Classification of infinitesimal symmetries in covariant classical mechanics

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Abstract

In the framework of general relativistic classical mechanics on a spacetime with absolute time, we classify the infinitesimal symmetries of the classical structure by means of distinguished Lie subalgebras of the Lie algebra of "special phase functions".

These subalgebras are crucial also for the classification of infinitesimal quantum symmetries, which will be analysed in a forthcoming paper.

Key words: covariant classical mechanics, Lie algebras and symmetries, jets, cosymplectic manifolds.
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Contents

Introduction

This paper is the first part of a sequence of two papers. They are aimed at classifying systematically the symmetries of classical and quantum mechanics within the geometric framework of "covariant classical and quantum mechanics".

This framework is a geometric formulation of classical and quantum mechanics on a curved spacetime with absolute time and spacelike Riemannian metric, expressed in a manifestly coordinate free and observer independent way. We assume minimal axioms describing just the fundamental classical interactions, namely the gravitational and electromagnetic fields. The goal of this theory is to combine the standard quantum mechanics with those ideas and methods of Einstein's general relativity that are not related to the Lorentz metric and the speed of light, in order to understand quantum mechanics in a general relativistic observer independent way, as far as possible.

This approach requires non standard methods based on fibred manifolds, jets, connections and the Lie algebra of special phase functions. On the other hand, in the flat case the theory yields just the standard Schrödinger equation and the quantum operators for all usual examples. This approach has some analogies with other well-known geometric formulations of quantum mechanics, in particular, with geometric quantisation (see, for instance, [1, 31, 73].) But, it presents several methodological novelties and results as well, by overcoming several typical difficulties in the theory of geometric quantisation.

This approach was proposed in [35, 36] and further developed by several authors (see, for instance [40, 38, 65, 66] and references therein). On the other hand, several authors have been involved with a formulation of classical and quantum mechanics in the framework of a curved Galileian background (see, for instance, [13, 20, 21, 22, 23, 24, 25, 26, 34, 43, 44, 45, 46, 47, 49, 57, 69, 70, 71]). One of the typical features of covariant classical mechanics is the role played by a cosymplectic 2–form. This is a concept more general than that of contact 2–form and appropriate to account for the covariance of the theory. Actually, the literature on symplectic geometry is much wider and known than that on cosymplectic geometry; however, several authors have analysed the second one (see, for instance, [2, 8, 15, 19, 37, 51] and references therein).

In the present paper, we start by introducing the basic objects of the theory, namely the spacetime fibred over absolute time, the spacelike Riemannian metric, the spacetime gravitational connection and the electromagnetic field.

Then, we introduce the classical phase space and the main geometric objects induced by the spacetime structure, namely the contact maps, the phase connection, the 2nd order phase connection, the phase 2–form and the phase 2–vector. An important role is played by the phase 2–form, which encodes all other objects and turns out to be cosymplectic. This set up allows us to introduce in a natural way the distinguished Lie algebra of special phase functions and their various lifts. The special Lie bracket is linked to the Poisson bracket and allows us to deal with spacetime functions, momentum and energy on the same footing. An essential feature of special phase functions is that they admit a lift to the tangent space of spacetime.

Eventually, we classify systematically the infinitesimal symmetries of the above classical objects and show, step by step, that they are generated by distinguished subalgebras of the Lie algebra of special phase functions.

We observe that the classical Lagrangian formalism, Nöther's theorem and the momentum map [1, 12, 16, 17, 29, 30, 32, 48, 61, 62, 64, 68] arise naturally in the present scheme ruled by a cosymplectic 2–form and by the Lie algebra of special phase functions. The literature dealing with Lie algebras associated with geometric structures of analytical mechanics from different perspectives is very wide (see, for instance, [3, 4, 5, 6, 7, 9, 11, 14, 27, 28, 33, 50, 52, 53, 54, 56, 55, 59]). On the other hand, the present paper is devoted to the specific setting of covariant classical mechanics of a particle effected by the fundamental classical fields.

This paper extends considerably the results obtained in [66, 40].

The above results will play an essential role in the subsequent paper devoted to infinitesimal quantum symmetries, where we achieve analogous results by a similar approach. In particular, we will prove in this forthcoming paper that the Lie algebra of special phase functions yields a Lie algebra of quantum currents; the conserved probability current is just a particular case of this construction.

Thus, throughout the two papers, a crucial role is played by the Lie algebra of special phase functions, which turn out to be the generators of the infinitesimal symmetries both of the classical and quantum theories.

For each manifolds $\boldsymbol{M}$, $\boldsymbol{N}$ and each fibred manifolds $\boldsymbol{F} \to \boldsymbol{B}$, $\boldsymbol{G} \to \boldsymbol{B}$, we denote the sheaf of local maps $f : \boldsymbol{M} \to \boldsymbol{N}$ by $\operatorname{map}(\boldsymbol{M}, \boldsymbol{N})$, the sheaf of local sections $s : \boldsymbol{B} \to \boldsymbol{F}$ by $\sec(\boldsymbol{B}, \boldsymbol{F})$ and the sheaf of local fibred morphisms $f : \boldsymbol{F} \to \boldsymbol{G}$ over $\boldsymbol{B}$ by $\operatorname{fib}(\boldsymbol{F}, \boldsymbol{G})$. The capitalised symbols $\operatorname{Map}(\boldsymbol{M}, \boldsymbol{N})$, $\operatorname{Sec}(\boldsymbol{B}, \boldsymbol{F})$ and $\operatorname{Fib}(\boldsymbol{F}, \boldsymbol{G})$ will denote the corresponding sets of global maps.

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1 Covariant classical mechanics

1.1 Spacetime and its structure

1.1.1 Scale spaces

In the covariant formulation of physical theories the independence from the choice of coordinates and of units of measurements appear on the same footing. Thus, a rigorous treatment of units of merasurement is necessary.

We introduce the fundamental *scale spaces* $\mathbb{U}$ as "positive 1-dimensional semi-vector spaces" over $\mathrm{I\!R}^+$. A detailed account for this notion can be found in [36, 39]. Roughly speaking, they have the same algebraic structure as $\mathrm{I\!R}^+$, but no distinguished generator over $\mathrm{I\!R}^+$. We can naturally define the tensor product between scale spaces and ordinary vector spaces. Moreover, we can naturally define the rational powers $\mathbb{U}^{p/q}$ of a scale space $\mathbb{U}$. Rules analogous to those of real numbers hold for scale spaces; accordingly, we adopt analogous notation. In particular, we shall write $\mathbb{U}^0 := \mathbb{U}$, $\mathbb{U}^{-1} := \mathbb{U}^*$, $\mathbb{U}^p := \otimes^p \mathbb{U}$.

These spaces will appear in the theory tensorialised with spacetime tensors. The scale spaces appearing in tensor products are not effected by differential operators, hence their elements can be treated as constants.

We introduce the space $\mathbb{T}$ of *future oriented time intervals*, the space $\mathbb{L}$ of *lengths* and the space $\mathbb{M}$ of *masses*.

We shall refer to particles of mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$.

Moreover, we will consider time units $u_0 \in \mathbb{T}$, or their duals $u^0 \in \mathbb{T}^*$.

In order to unscale some objects of the theory, we will need a scale with scale dimensions of the *Planck constant* $\hbar : \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$. Actually, in the classical theory any such scale would do; on the other hand, in the quantum theory, we have to assume just the actual value of this scale.

1.1.2 Spacetime

Our basic framework is spacetime with its fibring over absolute time [36, 39].

We assume *time* to be an affine space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes \mathrm{I\!R}$ and *spacetime* to be an oriented 4–dimensional manifold $\boldsymbol{E}$ fibred over time by the *absolute time map* $t : \boldsymbol{E} \to \boldsymbol{T}$.

We shall refer to *spacetime charts* $(x^\lambda) = (x^0, x^i)$ adapted to the time fibring, to the affine structure of time, to a time unit of measurement $u_0 \in \bar{\mathbb{T}}$ and to the orientation of spacetime.

We shall be concerned with the *tangent space* $T\boldsymbol{E}$ of spacetime and its *vertical tangent* subspace $\jmath : V\boldsymbol{E} \subset T\boldsymbol{E}$, consisting of the vectors tangent to the fibres, which are called *spacelike*. Moreover, we shall be concerned with the *cotangent space* $T^*\boldsymbol{E}$ of spacetime and its *horizontal subspace* $H^*\boldsymbol{E} := \boldsymbol{E} \times \bar{\mathbb{T}}^* \subset T^*\boldsymbol{E}$, consisting of forms vanishing on vertical vectors, which are called *timelike*. Furthermore, we shall be concerned with the *horizontal* *space* $H\boldsymbol{E} := \boldsymbol{E} \times \bar{\mathbb{T}}$ and the *cotangent vertical* space $V^*\boldsymbol{E}$. The local coordinate bases of $T\boldsymbol{E}$, $V\boldsymbol{E}$, $T^*\boldsymbol{E}$, $H\boldsymbol{E}$ and $V^*\boldsymbol{E}$ are denoted by

$$\partial_\lambda \in \sec(\boldsymbol{E}, T\boldsymbol{E}), \quad \partial_i \in \sec(\boldsymbol{E}, V\boldsymbol{E}), \quad d^\lambda \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}), \quad d^0 \in \sec(\boldsymbol{E}, H^*\boldsymbol{E})$$
$$u_0 \in \sec(\boldsymbol{E}, H\boldsymbol{E}), \qquad \check{d}^i \in \sec(\boldsymbol{E}, V^*\boldsymbol{E}).$$

We have the distinguished scaled *time form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$, with coordinate expression $dt = u_0 \otimes d^0$. It generates the horizontal subbundle $H^*\boldsymbol{E} \subset T^*\boldsymbol{E}$. We shall often make the natural identification $u^0 \simeq d^0$ via pullback.

We have the natural *timelike projection* and *spacelike projection*

$$dt : T\boldsymbol{E} \to H\boldsymbol{E} : X \mapsto dt(X) \qquad \text{and} \qquad \jmath^* : T^*\boldsymbol{E} \to V^*\boldsymbol{E} : \alpha \mapsto \overset{\vee}{\alpha} := \alpha \circ \jmath,$$

with coordinate expressions $dt(X) = X^0\, u_0$ and $\overset{\vee}{\alpha} = \alpha_i\, \check{d}^i$.

In general, the check symbol "$^\vee$" will denote the vertical restriction of spacetime forms.

We stress that we *do not* have natural inclusions and projections of the following type

$$V^*\boldsymbol{E} \subset T^*\boldsymbol{E}, \qquad H\boldsymbol{E} \subset T\boldsymbol{E} \qquad \text{and} \qquad T\boldsymbol{E} \to V\boldsymbol{E}, \qquad T^*\boldsymbol{E} \to H^*\boldsymbol{E}.$$

This is an important feature of our relativistic model; indeed, we need the choice of an "observer" in order to achieve such inclusions and projections.

We shall be involved with the Lie subalgebras

$$\operatorname{proj}(\boldsymbol{E}, T\boldsymbol{E}) \subset \sec(\boldsymbol{E}, T\boldsymbol{E}) \qquad \text{and} \qquad \operatorname{fine}(\boldsymbol{E}, T\boldsymbol{E}) \subset \operatorname{proj}(\boldsymbol{E}, T\boldsymbol{E})$$

of spacetime vector fields which are projectable on $\boldsymbol{T}$ and whose time component is constant, respectively. Their coordinate expressions are of the type

$$X = X^0\, \partial_0 + X^i\, \partial_i \in \operatorname{proj}(\boldsymbol{E}, T\boldsymbol{E}), \qquad \text{with} \qquad X^0 \in \operatorname{map}(\boldsymbol{T}, \mathrm{I\!R}), X^i \in \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R}),$$
$$X = X^0\, \partial_0 + X^i\, \partial_i \in \operatorname{fine}(\boldsymbol{E}, T\boldsymbol{E}), \qquad \text{with} \qquad X^0 \in \mathrm{I\!R}, \qquad X^i \in \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R}).$$

1.1.3 Observers

Observers are essential tools for performing physical measurements. In the standard literature, the measurements are usually described by coordinates. But what is essentially necessary is the observer underlying a system of coordinates.

Our relativistic model does not exhibit any distinguished observer.

The choice of an observer yields the observed inclusions and projections which are not provided by the time fibring.

An *observer* is defined to be a (local) section $o : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$, which projects on $\boldsymbol{1} \in \mathbb{T}^* \otimes \mathbb{T}$, i.e. a (local) section $o : \boldsymbol{E} \to J_1\boldsymbol{E}$, where $J_1\boldsymbol{E}$ denotes the 1st jet space of spacetime. Thus, an observer is just a connection of the spacetime fibring.

The coordinate expression of an observer o is of the type $o = u^0 \otimes (\partial_0 + o^i_0 \, \partial_i)$, where $o^i_0 \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$. The charts (x^λ) for which $o^i_0 = 0$ are said to be *adapted* to o . Conversely, each chart (x^λ) is adapted to a unique observer, whose coordinate expression turns out to be $o := u^0 \otimes \partial_0$.

Each observer o yields the *observed spacelike projection* $\nu[o] : T\boldsymbol{E} \to V\boldsymbol{E} : X \mapsto X - o(dt(X))$ and the *observed timelike projection* $o^* : T^*\boldsymbol{E} \to H^*\boldsymbol{E} : \alpha \circ o \simeq o \lrcorner \alpha$, whose coordinate expressions are $\nu[o] = (d^i - o^i_0 \, d^0) \otimes \partial_i$ and $o^* = d^0 \otimes (\partial_0 + o^i_0 \, \partial_i)$.

Moreover, an observer o yields the *observed spacelike inclusion* $\nu^*[o] : V^*\boldsymbol{E} \hookrightarrow T^*\boldsymbol{E} : \alpha \mapsto \alpha \circ \nu[o]$ and the *observed timelike inclusion* $o : H\boldsymbol{E} \hookrightarrow T\boldsymbol{E} : X \mapsto o(X)$, whose coordinate expressions are $\nu^*[o](\alpha) = \alpha_i \, (d^i - o^i_0 \, d^0)$ and $o(X) = X^0 \, (\partial_0 + o^i_0 \, \partial_i)$.

Thus, an observer o yields the *observed splittings* of the tangent and cotangent spaces of spacetime into the direct sum of their timelike and spacelike components

$$T\boldsymbol{E} = H\boldsymbol{E} \oplus V\boldsymbol{E} : X \mapsto dt(X) + \nu[o](X) \,, \quad T^*\boldsymbol{E} = H^*\boldsymbol{E} \oplus V^*\boldsymbol{E} : \alpha \mapsto o^*(\alpha) + \jmath^*(\alpha) \,.$$

1.1.4 Metric field

The fibres of spacetime are equipped with a given Riemannian metric [36, 39].

We assume spacetime to be equipped with a scaled Riemannian metric of the fibres $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$.

With reference to a particle of mass m , it is useful to define the *rescaled spacelike metric* $G := \frac{m}{\hbar} \, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$.

We denote the contravariant spacelike metric and the contravariant rescaled spacelike metric by $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{*2} \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})$ and $\bar{G} := \frac{\hbar}{m} \, \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})$.

The spacelike metric g and the spacetime orientation naturally yield the scaled *spacelike volume form* $\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E}$, with coordinate expression $\eta = \sqrt{|g|} \, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3$.

Moreover, the time form and the spacelike volume form yield the scaled *spacetime volume form* $\upsilon := dt \wedge \eta : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^*\boldsymbol{E}$, with coordinate expression $\upsilon = \sqrt{|g|} \, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3$.

For each $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, the Lie derivative $L[X] \, \bar{G} \in \sec\Big(\boldsymbol{E}, \; \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})\Big)$ has the coordinate expression $L[X] \, \bar{G} = (X^\lambda \, \partial_\lambda G^{ij}_0 - G^{hj}_0 \, \partial_h X^i - G^{ih}_0 \, \partial_h X^j) \, u^0 \otimes \partial_i \otimes \partial_j$.

For each $X \in \mathrm{proj}(\boldsymbol{E}, T\boldsymbol{E})$, the Lie derivative $L[X] \, G \in \sec\Big(\boldsymbol{E}, \; \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})\Big)$, has the coordinate expression $L[X] \, G = (X^\lambda \, \partial_\lambda G^0_{ij} + G^0_{hj} \, \partial_i X^h + G^0_{ih} \, \partial_j X^h) \, u_0 \otimes \check{d}^i \otimes \check{d}^j$.

For each $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, the *spacetime divergence* $\mathrm{div}_\upsilon X \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$ is well defined by the equality $L[X] \, \upsilon = (\mathrm{div}_\upsilon X) \, \upsilon$.

For each $X \in \mathrm{proj}(\boldsymbol{E}, T\boldsymbol{E})$, the *spacelike divergence* $\mathrm{div}_\eta X \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$ and the *timelike divergence* $\mathrm{div}_{\mathrm{dt}} X \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$ are well defined, respectively, by the equalities $L[X] \, \eta = (\mathrm{div}_\eta X) \, \eta$ and $L[X] \, dt = (\mathrm{div}_{\mathrm{dt}} X) \, dt$.

We have the coordinate expressions

$$\mathrm{div}_\upsilon X = \frac{\partial_\lambda (X^\lambda \sqrt{|g|})}{\sqrt{|g|}}, \quad \mathrm{div}_\eta X = X^0 \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_j (X^j \sqrt{|g|})}{\sqrt{|g|}}, \quad \mathrm{div}_{\mathrm{dt}} X = \partial_0 X^0 .$$

Hence, for each $X \in \mathrm{proj}(\boldsymbol{E}, T\boldsymbol{E})$, we obtain $\mathrm{div}_\upsilon X = \mathrm{div}_{\mathrm{dt}} X + \mathrm{div}_\eta X$ and, for each $X \in \mathrm{fine}(\boldsymbol{E}, T\boldsymbol{E})$, we obtain $\mathrm{div}_\upsilon X = \mathrm{div}_\eta X$.

Moreover, for each $X \in \mathrm{proj}(\boldsymbol{E}, T\boldsymbol{E})$, we obtain $\mathrm{div}_\eta X = \frac{1}{2} \langle \bar{G}, L[X]\, G \rangle$.

1.1.5 Gravitational and electromagnetic fields

Spacetime is equipped with given gravitational and electromagnetic fields [36].

We assume spacetime to be equipped with a *gravitational field*, i.e. with a linear connection $K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$, such that $\nabla^\natural dt = 0$ and $\nabla^\natural g = 0$, and such that its curvature tensor $R[K^\natural]$ fulfills the symmetry condition $R^\natural{}_{i\lambda j\mu} = R^\natural{}_{j\mu i\lambda}$.

1.1 Proposition. The coordinate expression of the gravitational field is of the type

$$\begin{aligned}
K^\natural{}_\lambda{}^0{}_\mu &= 0 \\
K^\natural{}_0{}^i{}_0 &= -G_0^{ij}\, 2\, \Phi_{0j} \\
K^\natural{}_0{}^i{}_h = K^\natural{}_h{}^i{}_0 &= -\tfrac{1}{2}\, G_0^{ij}\, (\partial_0 G^0_{jh} + 2\, \Phi_{hj}) \\
K^\natural{}_k{}^i{}_h = K^\natural{}_h{}^i{}_k &= -\tfrac{1}{2}\, G_0^{ij}\, (\partial_h G^0_{jk} - \partial_j G^0_{hk} + \partial_k G^0_{jh}),
\end{aligned}$$

where $\Phi^\natural[o] = \Phi^\natural{}_{\lambda\mu}\, d^\lambda \wedge d^\mu \in \sec(\boldsymbol{E}, \Lambda^2 T^*\boldsymbol{E})$ is a closed 2–form, which depends on the chosen chart only through the associated observer o. □

We shall denote by $A^\natural[o]$ the local potentials of $\Phi^\natural[o]$, according to $2\, dA^\natural[o] = \Phi^\natural[o]$.

We assume spacetime to be equipped with an *electromagnetic field*, i.e. with a closed scaled 2-form $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$ (see also [49]).

Given a particle with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$, it is convenient to consider, respectively, the unscaled field and the rescaled field

$$\tfrac{q}{\hbar}\, F : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \qquad \text{and} \qquad \tfrac{q}{m}\, \widehat{F} : \boldsymbol{E} \to \mathbb{T}^{-1} \otimes (V\boldsymbol{E} \otimes T^*\boldsymbol{E}),$$

where $\widehat{F} = g^{\sharp 2}(F) = g^{ih}\, F_{\lambda h}\, \partial_i \otimes d^\lambda$.

The electromagnetic field F can be "joined", in a covariant way, to the gravitational field yielding the *"joined" spacetime connection* $K = K^\natural - dt \otimes \frac{q}{2m}\, \widehat{F} - \frac{q}{2m}\, \widehat{F} \otimes dt$.

The joined K still fulfills the properties that we have assumed for $K^\natural$. Moreover, all objects derived from the joined connection split into components related to the gravitational and the electromagnetic fields.

In particular, the observed potential $A[o]$ of the joined connection splits into the sum of the gravitational and electromagnetic potentials as $A[o] = A^\natural[o] + \frac{q}{\hbar}\, A^\mathfrak{e}$, where

$A^{\mathfrak{e}} \in \sec\Big(\boldsymbol{E},\; (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}\Big)$ is a local (observer independent) potential of F, according to $2\, dA^{\mathfrak{e}} = F$.

Thus, from now on, with reference to a particle of mass m and charge q, we shall refer to the spacetime structure constituted by the 4-plet $(\boldsymbol{E},\, t,\, G,\, K)$, whose elements fulfill the properties mentioned above.

1.1.6 Basic model of spacetime

> The present paper deals with a curved spacetime. However, this model includes flat or partially flat spacetimes, as well [36]. Thus, the standard mechanics can be recovered as a particular case of our theory.

The simplest model of spacetime is given by the following construction.

We consider an oriented affine space $\boldsymbol{E}$ that is associated with the vector space $\bar{\boldsymbol{E}}$ and equipped with an affine map $t : \boldsymbol{E} \to \boldsymbol{T}$ of rank 1.

Let us consider the vector subspace $\boldsymbol{S} := \ker Dt \subset \bar{\boldsymbol{E}}$. We can easily see that all fibres of the fibring $t : \boldsymbol{E} \to \boldsymbol{T}$ are affine subspaces of $\boldsymbol{E}$ associated with the same vector space $\boldsymbol{S}$. Hence, the spacetime fibred space turns out to be an abelian principal bundle with structural group $\boldsymbol{S}$.

We assume as spacelike metric a constant Euclidean metric on $\boldsymbol{S}$.

Moreover, we assume as gravitational connection the connection induced by the affine structure of spacetime. Eventually, we assume a vanishing electromagnetic field.

Clearly, the above objects fulfill our axioms. We call such a spacetime a *special Newtonian spacetime*. In this model we can easily define the standard inertial motions and inertial observers.

We could consider also a little more complex model, which assumes the previous structure for background and adds a gravitational connection which is Ricci flat. This model accounts for the standard notions of classical mechanics, including Newton's law of gravitation.

The rigid body provides a further non trivial example of our model [63].

1.2 Phase space and the induced structure

1.2.1 Classical phase space

> We assume as phase space of a classical particle the 1st jet space of sections of spacetime [36, 39].

The 1st jet space ([42, 67]) $J_1\boldsymbol{E}$ of $t : \boldsymbol{E} \to \boldsymbol{T}$ is a fibred manifold $t^1 : J_1\boldsymbol{E} \to \boldsymbol{T}$ over $\boldsymbol{T}$ and an affine bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ over $\boldsymbol{E}$, associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$. Hence, the vertical space of $J_1\boldsymbol{E}$ with respect to $\boldsymbol{E}$ turns out to be $V_0 J_1\boldsymbol{E} = \mathbb{T}^* \otimes V\boldsymbol{E}$.

We denote the fibred charts of the phase space by (x^0, x^i, x^i_0).

The above affine structure yields the natural tensor $\nu : J_1\boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V J_1\boldsymbol{E})$, with $\nu = u_0 \otimes \check{d}^i \otimes \partial^0_i$.

We recall the natural *contact maps* $д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ and $\theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$, with $д = u^0 \otimes (\partial_0 + x^i_0\, \partial_i)$ and $\theta = \partial_i \otimes (d^i - x^i_0\, d^0)$. From now on, $J_1\boldsymbol{E}$ is considered as a subspace of $\mathbb{T}^* \otimes T\boldsymbol{E}$, according to the natural embedding $д$.

We shall be involved with the Lie subalgebras

$$\mathrm{proj}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \quad \text{and} \quad \mathrm{fine}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{proj}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$$

of vector fields of $J_1\boldsymbol{E}$, which are projectable on $\boldsymbol{E}$ and $\boldsymbol{T}$, and additionally whose time components are constant, respectively.

1.2.2 Holonomic prolongation of spacetime vector fields

We have a natural prolongation of spacetime vector fields to phase vector fields.

1.2 Proposition. [39, 58] There is a natural fibred morphism $r_1 : J_1T\boldsymbol{E} \to TJ_1\boldsymbol{E}$ over $J_1\boldsymbol{E} \underset{\boldsymbol{T}}{\times} J_1T\boldsymbol{T}$, with $(x^0, x^i, x^i_0;\, \dot{x}^0, \dot{x}^i, \dot{x}^i_0) \circ r_1 = (x^0, x^i, x^i_0;\, \dot{x}^0, \dot{x}^i, \dot{x}^i_0 - x^i_0\, \dot{x}^0_0)$.

Then, for each $X \in \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E})$, we obtain the vector field, called *1st holonomic prolongation* of X, $X_{(1)} := r_1 \circ J_1 X \in \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, which projects on X. Its coordinate expression is $X_{(1)} = X^0\, \partial_0 + X^i\, \partial_i + (\partial_0 X^i + \partial_j X^i\, x^j_0 - \partial_0 X^0\, x^i_0 - \partial_j X^0\, x^j_0\, x^i_0)\, \partial^0_i$.

The map $r_1 \circ J_1 : \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E}) \to \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : X \mapsto X_{(1)}$ turns out to be an injective $\mathbb{R}$–linear morphism of Lie algebras. □

In the particular case when the vector field X is projectable on $\boldsymbol{T}$, we recover the standard holonomic prolongation obtained through the 1st jet prolongation of the fibred flow of X. In fact, the above map restricts to the injective $\mathbb{R}$–linear map

$$\mathrm{proj}(\boldsymbol{E}, T\boldsymbol{E}) \to \mathrm{proj}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : X \mapsto X_{(1)},$$

with coordinate expression $X_{(1)} = X^0\, \partial_0 + X^i\, \partial_i + (\partial_0 X^i + \partial_j X^i\, x^j_0 - \partial_0 X^0\, x^i_0)\, \partial^0_i$.

Later, we shall use the following technical results.

1.3 Lemma. [39] For each $X \in \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E})$, we have the equalities

$$L[X_{(1)}]\,\theta = \Big(д.\big(dt(X)\big)\Big)\,\theta \qquad \text{and} \qquad L[X_{(1)}]\,д = -\Big(д.\big(dt(X)\big)\Big)\,д,$$

with coordinate expressions

$$L[X_{(1)}]\,\theta = (\partial_0 X^0 + \partial_j X^0\, x^j_0)\,(d^i - x^i_0\, d^0) \otimes \partial^0_i$$
$$L[X_{(1)}]\,д = -(\partial_0 X^0 + x^j_0\, \partial_j X^0)\,(\partial_0 + x^i_0\, \partial_i)\,. \;\square$$

For each $X^\uparrow \in \mathrm{proj}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, which projects on $X \in \mathrm{proj}(\boldsymbol{E}, T\boldsymbol{E})$, the *spacelike divergence* $\mathrm{div}_\eta X^\uparrow \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$ is well defined by the equality $L[X^\uparrow]\,\eta = (\mathrm{div}_\eta X^\uparrow)\,\eta$ and we obtain $\mathrm{div}_\eta X^\uparrow = \mathrm{div}_\eta X$.

1.2.3 Distinguished phase fields

The spacetime connection and the rescaled metric yield in a covariant way further objects on the phase space [36, 39].

The spacetime connection K yields a torsion free affine connection of the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, called *phase connection*, $\Gamma[K] : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$, with coordinate expression $\Gamma_{\lambda}{}^{i}_{0} := \Gamma_{\lambda}{}^{i}{}_{j}{}^{0}_{0}\, x^j_0 + \Gamma_{\lambda}{}^{i}{}_{0}{}^{0}_{0}$, where $\Gamma_{\lambda}{}^{i}{}_{\mu}{}^{0}_{0} = K_{\lambda}{}^{i}{}_{\mu}$. Conversely, Γ characterises K.

The phase connection Γ splits into the gravitational and electromagnetic components as $\Gamma = \Gamma^{\natural} + \Gamma^{\mathfrak{e}}$, where $\Gamma^{\mathfrak{e}} = -\frac{q}{2m}\,(\widehat{F} + д \lrcorner \widehat{F}) \in \sec\Big(J_1\boldsymbol{E},\ \ \mathbb{T}^* \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E})\Big)$.

We have $\Gamma^{i\,j}_{0\,0} - \Gamma^{j\,i}_{0\,0} = -G^{ih}_0 G^{jk}_0 \Big((\partial_h G^0_{kl} - \partial_k G^0_{hl})\, x^l_0 + \partial_h A_k - \partial_k A_h\Big)$, with $\Gamma^{hk}_{00} := G^{hl}_0\, \Gamma_{l}{}^{k}_{0}$.

Then, Γ yields the *2nd order connection* $\gamma[\Gamma] := д \lrcorner \Gamma : J_1\boldsymbol{E} \to J_2\boldsymbol{E} \subset \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$, with $\gamma = u^0 \otimes (\partial_0 + x^i_0\, \partial_i + \gamma_{0}{}^{i}_{0}\, \partial^0_i)$, where

$$
\begin{aligned}
\gamma_{0}{}^{i}_{0} &= K_h{}^i{}_k\, x^h_0\, x^k_0 + 2\, K_h{}^i{}_0\, x^h_0 + K_0{}^i{}_0 = \\
&= -\, G^{ij}_0 \Big((\partial_h G^0_{jk} - \tfrac{1}{2}\, \partial_j G^0_{hk})\, x^h_0\, x^k_0 + (\partial_0 G^0_{hj} + \partial_h A_j - \partial_j A_h)\, x^h_0 + \partial_0 A_j - \partial_j A_0\Big).
\end{aligned}
$$

Conversely, γ characterises Γ.

The 2nd order connection γ splits into the gravitational and electromagnetic components as $\gamma^{\natural} + \gamma^{\mathfrak{e}}$, where $\gamma^{\mathfrak{e}} = -\frac{q}{m}\, д \lrcorner \widehat{F} : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$ equals the Lorentz force.

Then, Γ and G yield the *phase 2–form* $\Omega[G, \Gamma] := G \lrcorner(\nu[\Gamma] \wedge \theta) : J_1\boldsymbol{E} \to \Lambda^2 J_1T\boldsymbol{E}$, with $\Omega = G^0_{ij}\, (d^i_0 - \gamma_{0}{}^{i}_{0} d^0 - \Gamma_{h}{}^{i}_{0}\, \theta^h) \wedge \theta^j$, and where $\nu[\Gamma]$ is the vertical valued form associated with Γ. Conversely, Ω characterises Γ and G [66].

The 2–form Ω splits into the gravitational and electromagnetic components as $\Omega = \Omega^{\natural} + \Omega^{\mathfrak{e}}$, where $\Omega^{\mathfrak{e}} = \frac{q}{2\hbar}\, F$.

Then, Γ and G yield the 2–vector $\Lambda[G, \Gamma] := \bar{G} \lrcorner(\check{\Gamma} \wedge \nu) : J_1\boldsymbol{E} \to \Lambda^2 VJ_1\boldsymbol{E}$, with $\Lambda = G^{ij}_0\, (\partial_i + \Gamma_{i}{}^{h}_{0}\, \partial^0_h) \wedge \partial^0_j$, where $\check{\Gamma} : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \otimes VJ_1\boldsymbol{E}$ is the vertical restriction of Γ.

The 2–vector Λ splits into the gravitational and electromagnetic components as $\Lambda = \Lambda^{\natural} + \Lambda^{\mathfrak{e}}$, where $\Lambda^{\mathfrak{e}} = \frac{q}{2\hbar}\, G^{\sharp}(F) : J_1\boldsymbol{E} \to \Lambda^2 VJ_1\boldsymbol{E}$.

We have the identities $\quad i(\gamma)\, dt = 1\,, \quad i(\gamma)\, \Omega = 0\,, \quad L[\gamma]\, \Lambda = 0\,, \quad [\Lambda, \Lambda] = 0\,.$

Therefore, $(J_1\boldsymbol{E},\, dt, \Omega)$ turns out to be a scaled cosymplectic manifold, where γ is the associated scaled Reeb vector field and Λ is the associated 2–vector.

We have the following results which will be used later.

1.4 Proposition. [66] For each $X \in \mathrm{proj}(\boldsymbol{E}, T\boldsymbol{E})$, the following implications holds

$$
\begin{aligned}
L[X_{(1)}]\, \Omega = 0 \qquad &\Leftrightarrow \qquad L[X_{(1)}]\, \Gamma = 0\,, \quad L[X]\, G = 0\,, \\
L[X_{(1)}]\, \Omega = 0 \qquad &\Rightarrow \qquad \mathrm{div}_\eta\, X = 0\,. \square
\end{aligned}
$$

1.3 Classical mechanics

The spacetime structure and the joined connection allow us to formulate the dynamics of a classical particle under the action of the gravitational and electromagnetic fields [36, 39].

1.3.1 Classical kinematics

A *motion* is defined to be a section $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$ and its *absolute velocity* is defined to be the 1st jet prolongation $j_1 s \in \sec(\boldsymbol{T}, J_1\boldsymbol{E})$.

An observer o yields the following objects:

- the affine fibred morphism $\nabla[o] := \mathrm{id} - (o \circ t^1_0) \in \mathrm{fib}(J_1\boldsymbol{E},\, \mathbb{T}^* \otimes V\boldsymbol{E})$ over $\boldsymbol{E}$,
- the observed *kinetic momentum* $\mathcal{Q}[o] := G^\flat \circ \nabla[o] : J_1\boldsymbol{E} \to V^*\boldsymbol{E}$ and the observed *kinetic energy* $\mathcal{K}[o] := \frac{1}{2}\, G \lrcorner (\nabla[o] \otimes \nabla[o]) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathrm{I\!R}$,
- for each motion s, the *observed velocity* $\nabla[o]s := j_1 s - o \circ s \in \sec(\boldsymbol{T},\, \mathbb{T}^* \otimes V\boldsymbol{E})$,

Their expressions, in adapted coordinates, are

$$\nabla[o] = x^i_0\, u^0 \otimes \partial_i\,, \quad \mathcal{Q}[o] = G^0_{ij}\, x^j_0 \check{d}^i\,, \quad \mathcal{K}[o] = \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0\, u^0\,, \quad \nabla[o]s = \partial_0 s^i\, u^0 \otimes \partial_i\,.$$

In the special Newtonian spacetime, we can also define the usual *observed angular momentum* $\mathcal{M}[o, c] := r[c] \times \mathcal{Q}[o]$ with respect to an inertial motion c as reference center.

For each motion s, the absolute gravitational *acceleration* is defined to be the section $\nabla[\gamma^\natural] j_1 s := j_2 s - (\gamma^\natural \circ j_1 s) \in \sec(\boldsymbol{T},\, \mathbb{T}^* \otimes \mathbb{T}^* \otimes V\boldsymbol{E})$ and the absolute joined *acceleration* is defined to be the section $\nabla[\gamma] j_1 s := j_2 s - (\gamma \circ j_1 s) \in \sec(\boldsymbol{T},\, \mathbb{T}^* \otimes \mathbb{T}^* \otimes V\boldsymbol{E})$. Clearly, we have $\nabla[\gamma] j_1 s = \nabla[\gamma^\natural] j_1 s - \gamma^\mathfrak{e} \circ j_1 s$. We have the coordinate expression

$$\nabla[\gamma] j_1 s = \Big(\partial_{00} s^i - (K_h{}^i{}_k \circ s)\, \partial_0 s^h\, \partial_0 s^k - 2\, (K_0{}^i{}_h \circ s)\, \partial_0 s^h - (K_0{}^i{}_0 \circ s)\Big)\, u^0 \otimes u^0 \otimes \partial_i\,.$$

1.3.2 Classical dynamics

We assume the *generalised Newton's law* as equation of motion for classical dynamics $\nabla[\gamma] j_1 s := j_2 s - \gamma \circ j_1 s = 0$.

A function $f \in \mathrm{map}(J_1\boldsymbol{E},\, \mathrm{I\!R})$ such that $\gamma.f = 0$ is said to be *conserved*. We denote the subsheaf of conserved functions by $\mathrm{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R}) \subset \mathrm{map}(J_1\boldsymbol{E},\, \mathrm{I\!R})$.

We can also obtain the classical dynamics by a Lagrangian formalism according to a cohomological procedure in the following way.

The phase 2-form Ω admits locally *horizontal potentials* $A^\uparrow \in \sec(J_1\boldsymbol{E},\, T^*\boldsymbol{E})$, which are defined up to a closed spacetime form $\alpha \in \sec(\boldsymbol{E},\, T^*\boldsymbol{E})$. Each horizontal potential $A^\uparrow$ splits, in a covariant way, as $A^\uparrow = \mathcal{L}[A^\uparrow] + \mathcal{P}[A^\uparrow]$, through the horizontal component $\mathcal{L}[A^\uparrow] := д \lrcorner A^\uparrow$, called *Lagrangian*, and the д–vertical component $\mathcal{P}[A^\uparrow] := \theta \lrcorner A^\uparrow$, called *momentum*. Moreover, we obtain $D\mathcal{L}[A^\uparrow] = \check{\mathcal{P}}[A^\uparrow]$. Hence, each horizontal potential $A^\uparrow$ turns out to be just the *Poincaré–Cartan form* Θ of the associated Lagrangian $\mathcal{L}[A^\uparrow]$.

Moreover, the horizontal component of Ω turns out to be the fibred morphism $\mathcal{E} = G^\flat(\nabla[\gamma]) : J_2\boldsymbol{E} \to \mathbb{T}^* \otimes V^*\boldsymbol{E}$. Indeed, $\mathcal{E}$ turns out to be just the Euler-Lagrange operator associated with the Lagrangian $\mathcal{L}(A^\uparrow)$, for each horizontal potential $A^\uparrow$.

Next, let us choose a horizontal potential $A^\uparrow$ and an observer o.

Then, we define the *observed potential* to be the spacetime 1-form $A[o] := o^* A^\uparrow \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$. This form turns out to be just an observed potential of the joined connection K. Moreover, we define the observed *Hamiltonian* to be the function $\mathcal{H}[o] := -o \lrcorner A^\uparrow$ and the observed *momentum* to be the form $\mathcal{P}[o] := \nu[o] \lrcorner A^\uparrow$.

We have the following expressions, in adapted coordinates,

$$A^\uparrow = -\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0\, d^0 + G^0_{ij}\, x^j_0\, d^i + A_\lambda\, d^\lambda\,,$$

$$\mathcal{L} = \mathcal{L}_0\, d^0 = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_i\, x^i_0 + A_0)\, d^0\,, \qquad \mathcal{P} = (G^0_{ij}\, x^j_0 + A_i)\,(d^i - x^i_0\, d^0)$$

$$A[o] = A_\lambda\, d^\lambda\,, \qquad \mathcal{H}[o] = \mathcal{H}_0\, d^0 = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0\,, \qquad \mathcal{P}[o] = (G^0_{ij}\, x^j_0 + A_i)\, d^i\,.$$

1.4 Hamiltonian methods

We devote this section to the basic recalls concerning the splitting of the tangent space of the phase space, the Hamiltonian lift of phase functions and the Poisson bracket of phase functions [36, 39].

1.4.1 Hamiltonian splitting

The time fibring and the 2nd order connection yield in a covariant way a splitting of the tangent and cotangent spaces of the phase space.

We have the natural dual splittings over $J_1\boldsymbol{E}$

$$TJ_1\boldsymbol{E} = H_\gamma J_1\boldsymbol{E} \oplus VJ_1\boldsymbol{E} \qquad \text{and} \qquad T^*J_1\boldsymbol{E} = H^*J_1\boldsymbol{E} \oplus V^*_\gamma J_1\boldsymbol{E}\,,$$

given by $X^\uparrow = dt(X^\uparrow)\,\gamma + \big(X^\uparrow - dt(X^\uparrow)\,\gamma\big)$ and $\phi^\uparrow = \phi^\uparrow(\gamma)\, dt + \big(\phi^\uparrow - \phi^\uparrow(\gamma)\, dt\big)$, where

- $VJ_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}$ is the vertical subbundle with respect to dt,
- $H^*J_1\boldsymbol{E} \subset T^*J_1\boldsymbol{E}$ is the horizontal subbundle generated by dt,
- $H_\gamma J_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}$ is the horizontal subbundle generated by γ,
- $V^*_\gamma J_1\boldsymbol{E} \subset T^*J_1\boldsymbol{E}$ is the vertical subbundle of forms which annihilate γ.

We define the *musical morphisms* to be the linear maps

$$\Omega^\flat : \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \to \sec(J_1\boldsymbol{E}, V^*_\gamma J_1\boldsymbol{E}) : X^\uparrow \mapsto i(X^\uparrow)\,\Omega\,,$$
$$\Lambda^\sharp : \sec(J_1\boldsymbol{E}, T^*J_1\boldsymbol{E}) \to \sec(J_1\boldsymbol{E}, VJ_1\boldsymbol{E}) : \phi^\uparrow \mapsto i(\phi^\uparrow)\,\Lambda\,.$$

The musical morphisms restrict to the mutually inverse linear maps

$$\Omega^\flat{}_0 : \sec(J_1\boldsymbol{E}, VJ_1\boldsymbol{E}) \to \sec(J_1\boldsymbol{E}, V^*_\gamma J_1\boldsymbol{E})$$
$$\Lambda^\sharp{}_0 : \sec(J_1\boldsymbol{E}, V^*_\gamma J_1\boldsymbol{E}) \to \sec(J_1\boldsymbol{E}, VJ_1\boldsymbol{E})\,.$$

For each $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and $\phi^{\uparrow} \in \sec(J_1\boldsymbol{E}, T^*J_1\boldsymbol{E})$, we obtain the equalities

$$(\Lambda^\sharp \circ \Omega^\flat)(X^{\uparrow}) = X^{\uparrow} - dt(X^{\uparrow})\,\gamma\,, \qquad \Omega^\flat(X^{\uparrow}) = (\Lambda^\sharp{}_0)^{-1}\Big(X^{\uparrow} - dt(X^{\uparrow})\,\gamma\Big)\,,$$
$$(\Omega^\flat \circ \Lambda^\sharp)(\phi^{\uparrow}) = \phi^{\uparrow} - \phi^{\uparrow}(\gamma)\,dt\,, \qquad \Lambda^\sharp(\phi^{\uparrow}) = (\Omega^\flat{}_0)^{-1}\Big(\phi^{\uparrow} - \phi^{\uparrow}(\gamma)\,dt\Big)\,.$$

Hence, we can write $X^{\uparrow} = dt(X^{\uparrow})\,\gamma + (\Lambda^\sharp \circ \Omega^\flat)\,(X^{\uparrow})$ and $\phi^{\uparrow} = \phi^{\uparrow}(\gamma)\,dt + (\Omega^\flat \circ \Lambda^\sharp)\,(\phi^{\uparrow})$.

Given a *time scale* $\tau \in \mathrm{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$, we define the τ–horizontal subbundle $H_\tau J_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}$ consisting of vectors whose time components are given by τ. Then, we obtain the mutually inverse affine maps

$$\Omega^\flat{}_\tau : \sec(J_1\boldsymbol{E}, H_\tau J_1\boldsymbol{E}) \to \sec(J_1\boldsymbol{E}, V^*_\gamma J_1\boldsymbol{E}) : X^{\uparrow} \mapsto i(X^{\uparrow})\,\Omega\,,$$
$$\Lambda^\sharp{}_\tau : \sec(J_1\boldsymbol{E}, V^*_\gamma J_1\boldsymbol{E}) \to \sec(J_1\boldsymbol{E}, H_\tau J_1\boldsymbol{E}) : \phi^{\uparrow} \mapsto \gamma(\tau) + i(\phi^{\uparrow})\,\Lambda\,.$$

For each $X^{\uparrow}, \bar{X}^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and $\phi^{\uparrow}, \bar{\phi}^{\uparrow} \in \sec(J_1\boldsymbol{E}, T^*J_1\boldsymbol{E})$, we have the following equivalences:

$$X^{\uparrow} = \bar{X}^{\uparrow} \quad \Leftrightarrow \quad dt(X^{\uparrow}) = dt(\bar{X}^{\uparrow})\,, \quad \Omega^\flat(X^{\uparrow}) = \Omega^\flat(\bar{X}^{\uparrow})$$
$$\phi^{\uparrow} = \bar{\phi}^{\uparrow} \quad \Leftrightarrow \quad \phi^{\uparrow}(\gamma) = \phi^{\uparrow}(\bar{\gamma})\,, \quad \Lambda^\sharp(\phi^{\uparrow}) = \Lambda^\sharp(\bar{\phi}^{\uparrow})\,.$$

1.4.2 Poisson bracket

We introduce the Poisson bracket in our cosymplectic framework by an approach which is rather analogous to that of symplectic manifolds [36, 39].

We define the *Poisson bracket* on $\mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$ by $\{f,\, g\} := i(df \wedge dg)\,\Lambda$, which has the coordinate expression $\{f, g\} = G^{ij}_0\,(\partial_i f\,\partial^0_j g - \partial_i g\,\partial^0_j f) - (\Gamma^{i\,j}_{00} - \Gamma^{j\,i}_{00})\,\partial^0_i f\,\partial^0_i g$.

For each $f, g \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$, we have [39] $\gamma.\{f, g\} = \{\gamma.f, g\} + \{f, \gamma.g\}$. Hence, the subsheaf $\mathrm{cons}(J_1\boldsymbol{E}, \mathbb{R}) \subset (J_1\boldsymbol{E}, \mathbb{R})$ is closed with respect to the Poisson bracket.

1.4.3 Hamiltonian lift of phase functions

In our cosymplectic framework we can introduce the vertical Hamiltonian lift, which partially resembles the usual Hamiltonian lift of symplectic manifolds. Moreover, we can introduce a further affine Hamiltonian lift which depends on an arbitrary choice of a time scale. Furthermore, we obtain a distinguished affine Hamiltonian lift through the distinguished time scale exhibited by each phase function [36, 39].

For each $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$, we define its *vertical Hamiltonian lift* to be the vertical vector field $\Lambda^\sharp(df) = (\Omega^\flat{}_0)^{-1}(df - \gamma.f) \in \sec(J_1\boldsymbol{E}, VJ_1\boldsymbol{E})$, with coordinate expression $\Lambda^\sharp(df) = -G^{ij}_0\,\partial^0_j f\,\partial_i + \Big(G^{ij}_0\,\partial_j f + (\Gamma^{i\,j}_{00} - \Gamma^{j\,i}_{00})\,\partial^0_j f\Big)\,\partial^0_i$.

For each $f, g \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$, we have $[\Lambda^\sharp(df),\, \Lambda^\sharp(dg)] = \Lambda^\sharp\Big(d\{f, g\}\Big)$. Hence, the map $\mathrm{map}(J_1\boldsymbol{E}, \mathbb{R}) \to \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto \Lambda^\sharp(df)$ turns out to be a morphism of Lie algebras.

For each $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we define its *Hamiltonian lift*, with respect to the *time scale* $\tau \in \mathrm{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$, to be the vector field $X^{\uparrow}{}_{\mathrm{ham}}[\tau, f] := \gamma(\tau) + \Lambda^\sharp(df) \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, with coordinate expression

$$H^{\uparrow}{}_{\mathrm{ham}}[\tau, f] = \tau^0\,\partial_0 + (\tau^0\,x_0^i - G_0^{ij}\,\partial_j^0 f)\,\partial_i + \Big(\tau^0\,\gamma_0{}_0^i + G_0^{ij}\,\partial_j f + (\Gamma_{0\,0}^{i\,j} - \Gamma_{0\,0}^{j\,i})\,\partial_j^0 f\Big)\,\partial_i^0\,.$$

We stress that we need the choice of the time scale τ because γ is a scaled vector field. This fact is not a minor point of our theory; instead, it plays an essential role throughout the classical and quantum theories.

Actually, each $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$ yields the time scale $f'' := \frac{1}{3}\langle \bar{G}, D^2 f\rangle$, where $D^2 f \in \mathrm{fib}\Big(J_1\boldsymbol{E}, \quad \mathbb{T}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})\Big)$ is the 2nd fibre derivative of f with respect to the affine fibre of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$. Thus, we have the coordinate expression $f'' = f^0\,u_0 = \frac{1}{3}\,G_0^{ij}\,\partial_i^0\partial_j^0 f\,u_0$. The map f'' is called the *time component* of f.

We define the *Hamiltonian lift* of each $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$ to be the vector field

$$X^{\uparrow}{}_{\mathrm{ham}}[f] := X^{\uparrow}_{\mathrm{ham}}[f'', f] = \gamma(f'') + \Lambda^\sharp(df) \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})\,.$$

1.5 Special phase functions

In this section we collect some basic facts on special phase functions, their Lie bracket and their tangent, Hamiltonian and holonomic lifts [36, 39].

1.5.1 The sheaf of special phase functions

A *special phase function* is defined to be a function $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$, such that $D^2 f = \tau \otimes G$, with $\tau \in \mathrm{map}(\boldsymbol{E}, \bar{\mathbb{T}})$. Clearly, if $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is a special phase function, then we obtain $\tau = f''$, hence $D^2 f = f'' \otimes G$, with $f'' \in \mathrm{map}(\boldsymbol{E}, \bar{\mathbb{T}})$.

The coordinate expression of a special phase function is of the type

$$f = f^0\,\tfrac{1}{2}\,G_{ij}^0\,x_0^i\,x_0^j + f^i\,G_{ij}^0\,x_0^j + \breve{f}\,, \qquad \text{with} \qquad f^0, f^i, \breve{f} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})\,.$$

Given an observer o, a special phase function f can be written as

$$f = f'' \lrcorner\, \mathcal{K}[o] + f'[o] \lrcorner\, \mathcal{Q}[o] + f[o]\,,$$

where, in adapted coordinates,

$$\begin{aligned} f'' &= \tfrac{1}{3}\,\langle \bar{G}, D^2 f\rangle = f^0\,u_0 \in \mathrm{map}(\boldsymbol{E}, \bar{\mathbb{T}})\,,\\ f'[o] &= G^\sharp(Df) \circ o \;= f^i\,\partial_j \;\in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})\\ f[o] &= f \circ o \qquad\quad\;\; = \breve{f} \qquad \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})\,. \end{aligned}$$

Given a horizontal potential $A^\uparrow$ and an observer o, a special phase function f can be written as $f = f'' \lrcorner\, \mathcal{H}[A^\uparrow, o] + f'[o] \lrcorner\, \mathcal{P}[A^\uparrow, o] + (f[o] + f^0\,A_0 - f^i\,A_i)$.

The subsheaf of special phase functions is denoted by $\mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

Moreover, we shall be involved with the distinguished subsheaves related to the affine structure of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$. Thus, we define the following subsheaves:

- the sheaf $\operatorname{proj}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ consisting of functions, called *projectable*, whose time component $f'' \in \operatorname{map}(\boldsymbol{T}, \bar{\mathbb{T}})$ depends only on $\boldsymbol{T}$;

- the sheaf $\operatorname{fine}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{proj}(J_1\boldsymbol{E}, \mathbb{R})$ consisting of projectable functions, called *fine*, whose time component $f'' \in \bar{\mathbb{T}}$ is constant;

- the sheaf $\operatorname{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{fine}(J_1\boldsymbol{E}, \mathbb{R})$ consisting of projectable functions, called *affine*, whose time component $f'' = 0$ vanishes, i.e. the subsheaf of affine functions with respect to the affine fibres of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$;

- the sheaf $\operatorname{map}(\boldsymbol{E}, \mathbb{R}) \subset \operatorname{aff}(J_1\boldsymbol{E}, \mathbb{R})$ consisting of affine functions such that $Df = 0$, i.e. the subsheaf of affine functions which depend only on $\boldsymbol{E}$.

1.5.2 Lifts of special phase functions

Let us analyse three distinguished lifts of special phase functions into vector fields: the Hamiltonian lift, the tangent lift and the holonomic lift.

Let us start with the Hamiltonian lift. The special phase functions are characterised by the following property.

1.5 Theorem. *Let $\tau \in \operatorname{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$ and $f \in \operatorname{map}(J_1\boldsymbol{E}, \mathbb{R})$. Then, the following conditions are equivalent:*

1) $X^{\uparrow}_{\text{ham}}[\tau, f] \in \operatorname{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ is projectable on a vector field $X[\tau, f] \in \operatorname{sec}(\boldsymbol{E}, T\boldsymbol{E})$,

2) $f \in \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ and $\tau = f''$.

If either of the above conditions is fulfilled, then we obtain

$$X^{\uparrow}_{\text{ham}}[\tau, f] = X^{\uparrow}{}_{\text{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df). \square$$

The *Hamiltonian lift* of special phase functions turns out to be the $\operatorname{map}(\boldsymbol{T}, \mathbb{R})$–linear map $X^{\uparrow}{}_{\text{ham}} : \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R}) \to \operatorname{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\text{ham}}[f] = X^{\uparrow}{}_{\text{ham}}[f'', f]$, with coordinate expression $X^{\uparrow}{}_{\text{ham}}[f] = f^0\,\partial_0 - f^i\,\partial_i + X^i_0\,\partial^0_i$, where

$$X^i_0 = G^{ij}_0\Big(\tfrac{1}{2}\,\partial_j f^0\,G^0_{hk}\,x^h_0\,x^k_0 + (\partial_j f^0_h + f^k\,(\partial_k G^0_{jh} - \partial_j G^0_{kh}) - f^0\,\partial_0 G^0_{jh})\,x^h_0 \\ + \partial_j \breve{f} + f^h\,(\partial_h A_j - \partial_j A_h) - f^0\,(\partial_0 A_j - \partial_j A_0)\Big).$$

Hence, the kernel of $X^{\uparrow}{}_{\text{ham}}$ is the subsheaf $\operatorname{map}(\boldsymbol{T}, \mathbb{R}) \subset \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$.

Then, let us analyse the tangent lift. We obtain the *tangent lift* of special phase functions defined as the map $X : \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R}) \to \operatorname{sec}(\boldsymbol{E}, T\boldsymbol{E}) : f \mapsto X[f] := X[f'', f]$, with coordinate expresssion $X[f] = f^0\,\partial_0 - f^i\,\partial_i$. As a consequence, X is surjective and its kernel is the subsheaf $\operatorname{map}(\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$. We obtain also the map $\operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})/\operatorname{map}(\boldsymbol{E}, \mathbb{R}) \to \operatorname{sec}(\boldsymbol{E}, T\boldsymbol{E}) : [f] \mapsto X[f]$, whose inverse has coordinate expression $X^0\,\partial_0 + X^i\,\partial_i \mapsto [X^0\,\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 - G^0_{ij}\,X^j\,x^i_0]$.

Eventually, let us introduce the holonomic lift [65].

We define the *holonomic lift* of special phase functions to be the $\mathrm{I\!R}$–linear map $X^{\uparrow}{}_{\mathrm{hol}} : \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\mathrm{hol}}[f] := (X[f])_{(1)}$, with coordinate expression $X^{\uparrow}{}_{\mathrm{hol}}[f] = f^0\,\partial_0 - f^i\,\partial_i - (\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0)\,\partial^0_i$.

The kernel of $X^{\uparrow}{}_{\mathrm{hol}}$ is the subsheaf $\mathrm{map}(\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

1.5.3 Special Lie bracket

The special phase functions are not closed under the Poisson bracket. Also for this reason, one introduces a non standard Lie bracket [36, 39].

We define the *special bracket* on $\mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ by $[\![f, g]\!] := \{f, g\} + \gamma(f'').g - \gamma(g'').f$, with coordinate expression

$$\begin{aligned}
[\![f,\, g]\!]^{\lambda} &= f^0\,\partial_0 g^{\lambda} - g^0\,\partial_0 f^{\lambda} - f^h\,\partial_h g^{\lambda} + g^h\,\partial_h f^{\lambda} \\
\breve{[\![f,\, g]\!]} &= f^0\,\partial_0 \breve{g} - g^0\,\partial_0 \breve{f} - f^h\,\partial_h \breve{g} + g^h\,\partial_h \breve{f} - (f^0\,g^h - g^0\,f^h)\,\Phi_{0h} + f^h\,g^k\,\Phi_{hk}\,.
\end{aligned}$$

The sheaf $\mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is an $\mathrm{I\!R}$–Lie algebra with respect to the special bracket.

The subsheaves

$$\mathrm{aff}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{fine}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{proj}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$$

are subsheaves of $\mathrm{I\!R}$–Lie subalgebras with respect to the special bracket. Moreover, the subsheaf $\mathrm{map}(\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{aff}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is a subsheaf of ideals.

At a first insight, the special bracket resambles the Jacobi bracket [51]. However, these brackets have essential differences.

In fact, in our context, the Jacobi bracket would be $[f, g] := \{f, g\} + f\,\gamma.g - g\,\gamma.f$ and not $[\![f, g]\!] := \{f, g\} + \gamma(f'').g - \gamma(g'').f$. Indeed, the special phase functions are not closed with respect to the Jacobi bracket; moreover, in our context, the Jacobi bracket is not well defined with respect to scale dimensions, as it is not invariant with respect to time scales. Except for this trouble, the Jacobi bracket could be defined for all phase functions, while the special bracket can be defined only for special phase functions, as it involves their "time components". We stress that the Jacobi bracket depends on the first jets of the functions, while the special bracket depends on the second jet, because the time component of the special phase functions depends on the second jet.

Furthermore, we observe that in our context we have $[\Lambda, \Lambda] = 0$ and not $[\Lambda, \Lambda] = 2\,\gamma \wedge \Lambda$, but, still, the special bracket fulfills the Jacobi property of Lie brackets. These facts do not conflict with the Lichnerowicz theorem concerning the classification of Lie algebras of functions (see, for instance, [51], p. 336), because the special phase functions are not closed with respect to the real multiplication.

1.5.4 Morphisms of Lie algebras

Let us analyse the relation between the special bracket of special phase functions and the Lie bracket of their prolongations.

The map $X^{\uparrow}{}_{\mathrm{ham}}$: $\mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ is *not* a morphism of Lie algebras, with respect to the special bracket and the Lie bracket, respectively. On the other hand, we have the following result.

1.6 Proposition. [39] For each $f, g \in \mathrm{proj}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we have

$$X^{\uparrow}{}_{\mathrm{ham}}[\,[\![f,\, g]\!]\,] = [X^{\uparrow}{}_{\mathrm{ham}}[f],\, X^{\uparrow}{}_{\mathrm{ham}}[g]]\,.$$

Thus, the sheaf of Hamiltonian lifts of projectable special phase functions is closed with respect to the Lie bracket and the map $X^{\uparrow}{}_{\mathrm{ham}}$: $\mathrm{proj}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ is a morphism of Lie algebras. $\square$

1.7 Proposition. [36] For each $f, g \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we have

$$X\Big[\,[\![f,\, g]\!]\,\Big] = \Big[X[f],\, X[g]\Big]\,.$$

Thus, the map $X : \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E})$ is a morphism of Lie algebras. $\square$

1.8 Proposition. For each $f, g \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we have

$$X^{\uparrow}{}_{\mathrm{hol}}\Big[\,[\![f, g]\!]\,\Big] = \Big[X^{\uparrow}{}_{\mathrm{hol}}[f],\, X^{\uparrow}{}_{\mathrm{hol}}[g]\Big]\,.$$

Hence, the map $\mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\mathrm{hol}}[f]$ is a morphism of Lie algebras. Its kernel equals $\mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$.

PROOF. If $f, g \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$, then, in virtue of Proposition 1.7 and Proposition 1.2, we obtain

$$\begin{aligned} X^{\uparrow}{}_{\mathrm{hol}}[\,[\![f, g]\!]\,] &:= \big(X[\,[\![f, g]\!]\,]\big)_{(1)} = \big[X[f], X[g]\big]_{(1)} = \\ &= \big[(X[f])_{(1)},\, (X[g])_{(1)}\big] := \big[X^{\uparrow}{}_{\mathrm{hol}}[f],\, X^{\uparrow}{}_{\mathrm{hol}}[g]\big]\,. \end{aligned}$$ QED

2 Classical symmetries

This section deals with the main aim of the paper. It is devoted to the analysis of the distinguished subalgebras of the algebra of special phase functions and to the classification of classical infinitesimal symmetries.

2.1 Subalgebras of special phase functions

We have distinguished subsheaves of the sheaf of special phase functions, which are closed with respect to the special bracket. These subalgebras will play an important role with respect to the infinitesimal symmetries of the classical structure.

2.1.1 Subalgebra of conserved special phase functions

We define the following sheaves

$$\begin{aligned}
\operatorname{cons}\operatorname{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) &:= \operatorname{cons}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \operatorname{spec}(J_1\boldsymbol{E}, \mathrm{I\!R}) \\
\operatorname{cons}\operatorname{proj}(J_1\boldsymbol{E}, \mathrm{I\!R}) &:= \operatorname{cons}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \operatorname{proj}(J_1\boldsymbol{E}, \mathrm{I\!R}) \\
\operatorname{cons}\operatorname{fine}(J_1\boldsymbol{E}, \mathrm{I\!R}) &:= \operatorname{cons}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \operatorname{fine}(J_1\boldsymbol{E}, \mathrm{I\!R}) \\
\operatorname{cons}\operatorname{aff}(J_1\boldsymbol{E}, \mathrm{I\!R}) &:= \operatorname{cons}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \operatorname{aff}(J_1\boldsymbol{E}, \mathrm{I\!R}) \\
\operatorname{cons}\operatorname{map}(\boldsymbol{E}, \mathrm{I\!R}) &:= \operatorname{cons}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R})\,.
\end{aligned}$$

We stress that the special bracket reduces to the Poisson bracket on the sheaf $\operatorname{cons}(J_1\boldsymbol{E}, \mathrm{I\!R})$, hence also on the above subsheaves.

2.1 Proposition. The sheaf $\operatorname{cons}\operatorname{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is closed with respect to the special bracket.

Proof. If $f, g \in \operatorname{cons}\operatorname{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$, then $\gamma.[\![f, g]\!] = \gamma.\{f, g\} = \{\gamma.f, g\} + \{f, \gamma.g\} = 0$. QED

2.2 Lemma. For each $\operatorname{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we have the coordinate expression

$$\begin{aligned}
\gamma_0.f \equiv \gamma(u_0).f = &\tfrac{1}{6}\,(\partial_i f^0\, G^0_{hk} + \partial_k f^0\, G^0_{ih} + \partial_h f^0\, G^0_{ki})\, x^i_0\, x^h_0\, x^k_0 \\
&+ \tfrac{1}{2}\Big(\partial_0 f^0\, G^0_{hk} - f^0\,\partial_0 G^0_{hk} + f^i\,\partial_i G^0_{hk} + \partial_h f^i\, G^0_{ik} + \partial_k f^i\, G^0_{ih}\Big)\, x^h_0\, x^k_0 \\
&- \Big(f^0\,(\partial_0 A_h - \partial_h A_0) + f^i\,(\partial_h A_i - \partial_i A_h) - \partial_0 f^i\, G^0_{ih} - \partial_h \breve{f}\Big)\, x^h_0 \\
&+ \partial_0 \breve{f} - f^i\,(\partial_0 A_i - \partial_i A_0)\,.
\end{aligned}$$

Proof. By taking into account the coordinate expressions

$$\begin{aligned}
\gamma_0{}^i_0 &= -G^{ij}_0\,\big((\partial_h G^0_{jk} - \tfrac{1}{2}\,\partial_j G^0_{hk})\, x^h_0\, x^k_0 + \big(\partial_0 G^0_{hj} + (\partial_h A_j - \partial_j A_h)\big)\, x^h_0 + \partial_0 A_j - \partial_j A_0\big) \\
f &= f^0\,\tfrac{1}{2}\, G^0_{hk}\, x^h_0\, x^k_0 + f^h\, G^0_{hk}\, x^k_0 + \breve{f}\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
\gamma_0.f &= \partial_0 f^0 \,\tfrac{1}{2}\, G^0_{hk}\, x^h_0\, x^k_0 + f^0\, \tfrac{1}{2}\partial_0 G^0_{hk}\, x^h_0\, x^k_0 + \partial_0 f^h\, G^0_{hk}\, x^k_0 + f^h\, \partial_0 G^0_{hk}\, x^k_0 + \partial_0 \breve{f} \\
&\quad + (\partial_i f^0 \,\tfrac{1}{2}\, G^0_{hk}\, x^h_0\, x^k_0 + f^0\, \tfrac{1}{2}\partial_i G^0_{hk}\, x^h_0\, x^k_0 + \partial_i f^h\, G^0_{hk}\, x^k_0 + f^h\, \partial_i G^0_{hk}\, x^k_0 + \partial_i \breve{f})\, x^i_0 \\
&\quad - G^{ij}_0\, (f^0\, G^0_{ir}\, x^r_0 + f^r\, G^0_{ir})\, (\partial_h G^0_{jk} - \tfrac{1}{2}\, \partial_j G^0_{hk})\, x^h_0\, x^k_0 \\
&\quad - G^{ij}_0\, (f^0\, G^0_{ir}\, x^r_0 + f^r\, G^0_{ir})\, \big((\partial_0 G^0_{hj} + (\partial_h A_j - \partial_j A_h))\, x^h_0 + \partial_0 A_j - \partial_j A_0\big)\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\gamma_0.f &= \partial_0 f^0 \,\tfrac{1}{2}\, G^0_{hk}\, x^h_0\, x^k_0 + f^0\, \tfrac{1}{2}\partial_0 G^0_{hk}\, x^h_0\, x^k_0 + \partial_0 f^h\, G^0_{hk}\, x^k_0 + f^h\, \partial_0 G^0_{hk}\, x^k_0 + \partial_0 \breve{f} \\
&\quad + (\partial_i f^0 \,\tfrac{1}{2}\, G^0_{hk}\, x^h_0\, x^k_0 + f^0\, \tfrac{1}{2}\partial_i G^0_{hk}\, x^h_0\, x^k_0 + \partial_i f^h\, G^0_{hk}\, x^k_0 + f^h\, \partial_i G^0_{hk}\, x^k_0 + \partial_i \breve{f})\, x^i_0 \\
&\quad - (f^0\, x^j_0 + f^j)\, (\partial_h G^0_{jk} - \tfrac{1}{2}\, \partial_j G^0_{hk})\, x^h_0\, x^k_0 \\
&\quad - (f^0\, x^j_0 + f^j)\, \big((\partial_0 G^0_{hj} + (\partial_h A_j - \partial_j A_h))\, x^h_0 + \partial_0 A_j - \partial_j A_0\big)
\end{aligned}$$

and thus

$$\begin{aligned}
\gamma_0.f &= \tfrac{1}{2}\, \partial_i f^0\, G^0_{hk}\, x^i_0\, x^h_0\, x^k_0 \\
&\quad + (\tfrac{1}{2}\, \partial_0 f^0\, G^0_{hk} - \tfrac{1}{2}\, f^0\, \partial_0 G^0_{hk} + \tfrac{1}{2}\, f^i\, \partial_i G^0_{hk} + \partial_h f^i\, G^0_{ik})\, x^h_0\, x^k_0 \\
&\quad - \big(f^0\, (\partial_0 A_h - \partial_h A_0) + f^i\, (\partial_h A_i - \partial_i A_h) - \partial_0 f^i\, G^0_{ih} - \partial_h \breve{f}\big)\, x^h_0 \\
&\quad + \partial_0 \breve{f} - f^i\, (\partial_0 A_i - \partial_i A_0)\,. \text{QED}
\end{aligned}$$

2.3 Proposition. The sheaf $\operatorname{cons spec}(J_1\boldsymbol{E}, \mathbb{R})$ is constituted by the special phase functions f such that

$$\begin{aligned}
\partial_i f^0\, G^0_{hk} + \partial_k f^0\, G^0_{ih} + \partial_h f^0\, G^0_{ki} &= 0 \\
\partial_0 f^0\, G^0_{hk} - f^0\, \partial_0 G^0_{hk} + f^i\, \partial_i G^0_{hk} + \partial_h f^i\, G^0_{ik} + \partial_k f^i\, G^0_{ih} &= 0 \\
f^0\, (\partial_0 A_h - \partial_h A_0) + f^i\, (\partial_h A_i - \partial_i A_h) - \partial_0 f^i\, G^0_{ih} - \partial_h \breve{f} &= 0 \\
\partial_0 \breve{f} - f^i\, (\partial_0 A_i - \partial_i A_0) &= 0\,. \square
\end{aligned}$$

A general analysis of the above system is beyond the scope of the present paper. Here, we just discuss some equivalences and simple examples in the basic model of spacetime. A similar remark holds for the systems of differential equations, which will appear in the forthcoming sections.

2.4 Proposition. We have the useful identities $\gamma_0.\mathcal{H}_0 = -\partial_0 \mathcal{L}_0$ and $\gamma_0.\mathcal{P}_i = \partial_i \mathcal{L}_0$. Moreover, we have the following equivalences:

$$\begin{aligned}
\gamma.x^i_0 = 0 \quad &\Leftrightarrow \quad K_\lambda{}^r{}_\mu = 0 \\
\gamma.\mathcal{K}_0 = 0 \quad &\Leftrightarrow \quad \partial_0 G^0_{hk} = 0\,, \;\; \partial_0 A_h - \partial_h A_0 = 0 \\
\gamma.\mathcal{H}_0 = 0 \quad &\Leftrightarrow \quad \partial_0 G^0_{hk} = 0\,, \;\; \partial_0 A_\lambda = 0 \\
\gamma.\mathcal{Q}_i = 0 \quad &\Leftrightarrow \quad \partial_i G^0_{hk} = 0\,, \;\; \partial_i A_\lambda - \partial_\lambda A_i = 0 \\
\gamma.\mathcal{P}_i = 0 \quad &\Leftrightarrow \quad \partial_i G^0_{hk} = 0\,, \;\; \partial_i A_\lambda = 0
\end{aligned}$$

$$\gamma.\mathcal{L}_0 = 0 \quad \Leftrightarrow \quad \nabla_h A_k + \nabla_k A_h = \partial_0 G^0_{hk}\,, \;\; \nabla_h A_0 = 0\,, \;\; A^i_0\,(\partial_0 A_i - \partial_i A_0) = \partial_0 A_0\,.\,\square$$

2.5 Example. In the special Newtonian spacetime, the sheaf $\operatorname{cons}\operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ is constituted by the special phase functions f such that

$$\begin{aligned} &\partial_i f^0 = 0 \\ &\partial_0 f^0 = -2\,\partial_1 f^1 = -2\,\partial_2 f^2 = -2\,\partial_3 f^3 \\ &\partial_1 f^2 = -\partial_2 f^1\,, \qquad \partial_1 f^3 = -\partial_3 f^1\,, \qquad \partial_2 f^3 = -\partial_3 f^2 \\ &\partial_0 f^i = -\partial_i \breve{f} \\ &\partial_0 \breve{f} = 0\,. \end{aligned}$$

A solution of this system is given by

$$f^0 = -a_0\,(x^0)^2 + d^0\,, \quad f^i = (a_0\,x^i + b^i_0)\,x^0\,, \quad \breve{f} = -(\tfrac{1}{2}\,a_0\,\sum_i (x^i)^2 + \sum_i b^i_0\,x^i + c)\,,$$

where $a_0,\, b^i_0,\, c, d^0 \in \mathbb{R}$. In particular, we obtain $\operatorname{cons}\operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{proj}(J_1\boldsymbol{E}, \mathbb{R})$.

For instance, the components of the kinetic energy, of the momentum and of the angular momentum with respect to an inertial observer are conserved special phase functions. $\square$

2.1.2 Subalgebra of holonomic functions

We can compare the holonomic and Hamiltonian lifts of a special phase function. The special phase functions whose holonomic and Hamiltonian lifts coincide constitute a subalgebra with respect to the special bracket.

We call $f \in \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ *holonomic* if $X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f]$.
We denote the subsheaf of holonomic functions by $\operatorname{hol}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$.
Accordingly, we set

$$\begin{aligned} \operatorname{hol}\operatorname{fine}(J_1\boldsymbol{E}, \mathbb{R}) &:= \operatorname{hol}(J_1\boldsymbol{E}, \mathbb{R}) \cap \operatorname{fine}(J_1\boldsymbol{E}, \mathbb{R}) \\ \operatorname{hol}\operatorname{aff}(J_1\boldsymbol{E}, \mathbb{R}) &:= \operatorname{hol}(J_1\boldsymbol{E}, \mathbb{R}) \cap \operatorname{aff}(J_1\boldsymbol{E}, \mathbb{R}) \\ \operatorname{hol}\operatorname{map}(\boldsymbol{E}, \mathbb{R}) &:= \operatorname{hol}(J_1\boldsymbol{E}, \mathbb{R}) \cap \operatorname{map}(\boldsymbol{E}, \mathbb{R})\,. \end{aligned}$$

2.6 Proposition. The sheaf $\operatorname{hol}(J_1\boldsymbol{E}, \mathbb{R})$ is constituted by the special phase functions f such that

$$\begin{aligned} \partial_i f^0 &= 0 \\ \partial_0 f^0\, G^0_{ij} - f^0\,\partial_0 G^0_{ij} + f^h\,\partial_h G^0_{ij} + \partial_j f^h\, G^0_{ih} + \partial_i f^h\, G^0_{jh} &= 0 \\ f^0\,(\partial_i A_0 - \partial_0 A_i) + \partial_0 f^h\, G^0_{ih} + f^h\,(\partial_h A_i - \partial_i A_h) + \partial_i \breve{f} &= 0\,. \end{aligned}$$

As a consequence, we obtain $\operatorname{hol}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{proj}(J_1\boldsymbol{E}, \mathbb{R})$.

Proof. By the coordinate expressions of the holonomic and Hamiltonian lifts of projectable special phase functions

$$
\begin{aligned}
X^{\uparrow}{}_{\text{hol}}[f] &= f^0\,\partial_0 - f^i\,\partial_i - (\partial_0 f^i + \partial_j f^i\,x_0^j + \partial_0 f^0\,x_0^i + \partial_j f^0\,x_0^j\,x_0^i)\,\partial_i^0 \\
X^{\uparrow}{}_{\text{ham}}[f] &= f^0\,\partial_0 - f^i\,\partial_i + G_0^{ij}\Big(\partial_j \breve{f} + \partial_j f^0\,\tfrac{1}{2}\,G^0_{hk}\,x_0^h\,x_0^k + \partial_j f^h\,G^0_{hk}\,x_0^k \\
&\quad - f^0\,\big(\partial_0 G^0_{hj}\,x_0^h + (\partial_0 A_j - \partial_j A_0)\big) + f^h\,\big(\partial_h G^0_{jk}\,x_0^h - (\partial_j A_h - \partial_h A_j)\big)\Big)\,\partial_i^0\,,
\end{aligned}
$$

we obtain the following coordinate expression of the condition $X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f]$

$$
\begin{aligned}
&- (\partial_0 f^i + \partial_j f^i\,x_0^j + \partial_0 f^0\,x_0^i + \partial_j f^0\,x_0^j\,x_0^i) \\
&= G_0^{ij}\Big(\partial_j \breve{f} + \partial_j f^0\,\tfrac{1}{2}\,G^0_{hk}\,x_0^h\,x_0^k + \partial_j f^h\,G^0_{hk}\,x_0^k \\
&- f^0\,\big(\partial_0 G^0_{hj}\,x_0^h + (\partial_0 A_j - \partial_j A_0)\big) + f^h\,\big(\partial_h G^0_{jk}\,x_0^h - (\partial_j A_h - \partial_h A_j)\big)\Big)\,.
\end{aligned}
$$

This is equivalent to the system

$$
\begin{aligned}
-\partial_h f^0\,\delta_k^i &= \partial_j f^0\,\tfrac{1}{2}\,G_0^{ij}\,G^0_{hk} \\
-(\partial_h f^i + \partial_0 f^0\,\delta_h^i) &= G_0^{ij}\,\big(\partial_j f^k\,G^0_{hk} - f^0\,\partial_0 G^0_{hj} + f^h\,\partial_h G^0_{jk}\big) \\
-\partial_0 f^i &= G_0^{ij}\,\big(\partial_j \breve{f} - f^0\,(\partial_0 A_j - \partial_j A_0) + f^h\,(\partial_h A_j - \partial_j A_h)\big)\,.
\end{aligned}
$$

By contracting the 1st equality with $G_0^{hk}\,G^0_{ir}$ and the 2nd and 3rd equalities with G^0_{ir}, we get

$$
\begin{aligned}
\partial_r f^0 &= 0 \\
-(\partial_h f^i\,G^0_{ir} + \partial_0 f^0\,G^0_{hr}) &= \partial_r f^k\,G^0_{hk} - f^0\,\partial_0 G^0_{hr} + f^k\,\partial_h G^0_{rk} \\
-\partial_0 f^i\,G^0_{ir} &= \partial_r \breve{f} - f^0\,(\partial_0 A_r - \partial_r A_0) + f^h\,(\partial_h A_r - \partial_r A_h)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\partial_r f^0 &= 0 \\
-(\partial_i f^h\,G^0_{hj} + \partial_0 f^0\,G^0_{ij}) &= \partial_j f^h\,G^0_{ih} - f^0\,\partial_0 G^0_{ij} + f^h\,\partial_i G^0_{jh} \\
-\partial_0 f^h\,G^0_{hi} &= \partial_i \breve{f} - f^0\,(\partial_0 A_i - \partial_i A_0) + f^h\,(\partial_h A_i - \partial_i A_h)\,.\ \text{QED}
\end{aligned}
$$

2.7 Proposition. The sheaf $\text{hol}\,(J_1\boldsymbol{E}, \mathrm{I\!R})$ is closed with respect to the special bracket.

Proof. If $f, g \in \text{hol}\,(J_1\boldsymbol{E}, \mathrm{I\!R})$, then, we obtain

$$
X^{\uparrow}{}_{\text{ham}}\big[\,[\![f, g]\!]\,\big] = \big[X^{\uparrow}{}_{\text{ham}}[f],\, X^{\uparrow}{}_{\text{ham}}[g]\big] = \big[X^{\uparrow}{}_{\text{hol}}[f],\, X^{\uparrow}{}_{\text{hol}}[g]\big] = X^{\uparrow}{}_{\text{hol}}\big[\,[\![f, g]\!]\,\big]\,.\ \text{QED}
$$

2.8 Example. In the special Newtonian spacetime, the sheaf $\text{hol}\,(J_1\boldsymbol{E}, \mathrm{I\!R})$ is constituted by the special phase functions f such that

$$\begin{aligned}
&\partial_i f^0 = 0 \\
&\partial_0 f^0 = -2\,\partial_1 f^1 = -2\,\partial_2 f^2 = -2\,\partial_3 f^3 \\
&\partial_1 f^2 = -\partial_2 f^1\,, \qquad \partial_1 f^3 = -\partial_3 f^1\,, \qquad \partial_2 f^3 = -\partial_3 f^2 \\
&\partial_0 f^i = -\partial_i \breve{f}\,.
\end{aligned}$$

A solution of this system is given by

$$f^0 = -2 \int a\, d^0\,, \qquad f^i = a\, x^i + b^i\,, \qquad \breve{f} = -(\tfrac{1}{2}\,\partial_0 a\, \sum_i (x^i)^2 + \sum_i \partial_0 b^i\, x^i + c)\,,$$

where a, b^i $c \in \operatorname{map}(\boldsymbol{T}, \mathrm{I\!R})$. □

2.1.3 Subalgebra of self–holonomic functions

Here, we consider special phase functions such that their holonomic prolongation is related to their differential through the cosymplectic 2–form. These special phase functions turn out to be conserved and holonomic.

We call $f \in \operatorname{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ *self–holonomic* if $i\big(X^{\uparrow}{}_{\mathrm{hol}}[f]\big)\,\Omega = df$.
We denote the subsheaf of self–holonomic functions by $\operatorname{self}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$.
Accordingly, we set

$$\begin{aligned}
\operatorname{self\,fine}(J_1\boldsymbol{E}, \mathrm{I\!R}) &:= \operatorname{self}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \operatorname{fine}(J_1\boldsymbol{E}, \mathrm{I\!R}) \\
\operatorname{self\,aff}(J_1\boldsymbol{E}, \mathrm{I\!R}) &:= \operatorname{self}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \operatorname{aff}(J_1\boldsymbol{E}, \mathrm{I\!R}) \\
\operatorname{self\,map}(\boldsymbol{E}, \mathrm{I\!R}) &:= \operatorname{self}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R})\,.
\end{aligned}$$

2.9 Lemma. If $f \in \operatorname{self}(J_1\boldsymbol{E}, \mathrm{I\!R})$, then $\gamma.f = 0$, hence

$$\operatorname{self}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{cons}(J_1\boldsymbol{E}, \mathrm{I\!R})\,.$$

Proof. We have $\gamma.f = i(\gamma)\, df = i(\gamma)\, i(X^{\uparrow}{}_{\mathrm{hol}}[f])\,\Omega = -i(X^{\uparrow}{}_{\mathrm{hol}}[f])\, i(\gamma)\,\Omega = 0$. QED

2.10 Lemma. If $f \in \operatorname{self}(J_1\boldsymbol{E}, \mathrm{I\!R})$, then $X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{ham}}[f]$, hence

$$\operatorname{self}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \operatorname{hol}(J_1\boldsymbol{E}, \mathrm{I\!R})\,.$$

Proof. The equality $i\big(X^{\uparrow}{}_{\mathrm{hol}}[f]\big)\,\Omega = df$ yields

$$X^{\uparrow}{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df) = \gamma(f'') + \Lambda^{\sharp}(\Omega^{\flat}(X^{\uparrow}{}_{\mathrm{hol}}[f]))\,.$$

Therefore, we have

$$\begin{aligned}
X^{\uparrow}{}_{\mathrm{ham}}[f] &= \gamma(f'') + X^{\uparrow}{}_{\mathrm{hol}}[f] - \gamma(X^{\uparrow}{}_{\mathrm{hol}}[f]) = \gamma(f'') + X^{\uparrow}{}_{\mathrm{hol}}[f] - \gamma(X[f]) \\
&= \gamma(f'') + X^{\uparrow}{}_{\mathrm{hol}}[f] - \gamma(f'') = X^{\uparrow}{}_{\mathrm{hol}}[f]\,. \text{ QED}
\end{aligned}$$

2.11 Proposition. For each $f, g \in \text{self}(J_1\boldsymbol{E}, \mathbb{R})$, we obtain

$$\Omega^\flat\Big(X^\uparrow{}_{\text{hol}}\big[\,[\![f, g]\!]\,\big]\Big) = d\,[\![f, g]\!]\ .$$

Hence, the subsheaf $\text{self}(J_1\boldsymbol{E},\ \mathbb{R}) \subset \text{spec}(J_1\boldsymbol{E},\ \mathbb{R})$ is closed with respect to the special bracket.

PROOF. It sufficies to prove that

$$\begin{aligned}\Lambda^\sharp\Big(\Omega^\flat\Big(X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,]\Big)\Big) &= \Lambda^\sharp\big(d\,[\![f, g]\!]\,\big)\\ i(\gamma)\,\Big(\Omega^\flat\Big(X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,]\Big)\Big) &= i(\gamma)\,\big(d\,[\![f, g]\!]\,\big)\,.\end{aligned}$$

In fact, we have

$$\begin{aligned}\Lambda^\sharp\Big(\Omega^\flat\Big(X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,]\Big)\Big) &= X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,] - \gamma\big(X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,]\big)\\ &= X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,] - \gamma\big(X\,[\,[\![f, g]\!]\,]\big)\\ &= X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,] - \gamma\big(\,[\![f, g]\!]''\big)\,.\end{aligned}$$

By Lemma 2.10 we have

$$\begin{aligned}\Lambda^\sharp\Big(\Omega^\flat\Big(X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,]\Big)\Big) &= X^\uparrow{}_{\text{ham}}\,[\,[\![f, g]\!]\,] - \gamma\big(\,[\![f, g]\!]''\big)\\ &= \Lambda^\sharp\big(d\,[\![f, g]\!]\,\big)\,.\end{aligned}$$

On the other hand, the identity $i(\gamma)\Omega = 0$ yields $i(\gamma)\,\Omega^\flat(X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,]) = 0$ and the definition of the special bracket and Lemma 2.9 yield

$$\begin{aligned}i(\gamma)\,d\,[\![f, g]\!] &= i(\gamma)\,d\big(\{f, g\} + \gamma(f'').g - \gamma(g'').f\big)\\ &= \{\gamma.f, g\} + \{f, \gamma.g\} + i(\gamma)\,d\big(\gamma(f'').g - \gamma(g'').f\big) = 0\,.\end{aligned}$$

Hence, $\Omega^\flat(X^\uparrow{}_{\text{hol}}\,[\,[\![f, g]\!]\,]) = d\,[\![f, g]\!]\ .\square$

Now, we state the conditions aimed at classifying the self–holonomic functions.

2.12 Lemma. For each $f \in \text{proj}(J_1\boldsymbol{E},\ \mathbb{R})$, we obtain

$$\begin{aligned}i\Big(X^\uparrow{}_{\text{hol}}\,[f]\Big)\,\Omega = {} & G^0_{ij}\,(f^0\,x^j_0 + f^j)\,d^i_0\\ & + \Big(f^0\,\Big((\partial_0 A_j - \partial_j A_0) + \partial_0 G^0_{jh}\,x^h_0 + \tfrac{1}{2}\,\partial_j G^0_{hk}\,x^h_0\,x^k_0\Big)\\ & - f^i\,\Big((\partial_i A_j - \partial_j A_i) + (\partial_i G^0_{jh} - \partial_j G^0_{ih})\,x^h_0\Big)\\ & - G^0_{ij}\,(\partial_0 f^i + \partial_h f^i\,x^h_0 + \partial_0 f^0\,x^i_0)\Big)\,d^j\\ & + \Big(f^j\,\Big((\partial_0 A_j - \partial_j A_0) + \partial_0 G^0_{jh}\,x^h_0 + \tfrac{1}{2}\,\partial_j G^0_{hk}\,x^h_0\,x^k_0\Big)\\ & + (\partial_0 f^i + \partial_j f^i\,x^j_0 + \partial_0 f^0\,x^i_0)\,G^0_{ij}\,x^j_0\Big)\,d^0\,.\end{aligned}$$

Proof. The coordinate expressions

$$X^{\uparrow}{}_{\mathrm{hol}}[f] = f^0\,\partial_0 - f^i\,\partial_i - (\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0)\,\partial^0_i$$
$$\Omega = G^0_{ij}\, d^i_0 \wedge (d^j - x^j_0\, d^0) + \big((\partial_0 A_j - \partial_j A_0) + \partial_0 G^0_{jh}\, x^h_0 + \tfrac{1}{2}\,\partial_j G^0_{hk}\, x^h_0\, x^k_0\big)\, d^0 \wedge d^j + \tfrac{1}{2}\,\big((\partial_i A_j - \partial_j A_i) + (\partial_i G^0_{jh} - \partial_j G^0_{ih})\, x^h_0\big)\, d^i \wedge d^j$$

yield

$$\begin{aligned} i\big(X^{\uparrow}{}_{\mathrm{hol}}[f]\big)\,\Omega = {} & f^0\, G^0_{ij}\, x^j_0\, d^i_0 + f^0\,\big((\partial_0 A_j - \partial_j A_0) + \partial_0 G^0_{jh}\, x^h_0 + \tfrac{1}{2}\,\partial_j G^0_{hk}\, x^h_0\, x^k_0\big)\, d^j \\ & + f^j\, G^0_{ij}\, d^i_0 + f^j\,\big((\partial_0 A_j - \partial_j A_0) + \partial_0 G^0_{jh}\, x^h_0 + \tfrac{1}{2}\,\partial_j G^0_{hk}\, x^h_0\, x^k_0\big)\, d^0 \\ & - f^i\,\big((\partial_i A_j - \partial_j A_i) + (\partial_i G^0_{jh} - \partial_j G^0_{ih})\, x^h_0\big)\, d^j \\ & - (\partial_0 f^i + \partial_h f^i\, x^h_0 + \partial_0 f^0\, x^i_0)\, G^0_{ij}\, d^j \\ & + (\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0)\, G^0_{ij}\, x^j_0\, d^0 . \end{aligned}$$

Therefore, we have

$$\begin{aligned} i\big(X^{\uparrow}{}_{\mathrm{hol}}[f]\big)\,\Omega = {} & G^0_{ij}\,(f^0\, x^j_0 + f^j)\, d^i_0 \\ & + \Big(f^0\,\big((\partial_0 A_j - \partial_j A_0) + \partial_0 G^0_{jh}\, x^h_0 + \tfrac{1}{2}\,\partial_j G^0_{hk}\, x^h_0\, x^k_0\big) \\ & - f^i\,\big((\partial_i A_j - \partial_j A_i) + (\partial_i G^0_{jh} - \partial_j G^0_{ih})\, x^h_0\big) \\ & - G^0_{ij}\,(\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0)\Big)\, d^j \\ & + \Big(f^j\,\big((\partial_0 A_j - \partial_j A_0) + \partial_0 G^0_{jh}\, x^h_0 + \tfrac{1}{2}\,\partial_j G^0_{hk}\, x^h_0\, x^k_0\big) \\ & + (\partial_0 f^i + \partial_h f^i\, x^h_0 + \partial_0 f^0\, x^i_0)\, G^0_{ij}\, x^j_0\Big)\, d^0 . \text{ QED} \end{aligned}$$

2.13 Lemma. For each $f \in \mathrm{proj}(J_1\boldsymbol{E}, \mathbb{R})$, we obtain

$$\begin{aligned} df = {} & (\partial_0 f^0\, \tfrac{1}{2}\, G^0_{hk}\, x^h_0\, x^k_0 + \partial_0 f^h\, G^0_{hk}\, x^k_0 + \partial_0 \breve{f} + f^0\, \tfrac{1}{2}\partial_0 G^0_{hk}\, x^h_0\, x^k_0 + f^h\, \partial_0 G^0_{hk}\, x^k_0)\, d^0 \\ & + (\partial_j f^h\, G^0_{hk}\, x^k_0 + \partial_j \breve{f} + f^0\, \tfrac{1}{2}\partial_j G^0_{hk}\, x^h_0\, x^k_0 + f^h\, \partial_j G^0_{hk}\, x^k_0)\, d^j \\ & + (f^0\, G^0_{ih}\, x^h_0 + f^h\, G^0_{ih})\, d^i_0 . \square \end{aligned}$$

2.14 Proposition. The sheaf $\mathrm{self}(J_1\boldsymbol{E}, \mathbb{R})$ is constituted by the special phase functions f such that

$$\begin{aligned} \partial_i f^0 &= 0 \\ f^0\, \partial_0 G^0_{ij} - \partial_0 f^0\, G^0_{ij} - f^h\, \partial_h G^0_{ij} - \partial_j f^h\, G^0_{ih} - \partial_i f^h\, G^0_{jh} &= 0 \\ f^0\,(\partial_0 A_j - \partial_j A_0) - f^i\,(\partial_i A_j - \partial_j A_i) - G^0_{ij}\, \partial_0 f^i - \partial_j \breve{f} &= 0 \\ f^i\,(\partial_i A_0 - \partial_0 A_i) + \partial_0 \breve{f} &= 0 . \end{aligned}$$

Proof. In virtue of Lemmas 2.12 and 2.13, for each $f \in \mathrm{proj}(J_1\boldsymbol{E}, \mathbb{R})$, we have $i\big(X^{\uparrow}{}_{\mathrm{hol}}[f]\big)\,\Omega = df$

if and only if

$$f^j\,\big((\partial_0 A_j - \partial_j A_0) + \partial_0 G^0_{jh}\,x^h_0 + \tfrac{1}{2}\,\partial_j G^0_{hk}\,x^h_0\,x^k_0\big) + (\partial_0 f^i + \partial_h f^i\,x^h_0 + \partial_0 f^0\,x^i_0)\,G^0_{ik}\,x^k_0 =$$
$$= \partial_0 f^0\,\tfrac{1}{2}\,G^0_{hk}\,x^h_0\,x^k_0 + \partial_0 f^h\,G^0_{hk}\,x^k_0 + \partial_0\breve{f} + f^0\,\tfrac{1}{2}\partial_0 G^0_{hk}\,x^h_0\,x^k_0 + f^h\,\partial_0 G^0_{hk}\,x^k_0$$

$$f^0\,\big((\partial_0 A_j - \partial_j A_0) + \partial_0 G^0_{jh}\,x^h_0 + \tfrac{1}{2}\,\partial_j G^0_{hk}\,x^h_0\,x^k_0\big)$$
$$-\,f^i\,\big((\partial_i A_j - \partial_j A_i) + (\partial_i G^0_{jh} - \partial_j G^0_{ih})\,x^h_0\big) - G^0_{ij}\,(\partial_0 f^i + \partial_h f^i\,x^h_0 + \partial_0 f^0\,x^i_0) =$$
$$= \partial_j f^h\,G^0_{hk}\,x^k_0 + \partial_j\breve{f} + f^0\,\tfrac{1}{2}\partial_j G^0_{hk}\,x^h_0\,x^k_0 + f^h\,\partial_j G^0_{hk}\,x^k_0\,.$$

By comparing the coefficients of the two above polynomial equalities, we obtain the following equivalent system

$$\begin{aligned} f^j\,(\partial_0 A_j - \partial_j A_0) - \partial_0\breve{f} &= 0\\ f^0\,(\partial_0 A_j - \partial_j A_0) - f^i\,(\partial_i A_j - \partial_j A_i) - G^0_{ij}\,\partial_0 f^i - \partial_j\breve{f} &= 0\\ f^0\,\partial_0 G^0_{jh} - \partial_0 f^0\,G^0_{hj} - f^i\,\partial_i G^0_{jh} - \partial_h f^i\,G^0_{ij} - \partial_j f^i\,G^0_{ih} &= 0\,.\ \text{QED} \end{aligned}$$

The system of the previous Proposition can be re–expressed in terms of an observer and the tangent lift of the projectable special phase function.

2.15 Proposition. Let us consider an observer o. Then, the sheaf $\mathrm{self}(J_1\boldsymbol{E}, I\!R)$ is constituted by the projectable special phase functions f such that

$$\begin{aligned} L\big[X[f]\big]\,G &= df'' \lrcorner\, G\\ X[f] \lrcorner\, \Phi[o] + \nu^*[o]\Big(G^\flat\big(L[o]\,X[f]\big)\Big) &= d(f[o])\,. \end{aligned}$$

Proof. We can write the system which characterises self–holonomic functions (Proposition 2.14) as

$$\begin{aligned} \partial_\mu G^0_{hk}\,X[f]^\mu + G^0_{ih}\,\partial_k(X[f]^i) + G^0_{ik}\,\partial_h(X[f]^i) &= \partial_0 f^0\,G^0_{hk}\\ X[f]^\mu\,(\partial_\mu A_j - \partial_j A_\mu) + G^0_{ij}\,\partial_0(X[f]^i) - \partial_j\breve{f} &= 0\\ X[f]^i\,(\partial_i A_0 - \partial_0 A_i) - \partial_0\breve{f} &= 0\,. \end{aligned}$$

On the other hand, we have the coordinate expressions

$$\begin{aligned} L\big[X[f]\big]\,G &= \big(\partial_\mu G^0_{hk}\,X[f]^\mu + G^0_{ih}\,\partial_k(X[f]^i) + G^0_{ik}\,\partial_h(X[f]^i)\big)\,u_0\otimes\check{d}^h\otimes\check{d}^k\\ df'' \lrcorner\, G &= \partial_0 f^0\,G^0_{hk}\,u_0\otimes\check{d}^h\otimes\check{d}^k\\ X[f] \lrcorner\, \Phi[o] + \nu^*[o]\Big(G^\flat\big(L[o]\,X[f]\big)\Big) &= X[f]^\mu\,\Phi_{\mu\nu}\,d^\nu + G^0_{ij}\,\partial_0(X[f]^i)\,d^j\\ d(f[o]) &= \partial_\mu\breve{f}\,.\ \text{QED} \end{aligned}$$

We have the following further intrinsic characterisations of self-holonomic special phase functions, which will play an important role in the classification of classical symmetries.

2.16 Theorem. *Let $f \in \mathrm{proj}(J_1\boldsymbol{E}, I\!R)$. Then, the following conditions are equivalent:*

1) $i\big[X^{\uparrow}{}_{\mathrm{hol}}[f]\big]\,\Omega = df$,

2) $L\Big[X^{\uparrow}{}_{\text{hol}}[f]\Big]\,\Omega = 0\,,$ *with* $f \in \text{cons}(J_1\boldsymbol{E},\, I\!R)\,,$
3) $X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f]\,,$ *with* $f \in \text{cons}(J_1\boldsymbol{E},\, I\!R)\,.$

Proof. 1) $\Rightarrow$ 2). Let $i\big(X^{\uparrow}{}_{\text{hol}}[f]\big)\,\Omega = df$. Then $L\big(X^{\uparrow}{}_{\text{hol}}[f]\big)\,\Omega := di\big(X^{\uparrow}{}_{\text{hol}}[f]\big)\,\Omega = ddf = 0$.
On the other hand, the identity $i(\gamma)\,\Omega = 0$ yields $i(\gamma)\,df = 0$, i.e. $f \in \text{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R})$.

2) $\Rightarrow$ 3). Let $L\big[X^{\uparrow}{}_{\text{hol}}[f]\big]\,\Omega = 0$, i.e. $di\big(X^{\uparrow}{}_{\text{hol}}[f]\big)\,\Omega = 0$.
Then, we have locally the equality $i\big(X^{\uparrow}{}_{\text{hol}}[f]\big)\,\Omega = dg$, with $g \in \text{map}(J_1\boldsymbol{E},\, \mathrm{I\!R})$.
On the other hand, the identity $i(\gamma)\,\Omega = 0$ yields $i(\gamma)\,dg = 0$, i.e. $g \in \text{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R})$.
As a consequence we obtain $(\Lambda^\sharp \circ \Omega^\flat)\big(X^{\uparrow}{}_{\text{hol}}[f]\big) = \Lambda^\sharp(dg)$ and from this we can deduce that $X^{\uparrow}{}_{\text{hol}}[f] - \gamma\big(X^{\uparrow}{}_{\text{hol}}[f]\big) = X^{\uparrow}{}_{\text{hol}}[f] - \gamma(f'') = \Lambda^\sharp(dg)$. Hence $X^{\uparrow}{}_{\text{hol}}[f] = \gamma(f'') + \Lambda^\sharp(dg) := X^{\uparrow}{}_{\text{ham}}[f'', g]$.
On the other hand, since $X^{\uparrow}{}_{\text{hol}}[f]$ is projectable on $\boldsymbol{E}$, Theorem 1.5 implies $g \in \text{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ and $g'' = f''$.
Hence, we obtain $X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[g]$, which yields $X[f] = X[g]$, hence $f = g + h$, with $h \in \text{map}(\boldsymbol{E}, \mathrm{I\!R})$.
On the other hand, $f, g \in \text{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R})$ implies $h \in \text{cons}\,\text{map}(\boldsymbol{E}, \mathrm{I\!R}) = \mathrm{I\!R}$.
Therefore, we obtain $X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f]$ in the domain of definition of g , But, if the above equality holds locally, then it holds in the domain of definition of f .

3) $\Rightarrow$ 1). Let $f \in \text{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R})$. Then, the identities concerning the linear musical isomorphisms and the identity $i(\gamma)\,\Omega = 0$ yield $i\big(X^{\uparrow}{}_{\text{ham}}[f]\big)\,\Omega := i\big(\gamma(f'') + \Lambda^\sharp(df)\big)\,\Omega = (\Omega^\flat \circ \Lambda^\sharp)(df) = df - i(\gamma)\,df = df$.
Hence, $X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f]$ implies $i\big(X^{\uparrow}{}_{\text{hol}}[f]\big)\,\Omega = df$. QED

2.17 Corollary. We have $\text{self}(J_1\boldsymbol{E},\, \mathrm{I\!R}) = \text{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R}) \cap \text{hol}\,(J_1\boldsymbol{E},\, \mathrm{I\!R})$. □

Indeed, an even stronger result holds.

2.18 Theorem. *We have* $\text{self}(J_1\boldsymbol{E},\, I\!R) = \text{cons}\,\text{proj}(J_1\boldsymbol{E},\, I\!R)$.

Proof. The classifying systems of Proposition 2.14 and of Proposition 2.3 coincide.
Hence, $\text{self}(J_1\boldsymbol{E},\, \mathrm{I\!R}) = \text{cons}\,\text{proj}(J_1\boldsymbol{E},\, \mathrm{I\!R})$. QED

2.1.4 Subalgebra of unimodular functions

Next, we consider the subalgebras of the algebra of projectable special phase functions related to the divergence of the tangent lift.

A vector field $X \in \text{proj}(\boldsymbol{E},\, T\boldsymbol{E})$ is called *conformal unimodular*, or *unimodular*, if we have, respectively, $d(\text{div}_\eta X) = 0$, or $\text{div}_\eta X = 0$.

For each $X, \bar{X} \in \text{proj}(\boldsymbol{E},\, T\boldsymbol{E})$, we have $\text{div}_\eta([X, \bar{X}]) = X.\,\text{div}_\eta \bar{X} - \bar{X}.\,\text{div}_\eta X$. Hence, the sheaves of conformal unimodular and unimodular vector fields of $T\boldsymbol{E}$ are closed with respect to the Lie bracket.

A function $f \in \text{spec}(J_1\boldsymbol{E},\, \mathrm{I\!R})$ is said to be *unimodular*, or *conformal unimodular* if, respectively, $\text{div}_\eta X[f] = 0$, or $d(\text{div}_\eta X[f]) = 0$. The subsheaves of unimodular and conformal unimodular projectable special phase functions are denoted, respectively, by $\text{unim}(J_1\boldsymbol{E},\, \mathrm{I\!R}) \subset \text{proj}(J_1\boldsymbol{E},\, \mathrm{I\!R})$ and $\text{c-unim}(J_1\boldsymbol{E},\, \mathrm{I\!R}) \subset \text{proj}(J_1\boldsymbol{E},\, \mathrm{I\!R})$.

In the above definition, we need to consider projectable special phase functions, because $\text{div}_\eta X$ is defined only for a projectable spacetime vector field X , due to the fact that η is a vertical form.

A vector field $X^{\uparrow} \in \mathrm{proj}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ is called *conformal unimodular*, or *unimodular*, if we have, respectively, $d(\mathrm{div}_\eta X^{\uparrow}) = 0$, or $\mathrm{div}_\eta X^{\uparrow} = 0$.

The sheaves of conformal unimodular and unimodular vector fields of $J_1\boldsymbol{E}$ are closed with respect to the Lie bracket.

2.19 Proposition. The sheaves of conformal unimodular and unimodular special phase functions are closed with respect to the special bracket.

PROOF. If $f, g \in \mathrm{proj}(J_1\boldsymbol{E}, \mathrm{I\!R})$, then we obtain

$$\mathrm{div}_\eta(X[\,[\![f, g]\!]\,]) = \mathrm{div}_\eta\,\big[X[f], X[g]\big] = X[g].\,\mathrm{div}_\eta(X[f]) - X[f].\,\mathrm{div}_\eta(X[g])\,.\ \mathrm{QED}$$

2.20 Proposition. If $f \in \mathrm{self\,fine}(J_1\boldsymbol{E}, \mathrm{I\!R})$, then $\mathrm{div}_\eta(X[f]) = 0$, hence

$$\mathrm{self\,fine}(J_1\boldsymbol{E}, \mathrm{I\!R}) = \mathrm{cons\,fine}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{unim}(J_1\boldsymbol{E}, \mathrm{I\!R})\,.$$

PROOF. The equality $i\big(X^{\uparrow}{}_{\mathrm{hol}}[f]\big)\,\Omega = df$ yields $L\big[X^{\uparrow}{}_{\mathrm{hol}}[f]\big]\,\Omega = 0$, hence, in virtue of Proposition 1.4, $\mathrm{div}_\eta(X[f]) = 0$. QED

2.1.5 Subalgebra of classic generators

Eventually, we consider the subalgebra of the algebra of special phase functions, which generates the infinitesimal symmetries of the full classical structure.

Each $f \in \mathrm{cons\,fine}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is called a *classical generator*. We denote the sheaf of classical generators by $\mathrm{clas}(J_1\boldsymbol{E}, \mathrm{I\!R}) := \mathrm{cons\,fine}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

2.21 Theorem. *We have* $\mathrm{clas}(J_1\boldsymbol{E}, I\!R) = \mathrm{self\,fine}(J_1\boldsymbol{E}, I\!R) \subset \mathrm{hol}\,(J_1\boldsymbol{E}, I\!R)$ *and* $\mathrm{clas}(J_1\boldsymbol{E}, I\!R) \subset \mathrm{unim}(J_1\boldsymbol{E}, I\!R)$.

PROOF. It follows immediately from Theorem 2.18, Lemma 2.10 and Proposition 2.20. QED

By reformulating a previous result, we have the following characterisation of the classical generators.

2.22 Corollary. The sheaf $\mathrm{clas}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is constituted by the special phase functions f such that

$$\begin{aligned}
\partial_\lambda f^0 &= 0 \\
f^0\,\partial_0 G^0_{ij} - f^h\,\partial_h G^0_{ij} - \partial_j f^h\,G^0_{ih} - \partial_i f^h\,G^0_{jh} &= 0 \\
f^0\,(\partial_0 A_j - \partial_j A_0) - f^i\,(\partial_i A_j - \partial_j A_i) - G^0_{ij}\,\partial_0 f^i - \partial_j \breve{f} &= 0 \\
f^i\,(\partial_i A_0 - \partial_0 A_i) + \partial_0 \breve{f} &= 0\,.\,\square
\end{aligned}$$

2.23 Example. In the special Newtonian spacetime, the sheaf $\mathrm{clas}(J_1\boldsymbol{E}, \mathbb{R})$ is constituted by the special phase functions f such that

$$\begin{gathered}
\partial_\lambda f^0 = 0 \\
\partial_1 f^1 = \partial_2 f^2 = \partial_3 f^3 = 0 \\
\partial_1 f^2 = -\partial_2 f^1\,, \qquad \partial_1 f^3 = -\partial_3 f^1\,, \qquad \partial_2 f^3 = -\partial_3 f^2 \\
\partial_0 f^i = -\partial_i \breve{f} \\
\partial_0 \breve{f} = 0\,.
\end{gathered}$$

For instance, a solution of this system is given by

$$\begin{aligned}
f^0 &= a^0 \\
f^i &= b^i_j\, x^j + c^i_0\, x^0 + d^i \\
\breve{f} &= -\sum_{1\le i\le 3} c^i_0\, x^i + e\,,
\end{aligned}$$

where a^0, b^i_j, c^i_0, d^i, $e \in \mathbb{R}$ and $b^i_j = -b^j_i$. □

2.2 Classical infinitesimal symmetries

We classify the vector fields of the phase space which are infinitesimal symmetries of spacetime and its structures.

2.2.1 Infinitesimal symmetries of geometric structures

We start by defining the infinitesimal symmetries of some typical geometric structures. All concepts below are defined in such a way that the corresponding local group of diffeomeorphisms act on the geometric structure and preserve it.

We introduce the following general concepts.

1. We define an *infinitesimal symmetry* of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ to be a projectable vector field X of $\boldsymbol{F}$.

2. We define an *infinitesimal symmetry* of a bundle $q : \boldsymbol{G} \to \boldsymbol{M}$, which is a natural prolongation of a manifold $\boldsymbol{M}$, to be the projectable vector field Y obtained by the corresponding natural lift of a vector field X of $\boldsymbol{M}$.

3. We define an *infinitesimal symmetry* of a bundle $q : \boldsymbol{G} \to \boldsymbol{F}$, which is a natural prolongation of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ to be the projectable vector field Y obtained by the corresponding natural lift of a vector field X of $\boldsymbol{F}$.

4. We define an *infinitesimal symmetry* of a tensor σ of a manifold $\boldsymbol{M}$ to be a vector field X of $\boldsymbol{M}$ such that $L[X]\,\sigma = 0$.

5. We define an *infinitesimal symmetry* of a covariant vertical tensor σ of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ to be a projectable vector field X of $\boldsymbol{F}$ such that $L[X]\,\sigma = 0$.

6. We define an *infinitesimal symmetry* of an affine space $\boldsymbol{A}$ to be a constant vector field of $\boldsymbol{A}$.

2.2.2 Infinitesimal symmetries of spacetime and phase space

According to the above guideline, we introduce the infinitesimal symmetries of time, of spacetime and of the phase space as the vector fields which preserve the affine structure of time, the time fibring of spacetime and the natural 1st jet functor.

An *infinitesimal symmetry of time* is defined to be an infinitesimal symmetry of the affine structure of $\boldsymbol{T}$, which can be regarded just as an element $\underline{X} \in \bar{\mathbb{T}}$ (i.e. a constant vector field of $\boldsymbol{T}$.)

An *infinitesimal symmetry of spacetime* is defined to be an infinitesimal symmetry of the time fibring t yielding also an infinitesimal symmetry of the affine structure of $\boldsymbol{T}$, i.e. a vector field $X \in \mathrm{fine}(\boldsymbol{E}, T\boldsymbol{E})$.

An *infinitesimal symmetry of the phase space* is defined to be the infinitesimal symmetry of the 1st jet prolongation of spacetime yielding also an infinitesimal symmetry of the time fibring t and an infinitesimal symmetry of the affine structure of $\boldsymbol{T}$, i.e. the holonomic prolongation of an infinitesimal symmetry of spacetime $X_{(1)} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, with $X \in \mathrm{fine}(\boldsymbol{E}, T\boldsymbol{E})$.

We define, respectiverly, a spacetime *infinitesimal symmetry* of dt and a phase *infinitesimal symmetry* of dt to be vector fields $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$ and $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, such that $L[X]\, dt = 0$ and $L[X^{\uparrow}]\, dt = 0$.

2.24 Proposition. The infinitesimal symmetries of dt are the vector fields of the type $X \in \mathrm{fine}(\boldsymbol{E}, T\boldsymbol{E})$ and $X^{\uparrow} \in \mathrm{fine}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$.

PROOF. In fact, we have $L[X]\, dt = d\, i(X)\, dt$ and $L[X^{\uparrow}]\, dt = d\, i(X^{\uparrow})\, dt$. QED

2.25 Corollary. A vector field $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$ is an infinitesimal symmetry of spacetime if and only if it is an infinitesimal symmetry of dt. $\square$

2.2.3 Infinitesimal symmetries of the cosymplectic 2–form

We define an *infinitesimal symmetry* of Ω to be a vector field $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, such that $L[X^{\uparrow}]\, \Omega = 0$.

2.26 Theorem. *The infinitesimal symmetries $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ of Ω are of the local type $X^{\uparrow} = X^{\uparrow}{}_{\mathrm{ham}}[\tau, f]$, with $\tau \in \mathrm{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$ and $f \in \mathrm{cons}(J_1\boldsymbol{E}, I\!\!R)$, where f is determined up to a constant.*

PROOF. Let us consider any $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and set $\tau := dt(X^{\uparrow}) \in \mathrm{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$.

Then, $X^{\uparrow}$ can be uniquely written as $X^{\uparrow} = \gamma(\tau) + \bar{X}^{\uparrow}$, with $\bar{X}^{\uparrow} \in \sec(J_1\boldsymbol{E}, VJ_1\boldsymbol{E})$.

Moreover, by recalling the identity $i(\gamma)\, \Omega = 0$, we obtain $L[\gamma(\tau)]\, \Omega = 0$.

Furthermore, by recalling the identity $d\Omega = 0$, we have $L[\bar{X}^{\uparrow}]\, \Omega = 0$ if and only if $di(\bar{X}^{\uparrow})\, \Omega = 0$, i.e. if and only if locally $i(\bar{X}^{\uparrow})\, \Omega = df$, with $\gamma.f = 0$, i.e., in virtue of the results of Section 1.4.1, if and only if locally $\bar{X}^{\uparrow} = \Lambda^{\sharp}(df)$, with $\gamma.f = 0$. QED

2.27 Corollary. The infinitesimal symmetries $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ of dt and Ω are of the local type $X^{\uparrow} = X^{\uparrow}{}_{\text{ham}}[\tau, f]$, with $\tau \in \bar{\mathbb{T}}$ and $f \in \text{cons}(J_1\boldsymbol{E}, \mathbb{R})$, where f is defined up to a constant. □

2.2.4 Infinitesimal symmetries of the classical structure

Next, we classify the infinitesimal symmetries of dt and Ω, which are projectable on $\boldsymbol{E}$. Indeed, the projectability condition yields an important consequence: namely, it implies that the vector field is generated by a special phase function.

2.28 Corollary. The infinitesimal symmetries $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ of Ω, which are projectable on $\boldsymbol{E}$, are of the local type $X^{\uparrow} = X^{\uparrow}{}_{\text{ham}}[f]$, with $f \in \text{cons}\,\text{spec}(J_1\boldsymbol{E}, \mathbb{R})$, where f is defined up to a constant.

Proof. It follows from Theorem 2.26 and Theorem 1.5. QED

2.29 Corollary. The infinitesimal symmetries $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ of dt and Ω, which are projectable on $\boldsymbol{E}$, are of the type $X^{\uparrow} = X^{\uparrow}{}_{\text{ham}}[f]$, with $f \in \text{cons}\,\text{fine}(J_1\boldsymbol{E}, \mathbb{R})$, where f is determined up to a constant.

Proof. It follows from the above Corollary 2.28 and Proposition 2.24. QED

2.30 Corollary. Let us consider a vector field $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$.
If its holonomic prolongation $X_{(1)} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ is an infinitesimal symmetry of dt and Ω, then we obtain locally $X_{(1)} = X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f]$, with $f \in \text{cons}\,\text{fine}(J_1\boldsymbol{E}, \mathbb{R})$ and $X = X[f]$.

Proof. In virtue of Corollary 2.29, we obtain $X_{(1)} = X^{\uparrow}{}_{\text{ham}}[f]$, with $f \in \text{cons}\,\text{fine}(J_1\boldsymbol{E}, \mathbb{R})$.
On the other hand, in virtue of Proposition 1.2, $X_{(1)}$ projects on X and, in virtue of Theorem 1.5, $X^{\uparrow}{}_{\text{ham}}[f]$ projects on $X[f]$. Hence, we obtain $X = X[f]$ and $X_{(1)} = X^{\uparrow}{}_{\text{hol}}[f]$. QED

We can reformulate the above result in a slightly stronger way.
An *infinitesimal symmetry of the classical structure* is defined to be a vector field $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, which is an infinitesimal symmetry of dt and Ω and which is projectable on $\boldsymbol{E}$.

2.31 Corollary. The infinitesimal symmetries $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ of the classical structure are of the local type $X^{\uparrow} = X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f]$, with $f \in \text{cons}\,\text{fine}(J_1\boldsymbol{E}, \mathbb{R})$.

Proof. In virtue of Corollary 2.29, we obtain $X^{\uparrow} = X^{\uparrow}{}_{\text{ham}}[f]$, with $f \in \text{cons}\,\text{fine}(J_1\boldsymbol{E}, \mathbb{R})$.
On the other hand, in virtue of Theorem 2.21, we have $\text{cons}\,\text{fine}(J_1\boldsymbol{E}, \mathbb{R}) = \text{self}\,\text{fine}(J_1\boldsymbol{E}, \mathbb{R})$, hence $X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f]$. QED

2.32 Proposition. The subsheaf of infinitesimal symmetries of the classical structure is a sheaf of Lie subalgebras.

Proof. It follows from the fact that the holonomic lift of special phase functions is a morphisms of Lie algebras and that the holonomic lift of projectable special phase functions is a morphisms of Lie algebras. QED

Of course, we can analogously prove that also the other subsheaves of infinitesimal symmetries considered above are subalgebras.

2.3 Classical currents

We devote this last section to the analysis of distinguished functions that are generated by symmetries of our structure.

2.3.1 Functions generated by a horizontal potential

Each pair consisting of a spacetime vector field X and a horizontal potential $A^{\uparrow}$ of Ω yield a special phase function. This construction turns out to be an important source of special phase functions in our classical and quantum theories. Indeed, the above simple definition encodes deep aspects relating the horizontal potentials of Ω and the classical and quantum symmetries.

Let us consider a spacetime vector field $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$ and a horizontal potential $A^{\uparrow} \in \mathrm{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ of Ω.
We define the *function generated* by X and $A^{\uparrow}$ as $-X \lrcorner A^{\uparrow} \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

2.33 Proposition. The function $-X \lrcorner A^{\uparrow}$ is a special phase function.
Its coordinate expression is $-X \lrcorner A^{\uparrow} = X^0 \, \frac{1}{2} \, G^0_{ij} \, x^i_0 \, x^j_0 - X^i \, G^0_{ij} \, x^j_0 - X^\lambda \, A_\lambda$ and, with reference to an observer o, we have $-X \lrcorner A^{\uparrow} = X \lrcorner \mathcal{K}[o] - \nu[o](X) \lrcorner \mathcal{Q}[o] - X \lrcorner A[o]$.

Proof. It follows from the coordinate expression of $A^{\uparrow}$. QED

Now, as a particular case, let us consider an $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ and its tangent prolongation $X[f] \in \sec(\boldsymbol{E}, T\boldsymbol{E})$.

2.34 Corollary. We obtain the special phase function $-X[f] \lrcorner A^{\uparrow} \in \mathrm{spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$, with observed expression $-X[f] \lrcorner A^{\uparrow} = f - f \circ o - X[f] \lrcorner A[o]$. In an adapted chart, we get

$$-X[f] \lrcorner A^{\uparrow} = f^0 \, (\tfrac{1}{2} \, G^0_{ij} \, x^i_0 \, x^j_0 - A_0) + f^i \, (G^0_{ij} \, x^j_0 + A_i) = f - (\breve{f} + f^0 \, A_0 - f^i \, A_i) \,.$$

Indeed, we obtain $X\Big[- X[f] \lrcorner A^{\uparrow} \Big] = X[f]$. □

2.35 Corollary. For each observer o, the function

$$\bar{f} := f \circ o + X[f] \lrcorner A[o] = f + X[f] \lrcorner A^{\uparrow} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R}) \,,$$

does not depend on the choice of the observer o. Therefore, the coordinate expression $\bar{f} = \breve{f} + f^0 \, A_0 - f^i \, A_i$ does not depend on the adapted chart.

Proof. In fact, f and $X[f] \lrcorner A^{\uparrow}$ do not depend on the choice of any observer. QED

For instance, we have $-X[\mathcal{L}_0] \lrcorner A^{\uparrow} = \mathcal{L}_0 - 2 \, A_0 + G^{ij}_0 \, A_i \, A_j$, $-X[\mathcal{H}_0] \lrcorner A^{\uparrow} = \mathcal{H}_0$ and $-X[\mathcal{P}_i] \lrcorner A^{\uparrow} = \mathcal{P}_i$.

2.3.2 Nöther's theorem

The previous results on infinitesimal symmetries can be applied to the Lagrangian formalism. Here, we call in mind some of results already presented in [66] and add new results as well.

Let us consider a horizontal potential $A^{\uparrow} \in \mathrm{fib}(J_1\boldsymbol{E},\, T^*\boldsymbol{E})$ of Ω and the associated Lagrangian $\mathcal{L} \in \sec(J_1\boldsymbol{E},\, H^*J_1\boldsymbol{E})$ and momentum $\mathcal{P} \in \sec(J_1\boldsymbol{E},\, T^*J_1\boldsymbol{E})$.

We define an *infinitesimal symmetry* of $A^{\uparrow}$ to be a vector field $X^{\uparrow} \in \sec(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$, such that $L[X^{\uparrow}]\, A^{\uparrow} = 0$.

2.36 Lemma. Each infinitesimal symmetry of $A^{\uparrow}$ is an infinitesimal symmetry of Ω. □

We can formulate the following (Nöther) theorem which relates holonomic infinitesimal symmetries of $A^{\uparrow}$ to conserved functions. For this, let us consider an $X \in \sec(\boldsymbol{E},\, T\boldsymbol{E})$.

We say that X is a *holonomic infinitesimal symmetry* of a tensor ϕ of the phase space if its holonomic prolongation $X_{(1)} \in \sec(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ is an infinitesimal symmetry of ϕ, i.e. if $L[X_{(1)}]\, \phi = 0$.

2.37 Theorem. *If $L[X_{(1)}]\, A^{\uparrow} = 0$, then the 1–form $i(X_{(1)})\, \Omega \in \sec(J_1\boldsymbol{E},\, T^*J_1\boldsymbol{E})$ is exact and the function $f := -X \lrcorner A^{\uparrow} \in \mathrm{map}(J_1\boldsymbol{E},\, I\!\!R)$ turns out to be a potential of $i(X_{(1)})\, \Omega$. Moreover, we obtain $f \in \mathrm{cons\,self}(J_1\boldsymbol{E},\, I\!\!R)$ and $X_{(1)} = X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{ham}}[f]$.*

PROOF. We have $i(X_{(1)})\, \Omega = i(X_{(1)})\, dA^{\uparrow} = L[X_{(1)}]\, A^{\uparrow} - di(X_{(1)})\, A^{\uparrow} = 0 - di(X)\, A^{\uparrow} = df$.
Hence, f is a potential of $i(X_{(1)})\, \Omega$. Moreover, in virtue of Lemma 2.36 and Corollary 2.28, we obtain $f \in \mathrm{cons\,self}(J_1\boldsymbol{E},\, I\!\!R)$ and $X_{(1)} = X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{ham}}[f]$ in the whole domain of $A^{\uparrow}$. QED

2.38 Corollary. If X is an infinitesimal symmetry of dt and a holonomic infinitesimal symmetry of $A^{\uparrow}$, then the potential $f := -X \lrcorner A^{\uparrow} = -(X \lrcorner \mathcal{P} + \underline{X} \lrcorner \mathcal{L})$ of $i(X_{(1)})\, \Omega$ is a classical generator.

PROOF. It follows from the above Theorem 2.37, Corollary 2.30, the definition of classical generators and Proposition 2.24. QED

2.39 Corollary. If an observer $o \in \sec(\boldsymbol{E},\, \mathbb{T}^* \otimes T\boldsymbol{E})$ is a (scaled) infinitesimal symmetry of $A^{\uparrow}$, then the associated (scaled) potential of $i(o_{(1)})\, \Omega$ is just the associated Hamiltonian $\mathcal{H}[A^{\uparrow}, o] := -o \lrcorner A^{\uparrow} \in \mathrm{map}(J_1\boldsymbol{E},\, \mathbb{T}^* \otimes I\!\!R)$. In particular, $\mathcal{H}[A^{\uparrow}, o]$ turns out to be a conserved (scaled) function. □

Next, we prove that the holonomic infinitesimal symmetries of dt and of the horizontal potential are just the holonomic infinitesimal symmetries of dt and of the Lagrangian.

2.40 Lemma. For each $X \in \mathrm{fine}(\boldsymbol{E},\, T\boldsymbol{E})$, we have the coordinate expressions

$$L[X_{(1)}]\, \mathcal{L} = (X^{\mu}\, \partial_{\mu}\mathcal{L}_0 + \partial_0 X^j\, \partial^0_j \mathcal{L}_0 + \partial_h X^j\, x^h_0\, \partial^0_j \mathcal{L}_0)\, d^0\,,$$

$$L[X_{(1)}]\, \mathcal{P} = \partial^0_i (X^{\mu}\, \partial_{\mu}\mathcal{L}_0 + \partial_0 X^j\, \partial^0_j \mathcal{L}_0 + \partial_h X^j\, x^h_0\, \partial^0_j \mathcal{L}_0)\, d^i$$

$$- \partial^0_i (X^\mu \, \partial_\mu \mathcal{L}_0 + \partial_0 X^j \, \partial^0_j \mathcal{L}_0 + \partial_h X^j \, x^h_0 \, \partial^0_j \mathcal{L}_0) \, x^i_0 \, d^0 \, .$$

Proof. We have

$$\begin{aligned} L[X_{(1)}] \, \mathcal{P} &= (X^\mu \, \partial_\mu \partial^0_i \mathcal{L}_0 + \partial_0 X^j \, \partial^0_j \partial^0_i \mathcal{L}_0 + \partial_h X^j \, x^h_0 \, \partial^0_j \partial^0_i \mathcal{L}_0 + \partial_i X^j \, \partial^0_j \mathcal{L}_0) \, d^i \\ &\quad - \big((X^\mu \, \partial_\mu \partial^0_i \mathcal{L}_0 + \partial_0 X^j \, \partial^0_j \partial^0_i \mathcal{L}_0 + \partial_h X^j \, x^h_0 \, \partial^0_j \partial^0_i \mathcal{L}_0) \, x^i_0 \\ &\qquad + \partial_0 X^i \, \partial^0_i \mathcal{L}_0 + \partial_h X^i \, x^h_0 \, \partial^0_i \mathcal{L}_0 - \partial_0 X^j \, \partial^0_j \mathcal{L}_0) \big) \, d^0 \\ &= (X^\mu \, \partial_\mu \partial^0_i \mathcal{L}_0 + \partial_0 X^j \, \partial^0_j \partial^0_i \mathcal{L}_0 + \partial_h X^j \, x^h_0 \, \partial^0_j \partial^0_i \mathcal{L}_0 + \partial_i X^j \, \partial^0_j \mathcal{L}_0) \, d^i \\ &\quad - \big((X^\mu \, \partial_\mu \partial^0_i \mathcal{L}_0 + \partial_0 X^j \, \partial^0_j \partial^0_i \mathcal{L}_0 + \partial_h X^j \, x^h_0 \, \partial^0_j \partial^0_i \mathcal{L}_0) \, x^i_0 + \partial_h X^i \, x^h_0 \, \partial^0_i \mathcal{L}_0) \big) \, d^0 \\ &= \partial^0_i (X^\mu \, \partial_\mu \mathcal{L}_0 + \partial_0 X^j \, \partial^0_j \mathcal{L}_0 + \partial_h X^j \, x^h_0 \, \partial^0_j \mathcal{L}_0 - \partial_i X^j \, \partial^0_j \mathcal{L}_0 + \partial_i X^j \, \partial^0_j \mathcal{L}_0) \, d^i \\ &\quad - \big(\partial^0_i (X^\mu \, \partial_\mu \mathcal{L}_0 + \partial_0 X^j \, \partial^0_j \mathcal{L}_0 + \partial_h X^j \, x^h_0 \, \partial^0_j \mathcal{L}_0) \, x^i_0 - \partial_i X^j \, x^i_0 \, \partial^0_j \mathcal{L}_0 + \partial_h X^i \, x^h_0 \, \partial^0_i \mathcal{L}_0) \big) \, d^0 \\ &= \partial^0_i (X^\mu \, \partial_\mu \mathcal{L}_0 + \partial_0 X^j \, \partial^0_j \mathcal{L}_0 + \partial_h X^j \, x^h_0 \, \partial^0_j \mathcal{L}_0) \, d^i \\ &\quad - \partial^0_i (X^\mu \, \partial_\mu \mathcal{L}_0 + \partial_0 X^j \, \partial^0_j \mathcal{L}_0 + \partial_h X^j \, x^h_0 \, \partial^0_j \mathcal{L}_0) \, x^i_0 \, d^0 \, . \text{QED} \end{aligned}$$

2.41 Proposition. For each $X \in \operatorname{fine}(\boldsymbol{E}, T\boldsymbol{E})$, we have the following implication

$$L[X_{(1)}] \, \mathcal{L} = 0 \qquad \Rightarrow \qquad L[X_{(1)}] \, \mathcal{P} = 0 \, . \square$$

2.42 Theorem. *[66] For each* $X \in \operatorname{fine}(\boldsymbol{E}, T\boldsymbol{E})$, *the following equivalence holds*

$$L[X_{(1)}] \, A^\uparrow = 0 \qquad \Leftrightarrow \qquad L[X_{(1)}] \, \mathcal{L} = 0 \, .$$

Proof. If $L[X_{(1)}] \, A^\uparrow = 0$, then, in virtue of Lemma 1.3, we have

$$L[X_{(1)}] \, \mathcal{L} := L[X_{(1)}] \, i(д) \, A^\uparrow = -i(д) \, L[X_{(1)}] \, A^\uparrow + i\big([X_{(1)}, д]\big) \, A^\uparrow = 0 + 0 \, .$$

If $L[X_{(1)}] \, \mathcal{L} = 0$, then, in virtue of Section 1.3.2 and Proposition 2.41

$$L[X_{(1)}] \, A^\uparrow = L[X_{(1)}] \, \big(\mathcal{L} + \mathcal{P}\big) = 0 + 0 \, . \text{QED}$$

2.3.3 Group of symmetries and momentum map

We define a momentum map for classical symmetries in our framework, by analogy with the standard symplectic and cosymplectic literature (see, for instance, [2, 18, 59, 61] and references therein).

Let us consider a Lie group $\boldsymbol{G}^\uparrow$ and its Lie algebra $\mathfrak{g}^\uparrow := T_{e^\uparrow} \boldsymbol{G}^\uparrow$, where $e^\uparrow \in \boldsymbol{G}$ denotes the unit element of $\boldsymbol{G}^\uparrow$. We refer to a chart (ζ^I) of $\boldsymbol{G}$, defined in an open neighbouroud of $e^{\uparrow}$, and to the induced basis $(b^{\uparrow}{}_I) := \big(\partial_I(e^{\uparrow})\big)$ of $\mathfrak{g}$.

Let us make the following assumptions, by postulating, step by step, subsequent additional hypotheses.

0) Let us start by assuming that $\boldsymbol{G}^{\uparrow}$ be a group of *symmetries of the phase space*. It means that $\boldsymbol{G}^{\uparrow}$ *acts on the phase space* through a map $\alpha^{\uparrow} : \boldsymbol{G}^{\uparrow} \times J_1\boldsymbol{E} \to J_1\boldsymbol{E}$.

By taking the partial tangent map of $\alpha^{\uparrow}$, with respect to $\boldsymbol{G}^{\uparrow}$, at $e^{\uparrow} \in \boldsymbol{G}^{\uparrow}$, we obtain the fibred morphism $\partial\alpha^{\uparrow} : \mathfrak{g}^{\uparrow} \times J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$ over $J_1\boldsymbol{E}$. Hence, for each $\xi^{\uparrow} \in \mathfrak{g}^{\uparrow}$, we obtain the vector field $X^{\uparrow} := \partial\alpha^{\uparrow}(\xi^{\uparrow}) := (\partial\alpha^{\uparrow})_{|\xi^{\uparrow}} : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$, with coordinate expression $\partial\alpha^{\uparrow}(\xi^{\uparrow}) = \xi^I\,(\partial_I\alpha^0\,\partial_0 + \partial_I\alpha^i\,\partial_i + \partial_I\alpha^i_0\,\partial^0_i)$. We can regard $\partial\alpha^{\uparrow}$ as a homomorphism of $\mathbb{R}$–Lie algebras $\partial\alpha^{\uparrow} : \mathfrak{g}^{\uparrow} \to \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$.

We define the *time component of* $\partial\alpha^{\uparrow}$ as $\partial\underline{\alpha}^{\uparrow} := dt \circ \partial\alpha^{\uparrow} : \mathfrak{g}^{\uparrow} \to \mathrm{Map}(J_1\boldsymbol{E},\, \bar{\mathbb{T}})$.

If $\alpha^{\uparrow}$ is *a free action*, then $\partial\alpha^{\uparrow} : \mathfrak{g}^{\uparrow} \to \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ turns out to be injective.

1) Let us suppose additionally that $\alpha^{\uparrow}$ be a group of *symmetries* of the phase space *fibred over time*. It means that

i) we have a Lie group $\underline{\boldsymbol{G}}$, which acts on $\boldsymbol{T}$ through a map $\underline{\alpha} : \underline{\boldsymbol{G}} \times \boldsymbol{T} \to \boldsymbol{T}$,

ii) we have a Lie group epimorphism $\underline{\pi}^{\uparrow} : \boldsymbol{G}^{\uparrow} \to \underline{\boldsymbol{G}}$,

iii) the following diagram commutes

$$\begin{tikzcd} \boldsymbol{G}^{\uparrow} \times J_1\boldsymbol{E} \arrow[r, "\alpha^{\uparrow}"] \arrow[d, "\underline{\pi}^{\uparrow}\times t^1"'] & J_1\boldsymbol{E} \arrow[d, "t^1"] \\ \underline{\boldsymbol{G}} \times \boldsymbol{T} \arrow[r, "\underline{\alpha}"] & \boldsymbol{T} \end{tikzcd}$$

Let $\underline{\mathfrak{g}}$ be the Lie algebra of the the Lie group $\underline{\boldsymbol{G}}$.

2) Let us suppose additionally that $\boldsymbol{G}^{\uparrow}$ be a *group of symmetries* of dt. It means that, for each $g^{\uparrow} \in \boldsymbol{G}^{\uparrow}$, we have $(\alpha^{\uparrow}{}_{|g^{\uparrow}})^*(dt) = dt$. This hypothesis means also that $\boldsymbol{G}^{\uparrow}$ is a group of *symmetries of the affine struture of* $\boldsymbol{T}$.

The Lie algebra $\mathfrak{g}^{\uparrow}$ is a *Lie algebra of infinitesimal symmetries* of dt, i.e., for each $\xi^{\uparrow} \in \mathfrak{g}^{\uparrow}$, we have $L\big[\partial\alpha^{\uparrow}(\xi^{\uparrow})\big]\,dt = 0$. Moreover, for each $\xi^{\uparrow} \in \mathfrak{g}^{\uparrow}$, the vector field $\partial\alpha^{\uparrow}(\xi^{\uparrow}) \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ has constant time component. Hence, $\partial\alpha^{\uparrow} : \mathfrak{g}^{\uparrow} \to \mathrm{Fine}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$.

Moreover, we obtain the equality $\partial\underline{\alpha}^{\uparrow} = \partial\underline{\alpha} : \mathfrak{g}^{\uparrow} \to \bar{\mathbb{T}}$, according to the following commutative diagram

$$\begin{tikzcd} \mathfrak{g}^{\uparrow} \arrow[r, "\partial\alpha^{\uparrow}"] \arrow[d, "T\underline{\pi}_{|e}"'] & \mathrm{Fine}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) \arrow[d] \\ \underline{\mathfrak{g}} \arrow[r, "\partial\underline{\alpha}"] & \bar{\mathbb{T}}\,. \end{tikzcd}$$

3) Let us suppose additionally that $\boldsymbol{G}^{\uparrow}$ be a group of *symmetries of* Ω. It means that, for each $g^{\uparrow} \in \boldsymbol{G}^{\uparrow}$, we have $(\alpha^{\uparrow}{}_{|g^{\uparrow}})^*(\Omega) = \Omega$.

The Lie algebra $\mathfrak{g}^\uparrow$ is a *Lie algebra of infinitesimal symmetries* of Ω, i.e., for each $\xi^\uparrow \in \mathfrak{g}^\uparrow$, we have $L\Big[\partial\alpha^\uparrow(\xi^\uparrow)\Big]\,\Omega = 0$.

Corollary 2.27 suggests the following definition.

We define a *momentum map* to be a map $J^\uparrow : \mathfrak{g}^\uparrow \to \mathrm{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R}) : \xi^\uparrow \mapsto f$, where $f \in \mathrm{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R})$ is a local potential of the 1–form $i\Big(\partial\alpha^\uparrow(\xi)\Big)\,\Omega$.

In general, a momentum map is not unique. In fact, if $i\Big(\partial\alpha^\uparrow(\xi^\uparrow)\Big)\,\Omega$ is not exact, then its potential is determined only locally and up to a real additive constant.

4) Let us suppose additionally that *we have chosen a momentum map*

$$J^\uparrow : \mathfrak{g}^\uparrow \to \mathrm{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R})\,.$$

2.43 Proposition. For each $\xi^\uparrow \in \mathfrak{g}^\uparrow$, we have

$$\partial\alpha^\uparrow(\xi^\uparrow) = X^\uparrow{}_{\mathrm{ham}}\Big[J^\uparrow(\xi^\uparrow)\Big] := X^\uparrow{}_{\mathrm{ham}}\Big[\partial\underline{\alpha}^\uparrow(\xi^\uparrow),\; J^\uparrow(\xi^\uparrow)\Big]\,.$$

Proof. The definition of momentum map yields the equality

$$d\big(J^\uparrow(\xi^\uparrow)\big) = i\big(\partial\alpha^\uparrow(\xi^\uparrow)\big)\,\Omega\,.$$

Hence, the results of Section 1.4.1 yield $\Lambda^\sharp\Big(d\big(J^\uparrow(\xi^\uparrow)\big)\Big) = \Lambda^\sharp\Big(\Omega^\flat\big(\partial\alpha^\uparrow(\xi^\uparrow)\big)\Big) = \partial\alpha^\uparrow(\xi^\uparrow) - \gamma\big(\partial\alpha^\uparrow(\xi^\uparrow)\big)$, i.e. $\partial\alpha^\uparrow(\xi) = \gamma\big(\partial\underline{\alpha}^\uparrow(\xi^\uparrow)\big) + \Lambda^\sharp\Big(d\big(J^\uparrow(\xi^\uparrow)\big)\Big) := X^\uparrow{}_{\mathrm{ham}}\big[\partial\underline{\alpha}^\uparrow(\xi^\uparrow),\, J^\uparrow(\xi^\uparrow)\big]$. QED

The above result suggests the following definition. We define the *extended momentum map* to be the map $\widetilde{J}^\uparrow : \mathfrak{g}^\uparrow \to \bar{\mathbb{T}} \times \mathrm{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R}) : \xi^\uparrow \mapsto \Big(\partial\underline{\alpha}^\uparrow(\xi^\uparrow),\; J^\uparrow(\xi^\uparrow)\Big)$.

We define the *time scale bracket* of $\mathrm{map}(J_1\boldsymbol{E},\, \bar{\mathbb{T}})$ by $[\tau, \sigma] := \gamma(\tau).\sigma - \gamma(\sigma).\tau$. It turns out to be a $\Big(\mathrm{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R})\Big)$–Lie bracket.

Then, we define the *extended Poisson bracket* of $\mathrm{map}(J_1\boldsymbol{E},\, \bar{\mathbb{T}} \times \mathrm{I\!R})$ by $\Big\{(\tau, f),\; (\sigma, g)\Big\} := \Big([\tau,\, \sigma],\; \{f, g\}\Big)$. It turns out to be an $\mathrm{I\!R}$–Lie bracket. Moreover, the subsheaf $\bar{\mathbb{T}} \times \mathrm{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R}) \subset \mathrm{map}(J_1\boldsymbol{E},\, \bar{\mathbb{T}} \times \mathrm{I\!R}) \times \mathrm{map}(J_1\boldsymbol{E},\, \bar{\mathbb{T}} \times \mathrm{I\!R})$ turns out to be an $\mathrm{I\!R}$–Lie subalgebra.

Next, we analyse the behaviour of the extended momentum map with respects to the Lie bracket of $\mathfrak{g}$ and of the extended Poisson bracket.

Proposition 2.43 can be reformulated by saying that the following diagram commutes locally

$$\begin{tikzcd}
\mathfrak{g}^\uparrow \arrow[rr, "\widetilde{J}^\uparrow"] \arrow[dr, "\partial\alpha^\uparrow"'] & & \bar{\mathbb{T}} \times \mathrm{cons}(J_1\boldsymbol{E},\, \mathrm{I\!R}) \arrow[dl, "X^\uparrow{}_{\mathrm{ham}}"] \\
& \mathrm{Fine}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) &
\end{tikzcd}$$

2.44 Proposition. For each $\xi^\uparrow,\ \xi'^\uparrow \in \mathfrak{g}^\uparrow$, we have

$$\widetilde{J}^\uparrow\big([\xi^\uparrow,\xi'^\uparrow]\big) = \Big[\widetilde{J}^\uparrow(\xi^\uparrow),\ \widetilde{J}^\uparrow(\xi'^\uparrow)\Big] + (0,\ k)\,, \qquad \text{where} \qquad k \in \mathbb{R}\,.$$

Proof. We have $\partial\alpha^\uparrow(\xi^\uparrow),\, \partial\alpha^\uparrow(\xi'^\uparrow) \in \mathrm{Fine}(J_1\boldsymbol{E},\ J_1T\boldsymbol{E})$, hence $\partial\underline{\alpha}^\uparrow\big([\xi^\uparrow,\xi'^\uparrow]\big) = 0$. Then, in virtue of Proposition 2.43, we obtain

$$\begin{aligned}\partial\alpha^\uparrow\big([\xi^\uparrow,\xi'^\uparrow]\big) &= X^\uparrow{}_{\mathrm{ham}}\Big[\partial\underline{\alpha}^\uparrow\big([\xi^\uparrow,\xi'^\uparrow]\big),\ J^\uparrow\big([\xi^\uparrow,\xi'^\uparrow]\big)\Big]\\ &= X^\uparrow{}_{\mathrm{ham}}\Big[0,\ J^\uparrow\big([\xi^\uparrow,\xi'^\uparrow]\big)\Big]\,.\end{aligned}$$

On the other hand, in virtue of Proposition 2.43, we obtain

$$\begin{aligned}\partial\alpha^\uparrow\big([\xi^\uparrow,\xi'^\uparrow]\big) &= \big[\partial\alpha^\uparrow(\xi^\uparrow),\ \partial\alpha^\uparrow(\xi'^\uparrow)\big]\\ &= \Big[X^\uparrow{}_{\mathrm{ham}}\big[\partial\underline{\alpha}^\uparrow(\xi^\uparrow),\ J^\uparrow(\xi^\uparrow)\big],\ X^\uparrow{}_{\mathrm{ham}}\big[\partial\underline{\alpha}^\uparrow(\xi'^\uparrow),\ J^\uparrow(\xi'^\uparrow)\big]\Big]\\ &= X^\uparrow{}_{\mathrm{ham}}\Big[\big[(\partial\underline{\alpha}^\uparrow(\xi^\uparrow),\ J^\uparrow(\xi^\uparrow)),\ (\partial\underline{\alpha}^\uparrow(\xi'^\uparrow),\ J(\xi'^\uparrow))\big]\Big]\\ &= X^\uparrow{}_{\mathrm{ham}}\Big[0,\ \big\{J^\uparrow(\xi^\uparrow),\ J^\uparrow(\xi'^\uparrow)\big\}\Big]\,.\end{aligned}$$

Therefore, by comparing the above equalities, we obtain

$$X^\uparrow{}_{\mathrm{ham}}\Big[0,\ J^\uparrow\big([\xi^\uparrow,\xi'^\uparrow]\big)\Big] = X^\uparrow{}_{\mathrm{ham}}\Big[0,\ \big\{J^\uparrow(\xi^\uparrow),\ J^\uparrow(\xi'^\uparrow)\big\}\Big]\,,$$

which, in virtue of Section 1.4.3, yields $\widetilde{J}^\uparrow\big([\xi^\uparrow,\xi'^\uparrow]\big) = \big[\widetilde{J}^\uparrow(\xi^\uparrow),\ \widetilde{J}^\uparrow(\xi'^\uparrow)\big] + (0,\ k)$, where $k \in \mathbb{R}$. QED

5) Let us suppose additionally that $\boldsymbol{G}^\uparrow$ be a group of *holonomic symmetries of spacetime*. It means that

i) the Lie group $\boldsymbol{G} \equiv \boldsymbol{G}^\uparrow$ acts on $\boldsymbol{E}$ through a map $\alpha : \boldsymbol{G} \times \boldsymbol{E} \to \boldsymbol{E}$,

ii) the following diagram commutes

$$\begin{tikzcd} \boldsymbol{G} \times \boldsymbol{E} \arrow[r, "\alpha"] \arrow[d, "\underline{\pi} \times t"'] & \boldsymbol{E} \arrow[d, "t"] \\ \underline{\boldsymbol{G}} \times \boldsymbol{T} \arrow[r, "\underline{\alpha}"] & \boldsymbol{T} \end{tikzcd}$$

where $\underline{\pi} \equiv \underline{\pi}^\uparrow : \boldsymbol{G} \to \underline{\boldsymbol{G}}$,

iii) for each $g \in \boldsymbol{G}$, we have $(\alpha^\uparrow{}_{|g})^*(dt) = dt$,

iv) the action $\alpha^\uparrow$ is the 1–jet prolongation of α with respect to $J_1\boldsymbol{E}$

$$\alpha^\uparrow = J_1\alpha : \boldsymbol{G} \times J_1\boldsymbol{E} \to J_1\boldsymbol{E}\,,$$

according to the following commutative diagram

$$\begin{tikzcd} \boldsymbol{G} \times J_1\boldsymbol{E} \arrow[r, "J_1\alpha"] \arrow[d, "\mathrm{id} \times t^1"'] & J_1\boldsymbol{E} \arrow[d, "t^1"] \\ \boldsymbol{G} \times \boldsymbol{E} \arrow[r, "\alpha"] & \boldsymbol{E} \end{tikzcd}$$

2.45 Note. By taking the partial tangent map of α, with respect to $\boldsymbol{G}$, at $e \in \boldsymbol{G}$, we obtain the fibred morphism $\partial\alpha : \mathfrak{g} \times \boldsymbol{E} \to T\boldsymbol{E}$ over $\boldsymbol{E}$.

We obtain $\partial\underline{\alpha} := dt \circ \partial\alpha = \partial\underline{\alpha} \circ T\underline{\pi} : \mathfrak{g} \to \bar{\mathbb{T}}$, according to the commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\partial\alpha} & \mathrm{Fine}(\boldsymbol{E}, T\boldsymbol{E}) \\ {\scriptstyle T\pi_{|e}}\Big\downarrow & & \Big\downarrow \\ \underline{\mathfrak{g}} & \xrightarrow{\partial\underline{\alpha}} & \bar{\mathbb{T}} \end{array}$$

Hence, for each $\xi \in \mathfrak{g}$, we obtain the vector field $X := \partial\alpha(\xi) := (\partial\alpha)_{|\xi} : \boldsymbol{E} \to T\boldsymbol{E}$, with constant time component. Its coordinate expression is $\partial\alpha(\xi) = \xi^I \, (\partial_I \alpha^0 \, \partial_0 + \partial_I \alpha^i \, \partial_i)$, with $\partial_I \alpha^i \in \mathrm{I\!R}$.

Thus, we can regard $\partial\alpha$ as a homorphism of $\mathrm{I\!R}$–Lie algebras $\partial\alpha : \mathfrak{g} \to \mathrm{Fine}(\boldsymbol{E}, T\boldsymbol{E})$. $\square$

2.46 Note. The map $\partial\alpha^\uparrow : \mathfrak{g} \times J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$ turns out to be the holonomic prolongation of $\partial\alpha : \mathfrak{g} \times \boldsymbol{E} \to T\boldsymbol{E}$, i.e., for each $\xi \in \mathfrak{g}$, we have $\partial\alpha^\uparrow(\xi) = \big(\partial\alpha^\uparrow(\xi)\big)_{(1)}$. $\square$

2.47 Proposition. For each $\xi^\uparrow \in \mathfrak{g}^\uparrow$, the following facts hold.

i) The time components of $\partial\alpha^\uparrow(\xi^\uparrow)$ and $J^\uparrow(\xi^\uparrow)$ coincide, i.e. $\partial\underline{\alpha}^\uparrow(\xi^\uparrow) = \big(J^\uparrow(\xi^\uparrow)\big)'' \in \bar{\mathbb{T}}$.

ii) The conserved phase function $J^\uparrow(\xi^\uparrow)$ turns out to be a special phase function; more precisely, $J^\uparrow(\xi^\uparrow) \in \mathrm{clas}(J_1\boldsymbol{E}, \mathrm{I\!R}) = \mathrm{cons\,fine}(J_1\boldsymbol{E}, \mathrm{I\!R}) = \mathrm{self\,fine}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

iii) We have $\partial\alpha^\uparrow(\xi^\uparrow) = X^\uparrow{}_{\mathrm{ham}}\big[J^\uparrow(\xi^\uparrow)\big] = X^\uparrow{}_{\mathrm{hol}}\big[J^\uparrow(\xi^\uparrow)\big]$.

iv) The spacetime vector field $X := \partial\alpha(\xi)$ associated with $\xi := \pi(\xi^\uparrow)$ is the tangent lift of $J^\uparrow(\xi^\uparrow)$, i.e. $\partial\alpha(\xi) = X\big[J^\uparrow(\xi^\uparrow)\big]$.

Proof. It follows from Note 2.45, Proposition 2.43, Theorem 1.5 and Theorem 2.21. QED

Thus, in the case of spacetime symmetries, the function $J^\uparrow$ characterises $\widetilde{J}^\uparrow$.

6) Let us consider a horizontal potential $A^\uparrow$ of Ω and suppose additionally that $\alpha^\uparrow$ preserves $A^\uparrow$. It means that $\boldsymbol{G}^\uparrow$ is a *group of symmetries* of $A^\uparrow$, i.e. that, for each $g^\uparrow \in \boldsymbol{G}^\uparrow$, $(\alpha^\uparrow{}_{|g^\uparrow})^*(A^\uparrow) = A^\uparrow$.

2.48 Note. The Lie algebra $\mathfrak{g}^\uparrow$ is a *Lie algebra of infinitesimal symmetries* of $A^\uparrow$, i.e., for each $\xi^\uparrow \in \mathfrak{g}^\uparrow$, we have $L\big[\partial\alpha^\uparrow(\xi^\uparrow)\big] \, A^\uparrow = 0$. $\square$

2.49 Proposition. The following implication holds

$$L\Big[\partial\alpha^\uparrow(\xi^\uparrow)\Big] \, A^\uparrow = 0 \qquad \Rightarrow \qquad L\Big[\partial\alpha^\uparrow(\xi^\uparrow)\Big] \, \Omega = 0 \, .$$

Proof. We have $L\big[\partial\alpha^\uparrow(\xi^\uparrow)\big] \, \Omega = L\big[\partial\alpha^\uparrow(\xi^\uparrow)\big] \, dA^\uparrow = d \, L\big[\partial\alpha^\uparrow(\xi^\uparrow)\big] \, A^\uparrow = 0$. QED

In this case, we obbtain a distinguished momentum map.

Let $\mathcal{L}$ and $\mathcal{P}$ be the Lagrangian and momentum associated with $A^\uparrow$, and $\mathcal{H}[o]$ and $\mathcal{P}[o]$ be the observed Hamiltonian and momentum associated with $A^\uparrow$ and an observer o.

2.50 Proposition. The map $J^\uparrow : \mathfrak{g}^\uparrow \to \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R}) : \xi^\uparrow \mapsto -\big(\partial\alpha^\uparrow(\xi^\uparrow)\big) \lrcorner A^\uparrow$ turns out to be a momentum map whose target is defined in the domain of $A^\uparrow$.

Hence, for each $\xi^\uparrow \in \mathfrak{g}^\uparrow$, we have the equalities

$$\begin{aligned} J^\uparrow(\xi^\uparrow) &= -\partial\alpha^\uparrow(\xi^\uparrow) \lrcorner \mathcal{L} - \partial\alpha^\uparrow(\xi^\uparrow) \lrcorner \mathcal{P} \\ &= \partial\alpha^\uparrow(\xi^\uparrow) \lrcorner \mathcal{H}[o] - \partial\alpha^\uparrow(\xi^\uparrow) \lrcorner \mathcal{P}[o] \,. \end{aligned}$$

For each $\xi^\uparrow = \xi^I\, b^\uparrow{}_I$, we have the coordinate expressions

$$J^\uparrow(\xi^\uparrow) = -\xi^I \Big((\partial_I \alpha^i - x^i_0\, \partial_I \alpha^0)\, \partial^0_i \mathcal{L}_0 + \partial_I \alpha^0\, \mathcal{L}_0 \Big) \,.$$

PROOF. We have $i\big(\partial\alpha^\uparrow(\xi^\uparrow)\big)\,\Omega = i\big(\partial\alpha^\uparrow(\xi^\uparrow)\big)\, dA^\uparrow = L\big[\partial\alpha^\uparrow(\xi^\uparrow)\big]\, A^\uparrow - di\big(\partial\alpha^\uparrow(\xi^\uparrow)\big)\, A^\uparrow = -di\big(\partial\alpha^\uparrow(\xi^\uparrow)\big)\, A^\uparrow$, hence $-\big(\partial\alpha^\uparrow(\xi^\uparrow)\big) \lrcorner A^\uparrow$ is a potential of $i\big(\partial\alpha^\uparrow(\xi^\uparrow)\big)\,\Omega$. Then, the expression of the above momentum map follows from the equalities $A^\uparrow = \mathcal{L} + \mathcal{P}$ and $A^\uparrow = -\mathcal{H}[o] + \mathcal{P}[o]$. QED

We can exhibit a close relation between the momentum map J and the momentum $\mathcal{P}$ in the following way.

7) Let us suppose additionally that $\boldsymbol{G}^\uparrow$ be a group of *vertical holonomic symmetries* of $A^\uparrow$. It means that the group $\underline{\boldsymbol{G}}$ is the identity group of $\boldsymbol{T}$.

2.51 Note. We have $i\Big(\partial\alpha^\uparrow(\xi^\uparrow)\Big)\, dt = 0$. □

2.52 Corollary. The momentum map of the above Proposition 2.50 becomes

$$J^\uparrow : \mathfrak{g}^\uparrow \to \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R}) : \xi^\uparrow \mapsto -\Big(\partial\alpha^\uparrow(\xi^\uparrow)\Big) \lrcorner \mathcal{P} \,.$$

PROOF. It follows immediately from the above Proposition 2.50 and Note 2.51. QED

Now, we apply the above results to a few examples.

2.53 Example. Let us consider a special Newtonian spacetime, the abelian group of *vertical translations* $\boldsymbol{G} = \boldsymbol{S}$ and the group action $\alpha : \boldsymbol{S} \times \boldsymbol{E} \to \boldsymbol{E} : (v, e) \mapsto (e + v)$.

Of course, the Lie algebra of $\boldsymbol{G} = \boldsymbol{S}$ is $\mathfrak{g} = \boldsymbol{S}$. A horizontal potential $A^\uparrow$ of Ω exists globally and $\boldsymbol{S}$ turns out to be a group of holonomic symmetries of $A^\uparrow$, through the 1–jet prolongation $\alpha^\uparrow := J_1\alpha : \boldsymbol{S} \times J_1\boldsymbol{E} \to J_1\boldsymbol{E}$ of the action α. The distinguished momentum map $J^\uparrow$ associated with $\alpha^\uparrow$ is the map $J^\uparrow : \boldsymbol{S} \to \mathrm{Aff\,Clas}(J_1\boldsymbol{E}, \mathrm{I\!R}) : v \mapsto -\mathcal{P}(v)$, with coordinate expression $J^\uparrow(v) = -v^i\, G^0_{ij}\, x^j_0$. □

2.54 Example. Let us consider a special Newtonian spacetime and a complete global observer o. Hence, the observer yields the fibred isomorphism $s[o] : \boldsymbol{E} \to \boldsymbol{T} \times \boldsymbol{P}$ over $\boldsymbol{T}$. Let us consider the abelian group of *time translations* $\boldsymbol{G} = \mathbb{T}$ and the group action

$\alpha : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E}$ given by the following commutative diagram

$$\begin{array}{ccc} \bar{\mathbb{T}} \times \boldsymbol{E} & \xrightarrow{\alpha} & \boldsymbol{E} \\ {\scriptstyle \mathrm{id} \times s[o]} \big\downarrow & & \big\uparrow {\scriptstyle s[o]^{-1}} \\ \bar{\mathbb{T}} \times (\boldsymbol{T} \times \boldsymbol{P}) & \longrightarrow & \boldsymbol{T} \times \boldsymbol{P} \end{array}$$

where we have considered the map $\bar{\mathbb{T}} \times (\boldsymbol{T} \times \boldsymbol{P}) \to \boldsymbol{T} \times \boldsymbol{P} : \big(v,\ (\tau, p)\big) \mapsto (v + \tau,\ p)$.

Of course, the Lie algebra of $\boldsymbol{G} = \bar{\mathbb{T}}$ is $\mathfrak{g} = \bar{\mathbb{T}}$. A horizontal potential $A^{\uparrow}$ of Ω exists globally and $\bar{\mathbb{T}}$ turns out to be a group of holonomic symmetries of $A^{\uparrow}$, through the 1–jet prolongation $\alpha^{\uparrow} := J_1\alpha : \bar{\mathbb{T}} \times J_1\boldsymbol{E} \to J_1\boldsymbol{E}$. The distinguished momentum map $J^{\uparrow}$ associated with $\alpha^{\uparrow}$ is the map $J^{\uparrow} : \bar{\mathbb{T}} \to \mathrm{Clas}(J_1\boldsymbol{E},\ \mathrm{I\!R}) : \lambda \mapsto -(o \lrcorner A^{\uparrow})(\lambda) = \mathcal{H}(\lambda)$. □

Another interesting example could be obtained by considering the angular momentum of a rigid body.

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