# Uniqueness results by covariance in covariant quantum mechanics

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**Abstract**

We show that the only requirement of general covariance essentially determines the quantum operators associated with a classical quantisable function and the Schrödinger operator.

Our framework is the covariant quantum mechanics of a scalar quantum particle in a curved spacetime, which is fibred over absolute time and equipped with given spacelike metric, gravitational field and electromagnetic field.

In particular, in the flat case, we recover the standard operators.

# Introduction

Since the very beginning of quantum mechanics, the quantum operators associated with classical quantisable functions and the Schrödinger operator have been approached in several ways; indeed, the related literature is huge.

The most usual approaches to quantum operators are based on Hamiltonian techniques, such as the Dirac's canonical quantisation and its refinement provided by geometric quantisation. Analogously, the standard approaches to the Schrödinger operator are based on the quantisation of the classical Hamiltonian.

On the other hand, there are several theories involved, in some respects, with different formulations of these operators (see, for instance, [2, 3, 4, 14, 21, 15] and further references therein).

In this paper, we claim that the quantum operators associated with a classical quantisable function and the Schrödinger operator can be essentially determined by the only requirement of general covariance, in a reasonable setting with given exterior fundamental fields and under weak hypotheses of the order of the operators.

These results are obtained in the framework of "covariant quantum mechanics" proposed by A. Jadczyk and M. Modugno [6, 7] and further developed in cooperation with other authors (see, for instance, [1, 5, 8, 9, 10, 11, 12, 18, 19, 22, 23, 24] and references therein). Some aspects of this formulation have been discussed in the previous sessions of the meeting "Lie Theory and Its Applications in Physics" [16, 17]. In a few words, we consider a classical spacetime, which is fibred over absolute time and is equipped with given Riemannian spacelike metric, gravitational field and electromagnetic field. Partially similar settings have been considered by several authors; in particular, we are indebted to C. Duval and K. Künzle (see, for instance, [3]). The above fundamental fields yield, in a covariant way, a cosymplectic 2–form. Then, we consider a complex bundle over spacetime equipped with a Hermitian metric and a universal, Hermitian connection, whose curvature is proportional to the cosymplectic 2–form. These objects, yield, in a covariant way, good candidates for the quantum operators, through an isomorphism between the Lie algebra of Hermitian quantum vector fields and a distinguished Lie algebra of classical functions. Moreover, these objects yield, in a covariant way, a good candidate for the Schrödinger operator, equivalently, via quantum covariant differentials and via a quantum Lagrangian.

Then, a natural question arises whether we could obtain, in a covariant way, further candidates for quantum operators and Schrödinger operator. In this paper, we answer this question showing that there are essentially no further covariant solutions of the problem. Even more, the covariance determines directly these operators, in a pure geometric way, independently of any Lagrangian or Hamiltonian approaches!

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# 1 Covariant quantum mechanics

Let us summarise briefly the setting of our model for covariant quantum mechanics. Further details can be found in [7, 5, 10, 12, 16, 17].

We assume the following "positive 1-dimensional semi-vector spaces" over $\mathbb{R}^+$ as fundamental unit spaces: the space $\mathbb{T}$ of *time intervals*, the space $\mathbb{L}$ of *lengths* and the space $\mathbb{M}$ of *masses*. Moreover, we assume the *Planck constant* to be an element $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

*Time* is an an affine space of dimension 1 associated with the vector space $\mathbb{T} \otimes \mathbb{R}$.

*Spacetime* is an oriented curved manifold $\boldsymbol{E}$ of dimension 4, which is equipped with:

- a *time fibring* $t : \boldsymbol{E} \to \boldsymbol{T}$,

- a scaled Riemannian *spacelike metric* $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes S^2 V^* \boldsymbol{E}$,

- a *gravitational field*, constituted by a linear connection $K^\natural$, such that $\nabla^\natural dt = 0$ and $\nabla^\natural g = 0$, and whose curvature fulfills the identity $R^{\natural i}{}_{\lambda}{}^{j}{}_{\mu} = R^{\natural j}{}_{\mu}{}^{i}{}_{\lambda}$,

- an *electromagnetic field* constituted by a closed 2–form $f : \boldsymbol{E} \to (\mathbb{L}\otimes\mathbb{M})^{1/2} \otimes \Lambda^2 T^* \boldsymbol{E}$.

With reference to a particle with mass $m$ and charge $q$, we define the *re–scaled* spacelike metric $G := \frac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes S^2 V^* \boldsymbol{E}$ and electromagnetic field $F := \frac{q}{\hbar}\, f : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E}$.

Then, $G$, $K^\natural$ and $F$ yield the *total connection* $K := K^\natural + (dt \otimes \widehat{F} + \widehat{F} \otimes dt)$ of $\boldsymbol{E}$ (where $\widehat{F} := G^\sharp(F)$), which fulfills the same properties of $K^\natural$.

Moreover, $g$ yields a *spacelike volume form* $\eta$ and a *spacetime volume form* $\upsilon = dt \wedge \eta$. Then, we obtain the *spacetime divergence* $\operatorname{div}_\upsilon X$, for each spacetime vector field $X$, and the *spacelike divergence* $\operatorname{div}_\eta X$, for each projectable spacetime vector field $X$.

We denote by $(x^\lambda) = (x^0, x^i)$ the *spacetime charts*, adapted to the time fibring the affine structure of time and a time unit of measurement $u^0 \in \mathbb{T}$.

We assume the first jet space $J_1\boldsymbol{E}$ as classical *phase space*. We denote by $(x^\lambda, x^i_0)$ the *phase charts* induced by a spacetime chart. We obtain, in a covariant way, the *contact map* $д = u^0 \otimes (\partial_0 + x^i_0\, \partial_i)$.

Then, $G$ and $K$ yield, in a covariant way, a *cosymplectic 2–form* $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E}$.

The Reeb *2nd order connection* $\gamma$ associated with $\Omega$ yields the generalised *Newton law* for classical motions effected by the given gravitational and electromagnetic fields.

The closed 2–form $\Omega$ admits locally a *horizontal potential* $\Theta$, defined up to a closed spacetime 1–form. The horizontal phase 1–form $\mathcal{L}[\Theta] := д \lrcorner \Theta$ is the *classical Lagrangian*; moreover, for each observer $o$, the horizontal phase 1–form $\mathcal{H}[\Theta, o] := -o \lrcorner \Theta$ is the observed *classical Hamiltonian*, the vertical phase 1–form $\mathcal{P}[\Theta, o] := \nu[o] \lrcorner \Theta$ is the observed *classical momentum*, and the spacetime 1–form $A[\Theta, o] := o^*\Theta$ is the observed *classical potential*. The Euler–Lagrange equation associated with $\mathcal{L}[\Theta]$ turns out to be just the Newton law.

A function of the type $f = \frac{1}{2}\, f^0\, G^0_{ij}\, x^i_0\, x^j_0 + f^i\, G^0_{ij}\, x^j_0 + \overset{o}{f}$, where $f^0, f^i, \overset{o}{f} \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$ is called *special quadratic* and $f'' := f^0\, u_0 \in \operatorname{map}(\boldsymbol{E}, \mathbb{T}\otimes\mathbb{R})$ is called the "time component" of $f$.

The sheaf of special quadratic functions $\operatorname{spec}(J_1\boldsymbol{E}, \mathbb{R})$ is a Lie algebra through the bracket $[\![ f, g ]\!] := \{f, g\} + \gamma(f'').g - \gamma(g'').f$.

The conditions $\partial_i f^0 = 0$, or $\partial_\lambda f^0 = 0$, or $f^0 = 0$, or $f^\lambda = 0$, yield, respectively, the Lie subalgebras of *projectable*, *fine*, *affine* and *spacetime* functions $\mathrm{spec}(J_1\boldsymbol{E}, \mathbb{R}) \supset \mathrm{proj}(J_1\boldsymbol{E}, \mathbb{R}) \supset \mathrm{fine}(J_1\boldsymbol{E}, \mathbb{R}) \supset \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \supset \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

In particular, the spacetime coordinates, the momentum and the Hamiltonian are special quadratic functions.

The map $\mathrm{spec}(J_1\boldsymbol{E}, \mathbb{R}) \to \sec(\boldsymbol{E}, T\boldsymbol{E}) : f \mapsto X[f] = f^0\,\partial_0 - f^i\,\partial_i$ is a covariant morphism of Lie algebras.

The *quantum bundle* is a 1–dimensional complex bundle $\boldsymbol{Q} \to \boldsymbol{E}$, which is equipped with:

- a *quantum metric*, which is a Hermitian metric valued in the space of spacelike volume forms $\mathsf{h} : \boldsymbol{Q} \to \mathbb{C} \otimes \Lambda^3 V^*\boldsymbol{E}$,

- a *quantum connection*, which is a "universal", Hermitian connection of the *extended quantum bundle* $\boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$, whose curvature is $R[\text{Ч}^\uparrow] = -2\,\Omega \otimes \mathbb{I}^\uparrow$, where $\mathbb{I}^\uparrow$ is the Liouville vector field of $\boldsymbol{Q}^\uparrow$.

We recall that the Planck constant $\hbar$ has been incorporated into $\Omega$ through $G$.

We shall refer to fibred *quantum charts* $(x^\lambda, z)$, where $(x^\lambda)$ is a spacetime chart and $z \in \mathrm{lin}\,(\boldsymbol{Q}, \mathbb{L}^{*3/2} \otimes \mathbb{R})$ is the complex fibre coordinate induced by a *quantum basis* $\mathsf{b} \in \sec(\boldsymbol{E}, \mathbb{L}^{3/2} \otimes \boldsymbol{Q})$ normalised as $\mathsf{h}(\mathsf{b}, \mathsf{b}) = \eta$. For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we set $\Psi = \psi\,\mathsf{b}$, with $\psi \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{*3/2} \otimes \mathbb{R})$.

The coordinate expression of the quantum connection is $\text{Ч}^\uparrow{}_0 = -\mathfrak{i}\,\mathcal{H}_0$, $\text{Ч}^\uparrow{}_i = -\mathfrak{i}\,\mathcal{P}_i$ and $\text{Ч}^{\uparrow}{}^0_i = 0$, where the gauge of the classical functions $\mathcal{H}_0$ and $\mathcal{P}_i$ is locally determined by the chosen quantum basis.

For each $f \in \mathrm{spec}(J_1\boldsymbol{E}, \mathbb{R})$, we obtain the *quantum vector field* defined by the covariant formula $Y[f] = f^0\,\partial_0 - f^i\,\partial_i + \mathfrak{i}\,(\overset{o}{f} + A_0\,f^0 - A_i\,f^i)\,\mathbb{I}$, where $A_\lambda\,d^\lambda$ is the classical potential determined by the quantum connection and the chosen quantum chart. The map $f \mapsto Y[f]$ is a Lie algebra isomorphism.

The *Hermitian quantum vector fields* are classified by the projectable functions through the covariant formula $H[f] = Y[f] - \frac{1}{2}\,\mathrm{div}_\eta X[f]\,\mathbb{I}$.

These vector fields act as Lie derivatives on quantum sections.

A criterion of projectability yields in a covariant way the *quantum Lagrangian* $\mathsf{L}_{(k)} = \frac{1}{4}\Big((z\,\bar{z}_0 - \bar{z}\,z_0) + \mathfrak{i}\,2\,A_0\,\bar{z}\,z - \mathfrak{i}\,G^{ij}_0\,(z_i\,\bar{z}_j + A_i\,A_j\,\bar{z}\,z) + A^i_0\,(\bar{z}\,z_i - z\,\bar{z}_i) + 4\,k\,\mathfrak{i}\,r_0\,\bar{z}\,z\Big)\upsilon^0$, where $r$ is the scalar spacelike curvature induced by $G$ and $k \in \mathbb{R}$.

The Euler–Lagrange operator associated with $\mathsf{L}_{(k)}$ yields, in a covariant way, the *Schrödinger operator* $\mathsf{S}_{(k)} = \Big(\overset{o}{\nabla}_0 + \frac{1}{2}\,(\mathrm{div}_\eta\,o)_0 - \frac{1}{2}\,\mathfrak{i}\,(\overset{o}{\Delta}_0 + k\,r_0)\Big)\,u^0$, where $\overset{o}{\nabla}$ is the the observed quantum covariant differential, $\mathrm{div}_\eta\,o$ is the divergence of the observer, $\overset{o}{\Delta}$ is the observed quantum Laplacian; indeed the above combination turns out to be observer independent.

Then, we define the (infinite dimensional) *functional quantum bundle* $\boldsymbol{H} \to \boldsymbol{T}$ to be the fibred set over $\boldsymbol{T}$, whose fibres are constituted by the compact support smooth sections, at fixed time, of the quantum bundle. The quantum metric $\mathsf{h}$ equips the functional quantum bundle with a fibred *pre–Hilbert metric* $\langle \, | \, \rangle$. Then, a true Hilbert bundle can be obtained by a completion procedure.

The Hermitian quantum vector fields $Y[f]$ (via Lie derivatives) and the Schrödinger operator $\mathsf{S}_{(k)}$ yield, in a covariant way, symmetric operators on the fibres of the functional quantum bundle. Moreover, the Schrödinger operator can be regarded as a Hermitian operator of the functional quantum bundle (actually, as the covariant differential associated with a Hermitian connection).

## 2 Uniqueness results by covariance

Let us consider the complex linear fibred automorphisms $\Phi : \boldsymbol{Q} \to \boldsymbol{Q}$ projectable on fibred automorphisms $\phi : \boldsymbol{E} \to \boldsymbol{E}$. The tangent prolongations of the $\Phi$,s and $\phi$,s, at fixed points of $\boldsymbol{E}$, yield the group bundles $\mathcal{G}(\boldsymbol{Q})$ and $\mathcal{G}(\boldsymbol{E})$ over $\boldsymbol{E}$. The fibred surjection $\mathcal{G}(\boldsymbol{Q}) \to \mathcal{G}(\boldsymbol{E})$ is a fibred central extension. Moreover, we have fibred actions of $\mathcal{G}(\boldsymbol{E})$ on $\boldsymbol{E} \to \boldsymbol{T}$ and of $\mathcal{G}(\boldsymbol{Q})$ on $\boldsymbol{Q} \to \boldsymbol{E}$ and on all other bundles that we have derived from them.

Moreover, we consider the trivial group bundle $\mathcal{G}(\mathbb{T}) := \boldsymbol{E} \times \mathbb{R}^+$ of changes of time scales. We have a fibred action of $\mathcal{G}(\mathbb{T})$ on time scaled bundles such as $\mathbb{T}^* \otimes T\boldsymbol{E}$ and $\mathbb{T}^* \otimes \boldsymbol{Q}$.

First, let us classify the covariant operators of Schrödinger type.

**2.1 Theorem.** *All 2nd order operators $\mathsf{O}(\Psi) : \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q}$, which depend on $dt$, $G$, $K$, $\mathsf{h}$, $\Upsilon^\uparrow$ and are covariant with respect to $\mathcal{G}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$, are of the type*

$$\mathsf{O}(\Psi) = \alpha \, \mathsf{S}_{(0)}(\Psi) + \beta \, r \, \Psi \,, \quad \text{with} \quad \alpha, \, \beta \in \mathbb{C} \,.$$

*Moreover, if we add the condition that the operators $\mathsf{O}(\Psi)$ yield Hermitian operators on the functional quantum bundle, then they are of the type*

$$\mathsf{O}(\Psi) = \mathsf{S}_{(k)}(\Psi) \,. \, \square$$

Next, let us classify the covariant operators of quantum Lagrangian type.

**2.2 Theorem.** *All 2nd order operators $\mathsf{L}(\Psi) : \boldsymbol{E} \to \mathfrak{i} \, \Lambda^4 T^* \boldsymbol{E}$, which depend on $dt$, $G$, $K$, $\mathsf{h}$, $\Upsilon^\uparrow$ and are covariant with respect to $\mathcal{G}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$, are of the type*

$$\begin{aligned} \mathsf{L}[\Psi] = a \, \mathsf{L}_{(k)}(\Psi) &+ b \, \tfrac{1}{2} \, dt \wedge \Big( \mathsf{h}(\Psi, \mathsf{S}_{(0)}[\Psi]) - \mathsf{h}(\mathsf{S}_{(0)}[\Psi], \Psi) \Big) \\ &+ c \, \mathfrak{i} \, \tfrac{1}{2} \, dt \wedge \Big( \mathsf{h}(\Psi, \mathsf{S}_{(0)}[\Psi]) + \mathsf{h}(\mathsf{S}_{(0)}[\Psi], \Psi) \Big) \,, \qquad a, b, c \in \mathbb{R} \,. \, \square \end{aligned}$$

The Euler–Lagrange operators associated with the above Lagrangians are just the above Schrödinger operators. Moreover, we observe that the 2nd order terms in the above

Lagrangians are not relevant because they yield again the Schrödinger operator.

Eventually, let us classify the covariant operators associated with special quadratic functions.

**2.3 Theorem.** *All 2nd order operators $\mathsf{O}[f](\Psi) : \boldsymbol{E} \to \boldsymbol{Q}$, which depend on $dt$, $G$, $K$, $\mathfrak{h}$, $Ч^{\uparrow}$, depend linearly on a special quadratic function $f$ and are covariant with respect to $\mathcal{G}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$, are of the type*

$$\mathsf{O}[f](\Psi) =$$
$$= \Big(\alpha_1\, f'' \lrcorner\, \mathsf{S} + \alpha_2\, L_{Y[f]} + \alpha_3\, L_{Y[д.f'']} + \alpha_4\, \mathrm{div}_\upsilon\, X[f] + \alpha_5\, \mathrm{div}_\eta\, X[д.f''] + \alpha_6\, f'' \lrcorner\, r\Big)\,(\Psi)\,,$$

*where $\alpha_1, \dots, \alpha_6 \in \mathbb{C}$.*

*Moreover, if we add the condition $\mathsf{O}[1](\Psi) = \Psi$, then we obtain $a_2 = \mathfrak{i}$.*

*Furthermore, if we add the condition that the operators $\mathsf{O}[f](\Psi)$ yield operators $\widehat{\mathsf{O}}[f]$ acting on the fibres of the functional quantum bundle, then we obtain $a_1 = -\mathfrak{i}$.*

*Eventually, if we add the condition that the induced operators $\widehat{\mathsf{O}}[f]$ are symmetric, then the special functions must be projectable and, for such functions, the corresponding operators are of the type*

$$\mathsf{O}[f](\Psi) = \Big(\mathfrak{i}\,(L_{H[f]} - f'' \lrcorner\, \mathsf{S}_{(k)}) + a\ \mathrm{div}_\upsilon\, X[f] + b\,\mathfrak{i}\, L_{Y[д.f'']} + c\ \mathrm{div}_\eta\, X[д.f'']\Big)\,(\Psi)\,,$$

*where $a, b, c, k \in \mathbb{R}$.* $\square$

In order to fix the undetermined coefficients $a, b, c, k \in \mathbb{R}$ we need further requirements.

**2.4 Corollary.** All operators as above associated, respectively, with $f \in \mathrm{fine}(\boldsymbol{E})$, $f \in \mathrm{aff}(\boldsymbol{E})$, $f \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$, are of the type

$$\begin{aligned}
&\mathsf{O}[f](\Psi) = \Big(\mathfrak{i}\,(L_{Y[f]} - f'' \lrcorner\, \mathsf{S}_{(k)}) + a\ \mathrm{div}_\upsilon\, X[f]\Big)\,(\Psi)\,, &&\text{for} \quad f \in \mathrm{fine}(J_1\boldsymbol{E},\, \mathbb{R})\,,\\
&\mathsf{O}[f](\Psi) = \Big(\mathfrak{i}\, L_{Y[f]} + a\ \mathrm{div}_\eta\, X[f]\Big)\,(\Psi)\,, &&\text{for} \quad f \in \mathrm{aff}(J_1\boldsymbol{E},\, \mathbb{R})\,,\\
&\mathsf{O}[f](\Psi) = f\,\Psi\,, &&\text{for} \quad f \in \mathrm{map}(\boldsymbol{E},\, \mathbb{R})\,.\ \square
\end{aligned}$$

**2.5 Example.** All operators as above associated with the classical Hamiltonian $\mathcal{H}_0 = \frac{1}{2}\, g_{ij}\, x_0^i\, x_0^j - A_0$ and the classical momentum $\mathcal{P}_i = g_{ij}\, x_0^j + A_i$ are, respectively, of the type

$$\mathsf{O}[\mathcal{H}_0] = -(\tfrac{1}{2}\, \overset{o}{\Delta}_0 + \tfrac{1}{2}\, k\, r_0 - A_0) + a\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\,,$$
$$\mathsf{O}[\mathcal{P}_j] = -\mathfrak{i}\,\Big(\partial_j + \tfrac{1}{2}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}}\Big) - a\, \frac{\partial_i \sqrt{|g|}}{\sqrt{|g|}}\,.\ \square$$

Let us sketch briefly the proofs of the above Theorems.

Proof. The proofs can be achieved by the following steps:

- in virtue of the "orbit reduction theorem" [13] and by taking into account the covariance with respect to $\mathcal{G}(\boldsymbol{Q})$, we can express the operators through the curvatures of the connections and the covariant derivatives of the sections up to 2nd order;

- the "fundamental identities" postulated for the fields $dt, G, K, \mathfrak{h}, \Psi^{\uparrow}$ make some of the above objects vanishing, or express some of them through the others;

- in virtue of the "homogeneous function theorem" [13] and by taking into account the covariance with respect to the subbundle of $\mathcal{G}(\boldsymbol{Q})$ constituted by homotheties, we can express the operators in terms of polynomials;

- in virtue of the "metric covariant function" [13], by taking into account the covariance with respect to $\mathcal{G}(\mathbb{T})$ and counting the covariant and contravariant indices occurring in the previous expression of the operators, we can compute the only contractions allowed and get the result. QED

We have assumed no distinguished time scale. But, if we suppose that the quantum system be involved with a distinguished $\tau \in \mathbb{T}$, then we could not require the covariance with respect to $\mathcal{G}(\mathbb{T})$ and we would obtain several additional solutions of our classification problems. For instance, for the Schrödinger operator we would obtain the additional covariant non–linear term

$$\mathsf{O}'(\Psi) = \tau^{1/2}\,(\hbar/m)^{3/2}\,\frac{\mathfrak{h}}{\eta}(\Psi,\Psi)\,\Psi\,.$$

Analogous considerations hold for length scales. In fact, distinguished time scales $\tau \in \mathbb{T}$ and distinguished length scales $\mathfrak{l} \in \mathbb{L}$ can be related by the equality $\tau = m/\hbar \otimes \mathfrak{l}^2$.

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