{\lambda_1\dots\lambda_r\,\mathrm{b}}\, w^{\mathrm{b}}\,, \qquad \text{with} \qquad \Xi^{\mathrm{a}}_{\lambda_1\dots\lambda_r\,\mathrm{b}} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})\,.$$

If Ξ is a (real) linear tangent valued form, then we have $[\Xi, \mathbb{I}] = 0$.

We denote the subsheaf of (real) linear tangent valued forms of degree r by

$$\mathrm{lin}_{\mathrm{I\!R}}(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q}) \subset \mathrm{proj}(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})\,.$$

The sheaf of (real) linear tangent valued forms is closed with respect to the FN bracket.

A (real) linear tangent valued form Ξ is said to be complex *linear* if it is a complex linear fibred morphism over its projection $\underline{\Xi}$.

Thus, a projectable tangent valued form Ξ is complex linear if and only if

$$\Xi^{z}_{\lambda_1\dots\lambda_r} = \Xi^{z}_{\lambda_1\dots\lambda_r\,z}\, z\,, \qquad \text{with} \qquad \Xi^{z}_{\lambda_1\dots\lambda_r\,z} \in \mathrm{map}(\boldsymbol{E}, \mathbb{C})\,,$$

i.e., if and only if

$$\Xi^{1}_{\lambda_1\dots\lambda_r\,1} = \Xi^{2}_{\lambda_1\dots\lambda_r\,2} \qquad \text{and} \qquad \Xi^{2}_{\lambda_1\dots\lambda_r\,1} = -\Xi^{1}_{\lambda_1\dots\lambda_r\,2}\,.$$

In such a case, we have

$$\Xi^{z}_{\lambda_1\dots\lambda_r\,z} = \Xi^{1}_{\lambda_1\dots\lambda_r\,1} + \mathfrak{i}\,\Xi^{2}_{\lambda_1\dots\lambda_r\,1} = \Xi^{2}_{\lambda_1\dots\lambda_r\,2} - \mathfrak{i}\,\Xi^{1}_{\lambda_1\dots\lambda_r\,2}\,.$$

If Ξ is a complex linear tangent valued form, then we have $[\Xi, \mathfrak{i}\,\mathbb{I}] = 0$.

We denote the subsheaf of complex linear tangent valued forms of degree r by

$$\mathrm{lin}_{\mathbb{C}}(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q}) \subset \mathrm{lin}_{\mathrm{I\!R}}(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})\,.$$

The sheaf of complex linear tangent valued forms is closed with respect to the FN bracket.

2.4 Hermitian tangent valued forms

Eventually, we introduce the notion of Hermitian tangent valued forms.

2.1 Lemma. If $\alpha \in \sec(\boldsymbol{Q}, V^*\boldsymbol{Q})$ and $\Xi \in \operatorname{proj}(\boldsymbol{Q}, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})$, then the Lie derivative $L(\Xi)\alpha$ is well defined, in spite of the fact that the form α is vertical valued, and has coordinate expression

$$L(\Xi)\alpha = (\Xi^{\mu}_{\lambda_1\dots\lambda_r}\,\partial_\mu\alpha_{\mathrm{a}} + \Xi^{\mathrm{b}}_{\lambda_1\dots\lambda_r}\,\partial_{\mathrm{b}}\,\alpha_{\mathrm{a}} + \alpha_{\mathrm{b}}\,\partial_{\mathrm{a}}\,\Xi^{\mathrm{b}}_{\lambda_1\dots\lambda_r})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes \check{d}^{\mathrm{a}}\,.$$

PROOF. If $\tilde{\alpha} \in \sec(\boldsymbol{Q}, T^*\boldsymbol{Q})$ is any extension of α (obtained, for instance through a connection of the line bundle), then let us prove that the vertical restriction "to one variable"

$$L(\Xi)\alpha := \big(L(\Xi)\tilde{\alpha}\big)^{\vee_1} \in \sec(\boldsymbol{Q}, \Lambda^r T^*\boldsymbol{E} \otimes V^*\boldsymbol{Q})$$

does not depend on the choice of the extension $\tilde{\alpha}$.

The coordinate expression of $\tilde{\alpha}$ is of the type $\tilde{\alpha} = \alpha_\mu\, d^\mu + \alpha_{\mathrm{a}}\, d^{\mathrm{a}}$.

Then, the expression $\Xi = d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes(\Xi^{\lambda}_{\lambda_1\dots\lambda_r}\,\partial_\lambda + \Xi^{\mathrm{a}}_{\lambda_1\dots\lambda_r}\,\partial_{\mathrm{a}})$, with $\partial_{\mathrm{b}}\,\Xi^{\lambda}_{\lambda_1\dots\lambda_r} = 0$, yields

$$\begin{aligned}L(\Xi)\,\tilde{\alpha} = d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\wedge&\\ \wedge\Big((\Xi^{\mu}_{\lambda_1\dots\lambda_r}\,\partial_\mu\alpha_\lambda + \Xi^{\mathrm{b}}_{\lambda_1\dots\lambda_r}\,\partial_{\mathrm{b}}\,\alpha_\lambda + \alpha_\mu\,\partial_\lambda\Xi^{\mu}_{\lambda_1\dots\lambda_r} + \alpha_{\mathrm{b}}\,\partial_\lambda\Xi^{\mathrm{b}}_{\lambda_1\dots\lambda_r})\,d^\lambda&\\ + (\Xi^{\mu}_{\lambda_1\dots\lambda_r}\,\partial_\mu\alpha_{\mathrm{a}} + \Xi^{\mathrm{b}}_{\lambda_1\dots\lambda_r}\,\partial_{\mathrm{b}}\,\alpha_{\mathrm{a}} + \alpha_{\mathrm{b}}\,\partial_{\mathrm{a}}\,\Xi^{\mathrm{b}}_{\lambda_1\dots\lambda_r})\,d^{\mathrm{a}}\Big)\,.&\end{aligned}$$

Eventually, by considering the natural map

$$^{\vee_1} : \otimes^{r+1}T^*\boldsymbol{Q} \to \otimes^r T^*\boldsymbol{Q}\otimes V^*\boldsymbol{Q} : \beta^1\otimes\dots\otimes\beta^{r+1} \mapsto \sum_{1\le i\le r+1} \beta^1\otimes\dots\otimes\check{\beta}^i\otimes\dots\otimes\beta^{r+1}\,,$$

we obtain the section

$$\big(L(\Xi)\,\tilde{\alpha}\big)^{\vee_1} = (\Xi^{\mu}_{\lambda_1\dots\lambda_r}\,\partial_\mu\alpha_{\mathrm{a}} + \Xi^{\mathrm{b}}_{\lambda_1\dots\lambda_r}\,\partial_{\mathrm{b}}\alpha_{\mathrm{a}} + \partial_{\mathrm{a}}\Xi^{\mathrm{b}}_{\lambda_1\dots\lambda_r}\,\alpha_{\mathrm{b}})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes\check{d}^{\mathrm{a}}\,,$$

which turns out to be valued in the subspace $\Lambda^r T^*\boldsymbol{E}\otimes V^*\boldsymbol{Q} \subset \otimes^r T^*\boldsymbol{Q}\otimes V^*\boldsymbol{Q}$. QED

A (real) linear tangent valued form Ξ is said to be *Hermitian* if $L(\Xi)\mathfrak{h} = 0$.

2.2 Lemma. For each $\Xi \in \operatorname{lin}_{\mathbb{R}}(\boldsymbol{Q}, \Lambda^r T^*\boldsymbol{E}\otimes T\boldsymbol{Q})$, we have the coordinate expression

$$\begin{aligned}L(\Xi)\,\mathfrak{h} =&\\ = \Big(2\,\Xi^1_{\lambda_1\dots\lambda_r 1}\,w^1 + (\Xi^2_{\lambda_1\dots\lambda_r 1} + \Xi^1_{\lambda_1\dots\lambda_r 2})\,w^2 - \mathfrak{i}\,\Xi^{\mathrm{a}}_{\lambda_1\dots\lambda_r\mathrm{a}}\,w^2\Big)\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes\check{d}^1&\\ + \Big(2\,\Xi^2_{\lambda_1\dots\lambda_r 2}\,w^2 + (\Xi^2_{\lambda_1\dots\lambda_r 1} + \Xi^1_{\lambda_1\dots\lambda_r 2})\,w^1 + \mathfrak{i}\,\Xi^{\mathrm{a}}_{\lambda_1\dots\lambda_r\mathrm{a}}\,w^1\Big)\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_r}\otimes\check{d}^2\,.\square&\end{aligned}$$

2.3 Proposition. Each Hermitian tangent valued form Ξ turns out to be complex linear. Moreover, $\Xi \in \operatorname{lin}_{\mathbb{R}}(\boldsymbol{Q}, \Lambda^r T^*\boldsymbol{E}\otimes T\boldsymbol{Q})$ is Hermitian if and only if it is (locally) of the type

$$\Xi = \chi[\mathfrak{b}](\underline{\Xi}) + \mathfrak{i}\,\breve{\Xi}[\mathfrak{b}]\otimes\mathbb{I}\,,\qquad\text{with}\qquad \breve{\Xi}[\mathfrak{b}] \in \sec(\boldsymbol{E}, \Lambda^r T^*\boldsymbol{E})\,,$$

where $\chi[\mathfrak{b}]$ is the (local) flat connection of $\boldsymbol{Q}\to\boldsymbol{E}$ induced by the basis b.

In other words, Ξ is Hermitian if and only if

$$\Xi^1_{\lambda_1 \dots \lambda_r 1} = \Xi^2_{\lambda_1 \dots \lambda_r 2} = 0 \qquad \text{and} \qquad \Xi^2_{\lambda_1 \dots \lambda_r 1} = -\Xi^1_{\lambda_1 \dots \lambda_r 2} \,,$$

i.e. if and only if its coordinate expression is of the type

$$\Xi = d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes (\Xi^\lambda_{\lambda_1 \dots \lambda_r} \, \partial_\lambda + \mathfrak{i} \, \breve{\Xi}_{\lambda_1 \dots \lambda_r} \, \mathbb{I}) \,,$$

with $\Xi^\lambda_{\lambda_1 \dots \lambda_r} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$, $\quad \breve{\Xi}_{\lambda_1 \dots \lambda_r} = \Xi^2_{\lambda_1 \dots \lambda_r 1} = -\Xi^1_{\lambda_1 \dots \lambda_r 2} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$. □

2.4 Corollary. In particular, the Hermitian vertical valued forms Ξ are of the type

$$\Xi = \mathfrak{i} \, \breve{\Xi} \otimes \mathbb{I} \,, \qquad \text{with} \qquad \breve{\Xi} \in \sec(\boldsymbol{E}, \, \Lambda^r T^* \boldsymbol{E}) \,.$$

Hence, the Hermitianity of vertical valued forms does not depend on the choice of the Hermitian metric $\mathfrak{h}$. Moreover, the form $\breve{\Xi}$ is global and does not depend on the choice of the basis $\mathfrak{b}$. □

We denote the subsheaf of Hermitian tangent valued forms of degree r by

$$\mathrm{her}\, (\boldsymbol{Q}, \, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{Q}) \subset \mathrm{lin}_{\mathbb{C}}(\boldsymbol{Q}, \, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{Q}) \,.$$

Each $\Xi \in \mathrm{her}\, (\boldsymbol{Q}, \, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{Q})$ can be written locally as sum of decomposable tangent valued forms of the type

$$\xi \otimes Y \,, \qquad \text{with} \qquad \xi \in \sec(\boldsymbol{E}, \, \Lambda^r T^* \boldsymbol{E}) \,, \quad Y \in \mathrm{her}\, (\boldsymbol{Q}, T\boldsymbol{Q}) \,.$$

However, in general this decomposition is not unique and holds only locally.
If $\Xi \in \mathrm{her}\, (\boldsymbol{Q}, \, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{Q})$ and $\alpha \in \sec(\boldsymbol{E}, \, \Lambda^s T^* \boldsymbol{E})$, then

$$\alpha \wedge \Xi \in \mathrm{her}\, (\boldsymbol{Q}, \, \Lambda^{r+s} T^* \boldsymbol{E} \otimes T\boldsymbol{Q}) \,.$$

2.5 Graded Lie algebra of Hermitian tangent valued forms

We show that the sheaf of Hermitian tangent valued forms is closed with respect to the FN bracket.

2.5 Lemma. For each $\breve{\Xi} \in \sec(\boldsymbol{E}, \, \Lambda^r T^* \boldsymbol{E})$ and $\breve{\Sigma} \in \sec(\boldsymbol{E}, \, \Lambda^s T^* \boldsymbol{E})$ we have

$$[\mathfrak{i} \, \breve{\Xi} \otimes \mathbb{I} \,, \ \mathfrak{i} \, \breve{\Sigma} \otimes \mathbb{I}] = 0 \,. \square$$

2.6 Lemma. For each $\underline{\Xi} \in \sec(\boldsymbol{E}, \, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{E})$ and $\underline{\Sigma} \in \sec(\boldsymbol{E}, \, \Lambda^s T^* \boldsymbol{E} \otimes T\boldsymbol{E})$, we have

$$\big[\chi[\mathfrak{b}](\underline{\Xi}) \,, \ \chi[\mathfrak{b}](\underline{\Sigma})\big] = \chi[\mathfrak{b}]\big([\underline{\Xi} \,, \ \underline{\Sigma}]\big) \,. \square$$

2.7 Lemma. For each $\underline{\Xi}, \in \sec(\boldsymbol{E}, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{E})$ and $\check{\Sigma} \in \sec(\boldsymbol{E}, \Lambda^s T^*\boldsymbol{E})$, we have

$$\Big[\chi[\mathfrak{b}](\underline{\Xi})\,,\ \mathfrak{i}\,\check{\Sigma}\otimes\mathbb{I}\Big] = \mathfrak{i}\,\Big(L(\underline{\Xi})\,\check{\Sigma}\Big)\otimes\mathbb{I}\,.\square$$

2.8 Theorem. *The sheaf of Hermitian tangent valued forms is closed with respect to the FN bracket.*

Indeed, for each $\Xi \in \mathrm{her}\,(\boldsymbol{Q},\ \Lambda^r T^*\boldsymbol{E}\otimes TQ)$ *and* $\Sigma \in \mathrm{her}\,(\boldsymbol{Q},\ \Lambda^s T^*\boldsymbol{E}\otimes TQ)$, *we have*

$$\Big[\chi[\mathfrak{b}](\underline{\Xi}) + \mathfrak{i}\,\check{\Xi}[\mathfrak{b}]\otimes\mathbb{I}\,,\ \chi[\mathfrak{b}](\underline{\Sigma}) + \mathfrak{i}\,\check{\Sigma}[\mathfrak{b}]\otimes\mathbb{I}\Big] =$$
$$= \chi[\mathfrak{b}]\Big([\underline{\Xi},\underline{\Sigma}]\Big) + \mathfrak{i}\,\Big(L(\underline{\Xi})\,\check{\Sigma}[\mathfrak{b}] - (-1)^{rs}L(\underline{\Sigma})\,\check{\Xi}[\mathfrak{b}]\Big)\otimes\mathbb{I}\,.\square$$

2.9 Corollary. The sheaf of vertical Hermitian tangent valued forms is an abelian subalgebra and an ideal of the algebra of Hermitian tangent valued forms.

Indeed, for each $\Xi \in \mathrm{her}\,(\boldsymbol{Q},\ \Lambda^r T^*\boldsymbol{E}\otimes TQ)$ and $\Sigma \in \mathrm{her}\,(\boldsymbol{Q},\ \Lambda^s T^*\boldsymbol{E}\otimes VQ)$, we have

$$\Big[\chi[\mathfrak{b}](\underline{\Xi}) + \mathfrak{i}\,\check{\Xi}[\mathfrak{b}]\otimes\mathbb{I}\,,\ \ \mathfrak{i}\,\check{\Sigma}\otimes\mathbb{I}\Big] = \mathfrak{i}\,\Big(L(\underline{\Xi})\,\check{\Sigma}\Big)\otimes\mathbb{I}\,.\square$$

2.10 Corollary. The sheaf of Hermitian vector fields turns out to be a subalgebra of the algebra of Hermitian tangent valued forms.

Indeed, for each $X, Y \in \mathrm{her}\,(\boldsymbol{Q}, T\boldsymbol{Q})$, we have

$$\Big[\chi[\mathfrak{b}](\underline{X}) + \mathfrak{i}\,\check{X}[\mathfrak{b}]\,\mathbb{I}\,,\ \chi[\mathfrak{b}](\underline{Y}) + \mathfrak{i}\,\check{Y}[\mathfrak{b}]\,\mathbb{I}\Big] = \chi[\mathfrak{b}]([\underline{X},\ \underline{Y}]) + \mathfrak{i}\,(\underline{X}.\check{Y}[\mathfrak{b}] - \underline{Y}.\check{X}[\mathfrak{b}])\,\mathbb{I}\,.\square$$

Each $Y \in \mathrm{lin}_{\mathbb{R}}(\boldsymbol{Q}, T\boldsymbol{Q})$ turns out to be Hermitian if and only if

$$L(\underline{Y})\,\mathfrak{h}(\Phi,\Psi) = \mathfrak{h}\Big(L(Y)\,\Phi\,,\ \Psi\Big) + \mathfrak{h}\Big(\Phi\,,\ L(Y)\,\Psi\Big)\,, \qquad \forall\,\Phi,\Psi\in\sec(\boldsymbol{E},\boldsymbol{Q})\,.$$

3 Classification of Hermitian tangent valued forms

The above results provide a local characterisation of Hermitian tangent valued forms in terms of a basis or of a fibred chart.

On the other hand, the choice of a global connection allows us to exhibit a global characterisation of Hermitian tangent valued forms in terms of tangent valued forms and forms of the base space.

3.1 Hermitian connections

In view of the above global characterisation, we recall a few basic properties of Hermitian connections.

Let us consider a connection of the line bundle $c : \boldsymbol{Q} \to T^*\boldsymbol{E}\otimes T\boldsymbol{Q}$, i.e., tangent valued 1–form, which is projectable on $\boldsymbol{1}_{\boldsymbol{E}}$.

Its coordinate expression is of the type $c = d^\lambda \otimes (\partial_\lambda + c_\lambda^{\mathrm{a}}\,\partial_{\mathrm{a}})$, where $c_\lambda^{\mathrm{a}} \in \mathrm{map}(\boldsymbol{Q}, \mathbb{R})$.

The connection c is characterised by vertical 1–form $\nu[c] : \boldsymbol{Q} \to T^*\boldsymbol{Q} \otimes V\boldsymbol{Q}$, which expresses the vertical projection associated with the tangent splitting induced by c. We have the coordinate expression $\nu[c] = (d^{\mathrm{a}} - c^{\mathrm{a}}_{\lambda}\, d^{\lambda}) \otimes \partial_{\mathrm{a}}$.

For each $\Xi \in \mathrm{proj}(\boldsymbol{E},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})$, we have the *covariant exterior differential* [8]

$$d[c]\Xi := [c, \Xi] : \mathrm{proj}(\boldsymbol{E},\, \Lambda^{r+1} T^*\boldsymbol{E} \otimes V\boldsymbol{Q}),$$

with coordinate expression

$$\begin{aligned} d[c]\Xi = \Big(&- \partial_{\lambda_1} \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}}\, c^{\mathrm{a}}_{\rho} - \partial_{\rho} c^{\mathrm{a}}_{\lambda_1}\, \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}} \\ &+ \partial_{\lambda_1} \Xi^{\mathrm{a}}_{\lambda_2 \dots \lambda_{r+1}} + c^{\mathrm{b}}_{\lambda_1}\, \partial_{\mathrm{b}} \Xi^{\mathrm{a}}_{\lambda_2 \dots \lambda_{r+1}} - \partial_{\mathrm{b}} c^{\mathrm{a}}_{\lambda_1}\, \Xi^{\mathrm{b}}_{\lambda_2 \dots \lambda_{r+1}} \Big)\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+1}} \otimes \partial_{\mathrm{a}}. \end{aligned}$$

We stress that the covariant differential of tangent valued forms through the FN bracket, with respect to a non linear connection, turns out to be a natural genalisation of the covariant differential of sections and of vector valued forms for linear connections on a vector bundle and of Lie algebra valued forms for principal connections on a principal bundle. Indeed, the generalisation of standard identities, including Bianchi identities, hold for the generalised covariant differential as a consequence of the graded Bianchi identities of the FN bracket [8, 7].

The *curvature* of c is defined to be the vertical valued 2–form [8]

$$R[c] := -d[c]c := -[c, c] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{Q},$$

with coordinate expression $R[c] = -2\, (\partial_{\lambda} c^{\mathrm{a}}_{\mu} + c^{\mathrm{b}}_{\lambda}\, \partial_{\mathrm{b}} c^{\mathrm{a}}_{\mu})\, d^{\lambda} \wedge d^{\mu} \otimes \partial_{\mathrm{a}}$.

3.1 Lemma. For each $\underline{\Xi} \in \mathrm{sec}(\boldsymbol{E},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{E})$ and $\underline{\Sigma} \in \mathrm{sec}(\boldsymbol{E},\, \Lambda^s T^*\boldsymbol{E} \otimes T\boldsymbol{E})$, we have

$$[c,\, c(\underline{\Xi})] = \underline{\Xi} \lrcorner R[c] \qquad \text{and} \qquad [c(\underline{\Xi}),\, c(\underline{\Sigma})] = c\Big([\underline{\Xi},\, \underline{\Sigma}]\Big) - R[c](\underline{\Xi}, \underline{\Sigma}),$$

where $\underline{\Xi} \lrcorner R[c]$ and $R[c](\underline{\Xi}, \underline{\Sigma})$ are defined, via decomposable tangent valued forms, by

$$\begin{aligned} (\xi \otimes X) \lrcorner R[c] &= (-1)^r \xi \wedge (X \lrcorner R[c]) \\ R[c](\xi \otimes X,\, \sigma \otimes Y) &= (\xi \wedge \sigma) \otimes (Y \lrcorner X \lrcorner R[c]). \end{aligned}$$

PROOF. The result can be obtained by a computation via the expression of the FN bracket for decomposable forms.

An alternative proof can be obtained in coordinates, as follows. We have

$$\begin{aligned} &[c, c(\Xi)] = \\ &= \Big(- \partial_{\lambda_1} \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}}\, c^{\mathrm{a}}_{\rho} - \partial_{\rho} c^{\mathrm{a}}_{\lambda_1}\, \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}} \\ &+ \partial_{\lambda_1} \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}}\, c^{\mathrm{a}}_{\rho} + \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}}\, \partial_{\lambda_1} c^{\mathrm{a}}_{\rho} + c^{\mathrm{b}}_{\lambda_1}\, \partial_{\mathrm{b}} c^{\mathrm{a}}_{\rho}\, \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}} - \partial_{\mathrm{b}} c^{\mathrm{a}}_{\lambda_1}\, c^{\mathrm{b}}_{\rho}\, \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}} \Big)\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+1}} \otimes \partial_{\mathrm{a}} \\ &= -(\partial_{\rho} c^{\mathrm{a}}_{\lambda_1} - \partial_{\lambda_1} c^{\mathrm{a}}_{\rho} + c^{\mathrm{b}}_{\rho}\, \partial_{\mathrm{b}} c^{\mathrm{a}}_{\lambda_1} - c^{\mathrm{b}}_{\lambda_1}\, \partial_{\mathrm{b}} c^{\mathrm{a}}_{\rho})\, \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}}\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+1}} \otimes \partial_{\mathrm{a}} \\ &= R^{\mathrm{a}}_{\rho \lambda_1}\, \Xi^{\rho}_{\lambda_2 \dots \lambda_{r+1}}\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+1}} \otimes \partial_{\mathrm{a}}. \end{aligned}$$

Moreover, we have

$$
\begin{aligned}
&[c(\Xi),\, c(\underline{\Sigma})] = \\
&= (\Xi^{\rho}_{\lambda_1\dots\lambda_r}\,\partial_\rho\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\,\Sigma^{\rho}_{\lambda_1\dots\lambda_s}\,\partial_\rho\Xi^{\mu}_{\lambda_{s+1}\dots\lambda_{r+s}} \\
&- r\,\Xi^{\mu}_{\lambda_1\dots\lambda_{r-1}\rho}\,\partial_{\lambda_r}\Sigma^{\rho}_{\lambda_{r+1}\dots\lambda_{r+s}} + (-1)^{rs}\,s\,\Sigma^{\mu}_{\lambda_1\dots\lambda_{s-1}\rho}\,\partial_{\lambda_s}\Xi^{\rho}_{\lambda_{s+1}\dots\lambda_{r+s}})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r+s}}\otimes\partial_\mu \\
&+ (\Xi^{\rho}_{\lambda_1\dots\lambda_r}\,\partial_\rho c^{\mathrm{a}}_\mu\,\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} + \Xi^{\rho}_{\lambda_1\dots\lambda_r}\,c^{\mathrm{a}}_\mu\,\partial_\rho\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} \\
&- (-1)^{rs}\,(\Sigma^{\rho}_{\lambda_1\dots\lambda_s}\,\partial_\rho c^{\mathrm{a}}_\mu\,\Xi^{\mu}_{\lambda_{s+1}\dots\lambda_{r+s}} + \Sigma^{\rho}_{\lambda_1\dots\lambda_s}\,c^{\mathrm{a}}_\mu\,\partial_\rho\Xi^{\mu}_{\lambda_{s+1}\dots\lambda_{r+s}}) \\
&+ c^{\mathrm{b}}_\rho\,\Xi^{\rho}_{\lambda_1\dots\lambda_r}\,\partial_{\mathrm{b}}c^{\mathrm{a}}_\mu\,\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\,c^{\mathrm{b}}_\rho\,\Sigma^{\rho}_{\lambda_1\dots\lambda_s}\,\partial_{\mathrm{b}}c^{\mathrm{a}}_\mu\,\Xi^{\mu}_{\lambda_{s+1}\dots\lambda_{r+s}} + \\
&- r\,c^{\mathrm{a}}_\mu\,\Xi^{\mu}_{\lambda_1\dots\lambda_{r-1}\rho}\,\partial_{\lambda_r}\Sigma^{\rho}_{\lambda_{r+1}\dots\lambda_{r+s}} + (-1)^{rs}\,s\,c^{\mathrm{a}}_\mu\,\Sigma^{\mu}_{\lambda_1\dots\lambda_{s-1}\rho}\,\partial_{\lambda_s}\Xi^{\rho}_{\lambda_{s+1}\dots\lambda_{r+s}})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r+s}}\otimes\partial_{\mathrm{a}} \\
&\\
&= (\Xi^{\rho}_{\lambda_1\dots\lambda_r}\,\partial_\rho\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\,\Sigma^{\rho}_{\lambda_1\dots\lambda_s}\,\partial_\rho\Xi^{\mu}_{\lambda_{s+1}\dots\lambda_{r+s}} \\
&- r\,\Xi^{\mu}_{\lambda_1\dots\lambda_{r-1}\rho}\,\partial_{\lambda_r}\Sigma^{\rho}_{\lambda_{r+1}\dots\lambda_{r+s}} + (-1)^{rs}\,s\,\Sigma^{\mu}_{\lambda_1\dots\lambda_{s-1}\rho}\,\partial_{\lambda_s}\Xi^{\rho}_{\lambda_{s+1}\dots\lambda_{r+s}})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r+s}}\otimes\partial_\mu \\
&+ c^{\mathrm{a}}_\mu\,(\Xi^{\rho}_{\lambda_1\dots\lambda_r}\,\partial_\rho\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\,\Sigma^{\rho}_{\lambda_1\dots\lambda_s}\,\partial_\rho\Xi^{\mu}_{\lambda_{s+1}\dots\lambda_{r+s}} \\
&- r\,\Xi^{\mu}_{\lambda_1\dots\lambda_{r-1}\rho}\,\partial_{\lambda_r}\Sigma^{\rho}_{\lambda_{r+1}\dots\lambda_{r+s}} + (-1)^{rs}\,s\,\Sigma^{\mu}_{\lambda_1\dots\lambda_{s-1}\rho}\,\partial_{\lambda_s}\Xi^{\rho}_{\lambda_{s+1}\dots\lambda_{r+s}})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r+s}}\otimes\partial_{\mathrm{a}} \\
&+ (\partial_\rho c^{\mathrm{a}}_\mu + c^{\mathrm{b}}_\rho\,\partial_{\mathrm{b}}c^{\mathrm{a}}_\mu)\,(\Xi^{\rho}_{\lambda_1\dots\lambda_r}\,\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\Sigma^{\rho}_{\lambda_1\dots\lambda_s}\,\Xi^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r+s}}\otimes\partial_{\mathrm{a}} \\
&\\
&= (\Xi^{\rho}_{\lambda_1\dots\lambda_r}\,\partial_\rho\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\,\Sigma^{\rho}_{\lambda_1\dots\lambda_s}\,\partial_\rho\Xi^{\mu}_{\lambda_{s+1}\dots\lambda_{r+s}} \\
&- r\,\Xi^{\mu}_{\lambda_1\dots\lambda_{r-1}\rho}\,\partial_{\lambda_r}\Sigma^{\rho}_{\lambda_{r+1}\dots\lambda_{r+s}} + (-1)^{rs}\,s\,\Sigma^{\mu}_{\lambda_1\dots\lambda_{s-1}\rho}\,\partial_{\lambda_s}\Xi^{\rho}_{\lambda_{s+1}\dots\lambda_{r+s}})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r+s}}\otimes\partial_\mu \\
&+ c^{\mathrm{a}}_\mu\,(\Xi^{\rho}_{\lambda_1\dots\lambda_r}\,\partial_\rho\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\,\Sigma^{\rho}_{\lambda_1\dots\lambda_s}\,\partial_\rho\Xi^{\mu}_{\lambda_{s+1}\dots\lambda_{r+s}} \\
&- r\,\Xi^{\mu}_{\lambda_1\dots\lambda_{r-1}\rho}\,\partial_{\lambda_r}\Sigma^{\rho}_{\lambda_{r+1}\dots\lambda_{r+s}} + (-1)^{rs}\,s\,\Sigma^{\mu}_{\lambda_1\dots\lambda_{s-1}\rho}\,\partial_{\lambda_s}\Xi^{\rho}_{\lambda_{s+1}\dots\lambda_{r+s}})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r+s}}\otimes\partial_{\mathrm{a}} \\
&+ (\partial_\rho c^{\mathrm{a}}_\mu + c^{\mathrm{b}}_\rho\,\partial_{\mathrm{b}}c^{\mathrm{a}}_\mu)\,(\Xi^{\rho}_{\lambda_1\dots\lambda_r}\,\Sigma^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} - \Xi^{\mu}_{\lambda_1\dots\lambda_r}\,\Sigma^{\rho}_{\lambda_{r+1}\dots\lambda_{r+s}})\,d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r+s}}\otimes\partial_{\mathrm{a}}\,.\ \mathrm{QED}
\end{aligned}
$$

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the covariant differentials

$$\nabla[c]\Psi \in \sec(\boldsymbol{E},\, T^*\boldsymbol{E}\otimes\boldsymbol{Q}) \qquad \text{and} \qquad d[c]\tilde{\Psi} \in \sec(\boldsymbol{Q},\, T^*\boldsymbol{E}\otimes V\boldsymbol{Q})\,,$$

with coordinate expressions

$$\nabla[c]\Psi = (\partial_\lambda\psi^{\mathrm{a}} - c^{\mathrm{a}}_\lambda\circ\Psi)\,d^\lambda\otimes\mathsf{b}_{\mathrm{a}} \qquad \text{and} \qquad d[c]\tilde{\Psi} = (\partial_\lambda\psi^{\mathrm{a}} - \partial_{\mathrm{b}}c^{\mathrm{a}}_\lambda\,\psi^{\mathrm{b}})\,d^\lambda\otimes\partial_{\mathrm{a}}\,.$$

Now, let us consider a (real) linear connection c.

The above covariant differentials $\nabla[c]\Psi$ and $d[c]\tilde{\Psi}$ can be naturally identified.

The connection c turns out to be *complex linear* if and only if $\nabla(\mathfrak{i}\,\Psi) = \mathfrak{i}\,\nabla\Psi$, for each $\Psi\in\sec(\boldsymbol{E},\boldsymbol{Q})$.

3.2 Lemma. $L(c)\mathsf{h} : \boldsymbol{Q}\to\mathbb{C}\otimes(T^*\boldsymbol{E}\otimes V^*\boldsymbol{Q})$ and $\nabla\mathsf{h} : \boldsymbol{E}\to\mathbb{C}\otimes(T^*\boldsymbol{E}\otimes\boldsymbol{Q}^*\otimes\boldsymbol{Q}^*)$ turn out to be equal, up to a natural isomorphism.

Proof. We have the coordinate expressions

$$L(c)\,\mathsf{h} = \big(2\,c^{1}_{\lambda\,1}\,w^1 + (c^{2}_{\lambda\,1} + c^{1}_{\lambda\,2})\,w^2 - \mathfrak{i}\,c^{\mathrm{a}}_{\lambda\,\mathrm{a}}\,w^2\big)\,d^\lambda \otimes \check{d}^1 + \big(2\,c^{2}_{\lambda\,2}\,w^2 + (c^{2}_{\lambda\,1} + c^{1}_{\lambda\,2})\,w^1 + \mathfrak{i}\,c^{\mathrm{a}}_{\lambda\,\mathrm{a}}\,w^1\big)\,d^\lambda \otimes \check{d}^2$$

$$\nabla(c)\,\mathsf{h} = d^\lambda \otimes \big(2\,c^{1}_{\lambda\,1}\,w^1 + (c^{2}_{\lambda\,1} + c^{1}_{\lambda\,2})\,w^2 - \mathfrak{i}\,c^{\mathrm{a}}_{\lambda\,\mathrm{a}}\,w^2\big) \otimes w^1 + d^\lambda \otimes \big(2\,c^{2}_{\lambda\,2}\,w^2 + (c^{2}_{\lambda\,1} + c^{1}_{\lambda\,2})\,w^1 + \mathfrak{i}\,c^{\mathrm{a}}_{\lambda\,\mathrm{a}}\,w^1\big) \otimes w^2\,. \text{ QED}$$

3.3 Proposition. The connection c turns out to be *Hermitian* (see also [2, 9]) if and only if $\nabla\,\mathsf{h} = 0$, i.e. if and only if

$$d\Big(\mathsf{h}(\Psi,\,\Phi)\Big) = \mathsf{h}\Big(\nabla\Psi,\,\Phi\Big) + \mathsf{h}\Big(\Psi,\,\nabla\Phi\Big)\,, \qquad \forall\,\Psi,\Phi \in \sec(\boldsymbol{E},\boldsymbol{Q})\,.$$

According to Proposition 2.3, c is Hermitian if and only if it is locally of the type

$$c = \chi[\mathsf{b}] + \mathfrak{i}\,A[\mathsf{b}] \otimes \mathbb{I}\,, \qquad \text{with} \qquad A[\mathsf{b}] \in \sec(\boldsymbol{E},\,T^*\boldsymbol{E})\,.$$

In other words, c is Hermitian if and only if $c^1_{\lambda 1} = c^2_{\lambda 2} = 0$ and $c^2_{\lambda 1} = -c^1_{\lambda 2}$, i.e. if and only if its coordinate expression is of the type

$$c = d^\lambda \otimes (\partial_\lambda + \mathfrak{i}\,A_\lambda\,\mathbb{I})\,, \qquad \text{with} \qquad A_\lambda = c^2_{\lambda 1} \in \mathrm{map}(\boldsymbol{E},\mathrm{I\!R})\,.\;\square$$

Now, let c be Hermitian.
We have the coordinate expression $\nabla\Psi = (\partial_\lambda\psi - \mathfrak{i}\,A_\lambda\,\psi)\,d^\lambda \otimes \mathsf{b}\,, \quad \forall\,\Psi \in \sec(\boldsymbol{E},\boldsymbol{Q})\,.$

3.4 Lemma. For each $\Xi \in \mathrm{her}\,(\boldsymbol{Q},\,\Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})$, we obtain

$$d[c]\Xi = \mathfrak{i}\,\big(d[c]\,\Xi\big)^{\smile} \otimes \mathbb{I}\,,$$

where $\big(d[c]\,\Xi\big)^{\smile} \in \sec(\boldsymbol{E},\,\Lambda^{r+1}T^*\boldsymbol{E})$ is given by

$$\big(d[c]\,\Xi\big)^{\smile} = L(\mathbf{1}_{\boldsymbol{E}})\breve{\Xi} - (-1)^r L(\underline{\Xi})\,A[\mathsf{b}] = d\breve{\Xi} - (-1)^r L(\underline{\Xi})\,A[\mathsf{b}]$$

and has coordinate expression

$$\big(d[c]\,\Xi\big)^{\smile} = \Big(\partial_{\lambda_1}\breve{\Xi}_{\lambda_2\ldots\lambda_{s+1}} - (A_\rho\,\partial_{\lambda_1}\Xi^\rho_{\lambda_2\ldots\lambda_{s+1}} + \partial_\rho A_{\lambda_1}\,\Xi^\rho_{\lambda_2\ldots\lambda_{s+1}})\Big)\,d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{s+1}}\,.\;\square$$

The curvature of c is

$$R[c] = -\mathfrak{i}\,\Phi[c] \otimes \mathbb{I}\,,$$

where $\Phi[c] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$ is the closed 2–form given locally by $\Phi[c] = 2dA[\mathsf{b}]$.
Thus, we have the coordinate expression $\Phi[c] = 2\,\partial_\mu A_\lambda\,d^\mu \wedge d^\lambda$.

3.2 Global classification

Eventually, we show that the choice of a Hermitian connection yields a global classification of the Lie algebra of Hermitian tangent valued forms of the line bundle.

Let us consider a Hermitian connection c.

3.5 Lemma. If $\Xi \in \sec(\boldsymbol{E}, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{E})$, then $c(\underline{\Xi}) \in \mathrm{her}\,(\boldsymbol{Q},\, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{Q})$. $\square$

3.6 Proposition. We have the following mutually inverse isomorphisms

$$\mathfrak{h}[c] : \mathrm{her}\,(\boldsymbol{Q},\, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{Q}) \to \sec(\boldsymbol{E},\, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{E}) \times \sec(\boldsymbol{E},\, \Lambda^r T^* \boldsymbol{E})$$
$$\mathfrak{j}[c] : \sec(\boldsymbol{E},\, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{E}) \times \sec(\boldsymbol{E},\, \Lambda^r T^* \boldsymbol{E}) \to \mathrm{her}\,(\boldsymbol{Q},\, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{Q}),$$

given by

$$\mathfrak{h}[c] : \Xi \mapsto \Big(\underline{\Xi},\, -\mathfrak{i}\ \mathrm{tr}\big(\nu[c](\Xi)\big)\Big) \qquad \text{and} \qquad \mathfrak{j}[c] : (\underline{\Xi}, \breve{\Xi}) \mapsto c(\underline{\Xi}) + \mathfrak{i}\,\breve{\Xi} \otimes \mathbb{I},$$

i.e., in coordinates

$$\mathfrak{h}[c](\Xi) = \Big(\Xi^{\mu}_{\lambda_1\dots\lambda_r}\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r} \otimes \partial_\mu\,, \quad (\breve{\Xi}_{\lambda_1\dots\lambda_r} - A_\rho\, \Xi^{\rho}_{\lambda_1\dots\lambda_r})\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r}\Big)$$
$$\mathfrak{j}[c](\underline{\Xi}, \breve{\Xi}) = d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r} \otimes \Big(\Xi^{\mu}_{\lambda_1\dots\lambda_r}\, \partial_\mu + \mathfrak{i}\,(A_\rho\, \Xi^{\rho}_{\lambda_1\dots\lambda_r} + \breve{\Xi}_{\lambda_1\dots\lambda_r}) \otimes \mathbb{I}\Big). \square$$

3.7 Lemma. Let us consider a closed 2-form Φ of $\boldsymbol{E}$ and define the bracket

$$\Big(\sec(\boldsymbol{E},\, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{E}) \times \sec(\boldsymbol{E},\, \Lambda^r T^* \boldsymbol{E})\Big) \times \Big(\sec(\boldsymbol{E},\, \Lambda^s T^* \boldsymbol{E} \otimes T\boldsymbol{E}) \times \sec(\boldsymbol{E},\, \Lambda^s T^* \boldsymbol{E})\Big)$$
$$\to \Big(\sec(\boldsymbol{E},\, \Lambda^r T^* \boldsymbol{E} \otimes T\boldsymbol{E}) \times \sec(\boldsymbol{E},\, \Lambda^{r+s} T^* \boldsymbol{E})\Big),$$

given by

$$\Big[(\underline{\Xi}_1, \breve{\Xi}_1)\,,\ (\underline{\Xi}_2, \breve{\Xi}_2)\Big]_\Phi := \Big([\underline{\Xi}_1, \underline{\Xi}_2]\,, \quad \Phi(\underline{\Xi}_1, \underline{\Xi}_2) + L(\underline{\Xi}_1)\, \breve{\Xi}_2 - (-1)^{rs} L(\underline{\Xi}_2)\, \breve{\Xi}_1\Big),$$

where $\Phi(\underline{\Xi}_1, \underline{\Xi}_2)$ is defined, via decomposable tangent valued forms, as

$$\Phi(\xi \otimes X, \sigma \otimes Y) := (\xi \wedge \sigma)\, \Phi(X, Y).$$

Then, the above bracket turns out to be a graded Lie bracket.

Proof. The graded commutativity of the 1st component follows from the fact that $[\underline{\Xi}_1, \underline{\Xi}_2]$ is the FN bracket, which is a graded Lie bracket.
Moreover, the anticommutativity of the 2nd component follows from the equality

$$\Phi(\underline{\Xi}_1, \underline{\Xi}_2) + L(\underline{\Xi}_1)\, \breve{\Xi}_2 - (-1)^{rs} L(\underline{\Xi}_2)\, \breve{\Xi}_1 = -(-1)^{rs}\big(\Phi(\underline{\Xi}_2, \underline{\Xi}_1) + L(\underline{\Xi}_2)\, \breve{\Xi}_1 - (-1)^{rs} L(\underline{\Xi}_1)\, \breve{\Xi}_2\big).$$

Next, let us prove the Jacobi property. Let us consider three pairs $\Pi_i := (\underline{\Xi}_i, \breve{\Xi}_i)$, with

$$\underline{\Xi}_i \in \sec(\boldsymbol{E}, \Lambda^{\bar{i}} T^* \boldsymbol{E} \otimes T\boldsymbol{E}) \qquad \text{and} \qquad \breve{\Xi}_i \in \sec(\boldsymbol{E}, \Lambda^{\bar{i}} T^* \boldsymbol{E}),$$

where $\bar{i}$ denotes the degree of the i–th form, and set

$$(\underline{\Sigma}, \breve{\Sigma}) := \big[\Pi_1,\ [\Pi_2,\ \Pi_3]_\Phi\big]_\Phi + (-1)^{\bar{1}(\bar{2}+\bar{3})}\big[\Pi_2,\ [\Pi_3,\ \Pi_1]_\Phi\big]_\Phi + (-1)^{\bar{3}(\bar{1}+\bar{2})}\big[\Pi_3,\ [\Pi_1,\ \Pi_2]_\Phi\big]_\Phi\,,$$

where

$$[\Pi_i, \Pi_j]_\Phi := \big([\Xi_i, \Xi_j]\,,\quad \Phi(\Xi_i, \Xi_j) + L(\Xi_i)\,\breve{\Xi}_j - (-1)^{\bar{i}\bar{j}} L(\Xi_j)\,\breve{\Xi}_i\big)\,.$$

Then, the Jacobi property of the 1st component follows from the Jacobi property of the FN bracket

$$\underline{\Sigma} := \big[\Xi_1,\ [\Xi_2,\ \Xi_3]\big] + (-1)^{\bar{1}(\bar{2}+\bar{3})}\big[\Xi_2,\ [\Xi_3,\ \Xi_1]\big] + (-1)^{\bar{3}(\bar{1}+\bar{2})}\big[\Xi_3,\ [\Xi_1,\ \Xi_2]\big] = 0\,.$$

Moreover, the Jacobi property of the 2nd component follows from the following facts.
We have

$$\begin{aligned}
\breve{\Sigma} &= \Phi\big(\Xi_1,\ [\Xi_2, \Xi_3]\big) + (-1)^{\bar{1}(\bar{3}+\bar{2})}\,\Phi\big(\Xi_2,\ [\Xi_3, \Xi_1]\big) + (-1)^{\bar{3}(\bar{1}+\bar{2})}\,\Phi\big(\Xi_3,\ [\Xi_1, \Xi_2]\big) \\
&\quad + L(\Xi_1)\Phi(\Xi_2, \Xi_3) + (-1)^{\bar{1}(\bar{3}+\bar{2})}\,L(\Xi_2)\Phi(\Xi_3, \Xi_1) + (-1)^{\bar{3}(\bar{1}+\bar{2})}\,L(\Xi_3)\Phi(\Xi_1, \Xi_2) \\
&\quad + \Big(L(\Xi_1)L(\Xi_2) - (-1)^{\bar{1}\bar{2}}\,L(\Xi_2)L(\Xi_1) - L\big([\Xi_1\,,\ \Xi_2]\big)\Big)\breve{\Xi}_3 \\
&\quad + (-1)^{\bar{1}(\bar{2}+\bar{3})}\Big(L(\Xi_2)L(\Xi_3) - (-1)^{\bar{2}\bar{3}}\,L(\Xi_3)L(\Xi_2) - L\big([\Xi_2\,,\ \Xi_3]\big)\Big)\breve{\Xi}_1 \\
&\quad + (-1)^{\bar{3}(\bar{1}+\bar{2})}\Big(L(\Xi_3)L(\Xi_1) - (-1)^{\bar{3}\bar{1}}\,L(\Xi_1)L(\Xi_3) - L\big([\Xi_3\,,\ \Xi_1]\big)\Big)\breve{\Xi}_2 \\
&= \Phi(\Xi_1,\ [\Xi_2, \Xi_3]) + (-1)^{\bar{1}(\bar{3}+\bar{2})}\,\Phi(\Xi_2,\ [\Xi_3, \Xi_1]) + (-1)^{\bar{3}(\bar{1}+\bar{2})}\,\Phi(\Xi_3,\ [\Xi_1, \Xi_2]) \\
&\quad + L(\Xi_1)\Phi(\Xi_2, \Xi_3) + (-1)^{\bar{1}(\bar{3}+\bar{2})}\,L(\Xi_2)\Phi(\Xi_3, \Xi_1) + (-1)^{\bar{3}(\bar{1}+\bar{2})}\,L(\Xi_3)\Phi(\Xi_1, \Xi_2)\,.
\end{aligned}$$

On the other hand, for decomposable tangent valued forms $\Xi_i = \xi_i \otimes X_i$ we obtain

$$\begin{aligned}
\breve{\Sigma} &= \Phi\big(X_1,\ [X_2, X_3]\big)\,\xi_1 \wedge \xi_2 \wedge \xi_3 + (-1)^{\bar{1}(\bar{3}+\bar{2})}\,\Phi\big(X_2,\ [X_3, X_1]\big)\,\xi_2 \wedge \xi_3 \wedge \xi_1 \\
&\quad + (-1)^{\bar{3}(\bar{1}+\bar{2})}\,\Phi\big(X_3,\ [X_1, X_2]\big)\,\xi_3 \wedge \xi_1 \wedge \xi_2 \\
&\quad + \Phi(X_1, X_3)\,\xi_1 \wedge \xi_2 \wedge L(X_2)\xi_3 + (-1)^{\bar{1}(\bar{3}+\bar{2})}\,\Phi(X_2, X_1)\,\xi_2 \wedge \xi_3 \wedge L(X_3)\xi_1 \\
&\quad + (-1)^{\bar{3}(\bar{1}+\bar{2})}\,\Phi(X_3, X_2)\,\xi_3 \wedge \xi_1 \wedge L(X_1)\xi_2 \\
&\quad - (-1)^{\bar{2}\bar{3}}\,\Phi(X_1, X_2)\,\xi_1 \wedge \xi_3 \wedge L(X_3)\xi_2 - (-1)^{\bar{1}\bar{2}}\,\Phi(X_2, X_3)\,\xi_2 \wedge \xi_1 \wedge L(X_1)\xi_3 \\
&\quad - (-1)^{\bar{1}\bar{2}+\bar{3}(\bar{1}+\bar{2})}\,\Phi(X_3, X_1)\,\xi_3 \wedge \xi_2 \wedge L(X_2)\xi_1 \\
&\quad + (-1)^{\bar{2}}\,\Phi(X_1, X_3)\,\xi_1 \wedge d\xi_2 \wedge i(X_2)\xi_3 + (-1)^{\bar{3}+\bar{1}(\bar{3}+\bar{2})}\,\Phi(X_2, X_1)\,\xi_2 \wedge d\xi_3 \wedge i(X_3)\xi_1 \\
&\quad + (-1)^{\bar{1}+\bar{3}(\bar{1}+\bar{2})}\,\Phi(X_3, X_2)\,\xi_3 \wedge d\xi_1 \wedge i(X_1)\xi_2 \\
&\quad - (-1)^{\bar{3}+\bar{2}\bar{3}}\,\Phi(X_1, X_2)\,\xi_1 \wedge d\xi_3 \wedge i(X_3)\xi_2 - (-1)^{\bar{1}+\bar{1}\bar{2}}\,\Phi(X_2, X_3)\,\xi_2 \wedge d\xi_1 \wedge i(X_1)\xi_3 \\
&\quad - (-1)^{\bar{2}+\bar{1}\bar{2}+\bar{3}(\bar{1}+\bar{2})}\,\Phi(X_3, X_1)\,\xi_3 \wedge d\xi_2 \wedge i(X_2)\xi_1 \\
&\quad + \xi_1 \wedge L(X_1)\big(\Phi(X_2, X_3)\,\xi_2 \wedge \xi_3\big) + (-1)^{\bar{1}(\bar{3}+\bar{2})}\,\xi_2 \wedge L(X_2)\big(\Phi(X_3, X_1)\,\xi_3 \wedge \xi_1\big) \\
&\quad + (-1)^{\bar{3}(\bar{1}+\bar{2})}\,\xi_3 \wedge L(X_3)\big(\Phi(X_1, X_2)\,\xi_1 \wedge \xi_2\big) \\
&\quad + (-1)^{\bar{1}} d\xi_1 \wedge i(X_1)\big(\Phi(X_2, X_3)\,\xi_2 \wedge \xi_3\big) + (-1)^{\bar{2}+\bar{1}(\bar{3}+\bar{2})}\,d\xi_2 \wedge i(X_2)\big(\Phi(X_3, X_1)\,\xi_3 \wedge \xi_1\big) \\
&\quad + (-1)^{\bar{3}+\bar{3}(\bar{1}+\bar{2})}\,d\xi_3 \wedge i(X_3)\big(\Phi(X_1, X_2)\,\xi_1 \wedge \xi_2\big)\,,
\end{aligned}$$

i.e.

$$
\begin{aligned}
\breve{\Sigma} = \Big(&\Phi(X_1, [X_2\ X_3]) + \Phi(X_2, [X_3\ X_1]) + \Phi(X_3, [X_1\ X_2]) \\
&+ X_1.\Phi(X_2, X_3) + X_2.\Phi(X_3, X_1) + X_3.\Phi(X_1, X_2)\Big)\, \xi_1 \wedge \xi_2 \wedge \xi_3 \\
&+ \Phi(X_1, X_3)\big(\xi_1 \wedge \xi_2 \wedge L(X_2)\xi_3 + (-1)^{\bar{1}\bar{2}+\bar{3}\bar{1}+\bar{3}\bar{2}}\, \xi_3 \wedge \xi_2 \wedge L(X_2)\xi_1 \\
&\qquad - (-1)^{\bar{1}\bar{3}+\bar{1}\bar{2}}\, \xi_2 \wedge L(X_2)(\xi_3 \wedge \xi_1)\big) \\
&+ \Phi(X_1, X_2)\big(- (-1)^{\bar{1}\bar{3}+\bar{1}\bar{2}}\, \xi_2 \wedge \xi_3 \wedge L(X_3)\xi_1 - (-1)^{\bar{3}\bar{2}}\, \xi_1 \wedge \xi_3 \wedge L(X_3)\xi_2 \\
&\qquad + (-1)^{\bar{1}\bar{3}+\bar{3}\bar{2}}\, \xi_3 \wedge L(X_3)(\xi_1 \wedge \xi_2)\big) \\
&+ \Phi(X_2, X_3)\big(- (-1)^{\bar{1}\bar{3}+\bar{3}\bar{2}}\, \xi_3 \wedge \xi_1 \wedge L(X_1)\xi_2 - (-1)^{\bar{1}\bar{2}}\, \xi_2 \wedge \xi_1 \wedge L(X_1)\xi_3 \\
&\qquad + \xi_1 \wedge L(X_1)(\xi_2 \wedge \xi_3)\big) \\
&+ \Phi(X_1, X_3)\big((-1)^{\bar{2}}\, \xi_1 \wedge d\xi_2 \wedge i(X_2)\xi_3 + (-1)^{\bar{2}+\bar{1}\bar{2}+\bar{3}\bar{1}+\bar{3}\bar{2}}\, \xi_3 \wedge d\xi_2 \wedge i(X_2)\xi_1 \\
&\qquad - (-1)^{\bar{2}+\bar{1}\bar{3}+\bar{1}\bar{2}}\, d\xi_2 \wedge i(X_2)(\xi_3 \wedge \xi_1)\big) \\
&+ \Phi(X_1, X_2)\big(- (-1)^{\bar{3}+\bar{1}\bar{3}+\bar{1}\bar{2}}\, \xi_2 \wedge d\xi_3 \wedge i(X_3)\xi_1 - (-1)^{\bar{3}+\bar{3}\bar{2}}\, \xi_1 \wedge d\xi_3 \wedge i(X_3)\xi_2 \\
&\qquad + (-1)^{\bar{3}+\bar{1}\bar{3}+\bar{3}\bar{2}}\, d\xi_3 \wedge i(X_3)(\xi_1 \wedge \xi_2)\big) \\
&+ \Phi(X_2, X_3)\big(- (-1)^{\bar{1}+\bar{1}\bar{3}+\bar{3}\bar{2}}\, \xi_3 \wedge d\xi_1 \wedge i(X_1)\xi_2 - (-1)^{\bar{1}+\bar{1}\bar{2}}\, \xi_2 \wedge d\xi_1 \wedge i(X_1)\xi_3 \\
&\qquad + (-1)^{\bar{1}} d\xi_1 \wedge i(X_1)(\xi_2 \wedge \xi_3)\big) \\
&= d\Phi(X_1, X_2, X_3)\, \xi_1 \wedge \xi_2 \wedge \xi_3\,,
\end{aligned}
$$

which vanishes for a closed Φ. QED

Now, let us refer to the 2–form $\Phi[c] := \mathfrak{i}\ \mathrm{tr}\, R[c]$ associated with the curvature of c. Then, we can prove that the Hermitian tangent valued forms of the quantum bundle, as a graded Lie algebra, are isomorphic to the pairs of tangent valued forms and forms of the base space with the bracket twisted by $\mathfrak{i}\ \mathrm{tr}\, R[c]$, according to the following result.

3.8 Theorem. *The map*

$$
\mathfrak{j}[c] : \mathrm{sec}(\boldsymbol{E},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{E}) \times \mathrm{sec}(\boldsymbol{E},\, \Lambda^r T^*\boldsymbol{E}) \to \mathrm{her}\,(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q})
$$

is a graded Lie algebra isomorphism with respect to the graded Lie bracket $[\,,]_{\Phi[c]}$ *and the FN bracket.*

PROOF. We have

$$
\begin{aligned}
[c(\Xi),\, c(\Sigma)] &= c\big([\Xi,\, \Sigma]\big) - R[c](\Xi, \Sigma) = c\big([\Xi,\, \Sigma]\big) + \mathfrak{i}\, \Phi[c](\Xi,\, \Sigma)\, \mathbb{I}\,, \\
\big[c(\Xi),\, \mathfrak{i}\, \breve{\Sigma}\, \mathbb{I}\big] &= \mathfrak{i}\, (L(\Xi)\, \breve{\Sigma})\, \mathbb{I}\,, \\
\big[c(\Sigma),\, \mathfrak{i}\, \breve{\Xi}\, \mathbb{I}\big] &= \mathfrak{i}\, (L(\Sigma)\, \breve{\Xi})\, \mathbb{I}\,, \\
[\mathfrak{i}\, \breve{\Xi}\, \mathbb{I},\, \mathfrak{i}\, \breve{\Sigma}\, \mathbb{I}] &= 0\,,
\end{aligned}
$$

which implies

$$\begin{aligned}
\big[\mathfrak{j}[c](\underline{\Xi}, \breve{\Xi})\,,\ \mathfrak{j}[c](\underline{\Sigma}, \breve{\Sigma})\big] &= \big[c(\underline{\Xi}) + \mathfrak{i}\,\breve{\Xi}\,\mathbb{I}\,, \quad c(\underline{\Sigma}) + \mathfrak{i}\,\breve{\Sigma}\,\mathbb{I}\big] \\
&= \big[c(\underline{\Xi}),\ c(\underline{\Sigma})\big] + \big[c(\underline{\Xi}),\ \mathfrak{i}\,\breve{\Sigma}\,\mathbb{I}\big] + \big[\mathfrak{i}\,\breve{\Xi}\,\mathbb{I},\ c(\underline{\Sigma})\big] + \big[\mathfrak{i}\,\breve{\Xi}\,\mathbb{I},\ \mathfrak{i}\,\breve{\Sigma}\,\mathbb{I}\big] \\
&= c([\underline{\Xi},\, \underline{\Sigma}]) + \mathfrak{i}\,\big(\Phi[c](\underline{\Xi},\, \underline{\Sigma}) + L(\underline{\Xi})\,\breve{\Sigma} - (-1)^{rs}\,L(\underline{\Sigma})\,\breve{\Xi}\big)\,\mathbb{I} \\
&= \mathfrak{j}[c]\big([\underline{\Xi},\, \underline{\Sigma}]\,,\ \ \Phi[c](\underline{\Xi},\, \underline{\Sigma}) + L(\underline{\Xi})\,\breve{\Sigma} - (-1)^{rs}\,L(\underline{\Sigma})\,\breve{\Xi}\big) \\
&= \mathfrak{j}[c]\big(\big[(\underline{\Xi}, \breve{\Xi})\,,\ (\underline{\Sigma}, \breve{\Sigma})\big]_{\Phi[c]}\big)\,.\ \text{QED}
\end{aligned}$$

3.9 Corollary. The map

$$\mathrm{her}\,(\boldsymbol{Q},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{Q}) \to \sec(\boldsymbol{E},\, \Lambda^r T^*\boldsymbol{E} \otimes T\boldsymbol{E}) : \Xi \mapsto \underline{\Xi}$$

is a central extension of graded Lie algebras by $\sec(\boldsymbol{E},\, \Lambda^r T^*\boldsymbol{E})\,.\,\square$

In the Galilei and Einstein frameworks, the base space $\boldsymbol{E}$ is equipped with a geometric structure consisting of a metric, a gravitational connection and an electromagnetic field. This tructure yields a "special" graded Lie algebra of forms on the classical phase space and a distinguished system of quantum connections [4, 5]. Then, we can use this system of connections for classifying the Hermitian tangent valued forms of the quantum bundle. Even more, we can show that the graded Lie algebra of Hermitian tangent valued forms is isomorphic to the special graded Lie algebra of forms of the classical phase space. Indeed, this is the source of a procedure for quantisation of forms. But this matter is beyond the scope of the present purely geometric paper.

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