Covariant quantum mechanics and quantum symmetries

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Abstract

We sketch the basic ideas and results on the covariant formulation of quantum mechanics on a curved spacetime with absolute time equipped with given gravitational and electromagnetic fields.

Moreover, we analyse the classical and quantum symmetries and show their relations.

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Contents

1 Introduction

Our fundamental picture of the physical world is due to the theory of general relativity and to the quantum field theory, which got great theoretical and experimental success.

The well established historical steps in classical theory have been: non relativistic theory, special relativity, general relativity. Analogously, the well established steps in quantum theory have been: non relativistic quantum mechanics, special relativistic quantum field theory.

Unfortunately, these theories deal with different objects, use partially incompatible mathematical methods and fulfill different requirements of covariance. In particular, the standard formulation of quantum theories is highly based on concepts and methods strictly related to a flat spacetime and inertial observers, which conflict with general covariance on a curved spacetime.

So, a consistent formulation of quantum field theories and general relativity is a still open problem. The problem has at least two faces:

- general relativistic covariant formulation of quantum theories in a curved spacetime,
- quantum theory of gravitational field.

The model of *covariant quantum mechanics* discussed in this paper is aimed at contributing to the first face of the problem, by means of new ideas and methods [23, 24, 25, 4, 26, 27, 52, 31, 65, 66, 33, 67, 68, 29, 51, 55]. Namely, we study a general relativistic covariant formulation of quantum mechanics on a classical background constituted by a curved spacetime fibred over absolute time and equipped with given spacelike Riemannian metric, and gravitational and electromagnetic fields. Thus, we restrict our investigation just to fundamental fields of classical and quantum mechanics, because we believe that this arena could possibly suggest us good ideas for unifying deeper fundamental theories of physics.

The framework of our model is allowed by the possible general relativistic formulation of classical physics in a curved spacetime with absolute time. This theory is well established in the literature [60, 61, 7, 8, 9, 10, 11, 12, 13, 19, 37, 38, 39, 40, 41, 42, 44, 45, 47, 56, 62, 63], even if it is much less popular than the Einstein theory of relativity. This theory is rigorous and self–consistent from a mathematical viewpoint and describes the phenomena of classical physics by an approximation which is intermediate between the classical theory and the Einstein theory of relativity.

The standard term "relativistic theory" links the special or general covariance with the Minkowski or Lorentz metric. This usage is clearly motivated by the historical developments of the Einstein theory. However, it would be more appropriate to refer the word "relativistic theory" only to its semantic meaning related to covariance. Indeed, the standard usage would be highly misleading in our context. In fact, our model is general relativistic, in the sense of covariance, but it is not Minkowskian or Lorentzian.

Clearly, the Minkowski or Lorentz metric is physically related to the distinguished constant c. Actually, in our model this constant does not occur. The classical limit of Einstein general relativity for $c \to \infty$ [13] is quite delicate, if we wish to understand the limit of the geometric structures of the model and not only the limit of some measurements. In a sense, our model could be regarded as the "true" classical limit of Einstein general relativity.

Our model can be regarded as an intermediate step between the standard non relativistic quantum mechanics and a possible fully general relativistic quantum theory. This framework allows us to focus our attention on the general relativistic covariance and the curved spacetime, detaching them from the difficulties due to the Lorentz metric. Actually, our choice seems to be quite fruitful.

The main new methods and achievements can be summarised as follows.

First of all, our basic guide is the *covariance* (even more, the manifest covariance) of the theory as heuristic requirement. Nowadays, the concept of "covariance" has been formulated in a rigorous mathematical way through the geometric concept of "naturality" [35]. According to the covariance of the theory, *time* is not just a parameter, but a fundamental object of the theory; moreover, the main objects of the theory are not assumed to be split into time and space components. As classical *phase space* we take the first jet space of spacetime and not its tangent space; indeed, this minimal choice allows us to skip anholonomic constraints. Another consequence of our choices is that classical mechanics is ruled not by a symplectic structure, but by a *cosymplectic structure* [46]; actually, we do get a symplectic structure, but this describes only the spacelike aspects of classical theory and is insufficient to account for classical dynamics. An achievement of our theory is the Lie algebra of "*special quadratic functions*" (different from the Poisson algebra), which allows us to treat energy, momentum and spacetime functions on the same footing. We emphasize the fact that classical mechanics can be formulated in a covariant way by a *Lagrangian approach*, but not by a Hamiltonian approach, because the Hamiltonian function depends essentially on an observer.

As far as quantum mechanics is concerned, all objects are derived, in a covariant way, from three minimal objects. Here, we have some novelties. The quantum bundle lives *on spacetime* and not on the phase space and the quantum connection is "*universal*". These assumptions allow us to skip all problems of polarisations [70]. In a sense, we obtain naturally a covariant polarisation and this is sufficient for our purposes. Indeed, we replace the problematic search for such inclusions with a *method of projectability*, which turns out to be our implementation of covariance in the quantum theory. Another new assumption concerns the *Hermitian metric* of the quantum bundle, which takes its values in the space of spacelike volume forms. This assumption allows us to skip the problems related to half–densities. The Schroedinger equation is obtained, in a covariant way, through a *Lagrangian approach* and not through the standard non covariant Hamiltonian approach. Indeed, we exhibit an *explicit expression* of the Schroedinger equation for any quantum system. The quantum operators arise automatically, in a covariant way, from the *classification* of distinguished first and second order differential operators of the quantum framework and not from a quantisation requirement of a classical system [70]. The seat for the covariant probabilistic interpretation of quantum mechanics is a *Hilbert bundle*, naturally yielded by the quantum bundle, and not just a Hilbert space. Our theory provides explicit expressions of all objects for any *accelerated observer* and yields, at the same time, an interpretation in terms of gravitational field, according to the principle of equivalence.

In a few words, we start with really minimal geometric structures representing physical fields and proceed along a thread naturally imposed by the only requirement of general covariance. We take the well established *results* of classical and quantum mechanics as touchstone of our model. On the other hand, according to the aims of our theory, we disregard those standard *methods* for deriving quantum objects, which are incompatible with general covariance. Indeed, in the flat case, the results of our model reduce to the results of the standard classical and quantum mechanics.

In this paper, we deal just with a given gravitational and electromagnetic field; this is sufficient as classical background for our covariant model of quantum mechanics. On the other hand, our classical model can be completed by adding, in a covariant way, the equations linking the gravitational and electromagnetic fields to their mass and charge sources [25]. These equations are a covariant reduced version of the Einstein and Maxwell equations. In fact, due to the spacelike nature of the metric, there is no way to couple fully the gravitational and electromagnetic fields with the energy–momentum tensor and the charge current, respectively. Just this is the main point which makes the Einstein model physically much more complete than ours. On the other hand, in our quantum model, the gravitational and electromagnetic fields are "external", hence the relation of these fields with their sources does not play an effective role in this context.

The reader might be puzzled by the fact that we do not mention explicitly the representations of the (finite dimensional and infinite dimensional) groups involved in our theory. In fact, our natural geometric constructions provide these representations automatically. This is an outproduct of our manifestly covariant approach.

In our model we never make an essential use of the fact that the dimension of spacetime is $n = 1 + 3$. We just need $n \geq 1 + 2$. In fact we have applied our machinery to the quantisation of a rigid body, whose configuration space has dimension $n = 1 + 3 + 3$ [67].

Even more, in our model we never make an essential use of the fact that the spacelike metric of spacetime is Riemannian; we just need that it is non degenerate on each fibre. So, we could, for instance, apply our machinery to a model of dimension 5, with a fibring on an extra parameter, whose fibres are four dimensional Lorentzian manifolds. Such a model would work pretty well mathematically, but we do not know any interesting physical interpretation.

The scheme developed for covariant quantum mechanics of a scalar particle can be easily and nicely extended to the case of a spin particle [4].

In spite of the differences of the starting scheme of spacetime, several steps of the above methodology appeared to be usefully translable to the *Einstein case*. In particular, so far, we have been able to apply to the Einstein case the methods concerning the classical phase space, the algebra of quantisable functions and the algebra of pre–quantum operators [26, 30, 31, 28, 34, 32].

We hope that the new methods arising in our model could yield fruitful hints for a possible generally covariant formulation of quantum field theory in an Einstein framework.

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2 Covariant quantum mechanics

2.1 Classical background

We start by sketching our covariant model of classical curved spacetime fibred over absolute time, and the related formulation of classical mechanics. We recall the basic elements of the model and present new results, as well.

Classical spacetime. According to [25, 22], we postulate:

(C.1) a *classical spacetime* $\boldsymbol{E}$, which is an oriented four dimensional manifold;

(C.2) the *absolute time* $\boldsymbol{T}$, which is an oriented one dimensional affine space, associated with the vector space $\bar{\mathbb{T}}$;

(C.3) a *time fibring* $t : \boldsymbol{E} \to \boldsymbol{T}$, which is a surjective map of rank 1;

(C.4) a "scaled" *spacelike metric* g, which is a "scaled" Riemannian metric of the fibres of spacetime;

(C.5) a *gravitational field* $K^{\natural}$, which is a linear connection of spacetime, which preserves the time fibring and the spacelike metric and whose curvature fulfills the typical symmetry of Riemannian connections;

(C.6) a "scaled" *electromagnetic field* f, which is a "scaled" closed 2–form of spacetime.

Here, the word "scaled" used for the spacelike metric and the electromagnetic field means that these objects are tensorialised by a suitable scale factor which accounts for the appropriate units of measurement.

A *time unit of measurement* will be denoted by $u^0 \in \bar{\mathbb{T}}$ and its dual by $u_0 \in \bar{\mathbb{T}}^*$.

We refer to charts of spacetime $(x^\lambda) = (x^0, x^i)$ adapted to the time fibring, to the affine structure of time and to a time unit of measurement $u_0 \in \bar{\mathbb{T}}$.

With reference to a given particle of mass m and charge q, in order to get rid of any choice of length and mass units of measurement, it is convenient to "normalise" the spacelike metric and the electromagnetic field, by considering the Planck constant $\hbar$.

Thus, we consider the "re–scaled" spacelike metric $G := \frac{m}{\hbar}\, g$, which takes its values in $\bar{\mathbb{T}}$. Its coordinate expression is

$$G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j\ ,$$

where $\check{d}^i$ is the spacelike differential of the coordinate x^i.

Analogously, we consider the "re–scaled" electromagnetic field $F := \frac{q}{\hbar}\, f$, which is a true form.

Accordingly, all objects derived from G and F will be re–scaled and will include the mass and the charge of the particle, and the Planck constant as well.

As *phase space* for the classical particle we take the first order jet space $J_1\boldsymbol{E}$ of the spacetime fibring [35]. We recall that $J_1\boldsymbol{E}$ can be naturally identified with the affine subspace of $\bar{\mathbb{T}}^* \otimes T\boldsymbol{E}$, whose elements v are normalised according to the condition $v^0_0 = 1$ (which is independent from the choice of a unit of measurement of time). The chart naturally induced on the phase space by a spacetime chart is denoted by (x^0, x^i, x^i_0).

We have assumed a projection of spacetime over time, but, according to the principle of general relativity, not a distinguished splitting of spacetime into space and time. In other words, for each spacetime vector X, we obtain, in a covariant way, its projection on time $X^0\, u_0$, but not a timelike and a spacelike component.

On the other hand, an *observer* is defined to be a section $o : \boldsymbol{E} \to J_1\boldsymbol{E}$. The coordinate expression of an observer o is of the type $o = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$. An observer o yields a splitting of each spacetime vector X into its *observed timelike and spacelike components* $v = v^0\, (\partial_0 + o^i_0\, \partial_i) + (v^i - v^0\, o^i_0)\, \partial_i$. A spacetime chart is said to be adapted to an observer if $o^i_0 = 0$. Conversely, each spacetime chart determines the observer, whose coordinate expression is $o = u^0 \otimes \partial_0$.

According to the principle of general relativity, we do not assume distinguished observers.

The above objects C.1, ... , C.6 yield in a covariant way [25, 22]:
- the scaled *time form* $dt : \boldsymbol{E} \to \bar{\mathbb{T}} \otimes T^*\boldsymbol{E}$ of spacetime;
- a *spacelike volume form* η and a *spacetime volume form* υ of spacetime;
- a *2–form* $\Omega^\natural : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ of the phase space;

- a dt–vertical *2–vector* $\Lambda^\natural : J_1\boldsymbol{E} \to \Lambda^2 V J_1\boldsymbol{E}$ of the phase space,
- a second order connection $\gamma^\natural : J_1\boldsymbol{E} \to \bar{\mathbb{T}}^* \otimes T J_1\boldsymbol{E}$ of spacetime,

Here, we have used the symbol $^\natural$ to label objects derived from the gravitational field. We obtain the following identities

$$i(\gamma^\natural)\, dt = 1\,, \qquad i(\gamma^\natural)\, \Omega^\natural = 0\,, \qquad dt \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural \not\equiv 0\,,$$
$$d\Omega^\natural = 0\,, \qquad L[\gamma^\natural]\, \Lambda^\natural = 0\,, \qquad [\Lambda^\natural,\, \Lambda^\natural] = 0\,.$$

Hence, the pair $(dt, \Omega^\natural)$ turns out to be a *cosymplectic structure* of the phase space [46].

Moreover, $\Lambda^\natural$ and $\Omega^\natural$ yield inverse linear isomorphisms between the vector spaces of dt–vertical vectors and $\gamma^\natural$–horizontal forms of the phase space.

The Lie derivative of the spacelike metric G and of the spacelike volume form η with respect to a vector field of $\boldsymbol{E}$ is well defined provided that the vector field is projectable on $\boldsymbol{T}$.

If X is a vector field of $\boldsymbol{E}$ projectable on $\boldsymbol{T}$, then we define its *spacelike divergence* by means of the equality $\operatorname{div}_\eta X = L[X]\, \eta$. We have the coordinate expression

$$\operatorname{div}_\eta X = X^0 \, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i (X^i \, \sqrt{|g|})}{\sqrt{|g|}}\,.$$

It is convenient to add an electromagnetic term to the gravitational field, in a covariant way [25, 22], according to the formula

$$K = K^\natural + K^\mathfrak{e} := K^\natural + \tfrac{1}{2}\, (dt \otimes \hat{F} + \hat{F} \otimes dt)\,,$$

i.e., in coordinates,

$$K_h{}^i{}_k = K^\natural{}_h{}^i{}_k\,, \qquad K_0{}^i{}_k = K^\natural{}_0{}^i{}_k + \tfrac{1}{2}\, G^{ij}_0\, F_{jk}\,, \qquad K_0{}^i{}_0 = K^\natural{}_0{}^i{}_0 + G^{ij}_0\, F_{j0}\,,$$

where $\hat{F} := G^{ij}_0\, F_{j\lambda}\, d^0 \otimes \partial_i \otimes d^\lambda$.

Then, the "total" object K turns out to be a connection of spacetime, which fulfills the same properties postulated in (C.5). Moreover, all main formulas in classical and quantum mechanics concerning the given particle and involving the gravitational and electromagnetic fields can be expressed through the "total" K and its derived objects, without the need of splitting it into its gravitational and electromagnetic components.

Proceeding with the total spacetime connection K as before, we obtain the "total" second order connection, 2–form and 2–vector

$$\gamma = \gamma^\natural + \gamma^\mathfrak{e}\,, \qquad \Omega = \Omega^\natural + \Omega^\mathfrak{e}\,, \qquad \Lambda = \Lambda^\natural + \Lambda^\mathfrak{e}\,,$$

where the electromagnetic terms $\gamma^\mathfrak{e}$, $\Omega^\mathfrak{e}$ and $\Lambda^\mathfrak{e}$ turn out to be, respectively, the (re–scaled) Lorentz force, $\frac{1}{2}$ the (re–scaled) electromagnetic field and $\frac{1}{2}$ the (re–scaled) contravariant spacelike electromagnetic field, i.e. the (re–scaled) magnetic field.

These total objects fulfill all properties fulfilled by the gravitational objects as above.

The total *cosymplectic 2–form* Ω encodes the full structure of spacetime (metric, gravitational field and electromagnetic field), hence it plays a central role in the theory.

We obtain the following coordinate expressions

$$\begin{aligned}\gamma_{00}{}^{i} &= K_h{}^i{}_k\, x_0^h x_0^k + 2\, K_0{}^i{}_k\, x_0^k + K_0{}^i{}_0\\ \Omega &= G^0_{ij}\,\big(d_0^i - \gamma_{00}{}^{i}\, d^0 - (K_h{}^i{}_0 + K_h{}^i{}_k\, x_0^k)\,(d^h - x_0^h\, d^0)\big) \wedge (d^j - x_0^j\, d^0)\\ \Lambda &= G_0^{ij}\,\big(\partial_i + (K_i{}^h{}_0 + K_i{}^h{}_k\, x_0^k)\,\partial_h^0\big) \wedge \partial_j^0\,.\end{aligned}$$

Classical mechanics. The classical mechanics can be achieved as follows.

The second order connection γ yields, in a covariant way, the *generalised Newton law* $\nabla j_1 s = 0$, for a motion $s : \boldsymbol{T} \to \boldsymbol{E}$. Clearly, this equation splits into its gravitational and electromagnetic components as $\nabla^{\natural} j_1 s = \gamma^{\mathfrak{e}} \circ j_1 s$.

Moreover, the classical dynamics can be derived from Ω, by a Lagrangian formalism, in the following covariant way [52, 22, 36].

2.1 Proposition. The closed 2–form Ω admits locally *horizontal potentials* $\Theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, which are defined up to closed 1–forms of spacetime.

The horizontal potentials Θ have coordinate expression of the type

$$\Theta = -(\tfrac{1}{2}\, G^0_{ij}\, x_0^i\, x_0^j - A_0)\, d^0 + (G^0_{ij}\, x_0^j + A_j)\, d^i\,, \quad \text{with} \quad A \in \operatorname{Sec}(\boldsymbol{E}, T^*\boldsymbol{E})\,. \square$$

A horizontal potential Θ and an observer o yield the *classical potential* $A := o^*\Theta : \boldsymbol{E} \to T^*\boldsymbol{E}$, which is defined locally up to a closed form and depends on the observer.

2.2 Proposition. Let us consider a given horizontal potential Θ; if o and $\bar{o} = o + v$ are two observers, then the associated potentials A and $\bar{A}$ are related, in a chart adapted to o, by the formula

$$\bar{A} = A - \tfrac{1}{2}\, G^0_{ij}\, v_0^i\, v_0^j\, d^0 + G^0_{ij}\, v_0^j\, d^i\,.$$

Therefore, each horizontal potential Θ determines a distinguished observer; in fact, there is a unique observer o, such that the spacelike component of the associated potential A vanishes. $\square$

An observer o yields the *observed 2–form* $\Phi := 2\, o^*\Omega : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$.

2.3 Proposition. We have $\Phi_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda$.

We obtain also $\Phi_{0k} := -\, G^0_{kj}\, K_0{}^j{}_0$ and $\Phi_{hk} := G^0_{hj}\, K_k{}^j{}_0 - G^0_{kj}\, K_h{}^j{}_0$. $\square$

2.4 Proposition. A horizontal potential Θ yields, in a covariant way, the *classical Lagrangian* $\mathcal{L} : J_1\boldsymbol{E} \to T^*\boldsymbol{T}$, with coordinate expression

$$\mathcal{L} = (\tfrac{1}{2}\, G^0_{ij}\, x_0^i\, x_0^j + A_i\, x_0^i + A_0)\, d^0\,,$$

where A_λ are the components of the potential A observed by the observer o associated with the chart. The Lagrangian is defined locally and up to a gauge, but does not depend on any observer. The Poincaré–Cartan form associated with the Lagrangian $\mathcal{L}$ turns out to be just Θ.

The Euler–Lagrange equation associated with $\mathcal{L}$ turns out to coincide with the generalised Newton law. □

2.5 Proposition. A horizontal potential Θ and an observer o yield the *classical Hamiltonian* $\mathcal{H} : J_1\boldsymbol{E} \to T^*\boldsymbol{T}$ and the *classical momentum* $\mathcal{P} : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, defined as the negative of the o–horizontal component and the o–vertical components of Θ, respectively. Thus, we can write

$$\Theta = -\mathcal{H} + \mathcal{P}\,.$$

In an adapted chart, we have the coordinate expressions

$$\mathcal{H} = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0\,, \qquad \mathcal{P} = (G^0_{ij}\, x^j_0 + A_i)\, d^i\,.$$

They are defined locally and up to a gauge, and depend on the choice of the observer. □

The Newton law can be achieved also through $\mathcal{H}$ and $\mathcal{P}$, by means of a Hamiltonian formalism; but this procedure is non covariant, as it depends on the choice of an observer.

Classical Lie algebras. Additionally, our structures yield further results on Lie algebras of functions and lifts of functions.

First of all, we obtain the *Poisson Lie bracket* $\{f,\, g\} := \Lambda^\sharp(df \wedge dg)$ for the functions of phase space.

A function f of phase space is *conserved* along the solutions of the Newton law if and only if $\gamma.f = 0$. We denote the space of conserved functions by $\mathrm{Con}(J_1\boldsymbol{E},\, \mathbb{R})$. This space turns out to be a subalgebra of the Poisson algebra.

The time fibring and the spacelike metric yield, in a covariant way, a distinguished subset of the set of functions of phase space [25]. Namely, we define a *special quadratic function* to be a function of phase space, whose second fibre derivative (with respect to the affine fibres of phase space over spacetime) is proportional to the spacelike metric. In other words, in coordinates, the special quadratic functions are the functions of the type

$$f = \tfrac{1}{2}\, f^0\, G^0_{ij}\, x^i_0\, x^j_0 + f^i\, G^0_{ij}\, x^j_0 + \overset{o}{f}\,, \quad \text{with} \quad f^0, f^i, \overset{o}{f} \in \mathrm{Map}(\boldsymbol{E},\, \mathbb{R})\,.$$

The *time component* of a special quadratic function f as above is defined to be the (coordinate independent) map $f'' := f^0\, u_0 : \boldsymbol{E} \to \bar{\mathbb{T}}$.

2.6 Proposition. The space of special quadratic functions $\mathrm{Spec}(J_1\boldsymbol{E},\, \mathbb{R})$ turns out to be a Lie algebra through the *special* Lie bracket

$$[\![f,\, g]\!] := \{f,\, g\} + \gamma(f'').g - \gamma(g'').f\,,$$

with coordinate expression

$$[\![f, g]\!]^0 = f^0 \partial_0 g^0 - g^0 \partial_0 f^0 - f^h \partial_h g^0 + g^h \partial_h f^0$$
$$[\![f, g]\!]^i = f^0 \partial_0 g^i - g^0 \partial_0 f^i - f^h \partial_h g^i + g^h \partial_h f^i$$
$$[\![\overset{o}{f}, \overset{o}{g}]\!] = f^0 \partial_0 \overset{o}{g} - g^0 \partial_0 \overset{o}{f} - f^h \partial_h \overset{o}{g} + g^h \partial_h \overset{o}{f} - (f^0 g^k - g^0 f^k)\, \Phi_{0k} + f^h g^k\, \Phi_{hk} \,. \square$$

2.7 Corollary. We have the following distinguished subalgebras of the special Lie algebra:

- the subalgebra $\mathrm{Quan}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{Spec}(J_1\boldsymbol{E}, \mathbb{R})$ of *quantisable functions* f, whose time components f'' depend only on time;

- the subalgebra $\mathrm{Time}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{Quan}(J_1\boldsymbol{E}, \mathbb{R})$ of *time functions* f, whose time components f'' are constant;

- the subalgebra $\mathrm{Aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{Time}(J_1\boldsymbol{E}, \mathbb{R})$ of *affine functions* f, whose time components f'' vanish;

- the subalgebra $\mathrm{Map}(\boldsymbol{E}, \mathbb{R}) \subset \mathrm{Aff}(J_1\boldsymbol{E}, \mathbb{R})$ of *spacetime functions*. $\square$

2.8 Example. We obtain

$$\mathcal{L}_0, \mathcal{H}_0 \in \mathrm{Time}(J_1\boldsymbol{E}, \mathbb{R}) \,, \qquad \mathcal{P}_i \in \mathrm{Aff}(J_1\boldsymbol{E}, \mathbb{R}) \,, \qquad x^\lambda \in \mathrm{Map}(\boldsymbol{E}, \mathbb{R}) \,.$$

Clearly, the special bracket and the Poisson bracket coincide on $\mathrm{Aff}(J_1\boldsymbol{E}, \mathbb{R})$. $\square$

We have distinguished lifts of special quadratic functions to vector fields of spacetime and of phase space. Let us denote by $\mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E}) \subset \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})$ the subalgebra of vector fields of $\boldsymbol{E}$ which are projectable on $\boldsymbol{T}$.

2.9 Proposition. The time fibring and the spacelike metric yield, in a covariant way, for each $f \in \mathrm{Spec}(J_1\boldsymbol{E}, \mathbb{R})$, the *tangent lift* $X[f] : \boldsymbol{E} \to T\boldsymbol{E}$, whose coordinate expression is

$$X[f] = f^0\, \partial_0 - f^i\, \partial_i \,.$$

The lift $\mathrm{Spec}(J_1\boldsymbol{E}, \mathbb{R}) \to \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E}) : f \mapsto X[f]$ turns out to be a Lie algebra morphism (with respect to the special bracket and the standard Lie bracket, respectively); its kernel is $\mathrm{Map}(\boldsymbol{E}, \mathbb{R})$. $\square$

2.10 Example. We obtain

$$X[\mathcal{L}_0] = \partial_0 - A^i_0\, \partial_i \,, \qquad X[\mathcal{H}_0] = \partial_0 \,, \qquad X[\mathcal{P}_i] = -\partial_i \,, \qquad X[x^\lambda] = 0 \,,$$

where $A^i_0 := G^{ij}_0\, A_j$.

We observe that $X[\mathcal{L}] := u^0 \otimes X[\mathcal{L}_0]$ turns out to be the unique observer for which the spacelike component of the observed potential A vanishes.

Moreover, $X[\mathcal{H}] := u^0 \otimes X[\mathcal{H}_0]$ turns out to be just the observer by which we have defined the Hamiltonian. $\square$

2.11 Proposition. For each vector field X of $\boldsymbol{E}$ projectable on $\boldsymbol{T}$, the spacetime fibring yields, in a covariant way [35], the *holonomic prolongation*

$$X^{\uparrow}{}_{\mathrm{hol}} := X_{(1)} : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}\,,$$

whose coordinate expression is

$$X^{\uparrow}{}_{\mathrm{hol}} = X^{\lambda}\,\partial_{\lambda} + (\partial_0 X^i + \partial_j X^i\, x^j_0 - \partial_0 X^0\, x^i_0)\,\partial^0_i\,.$$

This prolongation turns out to be an injective Lie algebra morphism. □

2.12 Corollary. For each $f \in \mathrm{Quan}(J_1\boldsymbol{E}, \mathrm{I\!R})$, the time fibring yields, in a covariant way, the *holonomic lift*

$$X^{\uparrow}{}_{\mathrm{hol}}[f] := \big(X[f]\big)_{(1)} : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}\,,$$

whose coordinate expression is

$$X^{\uparrow}{}_{\mathrm{hol}}[f] = f^0\,\partial_0 - f^i\,\partial_i - (\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0)\,\partial^0_i\,.$$

The lift $\mathrm{Quan}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\mathrm{hol}}[f]$ turns out to be a Lie algebra morphism (with respect to the special bracket and the standard Lie bracket, respectively); its kernel is $\mathrm{Map}(\boldsymbol{E}, \mathrm{I\!R})$. □

2.13 Example. We obtain

$$X^{\uparrow}{}_{\mathrm{hol}}[\mathcal{L}_0] = \partial_0 - A^i_0\,\partial_i - (\partial_0 A^i_0 + \partial_j A^i_0\, x^j_0)\,\partial^0_i\,,$$
$$X^{\uparrow}{}_{\mathrm{hol}}[\mathcal{H}_0] = \partial_0\,, \qquad X^{\uparrow}{}_{\mathrm{hol}}[\mathcal{P}_i] = -\partial_i\,, \qquad X^{\uparrow}{}_{\mathrm{hol}}[x^{\lambda}] = 0\,. \;\square$$

For each function f of phase space, we obtain, in a covariant way, the dt–vertical *Hamiltonian lift* $\Lambda^{\sharp}(df) : J_1\boldsymbol{E} \to VJ_1\boldsymbol{E}$.

More generally, for each function f of phase space and for each *time scale* $\tau : J_1\boldsymbol{E} \to \bar{\mathbb{T}}$, we obtain the τ*–Hamiltonian lift* $\gamma(\tau) + \Lambda^{\sharp}(df) : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$.

In particular, we obtain the following result.

2.14 Proposition. For each $f \in \mathrm{Spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$, the cosymplectic structure yields, in a covariant way, the *Hamiltonian lift*

$$X^{\uparrow}{}_{\mathrm{Ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df) : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}\,,$$

whose coordinate expression is

$$X^{\uparrow}{}_{\mathrm{Ham}}[f] = f^0\,\partial_0 - f^i\,\partial_i + X^i_0\,\partial^0_i\,,$$

where

$$X_0^i = G_0^{ij}\,\big(\tfrac{1}{2}\,\partial_j f^0 G^0_{hk} x_0^h x_0^k + (-f^0\partial_0 G^0_{jh} + f^k\,\partial_k G^0_{jh} + G^0_{hk}\,\partial_j f^k)\,x_0^h \\ + \partial_j \overset{o}{f} + \Phi_{hj} f^h + f^0 \Phi_{j0}\big)\,.$$

The lift $\mathrm{Quan}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\mathrm{Ham}}[f]$ turns out to be a Lie algebra morphism (with respect to the special bracket and the standard Lie bracket, respectively); its kernel is $\mathrm{Map}(\boldsymbol{T}, \mathrm{I\!R})$. □

2.15 Example. We obtain

$$X^{\uparrow}{}_{\mathrm{Ham}}[\mathcal{H}_0] = \partial_0 - G_0^{ij}\,\partial_0 \mathcal{P}_j\,\partial_i^0\,,$$
$$X^{\uparrow}{}_{\mathrm{Ham}}[\mathcal{P}_i] = -\partial_i + G_0^{hj}\,\partial_i \mathcal{P}_h\,\partial_j^0\,, \qquad X^{\uparrow}{}_{\mathrm{Ham}}[x^0] = 0\,, \qquad X^{\uparrow}{}_{\mathrm{Ham}}[x^i] = G_0^{ij}\,\partial_j^0\,. \square$$

The interest of the above Hamiltonian lift is due to the following result concerning the projectability, which will play an important role in quantum mechanics.

2.16 Proposition. [25] The τ-Hamiltonian lift of a function f of phase space is projectable on a vector field of spacetime if and only if $f \in \mathrm{Spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ and $\tau = f''$.

Moreover, if these conditions are fulfilled, then the τ-Hamiltonian lift projects on the tangent lift of f. □

2.2 Covariant quantum mechanics

We proceed by sketching our covariant model of quantum mechanics on a curved spacetime fibred over absolute time. We recall the basic elements of the model and present new results, as well.

Quantum structure. According to [25, 22], for quantum mechanics of a charged spinless particle in the above classical background (including the given gravitational and electromagnetic external fields), we postulate:

(Q.1) a *quantum bundle* $\boldsymbol{Q} \to \boldsymbol{E}$, which is a one dimensional complex vector bundle over spacetime;

(Q.2) a *Hermitian metric* $\mathtt{h} : \boldsymbol{E} \to \mathbb{C} \otimes (\boldsymbol{Q}^* \otimes_{\boldsymbol{E}} \boldsymbol{Q}^*) \otimes_{\boldsymbol{E}} \Lambda^3 V^* \boldsymbol{E}$ of the quantum bundle, with values in the complexified space of spacelike volume forms of spacetime.

Locally, we shall refer to a scaled complex *quantum basis* $(\mathtt{b})$ normalised by the condition $h(\mathtt{b}, \mathtt{b}) = \eta$. The associated scaled complex chart is denoted by (z). Then, we obtain the scaled real basis $(\mathtt{b}_1\,, \mathtt{b}_2) := (\mathtt{b},\, \mathfrak{i}\,\mathtt{b})$ and the associated scaled real chart (w^1, w^2).

If $\Psi \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$, then we write $\Psi = \Psi^1\,\mathtt{b}_1 + \Psi^2\,\mathtt{b}_2 = \psi\,\mathtt{b}$, where Ψ^1, Ψ^2 and ψ are, respectively, the scaled real and complex components of Ψ.

Moreover, we consider the *extended quantum bundle*, $\boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$, obtained by extending the base space of the quantum bundle to the classical phase space, which here plays the role of space of classical observers.

Each system of connections $\{\overset{o}{\text{Ч}}\}$ of the quantum bundle parametrised by the classical observers induces, in a covariant way, a connection Ч of the extended quantum bundle, which is said to be *universal* [17, 22]. The universal connections are characterised in coordinates by the condition $\text{Ч}_i^0 = 0$.

Then, we postulate:
(Q.3) a *quantum connection* Ч of the extended quantum bundle, which is Hermitian, universal and whose curvature is $R[\text{Ч}] = \mathfrak{i}\,\Omega \otimes \mathbb{I}$, where $\mathbb{I} = (w^1\,\partial w_1 + w^2\,\partial w_2)$ denotes the identity vertical vector field of the quantum bundle.
We recall that Ω incorporates the mass m of the particle and the Planck constant $\hbar$.

2.17 Proposition. The coordinate expression of the quantum connection, with respect to a quantum basis and a spacetime chart, turns out to be locally of the type

$$\text{Ч}_0 = -\mathfrak{i}\,\mathcal{H}_0\,, \qquad \text{Ч}_i = \mathfrak{i}\,\mathcal{P}_i\,, \qquad \text{Ч}_i^0 = 0\,.\,\square$$

The above classical Hamiltonian $\mathcal{H}$ and momentum $\mathcal{P}$ are referred to the observer o associated with the spacetime chart (x^λ) and to a classical horizontal potential Θ of Ω, which is locally determined by the quantum connection Ч and the quantum basis $\mathfrak{b}$.
Then, the gauge of the classical potential $A := o^*\Theta$ is determined by the quantum connection and the quantum basis. Moreover, we recall that A includes both the gravitational and the electromagnetic potential.

These minimal geometric objects Q.1, ... , Q.3 constitute the only source, in a covariant way, of all further objects of quantum mechanics.

Actually, the quantum connection lives on the extended quantum bundle, whose base space is the phase space; on the other hand, the covariance of the theory requires that the significant physical objects be independent from observers. This fact suggests a method of projectability, in order to get rid of the observers encoded in the phase space. Actually, we have already used this method in the classical theory, just in view of these developments of quantum mechanics. Indeed, this method turns out to be fruitful.

Quantum dynamics. The *quantum dynamics* can be obtained in the following way.
The method of projectability yields, in a covariant way, a distinguished quantum Lagrangian (hence, the generalised Schroedinger equation, the quantum momentum and the probability current) [25, 22].
Even more, the covariance implies the essential uniqueness of the above Lagrangian and of the Schroedinger equation [27, 29].

2.18 Proposition. The coordinate expression of the quantum Lagrangian is

$$\begin{aligned} \mathsf{L}[\Psi] = \tfrac{1}{2}\big(&\mathfrak{i}\,(\bar\psi\,\partial_0\psi - \psi\,\partial_0\bar\psi) + 2\,A_0\,\bar\psi\,\psi \\ &- G_0^{ij}\,(\partial_i\bar\psi\,\partial_j\psi + A_i\,A_j\,\bar\psi\,\psi) - \mathfrak{i}\,G_0^{ij}\,A_j\,(\bar\psi\,\partial_i\psi - \psi\,\partial_i\bar\psi) + k\,\rho_0\,\bar\psi\,\psi\big) \end{aligned}$$

$$\sqrt{|g|}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3 \,,$$

where ρ is the scalar curvature of the fibres of spacetime determined by the spacelike metric and $k \in \mathrm{I\!R}$ is a real constant (which is not determined by the covariance). □

2.19 Corollary. The coordinate expression of the generalised Schroedinger equation turns out to be

$$\big(\partial_0 - \mathfrak{i}\, A_0 + \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \tfrac{1}{2}\, \mathfrak{i}\ (\overset{o}{\Delta}_0 + k\, \rho_0)\,\big)\, \psi = 0\,,$$

where

$$\overset{o}{\Delta}_0 := G^{hk}_0\, (\partial_h - \mathfrak{i}\, A_h)\, (\partial_k - \mathfrak{i}\, A_k) + \frac{\partial_h (G^{hk}_0 \sqrt{|g|})}{\sqrt{|g|}}\, (\partial_k - \mathfrak{i}\, A_k)$$

is the spacelike *quantum Laplacian.* □

2.20 Corollary. We obtain the conserved *probability current* with coordinate expression

$$\Psi^* \mathsf{j} = (\bar\psi\, \psi)\, \upsilon^0_0 - G^{hk}_0\, \big(\mathfrak{i}\, \tfrac{1}{2}\, (\bar\psi\, \partial_h \psi - \psi\, \partial_h \bar\psi) + A_h\, \bar\psi\, \psi\big)\, \upsilon^0_k\,,$$

where $\upsilon^0_\lambda := i(\partial_\lambda)\, \sqrt{|g|}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3$. □

Quantum operators. We obtain distinguished operators acting on the sections of the quantum bundle in the following covariant way.

First of all, we have a distinguished family of *second order pre–quantum operators*.

2.21 Proposition. The Schroedinger operator yields, for each time scale $\tau : \boldsymbol{E} \to \bar{\mathbb{T}}$, the second order linear operator $\mathsf{S}(\tau) : J_2\boldsymbol{Q} \to \boldsymbol{Q}$, which acts on the sections Ψ of the quantum bundle, according to the coordinate expression

$$\mathsf{S}(\tau)[\Psi] = \mathfrak{i}\, \tau^0\, \big(\partial_0 - \mathfrak{i}\, A_0 + \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \tfrac{1}{2}\, \mathfrak{i}\ (\overset{o}{\Delta}_0 + k\, \rho_0)\,\big)\, \psi\, \mathsf{b}\,.$$

In particular, each $f \in \mathrm{Spec}(J_1\boldsymbol{E}, \mathrm{I\!R})$ yields, in a covariant way, the second order pre–quantum operator $\mathsf{S}[f] := \mathsf{S}(f'')$. □

Then, we obtain a distinguished family of first order operators, by classifying the vector fields of the quantum bundle which preserve the Hermitian metric.

A vector field Y of $\boldsymbol{Q}$ is said to be *Hermitian* if it is projectable on $\boldsymbol{E}$ and on $\boldsymbol{T}$, is real linear over its projection on $\boldsymbol{E}$, and fulfills $L[Y]\, \mathsf{h} = 0$.

We denote the space of Hermitian vector fields of $\boldsymbol{Q}$ by $\mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q})$.

2.22 Proposition. A vector field Y of $\boldsymbol{Q}$ is Hermitian if and only if its coordinate expression is of the type

$$Y \;\equiv\; Y[f] = f^0\, \partial_0 - f^i\, \partial_i + \big(\mathfrak{i}\, (\overset{o}{f} + A_0\, f^0 - A_i\, f^i) - \tfrac{1}{2}\, \mathrm{div}_\eta\, X[f]\big)\, \mathbb{I}\,,$$

where $f \in \text{Quan}(J_1\boldsymbol{E}, \mathbb{R})$. The above expression of $Y[f]$ turns out to be independent of the choice of coordinates.

The space of Hermitian vector fields $\text{Her}(\boldsymbol{E}, T\boldsymbol{Q})$ is closed with respect to the Lie bracket. Moreover, the map

$$\text{Quan}(J_1\boldsymbol{E}, \mathbb{R}) \to \text{Her}(\boldsymbol{Q}, T\boldsymbol{Q}) : f \mapsto Y[f]$$

is an isomorphism of Lie algebras (with respect to the special bracket and the standard Lie bracket, respectively).

Furthermore, the map $\text{Her}(\boldsymbol{Q}, T\boldsymbol{Q}) \to \text{Pro}(\boldsymbol{E}, T\boldsymbol{E}) : Y[f] \mapsto X[f]$ turns out to be a central extension of Lie algebras by $\text{Map}(\boldsymbol{E}, \mathfrak{i}\,\mathbb{R}) \otimes \mathbb{I}$. □

For each $f \in \text{Quan}(J_1\boldsymbol{E}, \mathbb{R})$, the vector field $Y[f] : \boldsymbol{Q} \to T\boldsymbol{Q}$ is said to be the *quantum lift* of f.

2.23 Example. We obtain

$$Y[\mathcal{L}_0] = \partial_0 - A^i_0\,\partial_i - \big(\mathfrak{i}\,A_i\,A^i_0 + \tfrac{1}{2}\,(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A^i_0\,\sqrt{|g|})}{\sqrt{|g|}})\big)\,\mathbb{I}\,,$$

$$Y[\mathcal{H}_0] = \partial_0 - \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}\,, \qquad Y[\mathcal{P}_i] = -\partial_i + \tfrac{1}{2}\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}\,, \qquad Y[x^\lambda] = \mathfrak{i}\,x^\lambda\,\mathbb{I}\,. \square$$

2.24 Corollary. Each quantisable function f yields, in a covariant way, the first order operator acting on the sections of the quantum bundle

$$Z[f] := \mathfrak{i}\,L\big[Y[f]\big]\,,$$

whose coordinate expression is, for each $\Psi \in \text{Sec}(\boldsymbol{E}, \boldsymbol{Q})$,

$$Z[f].\Psi = \mathfrak{i}\,\Big(f^0\,\partial_0\psi - f^i\,\partial_i\psi - \big(\mathfrak{i}\,(\overset{o}{f} + A_0\,f^0 - A_i\,f^i) - \tfrac{1}{2}\,\text{div}_\eta\,X[f]\big)\,\psi\Big)\,\mathsf{b}\,. \square$$

For each quantisable function f, we say $Z[f]$ to be the associated *first order pre–quantum operator*. We denote the space of the first order pre–quantum operators by $\text{Oper}_1(\boldsymbol{Q})$.

2.25 Proposition. The space $\text{Oper}_1(\boldsymbol{Q})$ turns out to be a Lie algebra through the bracket

$$\big[Z[f], Z[g]\big] := -\,\mathfrak{i}\,\big(Z[f] \circ Z[g] - Z[g] \circ Z[f]\big)\,.$$

Moreover, the map $\text{Quan}(J_1\boldsymbol{E}, \mathbb{R}) \to \text{Oper}_1(\boldsymbol{Q}) : f \mapsto Z[f]$ turns out to be an isomorphism of Lie algebras (with respect to the special bracket and the above Lie bracket, respectively). □

2.26 Example. We obtain

$$Z[\mathcal{H}_0].\Psi = \mathfrak{i}\,\big(\partial_0\psi + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi\big)\,\mathfrak{b}$$

$$Z[\mathcal{P}_i].\Psi = -\mathfrak{i}\,\big(\partial_i\psi + \tfrac{1}{2}\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\,\psi\big)\,\mathfrak{b}$$

$$Z[x^\lambda].\Psi = x^\lambda\,\psi\,\mathfrak{b}\,.\,\square$$

The above results appear to be a covariant "correspondence principle" yielding *pre–quantum operators* associated with quantisable functions.

However, we still need to introduce the Hilbert stuff carrying the standard probabilistic interpretation of quantum mechanics. It can be done in the following covariant way [25, 22].

Let us restrict our postulate C.3, by requiring that the fibring of spacetime over time makes spacetime a bundle. Thus, we postulate that the fibres of spacetime are each other isomorphic.

Then, we consider the infinite dimensional *functional quantum bundle* $\boldsymbol{H}_c \to \boldsymbol{T}$, whose fibres are constituted by the compact support smooth sections, at fixed time, of the quantum bundle ("regular sections"). The Hermitian metric $\mathfrak{h}$ equips this bundle with a pre–Hilbert metric $\langle\,,\,\rangle$. Then, a true Hilbert bundle $\boldsymbol{H} \to \boldsymbol{T}$ can be obtained by a completion procedure. This bundle has no distinguished splittings into time and type Hilbert fibre; such a splitting can be obtained by choosing a classical observer.

Each regular section Ψ of the quantum bundle can be regarded as a section $\widehat{\Psi}$ of the functional quantum bundle. Accordingly, each "regular" operator O acting on sections of the quantum bundle can be regarded as an operator $\widehat{\mathsf{O}}$ acting on the sections of the functional quantum bundle.

Our previous results yield, for each quantisable function f, two distinguished operators acting on the sections of the functional quantum bundle, namely $\widehat{Z[f]}$ and $\widehat{\mathsf{S}[f]}$. Actually, in general, both operators do not act on the fibres of the functional bundle (at fixed time), because they involve the partial derivative ∂_0.

On the other hand, we have the following results [25, 22, 51].

2.27 Proposition. Let $f \in \mathrm{Quan}(J_1\boldsymbol{E}, \mathbb{R})$. Then, the combination

$$\widehat{f} := \widehat{Z[f]} - \widehat{\mathsf{S}[f]}$$

acts on the fibres of the functional bundle. We have the following coordinate expression

$$\widehat{f}(\widehat{\Psi}) = \big(-\tfrac{1}{2}\,f^0\,(\overset{o}{\Delta}_0 + k\,\rho_0) - \mathfrak{i}\,f^j\,(\partial_j - \mathfrak{i}\,A_j) + \overset{o}{f} - \mathfrak{i}\,\tfrac{1}{2}\,\frac{\partial_j(f^j\sqrt{|g|})}{\sqrt{|g|}}\big)\,\psi\,\widehat{\mathfrak{b}}\,.$$

Moreover, $\widehat{f}$ is symmetric with respect to the Hermitian metric $\langle\,,\,\rangle$. $\square$

For the self–adjointness of $\widehat{f}$ further global conditions on f are needed.
For each $f \in \text{Quan}(J_1\boldsymbol{E}, \text{I\!R})$, we say $\widehat{f}$ to be the *quantum operator* associated with f.

2.28 Example. We obtain the following distinguished quantum operators

$$\widehat{\mathcal{H}_0}(\widehat{\Psi}) = -(\tfrac{1}{2}\overset{o}{\Delta}_0 + \tfrac{1}{2}\,k\,\rho_0 - A_0)\,\psi\,\widehat{\mathfrak{b}}\,,$$
$$\widehat{\mathcal{P}_j}(\widehat{\Psi}) = -\mathfrak{i}\,\big(\partial_j + \tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\big)\,\psi\,\widehat{\mathfrak{b}}\,,$$
$$\widehat{x^\lambda}(\widehat{\Psi}) = x^\lambda\,\psi\,\widehat{\mathfrak{b}}\,.\square$$

The space of the fibre preserving maps of the functional quantum bundle into itself becomes a Lie algebra through the bracket $[h,k] := -\,\mathfrak{i}\,(h\circ k - k\circ h)$.

2.29 Proposition. For each $f, g \in \text{Quan}(J_1\boldsymbol{E}, \text{I\!R})$, we obtain

$$[\widehat{f},\,\widehat{g}] = \widehat{[\![f,\,g]\!]} - \mathfrak{i}\,\big[\widehat{S[f]},\,\widehat{Z[g]}\big] + \mathfrak{i}\,\big[\widehat{S[g]},\,\widehat{Z[f]}\big]\,.$$

In particular, for each $f, g \in \text{Aff}(J_1\boldsymbol{E}, \text{I\!R})$, we obtain

$$[\widehat{f},\,\widehat{g}] = \widehat{[\![f,\,g]\!]} = \widehat{\{f,\,g\}}\,.\square$$

Thus, the above results suggest our covariant "equivalence principle".

The Feynmann path integral approach can be nicely formulated in our framework [25]. In fact, the quantum connection Ч yields, in a covariant way, a non linear connection of the extended quantum bundle over time; moreover, this connection allows us to interpret the Feynmann amplitudes through the parallel transport of this connection. However, unfortunately, our theory does not contribute so far to the hard problem of the measure arising in the Feynmann theory.

The case of a spin particle (generalised Pauli equation) can be approached in an analogous way, by considering a further quantum bundle of dimension two, with the only additional postulate of a suitable soldering form [4].

3 Symmetries

Next, we classify the infinitesimal symmetries of the classical and quantum structures. We show that these symmetries are controlled by the Lie algebra of quantisable functions and its distinguished subalgebras. Moreover, we discuss the strict relations between classical and quantum symmetries.

3.1 Classical symmetries

We start by discussing the main results concerning symmetries of the classical structure.

Subalgebras. First we analyse further distinguished subalgebras of the Lie algebra of quantisable functions.

3.1 Proposition. We have the following distinguished subalgebras of the algebra of quantisable functions:
– the subalgebra $\mathrm{Hol}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{Quan}(J_1\boldsymbol{E}, \mathrm{I\!R})$, which is constituted by the functions f such that $X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{Ham}}[f]$;
– the subalgebra $\mathrm{Unim}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{Quan}(J_1\boldsymbol{E}, \mathrm{I\!R})$, which is constituted by the functions f such that $\mathrm{div}_\eta X[f] = 0$;
– the subalgebra $\mathrm{Self}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{Quan}(J_1\boldsymbol{E}, \mathrm{I\!R})$, which is constituted by the functions f such that $i(X^{\uparrow}{}_{\mathrm{hol}}[f])\,\Omega = df$. □

If $f \in \mathrm{Hol}(J_1\boldsymbol{E}, \mathrm{I\!R})$, then we set

$$X^{\uparrow}[f] := X^{\uparrow}{}_{\mathrm{Ham}}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f]\,.$$

3.2 Proposition. We have

$$\mathrm{Time}(J_1\boldsymbol{E},\mathrm{I\!R}) \cap \mathrm{Con}(J_1\boldsymbol{E},\mathrm{I\!R}) = \mathrm{Time}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \mathrm{Self}(J_1\boldsymbol{E}, \mathrm{I\!R})\,. \square$$

Then, we set

$$\begin{aligned}\mathrm{Clas}(J_1\boldsymbol{E},\mathrm{I\!R}) &:= \mathrm{Time}(J_1\boldsymbol{E},\mathrm{I\!R}) \cap \mathrm{Con}(J_1\boldsymbol{E},\mathrm{I\!R})\\ &= \mathrm{Time}(J_1\boldsymbol{E}, \mathrm{I\!R}) \cap \mathrm{Self}(J_1\boldsymbol{E}, \mathrm{I\!R})\end{aligned}$$

and denote the space of the tangent lifts of elements of $\mathrm{Clas}(J_1\boldsymbol{E},\mathrm{I\!R})$ by

$$\mathrm{Clas}(\boldsymbol{E}, T\boldsymbol{E}) \subset \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})\,.$$

3.3 Proposition. We have

$$\begin{aligned}\mathrm{Time}(J_1\boldsymbol{E},\mathrm{I\!R}) \cap \mathrm{Con}(J_1\boldsymbol{E},\mathrm{I\!R}) &\subset \mathrm{Hol}(J_1\boldsymbol{E}, \mathrm{I\!R})\\ \mathrm{Time}(J_1\boldsymbol{E},\mathrm{I\!R}) \cap \mathrm{Con}(J_1\boldsymbol{E},\mathrm{I\!R}) &\subset \mathrm{Unim}(J_1\boldsymbol{E}, \mathrm{I\!R})\,. \square\end{aligned}$$

3.4 Proposition. The special and the Poisson brackets coincide in $\mathrm{Clas}(J_1\boldsymbol{E}, \mathrm{I\!R})$. Hence, this space turns out to be a subalgebra of the Poisson and of the special algebras.
Moreover, $\mathrm{Clas}(\boldsymbol{E}, T\boldsymbol{E})$ turns out to be closed with respect to the standard Lie bracket. □

We call the elements of $\mathrm{Clas}(J_1\boldsymbol{E},\mathrm{I\!R})$ *classical generators*. This name will be justified by Proposition 3.5, Corollary 3.6 and Corollary 3.7.

Symmetries. A vector field $X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ is said to be a *symmetry of the classical structure* if it is projectable on $\boldsymbol{E}$ and $\boldsymbol{T}$ and fulfills

$$L[X^{\uparrow}]\, dt = 0\,, \qquad L[X^{\uparrow}]\, \Omega = 0\,.$$

We denote the space of symmetries of the classical structure by $\mathrm{Clas}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$.

3.5 Proposition. [55] A vector field $X^{\uparrow}$ of $J_1\boldsymbol{E}$ projectable on $\boldsymbol{E}$ fulfills $L[X^{\uparrow}]\, dt = 0$ and $L[X^{\uparrow}]\, \Omega = 0$ if and only if, locally,

$$X^{\uparrow} = X^{\uparrow}[f]\,, \quad \text{with} \quad f \in \mathrm{Clas}(J_1\boldsymbol{E}, \mathrm{I\!R})\,,$$

where f is defined up to a constant. □

3.6 Corollary. If $f \in \mathrm{Clas}(J_1\boldsymbol{E}, \mathrm{I\!R})$, then we obtain

$$L\big[X[f]\big]\, G = 0\,, \qquad L\big[X[f]\big]\, \eta = 0\,, \qquad L\big[X^{\uparrow}[f]\big]\, \gamma = 0\,, \qquad L\big[X^{\uparrow}[f]\big]\, K = 0\,. \square$$

3.7 Corollary. If X is a vector field of $\boldsymbol{E}$ projectable on $\boldsymbol{T}$, such that $L[X^{\uparrow}{}_{\mathrm{hol}}]\, \mathcal{L} = 0$, then we obtain locally

$$X = X[f]\,, \qquad X^{\uparrow}{}_{\mathrm{hol}} = X^{\uparrow}[f]\,, \quad \text{with} \quad f \in \mathrm{Clas}(J_1\boldsymbol{E}, \mathrm{I\!R})\,,$$

where f is defined up to a constant. □

3.2 Quantum symmetries

Eventually, we classify the vector fields of the extended quantum bundle which preserve the full quantum structure: all fibrings (on quantum bundle, on phase space, on spacetime, on time), the Hermitian metric, the quantum connection. Moreover, we compare the symmetries of the quantum structure with the symmetries of the quantum Lagrangian.

Symmetries of the quantum structure. A vector field $Y^{\uparrow}$ of $\boldsymbol{Q}^{\uparrow}$ is said to be a *symmetry of the quantum structure* if it is projectable on $\boldsymbol{Q}$, $J_1\boldsymbol{E}$, $\boldsymbol{E}$, $\boldsymbol{T}$, is real linear over $J_1\boldsymbol{E}$ and fulfills

$$L[Y^{\uparrow}]\, dt = 0\,, \qquad L[Y^{\uparrow}]\, \mathfrak{h} = 0\,, \qquad L[Y^{\uparrow}]\, Ч = 0\,.$$

We denote the space of symmetries of the quantum structure by $\mathrm{Quan}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$.

For each $f \in \mathrm{Hol}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we define its *extended quantum lift* to be the vector field of the extended quantum bundle

$$Y^{\uparrow}[f] := Ч\big(X^{\uparrow}[f]\big) + \big(\mathfrak{i}\, f - \tfrac{1}{2}\, \mathrm{div}_{\eta}\, X[f]\big)\, \mathbb{I}\,.$$

3.8 Proposition. A vector field $Y^{\uparrow}$ of $\boldsymbol{Q}^{\uparrow}$ is a symmetry of the quantum structure if and only if it is of the type

$$Y^{\uparrow} = Y^{\uparrow}[f]\,, \quad \text{with} \quad f \in \mathrm{Clas}(J_1\boldsymbol{E},\, \mathrm{I\!R})\,.$$

The space $\mathrm{Quan}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow})$ is closed with respect to the Lie bracket. Moreover, the map $\mathrm{Clas}(J_1\boldsymbol{E},\, \mathrm{I\!R}) \to \mathrm{Quan}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}[f]$ is an isomorphism of Lie algebras (with respect to the special bracket and the standard Lie bracket, respectively).

Furthermore, the map $\mathrm{Quan}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow}) \to \mathrm{Clas}(\boldsymbol{E},\, T\boldsymbol{E}) : Y^{\uparrow}[f] \mapsto X[f]$ turns out to be a central extension of Lie algebras by $\mathfrak{i}\,\mathrm{I\!R} \otimes \mathbb{I}$. □

Symmetries of the quantum dynamics. Next, we compare the symmetries of the quantum connection and the symmetries of the quantum Lagrangian.

3.9 Proposition. For each $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, \mathrm{I\!R})$, we obtain, in a covariant way, the *holonomic quantum lift* of f, defined as the holonomic prolongation [35]

$$Y_{\mathrm{hol}}[f] := \big(Y[f]\big)_{(1)} : J_1\boldsymbol{Q} \to TJ_1\boldsymbol{Q}$$

of the quantum lift $Y[f]$, whose coordinate expression is

$$\begin{aligned}
&Y_{\mathrm{hol}}[f] = f^0\,\partial_0 - f^i\,\partial_i \\
&- \tfrac{1}{2}\,\mathrm{div}_\eta\, X[f]\,(w^1\,\partial_1 + w^2\,\partial_2 - w^1_\lambda\,\partial^\lambda_1 - w^2_\lambda\,\partial^\lambda_2) - \tfrac{1}{2}\,\partial_\lambda\,\mathrm{div}_\eta\, X[f]\,(w^1\,\partial^\lambda_1 + w^2\,\partial^\lambda_2) \\
&+ (f^0\,A_0 - f^i\,A_i + \overset{o}{f})\,(w^1\,\partial_2 - w^2\,\partial_1 + w^1_\lambda\,\partial^\lambda_2 - w^2_\lambda\,\partial^\lambda_1) \\
&+ \partial_\lambda(f^0\,A_0 - f^i\,A_i + \overset{o}{f})\,(w^1\,\partial^\lambda_2 - w^2\,\partial^\lambda_1) \\
&- \partial_0 f^0\,(w^1_0\,\partial^0_1 + w^2_0\,\partial^0_2) + \partial_\lambda f^i\,(w^1_i\,\partial^\lambda_1 + w^2_i\,\partial^\lambda_2)\,. \square
\end{aligned}$$

3.10 Proposition. Let $f \in \mathrm{Time}(J_1\boldsymbol{E}, \mathrm{I\!R})$. Then, the following conditions are equivalent:

$$1)\; L\big[Y^{\uparrow}{}_{\mathrm{hol}}[f]\big]\,\mathsf{Ч} = 0\,, \qquad 2)\; L\big[Y_{\mathrm{hol}}[f]\big]\,\mathsf{L} = 0\,,$$
$$3)\; i(X^{\uparrow}{}_{\mathrm{hol}}[f])\,\Omega = df\,, \qquad 4)\; \gamma.f = 0\,, \qquad 5)\; f \in \mathrm{Clas}(J_1\boldsymbol{E},\, \mathrm{I\!R})\,. \square$$

Eventually, we consider the conserved currents associated with symmetries of the quantum Lagrangian, according to the standard Noether theorem. Additionally, our results allow us to associate such currents with classical quantisable functions.

For each $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, \mathrm{I\!R})$, we define the associated *quantum current* to be the 3–form

$$\mathrm{J}[f] := -\,i\big(Y[f]\big)\,\Pi : J_1\boldsymbol{Q} \to \Lambda^3 T^*\boldsymbol{Q}\,,$$

where Π is the Poincaré–Cartan form [52] associated with the quantum Lagrangian.

3.11 Corollary. For each $f \in \mathrm{Clas}(J_1\boldsymbol{E},\, \mathrm{I\!R})$, the current $\mathrm{J}[f]$ is conserved along the solutions $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ of the Schroedinger equation. □

3.12 Example. The current associated with the constant function $1 \in \mathrm{Clas}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is just the conserved probability current. □

Moreover, for each affine function and quantum section, we obtain, in a covariant way, a spacelike 3–form (which can be integrated on the fibres of spacetime), according to the following result.

3.13 Proposition. Let $f \in \mathrm{Aff}(J_1\boldsymbol{E},\, \mathrm{I\!R})$. Then, for each $\Psi \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$ we obtain

$$\big(\Psi^*(\mathrm{J}[f])\big)^{\vee} = \tfrac{1}{2}\,\big(\mathsf{h}(Z[f].\Psi,\, \Psi) - \mathsf{h}(\Psi,\, Z[f].\Psi)\big)$$

where $^{\vee}$ denotes the vertical restriction. We have the coordinate expression

$$\big(\Psi^*(\mathrm{J}[f])\big)^{\vee} = \big(f^i\,(\Psi^1\,\partial_i\Psi^2 - \Psi^2\,\partial_i\Psi^1) + \overset{o}{f}\,(\Psi^1\,\Psi^1 + \Psi^2\,\Psi^2)\big)\,\sqrt{|g|}\,\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3\,. \square$$

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