Covariant Schrödinger operator

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Abstract

We analyse the Schrödinger operator for a quantum scalar particle in a curved spacetime which is fibred over absolute time and is equipped with given spacelike metric, gravitational field and electromagnetic field.

We approach the Schrödinger operator by three independent ways: in terms of covariant differentials induced by the quantum connection, via a quantum Lagrangian and directly by the only requirement of general covariance.

In particular, in the flat case, our Schrödinger operator coincides with the standard one.

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Contents

Introduction 3

1 Classical background 8
1.1 Units of measurement . . . 8
1.2 Classical spacetime . . . 8
1.3 Observers . . . 9
1.4 Metric field . . . 10
1.5 Metric extensions . . . 11
1.6 Galileian connections . . . 12
1.7 Classical phase space . . . 14
1.8 Gravitational and electromagnetic fields . . . 16

2 Quantum structure 17
2.1 Quantum bundle . . . 17
2.2 Quantum metric . . . 18
2.3 Quantum connection . . . 19
2.3.1 Connections of the quantum bundle . . . 19
2.3.2 Universal connections of the extended quantum bundle . . . 20
2.3.3 Quantum connections . . . 21
2.3.4 Quantum differentials . . . 23

3 Quantum dynamics 25
3.1 Differential approach . . . 25
3.2 Lagrangian approach . . . 27
3.3 Uniqueness by covariance . . . 28
3.3.1 Covariant operators . . . 29
3.3.2 Schrödinger operator . . . 30
3.3.3 Quantum Lagrangian . . . 32
3.3.4 Schrödinger operator on the functional quantum bundle . . . 33

References 35

Introduction

Since the very beginning of quantum mechanics, the Schrödinger operator has been approached in several ways and investigated in many respects; the related literature is extremely huge. Indeed, the Schrödinger equation is one of the greatest successes of physics, hence it should be taken as a touchstone for any possible variation on this subject.

Certainly, the most usual approaches are based on Hamiltonian techniques related to the quantisation of the classical Hamiltonian and the standard formalism is basically analytical.

Among the geometric Hamiltonian approaches, we must mention "geometric quantisation" [1, 21, 44, 90, 91, 100]. In this context one can achieve a Schrödinger operator (which includes the spacelike scalar curvature) for quantum systems admitting a suitable polarisation.

The Feynmann path integral is another possible method of achieving the Schrödinger operator (see, for instance, [39, 20]). However, in spite of the fundamental importance of this formalism, its formal theoretical troubles are well known.

W. M. Tulczjew has achieved the Schrödinger operator by considering a five dimensional spacetime [94].

Possible modifications of the Schrödinger operator have been investigated as well. Possible non linear variations have been discussed by H.-D. Döbner, G. A. Goldin et al. [24, 25, 26, 27, 28, 29, 30] in the framework of non–linear quantum mechanics. A modification of the quantum dynamics has been proposed by G. C. Ghirardi in view of a unified picture of microscopic and macroscopic systems [43].

Another interesting aspect of the Schrödinger operator concerns its transformation rule with respect to the change of frame of reference and coordinates [89].

An even deeper problem would be a formulation of the Schrödinger operator on a curved spacetime and an analysis of its relation with the general principle of relativity and the principle of equivalence. A true solution of this problem is today too ambitious because it would involve well established quantum gravity and quantum field theory on a curved spacetime.

Among the attempts to formulate a quantum theory in the framework of general relativity, we quote E. Prugovečki [84, 85]. On the other hand, several authors have discussed the problem of equivalence principle for a quantum system (see, for instance, A. Camacho [9, 10, 11, 12, 13, 14, 15, 16]) and analysed the behaviour of a quantum system with respect to accelerated observers (see, for instance, I. Bialynicki–Birula, M. Kalinski, B. Mashhoon et al. [4, 5, 62, 77, 86]).

A more limited, but interesting, goal is the formulation of the Schrödinger operator on a curved spacetime fibred over absolute time and equipped with a spacelike Riemannian metric, which will be referred to as "Galilei curved spacetime". The idea of such a spacetime goes back to E. Cartan [18, 19] and has been further analysed by several authors, including V. Bargmann [3], G. Dautcourt [22], H. D. Dombrowski and K. Horneffer [31], M. Le Bellac and J. M. Levy–Leblond [73, 74], J. Ehlers [37], P. Havas [45], M. Mangiarotti [76], A. Trautman [92, 93]. This model of spacetime is intermediate between the Newton

spacetime and the Einstein spacetime. Hence, it provides an intermediate setting between the standard non–relativistic quantum mechanics and a possible general relativistic quantum theory. In this framework we can discuss the behaviour of the quantum system in accelerated frames, but we loose the description of phenomena related to the speed of light and to Lorentz metric. Among the approaches to the Schrödinger operator in such a curved spacetime, we would like to mention the pioneering paper by B. S. DeWitt [23], the paper by K. Kuchař [65], who approaches the Schrödinger operator via Dirac's constraint method in a coordinate independent way, and the papers by J. R. Fanchi [38] and A. Kyprianidis [71]. A special mention deserves the Lagrangian approach in a Bargmann framework due to C. Duval and K. Künzle [32, 33, 34, 35, 36, 66, 67, 68, 69, 70].

The standard concepts of "relativistic" and "non relativistic" theory are not adequate for our discussions. We need a more careful and refined use of these words. In this paper, the word "general (special) relativity" refers both to the Einstein general (special) relativity and to the Galilei general (special) relativity. In fact, both theories are formulated in a way which is independent of the choice of an accelerated (inertial) observer. In both cases, an observer is described by a normalised timelike vector field, but the mathematical definition of the "timelike" character and of the "normalisation" is different in the two models.

The present paper is aimed at analysing the Schrödinger operator on a curved spacetime in the framework of a covariant formulation of quantum mechanics that here will be referred to as "covariant quantum mechanics". This formulation has been originally proposed for scalar particles by A. Jadczyk and M. Modugno [47, 48, 49] and further developed in cooperation with J. Janyška, D. Saller, C. Tejero Prieto and R. Vitolo [46, 50, 51, 53, 55, 57, 59, 60, 61, 80, 81, 82, 87, 96, 98, 99]. It has been extended to spin particles in cooperation with D. Canarutto [17] and partially (up to pre–quantum operators) to a Lorentzian setting in cooperation with J. Janyška and R. Vitolo [52, 54, 56, 58, 97]. This approach is based on non standard geometric methods, such as fibred manifolds, jet spaces, non–linear connections, systems of connections, cosymplectic structures and Frölicher smooth spaces.

In a few words, the scheme of this model is the following. The classical spacetime is constituted by a manifold fibred over absolute time and equipped with a spacelike Riemannian metric, a time preserving and metric preserving connection (gravitational field) and a closed 2–form (electromagnetic field). As classical phase space we assume the 1st jet space of spacetime. The above fundamental fields yield, in a covariant way, a 2nd order connection, hence a covariant formulation of the law of motion. Moreover, they yield, in a covariant way, a global cosymplectic 2–form, which induces locally a gauge dependent Lagrangian and a gauge and observer dependent Hamiltonian. We show a distinguished Lie algebra of quadratic functions (including the Hamiltonian) different from the Poisson Lie algebra. Then, we formulate the covariant quantum theory for a scalar particle, affected by the given gravitational and electromagnetic fields, by considering a complex bundle over spacetime equipped with a Hermitian metric with values in the space of spacelike volume forms and a universal, Hermitian connection, whose curvature is proportional to

the cosymplectic 2–form. From these two objects, we derive, in a covariant way, all other quantum objects. In order to get rid of the observers we use a criterion of projectability. In particular, this method yields good candidates for the quantum operators, through the isomorphism between the Lie algebra of Hermitian quantum vector fields and the Lie algebra of projectable quadratic classical functions. Moreover, this method yields a good candidate for the Schrödinger operator, equivalently, via quantum covariant differentials and via a quantum Lagrangian.

Let us summarise the main features of "covariant quantum mechanics", for a scalar particle, in view of a comparison with more standard formulations.

We are concerned only with "fundamental fields". Therefore, we consider only external gravitational and electromagnetic fields for the classical background of the quantum system. In fact, we think that only a fundamental setting has the right to demand a covariance principle and has a chance to produce results possibly interesting for further covariant developments.

Our quantum theory is not intended as a "quantisation procedure" of a classical system into a quantum system. We just propose a direct approach to quantum mechanics. The classical theory is involved essentially through the spacetime structure and not really through the classical particle mechanics. Actually, the classical spacetime plays the role of background space, as it carries the information of the external classical gravitational and electromagnetic fields affecting the quantum particle. More precisely, this information is carried by the classical spacetime itself as base space of the quantum bundle and by the classical phase space, which plays here the role of space of classical observers. The main information of the given classical gravitational and electromagnetic fields is encoded in the cosymplectic 2–form of the classical phase space, which plays the role of curvature of the quantum connection.

We take the well established *results* of quantum mechanics, such as the standard Schrödinger equation and quantum operators, as touchstone of our model. On the other hand, according to the aims of our theory, we disregard those standard *methods* for deriving quantum objects, which are incompatible with general covariance.

Our basic guide is the *covariance* of the theory as heuristic requirement. Even more, we look for manifest covariance. Nowadays, the concept of "covariance" has been formulated in a rigorous mathematical way through the geometric concept of "naturality" [63]. Our theory provides explicit expressions of all objects for any *accelerated observer* and yields, at the same time, an interpretation in terms of gravitational field, according to the principle of equivalence.

According to the covariance of the theory, *time* is not just a parameter, but a fundamental object. Moreover, the main objects of the theory are not assumed to be split into time and space components.

As classical *phase space* we take the 1st jet space of spacetime and not its tangent space; indeed, this minimal choice allows us to skip anholonomic constraints.

Another consequence of our approach is that classical mechanics is ruled by a *cosymplectic structure* [75] and not by a symplectic structure. Actually, a spacelike symplectic

structure arises in our model, but it describes only the spacelike aspects of the classical theory and is insufficient to account, in a covariant way, for classical dynamics.

We emphasize the fact that classical mechanics can be formulated in a covariant way by a *2nd order connection* and by a *Lagrangian approach*, but not by a Hamiltonian approach, because the Hamiltonian function depends essentially on an observer.

The *Hamiltonian* is not postulated as an additional object of our classical setting, but it can be "extracted" by an observer, from a local potential of the cosymplectic 2–form.

An achievement of our theory is the Lie algebra of "*special quadratic functions*" (different from the Poisson algebra), which allows us to treat energy, momentum and spacetime functions on the same footing. This algebra and its subalgebras control fully the classical and quantum symmetries.

All objects of quantum mechanics are derived, in a covariant way, from three *minimal objects*. Here, we have some novelties.

The quantum bundle lives *on spacetime* and not on the phase space and the quantum connection is "*universal*". These assumptions allow us to skip all troubles of polarisations [100]. In a sense, we obtain naturally a covariant polarisation and this is sufficient for our purposes. Indeed, we replace the problematic search for such inclusions with a *method of projectability*, which turns out to be our implementation of covariance in the quantum theory.

Another new feature concerns the *Hermitian metric* of the quantum bundle, which takes its values in the space of spacelike volume forms. This assumption allows us to skip the problems related to half–densities. On the other hand, this assumption requires the projectability of Hermitian vector fields. Indeed, this feature turns out to be an opportunity in our approach.

We show the strict relation of the quantum connection with the classical *Poincaré–Cartan form*. In a sense, the quantum connection can be regarded as a global and gauge independent version of the above local and gauge dependent form.

The Schrödinger equation is obtained, in a covariant way, both through a *differential approach* and a *Lagrangian approach*.

The quantum operators arise, in a covariant way, from the *classification* of distinguished 1st and 2nd order differential operators of the quantum framework and not from a quantisation requirement of a classical system.

The seat for the covariant probabilistic interpretation of quantum mechanics is the *Hilbert bundle* over time, which arises naturally from the quantum bundle. This bundle is observer independent and each observer splits it into time and a Hilbert space.

We do not postulate transition maps for the classical and quantum theory (concerning transition of coordinates, potentials, and so on), but we start from simple intrinsic axioms and derive the transition rules from them. Thus, the groups involved in the theory arise as groups of automorphisms of the original geometric structures.

In a few words, we start with really minimal geometric structures representing fundamental physical fields and proceed along a thread naturally imposed by the only requirement of general covariance.

In the flat case, the results of our model fit exactly the corresponding results of standard quantum mechanics. Thus, in the flat case, we do need to exhibit successful examples, as it is sometimes required in alternative theories.

The specific aim of the present paper is to prove that, under weak assumptions on the classical background and minimal assumptions on the quantum bundle, the (2nd order) Schrödinger operator is determined by the only requirements of covariance and invariance with respect to time scales. Under these hypotheses, the Schrödinger operator turns out to be linear and of Lagrangian type.

Thus, we achieve the Schrödinger operator in three independent covariant ways:

- in terms of covariant differential and codifferential induced by the quantum connection, by using a criterion of projectability (hence, the independence from observers),

- in terms of a quantum Lagrangian associated with the quantum connection and the quantum metric, by using a criterion of projectability (hence, the independence from observers),

- by postulating just the covariance of the operator (this approach yields also the uniqueness).

At the first instance, our approach to Schrödinger operator has no direct relation with Hamiltonian techniques and with the quantisation of the classical Hamiltonian. Actually, the Schrödinger operator is global, gauge independent and observer independent, while the classical Hamiltonian is local, gauge dependent and observer dependent. On the other hand, the quantum Hamiltonian operator can be achieved a posteriori from the Schrödinger operator by taking into account the pre–quantum operator associated, in a covariant way, with the classical Hamiltonian [49, 46].

The covariance with respect to the time scales plays a role on the same footing of covariance with respect to the change of spacetime coordinates, to the change of observers and to the change of quantum bases.

If we would assume a distinguished time scale, then the requirement of covariance would be compatible with many additional terms. Actually, we could take into account some time scales arising from constants which have a room in our model, such as the Planck constant, the gravitational constant, the mass of the particle and so on. It seems that these time scales yield additional terms which are too big to be physically reasonable. But, we cannot exclude that one could find a reasonable distinguished time scale which breaks the covariance with respect to time scales. We could make similar considerations for length scales; but nothing essentially new would arise, because time scales and length scales are related, in the framework of our model, by $\hbar/m$.

We denote the sheaf of local sections of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and the sheaf of local fibred morphisms between the fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p' : \boldsymbol{F}' \to \boldsymbol{B}$, respectively, by $\mathrm{sec}(\boldsymbol{B}, \boldsymbol{F})$ and $\mathrm{fib}(\boldsymbol{F}, \boldsymbol{F}')$.

1 Classical background

In this section we summarise [49, 46] the basic concepts and results concerning the classical spacetime, in view of the quantum theory. With respect to previous papers, here we add a new discussion on the covariance group bundles, on metric extensions, Galileian connections and classical potentials.

1.1 Units of measurement

Covariance requires a rigorous treatment of units of measurement.

We assume [46, 49] the following "positive 1-dimensional semi-vector spaces" over $\mathbb{R}^+$ as fundamental unit spaces: the space $\mathbb{T}$ of *time intervals*, the space $\mathbb{L}$ of *lengths* and the space $\mathbb{M}$ of *masses*. Moreover, we assume the *Planck constant* $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

We refer to particles with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^* \otimes \mathbb{L}^{\frac{3}{2}} \otimes \mathbb{M}^{\frac{1}{2}}$.

We define a *time unit* to be an element $u_0 \in \mathbb{T}$ or to its dual $u^0 \in \mathbb{T}^*$.

1.2 Classical spacetime

We start with our postulate concerning the classical curved Galileian spacetime.

C.1 Postulate. We assume *time* to be an affine space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes \mathbb{R}$. We assume *spacetime* to be an oriented $(3+1)$-dimensional manifold $\boldsymbol{E}$ fibred over time by the *time map* $t : \boldsymbol{E} \to \boldsymbol{T}$. □

Thus, the time fibring yields the *time form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$.

1.1 Definition. We define the *spacetime charts* to be charts $(x^\lambda) \equiv (x^0, x^i)$ of spacetime, which are adapted to the time fibring, to the affine structure of time, to the orientation of time and of spacetime and to a time unit u_0. □

We shall always refer to spacetime charts.

If (x^λ) is a spacetime chart, then the induced local bases of $T\boldsymbol{E}$, $V\boldsymbol{E}$, $T^*\boldsymbol{E}$ and $V^*\boldsymbol{E}$ are denoted, respectively, by (∂_λ), (∂_i), (d^λ) and $(\check{d}^i)$.

The coordinate expression of the time form is $dt = u_0 \otimes d^0$.

1.2 Proposition. Let (x^λ) and $(\acute{x}^\lambda)$ be two spacetime charts, and set

$$\sigma^\lambda_{\mu_1 \dots \mu_r} := \acute{\partial}_{\mu_1} \dots \acute{\partial}_{\mu_r} x^\lambda , \qquad \acute{\sigma}^\lambda_{\mu_1 \dots \mu_r} := \partial_{\mu_1} \dots \partial_{\mu_r} \acute{x}^\lambda .$$

Then, we obtain $\acute{x}^0 = \acute{\sigma}^0_0 x^0 + \acute{\sigma}^0$, with $\acute{\sigma}^0 \in \mathbb{R}$, $\acute{\sigma}^0_0 \in \mathbb{R}^+$, $\acute{\sigma}^0_i = 0$, $\det(\acute{\sigma}^i_j) > 0$. □

1.3 Definition. We define the *time scale covariance bundle* to be the trivial bundle $\mathcal{G}(\mathbb{T}) := \boldsymbol{E} \times \mathbb{R}^+$ of $\mathbb{R}^+$–linear automorphisms of $\mathbb{T}$. □

1.4 Definition. We define:
- a *time automorphism* to be an orientation preserving affine isomorphism $\underline{\phi} : \boldsymbol{T} \to \boldsymbol{T}$;
- a *spacetime automorphism* to be a local, orientation preserving fibred diffeomorphism $\phi : \boldsymbol{E} \to \boldsymbol{E}$ over a time automorphism $\underline{\phi} : \boldsymbol{T} \to \boldsymbol{T}$. □

1.5 Proposition. The coordinate expression of a spacetime automorphism ϕ is of the type $\phi^0 = a^0_0\, x^0 + a^0,\ \phi^i \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$, with $a^0_0 \in \mathbb{R}^+,\ a^0 \in \mathbb{R},\ \det(\partial_j \phi^i) > 0$. □

If ϕ is a spacetime automorphism, then we set $\phi^\lambda_{\mu_1 \dots \mu_r} := \partial_{\mu_1} \dots \partial_{\mu_r}\, \phi^\lambda$.

1.6 Note. If $x \equiv (x^\lambda)$ and $\acute{x} \equiv (\acute{x}^\lambda)$ are two spacetime charts, then we obtain the spacetime automorphism $\phi := x^{-1} \circ \acute{x}$. Hence, we obtain $x \circ \phi \circ x^{-1} = \acute{x} \circ x^{-1}$, which yields $\phi^\lambda_{\mu_1 \dots \mu_r} = \acute{\sigma}^\lambda_{\mu_1 \dots \mu_r}$. □

Let $\mathrm{aut}(\boldsymbol{E}) \subset \mathrm{fib}(\boldsymbol{E}, \boldsymbol{E})$ be the sheaf of spacetime automorphisms. Then, for each $s \geq 0$ and $e \in \boldsymbol{E}$, we define the set

$$\mathcal{G}^s_e(\boldsymbol{E}) := \Big\{ (j_s\phi)(e) \;\Big|\; \phi \in \mathrm{aut}(\boldsymbol{E}),\, \phi(e) = e \Big\}.$$

1.7 Definition. For each integer $s \geq 0$, we define the *spacetime covariance bundle*, of order s, to be the bundle

$$\mathcal{G}^s(\boldsymbol{E}) := \bigsqcup_{e \in \boldsymbol{E}} \mathcal{G}^s_e(\boldsymbol{E}) \to \boldsymbol{E}\,. \,\square$$

Clearly, the fibre of $\mathcal{G}^0(\boldsymbol{E})$ has dimension 0. We denote the fibred chart of $\mathcal{G}^s(\boldsymbol{E})$ induced by a spacetime chart (x^λ) by $(x^\lambda;\ \mathfrak{g}^0_0, \mathfrak{g}^i_\lambda;\ \dots;\ \mathfrak{g}^i_{\lambda_1 \dots \lambda_s})$.

1.8 Proposition. The bundle $\mathcal{G}^s(\boldsymbol{E})$ turns out a group bundle through the composition fibred morphism over $\boldsymbol{E}$

$$\mu^s : \mathcal{G}^s(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathcal{G}^s(\boldsymbol{E}) \to \mathcal{G}^s(\boldsymbol{E}) : \Big((j_s \overset{1}{\phi})(e),\ (j_s \overset{2}{\phi})(e) \Big) \mapsto \Big(j_s (\overset{1}{\phi} \circ \overset{2}{\phi}) \Big)(e)\,. \,\square$$

1.3 Observers

Then, we introduce the notion of observer and the associated transition maps.

1.9 Definition. An *observer* is defined to be a section $o \in \mathrm{sec}(\boldsymbol{E},\, J_1\boldsymbol{E})$, i.e. a section $o \in \mathrm{sec}(\boldsymbol{E},\, \mathbb{T}^* \otimes T\boldsymbol{E})$, which projects on $\mathbf{1} \in \mathbb{T}^* \otimes \mathbb{T}$. □

The coordinate expression of an observer is of the type $o = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$, with $o^i_0 \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

The charts (x^λ) for which $o^i_0 = 0$ are said to be *adapted* to o. Each chart (x^λ) determines the observer $o := u^0 \otimes \partial_0$.

Each observer o yields the *spacelike projection* $T\boldsymbol{E} \to V\boldsymbol{E}$ through the map

$$\nu[o] = (d^i - o_0^i\, d^0) \otimes \partial_i \in \sec(\boldsymbol{E},\, T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E})\,.$$

If o and $\acute{o}$ are two observers, then we can write $\acute{o} = o + v$, with $v \in \sec(\boldsymbol{E},\, \mathbb{T}^* \otimes V\boldsymbol{E})$.

1.10 Definition. The abelian group bundle $\mathfrak{T}(\boldsymbol{E}) := \mathbb{T}^* \otimes V\boldsymbol{E} \to \boldsymbol{E}$ is called the *observer transition bundle*. □

Its sections will yield the *transition maps* for "observed objects".

1.4 Metric field

Then, we discuss the metric structure of spacetime.

1.11 Definition. We define a *spacelike metric* to be a scaled Riemannian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes S^2V^*\boldsymbol{E}$ of the fibres of $\boldsymbol{E}$.
With reference to a mass $m \in \mathbb{M}$, we define the *re–scaled spacelike metric*

$$G := \tfrac{m}{\hbar}\, g = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j : \boldsymbol{E} \to \mathbb{T} \otimes S^2V^*\boldsymbol{E}\,. \,\square$$

We denote the contravariant spacelike metric and the contravariant re–scaled spacelike metric by $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{2*} \otimes S^2V\boldsymbol{E}$ and $\bar{G} := \frac{\hbar}{m}\, \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes S^2V\boldsymbol{E}$.
We denote the natural bundle of re–scaled spacelike metrics by $\mathrm{Met}(\boldsymbol{E}) \subset \mathbb{T} \otimes S^2V^*\boldsymbol{E}$.

C.2 Postulate. We assume a spacelike metric g. □

1.12 Proposition. The spacelike metric g and the spacetime orientation naturally yield the scaled *spacelike volume form* and the *spacetime volume form*

$$\eta = \sqrt{|g|}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3 : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3V^*\boldsymbol{E}\,,$$
$$\upsilon := dt \wedge \eta = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4T^*\boldsymbol{E}\,. \,\square$$

We set $\upsilon^0 := \upsilon(u^0) = \sqrt{|g|}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3$.

1.13 Definition. For each projectable vector field X of $\boldsymbol{E}$, we define the *timelike divergence* and the *spacelike divergence*, respectively,

$$\mathrm{div}_{\mathrm{dt}}\, X \in \mathrm{map}(\boldsymbol{E},\, \mathbb{R})\,, \quad \text{by the equality} \quad L[X]\, dt = (\mathrm{div}_{\mathrm{dt}}\, X)\, dt\,,$$
$$\mathrm{div}_{\eta}\, X \in \mathrm{map}(\boldsymbol{E},\, \mathbb{R})\,, \quad \text{by the equality} \quad L[X]\, \eta = (\mathrm{div}_{\eta}\, X)\, \eta\,.$$

For each vector field X of $\boldsymbol{E}$, we define the *spacetime divergence*

$$\mathrm{div}_{\upsilon}\, X \in \mathrm{map}(\boldsymbol{E},\, \mathbb{R})\,, \quad \text{by the equality} \quad L[X]\, \upsilon = (\mathrm{div}_{\upsilon}\, X)\, \upsilon\,. \,\square$$

1.14 Proposition. We have the coordinate expressions

$$\mathrm{div}_{\mathrm{dt}}\, X = \partial_0 X^0\,, \quad \mathrm{div}_{\eta}\, X = X^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}}\,, \quad \mathrm{div}_{v}\, X = \frac{\partial_\lambda(X^\lambda\sqrt{|g|})}{\sqrt{|g|}}\,. \square$$

1.15 Proposition. The re–scaled spacelike metric G naturally yields:
- the fibre–wise Riemannian connection $\varkappa[G] : V\boldsymbol{E} \to V^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} VV\boldsymbol{E}$,
- the fibre–wise curvature tensor $R[\varkappa] : V\boldsymbol{E} \to \Lambda^2 V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}$,
- the fibre–wise Ricci tensor $\mathrm{Ricci}[\varkappa] := C^1_1 R[\varkappa] : \boldsymbol{E} \to V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}$,
- the fibre–wise scaled scalar curvature $r[G] := \langle \bar{G},\, \mathrm{Ricci}[\varkappa]\rangle : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R}$. $\square$

1.5 Metric extensions

The extensions of the spacelike metric to spacetime metrics will play an interesting role in understanding the Galileian connections and their potentials.

1.16 Definition. A section $\widetilde{G} \in \mathrm{sec}(\boldsymbol{E},\ \mathbb{T} \otimes S^2 T^*\boldsymbol{E})$, whose vertical restriction is G, is called a *metric extension*. $\square$

The coordinate expression of a metric extension is of the type

$$\widetilde{G} = u_0 \otimes (G^0_{ij}\, d^i \otimes d^j + G^0_{0j}\, d^0 \otimes d^j + G^0_{i0}\, d^i \otimes d^0 + G^0_{00}\, d^0 \otimes d^0)\,,$$

where $G^0_{ij} \equiv G\,(\partial_i, \partial_j)$ and $G^0_{0j}, G^0_{i0}, G^0_{00} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ with $G^0_{0i} = G^0_{i0}$.

In particular, each observer o yields the metric extension $\widetilde{G}[o] := \nu^*[o]G$, whose expression, in an adapted chart, is $\widetilde{G}[o] = G^0_{ij}\, u_0 \otimes d^i \otimes d^j$.

1.17 Proposition. Let us consider a metric extension $\widetilde{G}$ and an observer o, and let us refer to an adapted chart. Then, we obtain the section

$$A[\widetilde{G}, o] := o \rfloor \widetilde{G} - \tfrac{1}{2}\, o \rfloor o \rfloor \widetilde{G} = \tfrac{1}{2}\, G^0_{00}\, d^0 + G^0_{i0}\, d^i \in \mathrm{sec}(\boldsymbol{E},\, T^*\boldsymbol{E})\,.$$

Moreover, we obtain $\widetilde{G} = \nu^*[o]G + dt \otimes A[\widetilde{G}, o] + A[\widetilde{G}, o] \otimes dt$. Clearly, we have $o \rfloor A[\widetilde{G}, o] = \frac{1}{2}\, o \rfloor o \rfloor \widetilde{G}$. $\square$

1.18 Corollary. Let us consider a metric extension $\widetilde{G}$ and two observers o, $\acute{o} + v$, and let us refer to a chart adapted to $\acute{o}$. Then, we obtain

$$A[\widetilde{G}, \acute{o}] - A[\widetilde{G}, o] = -\tfrac{1}{2}\, G(v, v) + \nu[o] \rfloor G^\flat(v) = \tfrac{1}{2}\, G^0_{ij}\, v^i_0\, v^j_0\, d^0 + G^0_{ij}\, v^j_0\, d^i\,. \square$$

1.19 Proposition. Let us consider a form $A \in \mathrm{sec}(\boldsymbol{E},\, T^*\boldsymbol{E})$ and an observer o, and let us refer to an adapted chart. Then, we obtain the metric extension

$$\widetilde{G}[A, o] := \nu^*[o]G + dt \otimes A + A \otimes dt\,.$$

Moreover, we obtain $A\Big[\widetilde{G}[A,o],\, o\Big] = A$. Clearly, we have $\frac{1}{2}\, o \lrcorner o \lrcorner \widetilde{G}[A,o] = o \lrcorner A$. □

1.20 Corollary. Let us consider an observer o. Then, the maps $\widetilde{G} \mapsto A[\widetilde{G}, o]$ and $A \mapsto \widetilde{G}[A,o]$ are inverse bijections. □

1.6 Galileian connections

Galileian connections are fundamental objects of our theory.

1.21 Definition. We define a *spacetime connection* to be a connection K of the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$, which is linear, torsion free and such that $\nabla dt = 0$.

A spacetime connection K is said to be *metric* if $\nabla G = 0$.

A metric spacetime connection K is said to be *Galileian* if its curvature tensor fulfills the condition $R^j_{0\lambda}{}^i{}_\mu = R^i_{0\mu}{}^j{}_\lambda$. □

1.22 Proposition. The coordinate expression of a spacetime connection is of the type $K = d^\lambda \otimes \partial_\lambda + K_\lambda{}^\mu{}_\nu\, \dot{x}^\nu\, d^\lambda \otimes \dot{\partial}_\mu$, with $K_\lambda{}^0{}_\nu = 0$ and $K_\mu{}^i{}_\nu = K_\nu{}^i{}_\mu$. The spacetime connections are the sections of a 2nd order natural bundle $\mathrm{Con}(\boldsymbol{E}) \to \boldsymbol{E}$. □

1.23 Lemma. Let us consider a spacetime connection K, an observer o and let us refer to an adapted chart. Then, we obtain

$$\nabla o = -K_\lambda{}^i{}_0\, u^0 \otimes d^\lambda \otimes \partial_i \in \sec\Big(\boldsymbol{E},\ \mathbb{T}^* \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E})\Big)\,.$$

$$\Phi[K,o] := \mathrm{Alt}\left(\nu^*[o]\Big(G^\flat(\nabla o)\Big)\right) = -K_{\lambda j0}^0\, d^\lambda \wedge d^j \in \sec(\boldsymbol{E},\ \Lambda^2 T^*\boldsymbol{E})\,,$$

where Alt is the antisymmetrization operator and $K_{\lambda j0}^0 := G^0_{ij} K_\lambda{}^i{}_0$. □

1.24 Lemma. [60] If o is an observer, then there is a unique metric spacetime connection $K[o]$, such that $\Phi[K,o] = 0$. Indeed, $K[o]$ is given by

$$K[o] = o_\tau + \nu^*[o_\tau](\varkappa) - \mathrm{Sym}\left(G^{\sharp 1}\Big(\nu^*[o](L[o]\, G)\Big)\right),$$

where o_τ is the tangent prolongation of the observer, $\nu^*[o_\tau](\varkappa)$ the pullback of $\varkappa$ induced by the observer, $\nu^*[o](L[o]\, G)$ the pullback of $L[o]\, G$ induced by the observer and $G^{\sharp 1}$ the musical morphism applied to the first component of the tensor $\nu^*[o](L[o]\, G)$.

The coordinate expression of $K[o]$, in an adapted chart, is

$$\begin{aligned} K_0{}^i{}_0 &= 0\,,\\ K_h{}^i{}_0 = K_0{}^i{}_h &= -\tfrac{1}{2}\, G_0^{ij}\, \partial_0 G^0_{jh}\,,\\ K_h{}^i{}_k = K_k{}^i{}_h &= -\tfrac{1}{2}\, G_0^{ij}\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\,. \end{aligned}$$ □

For each 2–form $\Phi \in \sec(\boldsymbol{E},\ \Lambda^2 T^*\boldsymbol{E})$, we set

$$\widehat{\Phi} := G^{\sharp 1}(\Phi) = \Phi^i_{0\mu}\, u^0 \otimes \partial_i \otimes d^\mu \in \sec\left(\boldsymbol{E},\ \mathbb{T}^* \otimes (V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E})\right).$$

1.25 Theorem. *Let us consider an observer o and refer to an adapted chart.*
1) If K is a metric spacetime connection, then we have

$$K = K[o] + dt \otimes \widehat{\Phi}[K,o] + \widehat{\Phi}[K,o] \otimes dt\,.$$

2) If $\Phi \in \sec(\boldsymbol{E},\ \Lambda^2 T^*\boldsymbol{E})$, then we obtain the metric spacetime connection

$$K[o,\Phi] := K[o] + dt \otimes \widehat{\Phi} + \widehat{\Phi} \otimes dt\,.$$

The coordinate expression of the above connections K is

$$\begin{aligned} K_0{}^i{}_0 &= -G_0^{ij}\, 2\,\Phi_{oj}\,, \\ K_0{}^i{}_h = K_h{}^i{}_0 &= -\tfrac{1}{2}\, G_0^{ij}\, (2\,\Phi_{hj} + \partial_0 G^0_{hj})\,, \\ K_k{}^i{}_h = K_h{}^i{}_k &= -\tfrac{1}{2}\, G_0^{ij}\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\,. \square \end{aligned}$$

Thus, an observer yields a local bijection between metric spacetime connections and spacetime 2–forms.

1.26 Theorem. [46, 60] *Let K be a metric spacetime connection and let us refer to an observer o. Then, K is Galileian if and only if $d\Phi[K,o] = 0$.* $\square$

1.27 Definition. If K is a Galileian connection and o an observer, then each local potential $A[K,o] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ of $\Phi[K,o]$ is called a *potential* of K. $\square$

The potential $A[K,o]$ is locally defined up to a gauge of the type df, with $f \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

1.28 Corollary. Let us consider an observer o and refer to an adapted chart.
1) If K is a Galileian connection, then we have

$$K = K[o] + dt \otimes \widehat{dA}[K,o] + \widehat{dA}[K,o] \otimes dt\,.$$

2) If $A \in \sec(\boldsymbol{E},\ T^*\boldsymbol{E})$, then we obtain the Galileian connection

$$K[o,A] := K[o] + dt \otimes \widehat{dA} + \widehat{dA} \otimes dt\,.$$

The coordinate expression of the above connections K is

$$\begin{aligned} K_0{}^i{}_0 &= -G_0^{ij}\, (\partial_0 A_j - \partial_j A_0)\,, \\ K_0{}^i{}_h = K_h{}^i{}_0 &= -\tfrac{1}{2}\, G_0^{ij}\, (\partial_h A_j - \partial_j A_h + \partial_0 G^0_{hj})\,, \\ K_k{}^i{}_h = K_h{}^i{}_k &= -\tfrac{1}{2}\, G_0^{ij}\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\,, \end{aligned}$$

i.e.

$$K_\lambda{}^i{}_\mu = -\tfrac{1}{2}\, G_0^{ij}\, (\partial_\lambda G^0_{j\mu} + \partial_\mu G^0_{j\lambda} - \partial_j G^0_{\lambda\mu})\,,$$

where we refer to the metric extension $\widetilde{G}\big[A[o], o\big]$.

Thus, an observer yields a local bijection between Galileian connections and spacetime 1–forms, modulo differentials of spacetime functions.

1.7 Classical phase space

We sketch briefly the classical phase space and the objects induced on it by a Galileian connection K.

1.29 Definition. The *phase space* is defined to be the 1st jet space $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ of spacetime [46, 49, 55, 88]. □

We denote the fibred charts of $J_1\boldsymbol{E}$ induced by a spacetime chart (x^λ) by (x^λ, x^i_0).

We recall that $J_1\boldsymbol{E} \to \boldsymbol{E}$ is an affine bundle associated with the vector bundle $\mathbb{T}^*\otimes V\boldsymbol{E}$. Hence, the vertical space of $J_1\boldsymbol{E}$ with respect to $\boldsymbol{E}$ turns out to be $V_0J_1\boldsymbol{E} = \mathbb{T}^* \otimes V\boldsymbol{E}$.

We recall the natural *contact maps*

$$\begin{aligned} &д = u^0 \otimes (\partial_0 + x^i_0\, \partial_i) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}\,,\\ &\theta = (d^i - x^i_0\, d^0) \otimes \partial_i : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}\,. \end{aligned}$$

1.30 Proposition. The metric G and a spacetime connection K yield in a covariant way [49, 46, 60] the following objects, which are global, gauge independent and observer independent:

- the *phase connection*

$$\Gamma = d^\lambda \otimes \partial_\lambda + (\Gamma_\lambda{}^{i\,0}_{00} + \Gamma_\lambda{}^{i\,0}_{0j}\, x^j_0)\, d^\lambda \otimes \partial^0_i : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E}\,,$$

where $\Gamma_\lambda{}^{i\,0}_{0\mu} = K_\lambda{}^i{}_\mu$;

- the *phase 2–form*

$$\Omega := \nu[\Gamma]\bar{\wedge}\theta = G^0_{ij}\left(d^i_0 - (\Gamma_\lambda{}^{i\,0}_{00} + \Gamma_\lambda{}^{i\,0}_{0j}\, x^j_0)\, d^\lambda\right) \wedge \theta^j : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}\,,$$

where $\nu[\Gamma]$ is the vertical valued form associated with Γ and $\bar{\wedge}$ is the wedge product followed by the contraction through G;

- the *2nd order phase connection*

$$\gamma := д \lrcorner \Gamma = u^0 \otimes (\partial_0 + x^i_0\, \partial_i + \gamma^i_{00}\, \partial^0_i) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}\,,$$

where $\gamma^i_{00} = \Gamma_h{}^{i\,0}_{0k}\, x^h_0\, x^k_0 + 2\,\Gamma_h{}^{i\,0}_{00}\, x^h_0 + \Gamma_0{}^{i\,0}_{00}$;

- the *phase 2–vector*

$$\Lambda := \check{\Gamma} \barwedge \nu = G_0^{ij} \left(\partial_i + (\Gamma_{i00}^{h0} + \Gamma_{i0k}^{h0}\, x_0^k)\, \partial_h^0\right) \wedge \partial_j^0 : J_1\boldsymbol{E} \to \Lambda^2 V J_1\boldsymbol{E}\,,$$

where $\check{\Gamma}$ is the vertical restriction of Γ. □

1.31 Theorem. [46] *Let K be a spacetime connection. Then, $d\Omega = 0$ if and only if K is Galileian.* □

1.32 Proposition. [49, 57] If K is a Galileian connection, then we obtain

$$dt \wedge \Omega^3 \not\equiv 0\,, \qquad d\Omega = 0\,,$$
$$i(\gamma)\,\Omega = 0\,, \qquad L(\gamma)\,\Omega = 0\,, \qquad L(\gamma)\,\Lambda = 0\,, \qquad [\Lambda,\, \Lambda] = 0\,. \square$$

Hence, $(J_1\boldsymbol{E},\, dt, \Omega)$ turns out to be a scaled cosymplectic manifold.
Now, let K be a Galileian connection.

1.33 Theorem. *The cosymplectic 2–form Ω admits locally a* horizontal potential $\Theta \in \mathrm{fib}(J_1\boldsymbol{E},\, T^*\boldsymbol{E})$*, defined up to a closed spacetime 1–form.* □

Next, let Θ be a horizontal potential of Ω and o an observer.

1.34 Definition. We define the following *classical* objects:
- the *Lagrangian* as the t–horizontal phase 1–form $\mathcal{L}[\Theta] := \text{д} \lrcorner\, \Theta$;
- the *momentum* as the contact phase 1–form $\mathcal{P}[\Theta] := \theta \lrcorner\, \Theta$;
- the observed *Hamiltonian* as the t–horizontal phase 1–form $\mathcal{H}[\Theta, o] := -o \lrcorner\, \Theta$;
- the observed *momentum* as the $\boldsymbol{E}$–vertical phase 1–form $\mathcal{P}[\Theta, o] := \nu[o] \lrcorner\, \Theta$;
- the observed *potential* as the phase 1–form $A[\Theta, o] := o^*\Theta$. □

1.35 Proposition. We obtain

$$\Theta = \mathcal{L}[\Theta] + \mathcal{P}[\Theta] = -\mathcal{H}[\Theta, o] + \mathcal{P}[\Theta, o]\,,$$
$$\mathcal{L}[\Theta] = -\mathcal{H}[\Theta, o] + \text{д} \lrcorner\, \mathcal{P}[\Theta, o]\,, \qquad \mathcal{P}[\Theta, o] = \nu[o] \lrcorner\, \mathcal{P}[\Theta]\,.$$

Moreover, we obtain $dA[\Theta, o] = \Phi[K, o] := o^*\Omega$. Hence, $A[\Theta, o]$ turns out to be one of the potentials $A[K, o]$; thus, the choice of Θ fixes the gauge of $A[K, o]$.

Furthermore, the Euler–Lagrange equation associated with $\mathcal{L}[\Theta]$ turns out to be just the Newton law $\nabla[\gamma] j_1 s = 0$.

In a chart adapted to o we have the following coordinate expressions

$$\Theta = -\tfrac{1}{2}\, G_{ij}^0\, x_0^i\, x_0^j\, d^0 + G_{ij}^0\, x_0^j\, d^i + A_\lambda\, d^\lambda\,, \qquad A[\Theta, o] = A_\lambda\, d^\lambda\,,$$
$$\mathcal{L}[\Theta] = (\tfrac{1}{2}\, G_{ij}^0\, x_0^i\, x_0^j + A_i\, x_0^i + A_0)\, d^0\,, \qquad \mathcal{P}[\Theta] = (G_{ij}^0\, x_0^j + A_i)\, (d^i - x_0^i\, d^0)\,,$$
$$\mathcal{H}[\Theta, o] = (\tfrac{1}{2}\, G_{ij}^0\, x_0^i\, x_0^j - A_0)\, d^0\,, \qquad \mathcal{P}[\Theta, o] = (G_{ij}^0\, x_0^j + A_i)\, d^i\,. \square$$

1.8 Gravitational and electromagnetic fields

Now, we introduce the gravitational and electromagnetic fields.

1.36 Definition. The *gravitational field* and the *electromagnetic field* are defined to be, respectively, a Galileian spacetime connection and a scaled closed 2–form of spacetime

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E} \quad \text{and} \quad f : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}\,. \square$$

With reference to a charge q, we define the *re–scaled electromagnetic field*

$$F = \frac{q}{\hbar}\, f : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}\,.$$

C.3 Postulate. We assume a gravitational field $K^\natural$ and an electromagnetic field F. $\square$

Then, it is convenient to introduce the *total connection*

$$K := K^\natural + K^e = K^\natural + (dt \otimes \widehat{F} + \widehat{F} \otimes dt)\,, \quad \text{where} \quad \widehat{F} = G_0^{ih}\, F_{h\mu}\, u^0 \otimes \partial_i \otimes d^\mu\,.$$

1.37 Proposition. The total connection K turns out to be Galileian. $\square$

In the following we shall refer to the total connection K.

Clearly, the objects induced on the phase space by the total connection K split into their gravitational and electromagnetic components.

2 Quantum structure

We are concerned with quantum mechanics of a spinless charged particle on a curved spacetime fibred over absolute time and equipped with a given spacelike metric, gravitational field and electromagnetic field.

This chapter is devoted to our postulates on the quantum theory and to their first consequences. We just assume a quantum bundle, a quantum metric and a quantum connection. All further objects are derived from them by a covariant procedure.

2.1 Quantum bundle

We start with the quantum bundle over spacetime.

Q.1 Postulate. We assume the *quantum bundle* to be a 1–dimensional complex bundle over spacetime $\pi : \boldsymbol{Q} \to \boldsymbol{E}$. □

Each section $\Psi \in \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q})$ is called a *quantum section*.

Let us consider the trivial bundle $\mathcal{L}(\boldsymbol{Q}) = \boldsymbol{E} \times \left(\mathbb{R}^+ \times U(1)\right)$ over $\boldsymbol{E}$ of *complex linear automorphisms* $\lambda : \boldsymbol{Q} \to \boldsymbol{Q}$ over $\boldsymbol{E}$.

2.1 Lemma. For each $s \geq r \geq 0$, we obtain the natural fibred action

$$\nu_{s,r} : \mathcal{G}^s(\boldsymbol{E}) \underset{\boldsymbol{E}}{\rtimes} J_r\mathcal{L}(\boldsymbol{Q}) \to J_r\mathcal{L}(\boldsymbol{Q}) : \left(j_s\phi(e),\, j_r\lambda(e)\right) \mapsto j_r(\lambda \circ \phi)(e)\,. \,\square$$

2.2 Definition. For each $s \geq r \geq 0$, we define the *quantum covariance bundle*, of order (s, r), to be the bundle

$$\mathcal{W}^{(s,r)}(\boldsymbol{Q}) := \mathcal{G}^s(\boldsymbol{E}) \underset{\boldsymbol{E}}{\rtimes} J_r\mathcal{L}(\boldsymbol{Q}) \to \boldsymbol{E}\,. \,\square$$

2.3 Proposition. The bundle $\mathcal{W}^{(s,r)}(\boldsymbol{Q})$ turns out a group bundle through the composition fibred morphism over $\boldsymbol{E}$

$$\mu^{(s,r)} : \mathcal{W}^{(s,r)}(\boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} \mathcal{W}^{(s,r)}(\boldsymbol{Q}) \to \mathcal{W}^{(s,r)}(\boldsymbol{Q}) :$$
$$\left(\left(j_s\overset{1}{\phi}(e),\, j_r\overset{1}{\lambda}(e)\right),\, \left(j_s\overset{2}{\phi}(e),\, j_r\overset{2}{\lambda}(e)\right)\right) \mapsto \left(\left(j_s(\overset{1}{\phi} \circ \overset{2}{\phi})\right)(e)\,,\; j_r((\overset{2}{\lambda} \circ \overset{1}{\phi}) \cdot \overset{1}{\lambda})\right)(e)\right)\,. \,\square$$

For $s \geq r \geq 0$ and $s - k \geq r - h \geq 0$, we obtain a natural fibred group epimorphism $\mathcal{W}^{(s,r)}(\boldsymbol{Q}) \to \mathcal{W}^{(s-k,\,r-h)}(\boldsymbol{Q})$ over $\boldsymbol{E}$.

We denote the fibred chart of $\mathcal{W}^{(s,r)}(\boldsymbol{Q})$ induced by a spacetime chart (x^λ) by $(x^\lambda;\; \mathsf{g}^0_0, \mathsf{g}^i_\lambda,\; \ldots,\; \mathsf{g}^i_{\lambda_1\ldots\lambda_s};\; c,\; c_\lambda,\; c_{\lambda_1\ldots\lambda_r})$.

For instance, we obtain the following coordinate expressions

$$(x^\lambda;\, c,\, c_\lambda) \circ \nu_{1,1} = (x^\lambda;\, c,\, c_\mu\, \mathsf{g}^\mu_\lambda)$$
$$(x^\lambda;\, c,\, c_\lambda,\, c_{\lambda\mu}) \circ \nu_{2,2} = (x^\lambda;\, c,\, c_\mu\, \mathsf{g}^\mu_\lambda,\, c_{\rho\sigma}\, \mathsf{g}^\rho_\lambda\, \mathsf{g}^\sigma_\mu + c_\nu\, \mathsf{g}^\nu_{\lambda\mu})\,.$$

In view of the quantum connection, we consider also the following bundle.

2.4 Definition. The *extended quantum bundle* is defined to be the pullback bundle $\pi^\uparrow : \boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$ of the quantum bundle with respect to the map $J_1\boldsymbol{E} \to \boldsymbol{E}$. □

Here, the extended base space $J_1\boldsymbol{E}$ plays the role of space of observers.

2.2 Quantum metric

Next, we introduce the Hermitian metric on the quantum bundle. In view of the integration on the fibres of spacetime, this metric has values in the space of spacelike volume forms.

2.5 Definition. We define a *quantum metric* to be a fibred Hermitian metric with values in the complexified space of vertical volume forms $\mathsf{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{C} \otimes \Lambda^3 V^*\boldsymbol{E}$. □

2.6 Proposition. The quantum metrics are the sections of a trivial principal bundle $\mathrm{Met}(\boldsymbol{Q}) \to \boldsymbol{E}$, whose structure group is $\mathbb{R}^+$.
Moreover, we have a natural fibred action $\mathcal{W}^{(1,0)}(\boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{Q}) \to \mathrm{Met}(\boldsymbol{Q})$. □

Q.2 Postulate. We assume a quantum metric h. □

Clearly, $\mathsf{h}' := \mathsf{h}/\eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{*3} \otimes \mathbb{C}$ is a standard scaled Hermitian metric.

2.7 Definition. We define a *quantum basis*, to be a section $\mathfrak{b} \in \mathrm{sec}(\boldsymbol{E},\, \mathbb{L}^{3/2} \otimes \boldsymbol{Q})$ such that $\mathsf{h}(\mathfrak{b}, \mathfrak{b}) = \eta$.
If $\mathfrak{b}$ is a quantum basis, then an associated *quantum chart* is defined to be a complex linear fibred chart (x^λ, z), where (x^λ) is a spacetime chart and $z \in \mathrm{map}(\boldsymbol{Q},\, \mathbb{C} \otimes \mathbb{L}^{*3/2})$ is the complex dual of $\mathfrak{b}$. □

We shall refer to quantum bases and quantum charts.
Each quantum basis $\mathfrak{b}$ yields the *real quantum basis* $(\mathfrak{b}_1,\, \mathfrak{b}_2) := (\mathfrak{b},\, \mathrm{i}\,\mathfrak{b})$ and the *real quantum chart* $(w^1,\, w^2) = \Big(\frac{1}{2}\,(z + \bar{z}),\, \frac{1}{2}\,(\bar{z} - z)\Big)$.

If $\Psi \in \mathrm{sec}(\boldsymbol{E},\, \boldsymbol{Q})$, then we write $\Psi = \psi\mathfrak{b}$ and obtain $\mathsf{h}(\Phi,\, \Psi) = \bar{\phi}\,\psi\,\eta$.

Let us consider two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}}$ and the associated fibre coordinates z and $\acute{z}$. Then, we obtain $\acute{\mathfrak{b}} = e^{\mathrm{i}\,\vartheta}\,\mathfrak{b}$ and $\acute{z} = e^{-\mathrm{i}\,\vartheta}\,z$, with $\vartheta \in \mathrm{map}(\boldsymbol{E},\, \mathbb{R})$.

Hence, the quantum bundle turns out to be a bundle associated with a principal bundle, whose structure group is $U(1)$.

The quantum metric $\mathfrak{h}$ yields, by pullback, the *extended quantum metric*

$$\mathfrak{h}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} \boldsymbol{Q}^{\uparrow} \to \mathbb{C} \otimes \Lambda^3 V^*\boldsymbol{E}\,.$$

2.3 Quantum connection

Eventually, we complete the quantum framework by introducing the quantum connection. We introduce this notion by three steps.

In our formulation we use some notions (for instance, universal connections) which are non standard, hence deserve a clear analysis. Other notions are essentially standard, but our setting involves some delicate aspects which need an explicit mention.

2.3.1 Connections of the quantum bundle

First, we analyse the Hermitian connections of the quantum bundle.

2.8 Proposition. Let us refer to a quantum basis $\mathfrak{b}$ and to an adapted quantum chart (x^λ, z). The complex linear connections Υ of $\boldsymbol{Q}$ are of the type [46]

$$\Upsilon = \chi[\mathfrak{b}] + \mathfrak{q}[\mathfrak{b}] \otimes \mathbb{I} = d^\lambda \otimes \partial_\lambda + \mathfrak{q}_\lambda\, z\, d^\lambda \otimes \mathfrak{b}\,,$$

where $\chi[\mathfrak{b}]$ is the flat connection induced by $\mathfrak{b}$, $\mathbb{I} = z \otimes \mathfrak{b}$ is the Liouville vector field of $\boldsymbol{Q}$ and $\mathfrak{q}[\mathfrak{b}] \equiv \mathfrak{q}_\lambda\, d^\lambda \in \mathrm{sec}(\boldsymbol{E}, \mathbb{C} \otimes T^*\boldsymbol{E})$ is a complex spacetime form. □

2.9 Corollary. The complex linear connections of $\boldsymbol{Q}$ are the sections of a bundle $\mathrm{Con}(\boldsymbol{Q}) \to \boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields a local fibred isomorphism $\mathrm{Con}(\boldsymbol{Q}) \to \mathbb{C} \otimes T^*\boldsymbol{E}$ over $\boldsymbol{E}$. □

2.10 Definition. A complex linear connection Υ of $\boldsymbol{Q}$ is said to be *Hermitian* if $\nabla\mathfrak{h} = 0$, where ∇ is the covariant differential induced by Υ and K. □

Indeed, we are involved also with K because $\mathfrak{h}$ has values in $\mathbb{C} \otimes \Lambda^3 V^*\boldsymbol{E}$.

2.11 Proposition. Let us refer to a quantum basis $\mathfrak{b}$ and to an adapted quantum chart (x^λ, z). The Hermitian connections Υ of $\boldsymbol{Q}$ are of the type [46]

$$\Upsilon = \chi[\mathfrak{b}] + \mathfrak{i}\, A[\mathfrak{b}] \otimes \mathbb{I} = d^\lambda \otimes \partial_\lambda + \mathfrak{i}\, A_\lambda\, d^\lambda \otimes \mathbb{I}\,,$$

where $A[\mathfrak{b}] \equiv A_\lambda\, d^\lambda \in \mathrm{sec}(\boldsymbol{E}, T^*\boldsymbol{E})$ is a spacetime form, called the *potential* of Υ. □

2.12 Corollary. Let us consider a Hermitian connection Υ of $\boldsymbol{Q}$. Then, with reference to two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \exp(\mathfrak{i}\,\vartheta)\,\mathfrak{b}$, we obtain the transition map $A[\acute{\mathfrak{b}}] = A[\mathfrak{b}] - d\vartheta$.

Hence, with reference to adapted quantum charts (x^λ, z) and $(\acute{x}^\lambda, \acute{z})$, we obtain the transition maps $\acute{A}_i = (A_j - \partial_j\vartheta)\,\sigma^j_i$ and $\acute{A}_0 = (A_0 - \partial_0\vartheta)\,\sigma^0_0 + (A_j - \partial_j\vartheta)\,\sigma^j_0$. □

2.13 Corollary. The Hermitian connections of $\boldsymbol{Q}$ are the sections of a bundle $\mathrm{Her}\,(\boldsymbol{Q}) \to \boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields locally a fibred isomorphism $\mathrm{Her}\,(\boldsymbol{Q}) \to T^*\boldsymbol{E}$ over $\boldsymbol{E}$. □

Analogous notions and constructions hold for the connections of the extended quantum bundle.

2.3.2 Universal connections of the extended quantum bundle

Next, we introduce the notion of universal connection [41, 46].

2.14 Proposition. Let $\{\overset{o}{\Upsilon}\} \equiv \{\Upsilon[o]\}$ be a "system" of complex linear connections of the quantum bundle parametrised by the observers $o \in \mathrm{sec}(\boldsymbol{E},\, J_1\boldsymbol{E})$.

Then, there is a unique complex linear connection $\Upsilon^\uparrow$ of the extended quantum bundle, called *universal*, such that $\Upsilon[o] = o^*\Upsilon^\uparrow$, for each o.

Now, let us consider a quantum basis $\mathfrak{b}$ and denote by $\upsilon[\mathfrak{b}, o] \in \mathrm{sec}(\boldsymbol{E},\, \mathbb{C} \otimes T^*\boldsymbol{E})$ the complex form associated with the connection $\Upsilon[o]$. Then, there is a unique horizontal complex phase form $\upsilon^\uparrow[\mathfrak{b}] = \upsilon^\uparrow{}_\lambda\, d^\lambda \in \mathrm{fib}(J_1\boldsymbol{E},\, \mathbb{C} \otimes T^*\boldsymbol{E})$, such that $\upsilon[\mathfrak{b}, o] = \upsilon^\uparrow[\mathfrak{b}] \circ o$.

Hence, the expression of $\Upsilon^\uparrow$ is

$$\Upsilon^\uparrow = \chi^\uparrow[\mathfrak{b}] + \upsilon^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \upsilon^\uparrow{}_\lambda\, d^\lambda \otimes \mathbb{I}^\uparrow\,,$$

where $\chi^\uparrow[\mathfrak{b}]$ is the flat Hermitian connection induced by $\mathfrak{b}$ and $\mathbb{I}^\uparrow = z \otimes \mathfrak{b}$ is the Liouville vector field of $\boldsymbol{Q}^\uparrow$.

Conversely, each complex linear connection $\Upsilon^\uparrow$ of $\boldsymbol{Q}^\uparrow$ of the above type yields a system of complex linear connections of $\boldsymbol{Q}$, whose universal connection is $\Upsilon^\uparrow$. □

2.15 Proposition. If $\{\Upsilon[o]\}$ is a system of complex linear connections of the quantum bundle, then we obtain $R\Big[\Upsilon[o]\Big] = o^*R[\Upsilon^\uparrow]$, for each observer o. □

2.16 Corollary. The universal connections of $\boldsymbol{Q}^\uparrow$ are the sections of a bundle $\mathrm{Uni}(\boldsymbol{Q}^\uparrow) \to J_1\boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields a local fibred isomorphism $\mathrm{Uni}(\boldsymbol{Q}^\uparrow) \to \mathbb{C} \otimes (J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E})$ over $J_1\boldsymbol{E}$. □

2.17 Definition. A complex linear connection $\Upsilon^\uparrow$ of $\boldsymbol{Q}^\uparrow$ is said to be *Hermitian* if $\nabla\mathfrak{h}^\uparrow = 0$, where ∇ is the covariant differential induced by $\Upsilon^\uparrow$ and K. □

2.18 Corollary. The Hermitian connections of $\boldsymbol{Q}^\uparrow$ are the sections of a bundle $\mathrm{Her}\,(\boldsymbol{Q}^\uparrow) \to J_1\boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields a local fibred isomorphism $\mathrm{Her}\,(\boldsymbol{Q}^\uparrow) \to T^*J_1\boldsymbol{E}$ over $J_1\boldsymbol{E}$. □

2.19 Proposition. Let $\{\Upsilon[o]\}$ be a system of complex linear connections of $\boldsymbol{Q}$ and $\Upsilon^\uparrow$ its universal connection. Then, the connections $\{\Upsilon[o]\}$ are Hermitian if and only if $\Upsilon^\uparrow$ is Hermitian. □

2.20 Corollary. Let us refer to a quantum basis $\mathfrak{b}$ and to an adapted quantum chart (x^λ, z). Then, the Hermitian universal connections $\Upsilon^\uparrow$ of $\boldsymbol{Q}^\uparrow$ are of the type [46]

$$\Upsilon^\uparrow = \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\, A^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}\, A^\uparrow{}_\lambda\, d^\lambda \otimes \mathbb{I}^\uparrow\,,$$

where $A^\uparrow[\mathfrak{b}] \equiv A^\uparrow{}_\lambda\, d^\lambda \in \mathrm{sec}(J_1\boldsymbol{E},\, T^*\boldsymbol{E})$ is a horizontal phase form. □

2.21 Corollary. Let us consider a universal Hermitian connection $\Upsilon^\uparrow$ of $\boldsymbol{Q}^\uparrow$. Then, two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \exp(\mathfrak{i}\,\vartheta)\,\mathfrak{b}$ yield the transition map $A^\uparrow[\acute{\mathfrak{b}}] = A^\uparrow[\mathfrak{b}] - d\vartheta$. Hence, with reference to adapted quantum charts (x^λ, z) and $(\acute{x}^\lambda, \acute{z})$, we obtain the transition maps $\acute{A}^\uparrow{}_i = (A^\uparrow{}_j - \partial_j\vartheta)\,\sigma^j_i$ and $\acute{A}^\uparrow{}_0 = (A^\uparrow{}_0 - \partial_0\vartheta)\,\sigma^0_0 + (A^\uparrow{}_j - \partial_j\vartheta)\,\sigma^j_0$. □

2.22 Corollary. The universal Hermitian connections of $\boldsymbol{Q}^\uparrow$ are the sections of the bundle $\mathrm{Uni}(\boldsymbol{Q}^\uparrow) \cap \mathrm{Her}\,(\boldsymbol{Q}^\uparrow) \to J_1\boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields a local fibred isomorphism $\mathrm{Uni}(\boldsymbol{Q}^\uparrow) \cap \mathrm{Her}\,(\boldsymbol{Q}^\uparrow) \to J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E}$ over $J_1\boldsymbol{E}$. □

2.3.3 Quantum connections

Eventually, we introduce the notion of quantum connection.

Now, we take into account the vertical metric G the total connection K and the induced cosymplectic 2–form Ω.

2.23 Definition. We define [46, 49] a *quantum connection* to be a connection $\Upsilon^\uparrow$ of the extended quantum bundle which is universal, Hermitian and whose curvature is

$$R[\Upsilon^\uparrow] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^\uparrow\,. \,\square$$

We recall that Ω includes $m/\hbar$ through the re–scaled metric G and that it includes both the gravitational and electromagnetic fields through the total connection K.

2.24 Proposition. Let $\{\Upsilon[o]\}$ be a system of Hermitian connections and $\Upsilon^\uparrow$ its universal connection. Then, the following conditions are equivalent:

a) $R[\Upsilon^\uparrow] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^\uparrow$;

b) $R\Big[\Upsilon[o]\Big] = -2\,\mathfrak{i}\,o^*\Omega \otimes \mathbb{I}$, for each observer o. □

2.25 Proposition. A quantum connection $\Upsilon^\uparrow$ exists locally. Indeed, with reference to a quantum basis $\mathfrak{b}$, its local expression is of the type

$$\Upsilon^\uparrow = \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\, A^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}\, A^\uparrow{}_\lambda\, d^\lambda \otimes \mathbb{I}^\uparrow\,, \quad \text{with} \quad dA^\uparrow[\mathfrak{b}] = \Omega\,.$$

Hence, for each observer o, if we set $A[\mathfrak{b}, o] := o^* A^\uparrow[\mathfrak{b}] = A_\lambda\, d^\lambda$, then we obtain

$$A^\uparrow[\mathfrak{b}] \;\equiv\; A^\uparrow{}_\lambda\, d^\lambda = -(\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i\,, \quad \text{with} \quad dA[\mathfrak{b}, o] = \Phi[o]\,.$$

2.26 Corollary. If $\Upsilon^\uparrow$ is a quantum connection, then, for each quantum basis $\mathfrak{b}$, $A^\uparrow[\mathfrak{b}]$ turns out to be a horizontal potential of Ω, whose gauge is determined by $\mathfrak{b}$. Moreover, for each observer o, $A[\mathfrak{b}, o]$ turns out to be a potential of K. □

2.27 Corollary. Let us consider a quantum connection $\Upsilon^\uparrow$, two quantum bases $\mathfrak{b}$, $\acute{\mathfrak{b}} = \exp(\mathfrak{i}\,\vartheta)\,\mathfrak{b}$ and two observers o, $\acute{o} = o + v$. Then, we obtain the transition map

$$A[\acute{\mathfrak{b}}, \acute{o}] = A[\mathfrak{b}, o] - \tfrac{1}{2}\, G(v, v) + \nu[o] \lrcorner\, G^\flat(v) - d\theta\,.$$

Thus, each connection of the system determines all others by the above formula. □

2.28 Corollary. The transition maps of Corollary 2.27 well define a fibred group action $\mathfrak{T}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \big(J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathrm{Her}\,(\boldsymbol{Q})\big) \;\to\; J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathrm{Her}\,(\boldsymbol{Q})$ over $\boldsymbol{E}$. □

2.29 Corollary. The quantum connections are the sections of a bundle $\mathrm{Con}(\boldsymbol{Q}^\uparrow)$ *over spacetime*. We have a natural fibred group action $\mathcal{W}^{(1,1)}(\boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} \mathrm{Con}(\boldsymbol{Q}^\uparrow) \to \mathrm{Con}(\boldsymbol{Q}^\uparrow)$. □

2.30 Note. The condition $R[\Upsilon^\uparrow] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^\uparrow$ has reduced the base space of the bundle of universal Hermitian connections from the phase space to spacetime. □

2.31 Corollary. For each quantum basis $\mathfrak{b}$ and observer o, we have the fibred isomorphism $\mathrm{Con}(\boldsymbol{Q}^\uparrow) \to T^*\boldsymbol{E} : \Upsilon^\uparrow \mapsto A[\mathfrak{b}, o]$. □

Thus, given a quantum connection, the choice of a quantum basis $\mathfrak{b}$ turns out to be another way to control the gauge of classical potentials.

2.32 Proposition. Let $\Upsilon^\uparrow$ be a quantum connection and $\mathfrak{b}$ a quantum basis. Then, there is a unique observer $o[\mathfrak{b}]$, such that the vertical restriction $\overset{\vee}{A}\big[\mathfrak{b}, o[\mathfrak{b}]\big]$ vanishes. Indeed, if o is any observer, then we obtain $o[\mathfrak{b}] = o - G^\sharp(A[\mathfrak{b}, o])$. □

2.33 Corollary. Let $\Upsilon^\uparrow$ be a quantum connection and Ψ a quantum section. Then, in the domain where $\Psi \neq 0$, we obtain the distinguished observer $o[\Psi] := o[\mathfrak{b}]$, where $\mathfrak{b} := \Psi/\sqrt{\mathfrak{h}'(\Psi, \Psi)}$. □

2.34 Theorem. [99] *A quantum connection* $\Psi^{\uparrow}$ *exists globally if and only if the cohomology class of* Ω *is integer.* □

Q.3 Postulate. We suppose that the quantum bundle admits quantum connections and assume a quantum connection $\Psi^{\uparrow}$. □

2.35 Note. The equality $d\,\Omega = 0$, which is a consequence of our assumption that K be Galileian, turns out to be an integrability condition for the equation $R[\Psi^{\uparrow}] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^{\uparrow}$, in virtue of the Bianchi identity for $R[\Psi^{\uparrow}]$.

On the other hand, if we would have just assumed that K be a metric spacetime connection, then the assumption of a quantum connection would imply that K be Galileian. □

2.3.4 Quantum differentials

The quantum connection $\Psi^{\uparrow}$, the Galileian connection K and the spacelike metric G yield distinguished differential operators.

2.36 Proposition. Given $\Psi \in \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q})$ and $o \in \mathrm{sec}(\boldsymbol{E}, J_1\boldsymbol{E})$, we obtain the following objects:

- the *quantum differential* related to $\Psi^{\uparrow}$

$$\nabla\Psi = (\nabla_\lambda \psi)\, d^\lambda \otimes \mathfrak{b}\,, \quad \text{with} \quad \nabla_\lambda \psi := (\partial_\lambda - \mathfrak{i}\, A^{\uparrow}{}_\lambda)\, \psi\,;$$

- the *observed quantum differential* related to $\Psi^{\uparrow}$ and o

$$\overset{o}{\nabla}\Psi = (\overset{o}{\nabla}_\lambda \psi)\, d^\lambda \otimes \mathfrak{b}\,, \quad \text{with} \quad \overset{o}{\nabla}_\lambda \psi := (\partial_\lambda - \mathfrak{i}\, A_\lambda)\, \psi\,;$$

- the *time–like quantum differential* related to $\Psi^{\uparrow}$

$$\widehat{\nabla}\Psi := \text{д} \lrcorner \nabla\Psi = \Big(\partial_0 + x^j_0\, \partial_j - \mathfrak{i}\, \mathcal{L}_0\Big[A^{\uparrow}[\mathfrak{b}]\Big]\Big)\, \psi\, u^0 \otimes \mathfrak{b}\,;$$

- the *space–like quantum differential* related to $\Psi^{\uparrow}$

$$\overset{\vee}{\nabla}\Psi = \Big(\partial_j - \mathfrak{i}\, \mathcal{P}_j\Big[A^{\uparrow}[\mathfrak{b}]\Big]\Big)\, \psi\, \check{d}^j \otimes \mathfrak{b}\,;$$

- the *observed 2nd quantum differential* related to $\Psi^{\uparrow}$, K and o

$$\overset{o}{\nabla}\overset{o}{\nabla}\Psi = \overset{o}{\nabla}_{\lambda\mu} \psi\, d^\lambda \otimes d^\mu \otimes \mathfrak{b}\,, \quad \text{with} \quad \overset{o}{\nabla}_{\lambda\mu}\psi = (\overset{o}{\nabla}_\lambda \overset{o}{\nabla}_\mu + K_\lambda{}^j{}_\mu\, \overset{o}{\nabla}_j)\, \psi\,;$$

- the *observed quantum Laplacian* related to G, $\Psi^{\uparrow}$, K and o

$$\overset{o}{\Delta}\Psi := \langle \bar{G},\ \overset{o}{\nabla}\overset{o}{\nabla}\Psi\rangle = (\overset{o}{\Delta}_0\, \psi)\, u^0 \otimes \mathfrak{b}\,, \quad \text{with} \quad \overset{o}{\Delta}_0\psi = G^{hk}_0\, (\overset{o}{\nabla}_h \overset{o}{\nabla}_k + \varkappa_h{}^j{}_k\, \overset{o}{\nabla}_j)\, \psi\,,$$

where we refer to a quantum basis $\mathfrak{b}$, to an observer o and to an adapted chart (x^λ, z). □

Analogously, we obtain the quantum differentials and quantum Laplacians of sections of the real dual quantum bundle $\boldsymbol{Q}^* \to \boldsymbol{E}$.

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we have $(\mathrm{re}\,\mathfrak{h})^\flat(\overset{o}{\Delta}\Psi) = \overset{o}{\Delta}\Big((\mathrm{re}\,\mathfrak{h})^\flat\Psi\Big)$.

Later, we shall be involved with the following technical result.

2.37 Lemma. Given two observers o and $ó = o + v$, we obtain

$$\begin{aligned}
\overset{ó}{\nabla} &= \overset{o}{\nabla} + \mathfrak{i}\,\tfrac{1}{2}\,G\,(v,v) - \mathfrak{i}\,\nu^*[o]\,(G^\flat(v)) \\
ó \lrcorner \overset{ó}{\nabla} &= o \lrcorner \overset{o}{\nabla} - \mathfrak{i}\,\tfrac{1}{2}\,G(v,v) + v \lrcorner \overset{o}{\nabla} \\
\overset{ó}{\Delta} &= \overset{o}{\Delta} - 2\,\mathfrak{i}\,v \lrcorner \overset{o}{\nabla} - \mathfrak{i}\,(C_1^1\,\nabla v) - G(v,v)\,.\,\square
\end{aligned}$$

3 Quantum dynamics

This chapter is devoted to the derivation of the Schrödinger operator and the related stuff from the quantum metric and the quantum connection.

We follow two approaches via covariant differentials and covariant Lagrangian.

In both cases, we use a projectability criterion as heuristic method. Namely, we start from objets living on the classical phase space, because this is the seat of the quantum connection; then, we look for the combinations of the above objects which project on spacetime. This procedure means that we get rid of observers. Indeed, this turns out to be an implementation of the principle of general relativity.

3.1 Differential approach

The quantum connection and the method of projectability yield good candidates for the quantum momentum, the Schrödinger operator and the probability current.

3.1 Proposition. For each $\Psi \in \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q})$, we have the two distinguished maps leaving on the classical phase space:

$$д \otimes \Psi = \psi\, u^0 \otimes (\partial_0 + x^i_0\, \partial_i) \otimes \mathfrak{b} \tag{1}$$

$$G^\sharp(\nabla\Psi) = \Big(G^{ij}_0\, \partial_j - \mathfrak{i}\,(x^i_0 + A^i_0)\Big)\, \psi\, u^0 \otimes \partial_i \otimes \mathfrak{b}\,. \tag{2}$$

Then, we obtain the section on spacetime

$$\mathsf{P}[\Psi] := д \otimes \Psi - \mathfrak{i}\, G^\sharp(\nabla\Psi) = u^0 \otimes \Big(\psi\, \partial_0 - \mathfrak{i}\, G^{ij}_0\, (\partial_j - \mathfrak{i}\, A_j)\, \psi\, \partial_i\Big) \otimes \mathfrak{b}\,.$$

Thus, we obtain, in a covariant way, a global, gauge independent and observer independent 1st order operator $\mathsf{P} : J_1\boldsymbol{Q} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q})$, called *quantum momentum*. □

3.2 Note. If $\Psi \neq 0$, then we obtain the section

$$W[\Psi] := \frac{\mathsf{P}[\Psi]}{\Psi} = \frac{\mathfrak{h}(\Psi,\, \mathsf{P}[\Psi])}{\mathfrak{h}(\Psi,\, \Psi)} = o[\Psi] + \frac{G^\sharp(\overset{\vee}{\overset{o}{\nabla}}\Psi)}{\Psi} = u^0 \otimes (\partial_0 - A^i_0\, \partial_i) - \mathfrak{i}\, \frac{G^{ij}_0\, \partial_j\psi}{\psi}\, \partial_i\,.$$

The real component $\mathrm{re}\, W[\Psi] = u^0 \otimes (\partial_0 - A^i_0\, \partial_i) + \mathfrak{i}\, \frac{1}{2}\, G^{ij}_0\, \frac{\psi\, \partial_j\bar\psi - \bar\psi\, \partial_j\psi}{\bar\psi\, \psi}\, u^0 \otimes \partial_i$ turns out to be a distinguished observer associated with Ψ (see also Corollary 2.33). □

3.3 Proposition. For each quantum section Ψ, we have the two distinguished maps

leaving on the classical phase space:

$$\text{д} \lrcorner \nabla\Psi = (\partial_0\psi + x_0^h\,\partial_h\psi - \mathfrak{i}\,\mathcal{L}_0\big[A^\uparrow[\mathfrak{b}]\big])\,\psi\,u^0\otimes\mathfrak{b} \tag{1}$$

$$\delta\mathsf{P}[\Psi] = \Big(\partial_0 - 2\,\mathfrak{i}\,A_0 + \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \mathfrak{i}\,\overset{o}{\Delta} - x_0^h\,\partial_h + \mathfrak{i}\,\mathcal{L}_0\big[A^\uparrow[\mathfrak{b}]\big]\Big)\,\psi\otimes u^0\otimes\mathfrak{b}\,, \tag{2}$$

where δ denotes the quantum codifferential. Then, we obtain the section on spacetime

$$\mathsf{S}[\Psi] := \tfrac{1}{2}\,(\text{д} \lrcorner \nabla\Psi + \delta\mathsf{P}[\Psi]) = \Big(\overset{o}{\nabla}_0 + \tfrac{1}{2}\,(\mathrm{div}_\eta\, o)_0 - \mathfrak{i}\,\tfrac{1}{2}\,\overset{o}{\Delta}_0\Big)\,\psi\,u^0\otimes\mathfrak{b}\,,$$

where $\mathrm{div}_\eta\, o = \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,u^0$.

Thus, we obtain, in a covariant way, a global, gauge independent and observer independent 2nd order operator $\mathsf{S} : J_2\boldsymbol{Q} \to \mathbb{T}^*\otimes\boldsymbol{Q}$, called *Schrödinger operator*. □

3.4 Note. We can write $\mathsf{S}[\Psi] = (\mathrm{re}\,\mathfrak{h})^\sharp\Big(\overset{o}{\delta} - \frac{1}{2}\,\mathfrak{i}\,\overset{o}{\Delta}\Big)\,(\mathrm{re}\,\mathfrak{h})^\flat(\Psi)$, where $\overset{o}{\delta}$ denotes the Lie derivative with respect to the quantum prolongation of the observer o. Clearly, $\overset{o}{\delta}$ and $\overset{o}{\Delta}$ depend on o, but the above combination turns out to be observer independent. □

3.5 Note. Given any observer o, the Schrödinger operator can be written as

$$\mathsf{S}[\Psi] = \tfrac{1}{2}\,(\text{д} \lrcorner \nabla\Psi + \delta\mathsf{P}[\Psi]) \circ o\,,$$

where $(\text{д} \lrcorner \nabla\Psi) \circ o = o \lrcorner \overset{o}{\nabla}\Psi$ and $(\delta\mathsf{P}[\Psi]) \circ o = -\mathfrak{i}\,\overset{o}{\Delta}\Psi + o \lrcorner \overset{o}{\nabla}\Psi + C_1^1\,(\nabla o)\,\Psi$.

Clearly, $(\text{д} \lrcorner \nabla\Psi) \circ o$ and $(\delta\mathsf{P}[\Psi]) \circ o$ depend on o, but the above combination turns out to be observer independent. □

We recall that the observed quantum Laplacian includes $m/\hbar$ through the re–scaled metric G. Moreover, we recall that the quantum potential $A[\mathfrak{b}, o]$ includes both the gravitational and electromagnetic potentials.

Clearly, in the flat case, S reduces to the standard Schrödinger operator.

3.6 Proposition. The scalar curvature of G yields, in a covariant way, the time scale $r : \boldsymbol{E} \to \bar{\mathbb{T}}^*$. Hence, for each $k \in \mathbb{R}$, we obtain the extended map

$$\mathsf{S}_{(k)}[\Psi] := \mathsf{S}[\Psi] - \mathfrak{i}\,\tfrac{1}{2}\,k\,r\otimes\Psi\,.$$

Thus, we obtain, in a covariant way, a global, gauge independent and observer independent 2nd order operator $\mathsf{S}_{(k)} : J_2\boldsymbol{Q} \to \mathbb{T}^*\otimes\boldsymbol{Q}$, called *extended Schrödinger operator*. □

3.7 Note. We have considered an imaginary coefficient $-\mathfrak{i}\,\frac{1}{2}\,k$ in order to obtain later a Hermitian connection on the quantum Hilbert bundle from the extended Schrödinger operator. There is no way to determine k by means of covariance arguments.

We stress that S involves only the 1st derivatives of the metric, while $\mathsf{S}_{(k)}$ involves the 2nd derivatives of the metric. The above additional term arises also in the Feynmann integral approach to the Schrödinger operator. It seems that in this context the coefficient k could be determined. However, there is no definite agreement on its value. On the other hand, if the scalar curvature is constant, then the additional term produces an overall shift in the energy spectrum, which can be hardly detected.

Clearly, in the flat case the additional term $-\mathfrak{i}\,\frac{1}{2}\,k\,r\otimes\Psi$ vanishes. □

3.8 Proposition. For each quantum section Ψ, we obtain the scaled vector field

$$\begin{aligned}\mathsf{j}[\Psi] &:= \tfrac{1}{2}\Big(\mathsf{h}'\Big(\Psi,\mathsf{P}[\Psi]\Big)+\mathsf{h}'\Big(\mathsf{P}[\Psi],\Psi\Big)\Big)\\ &= (\bar\psi\,\psi)\,u^0\otimes\partial_0+G_0^{hk}\Big(\mathfrak{i}\,\tfrac{1}{2}\,(\psi\,\partial_h\bar\psi-\bar\psi\,\partial_h\psi)-A_h\,\bar\psi\,\psi)\Big)\,u^0\otimes\partial_k\,.\end{aligned}$$

Thus, we obtain, in a covariant way, a global, gauge independent and observer independent 1st order operator $\mathsf{j}: J_1\boldsymbol{Q}\to(\mathbb{T}^*\otimes\mathbb{L}^{*3})\otimes T\boldsymbol{E}$, called *probability current*. □

3.9 Proposition. For each quantum section Ψ and for each $k\in\mathbb{R}$, we obtain $\delta\mathsf{j}[\Psi]=\mathsf{h}'\Big(\Psi,\mathsf{S}_{(k)}[\Psi]\Big)+\mathsf{h}'\Big(\mathsf{S}_{(k)}[\Psi],\Psi\Big)$, where δ denotes the codifferential of $\mathsf{j}[\Psi]$.

Hence, the quantum probability current is conserved along the solutions of the extended Schrödinger equation. □

3.2 Lagrangian approach

The quantum connection and the method of projectability yield good candidates for the quantum Lagrangian, hence for the derived quantum momentum, Schrödinger operator and probability current.

3.10 Proposition. For each $\Psi\in\mathrm{sec}(\boldsymbol{E},\boldsymbol{Q})$, we have the two distinguished maps leaving on the classical phase space:

$$\widehat{\mathsf{L}}[\Psi] := dt\wedge\Big(\mathsf{h}(\widehat\nabla\Psi,\Psi)-\mathsf{h}(\Psi,\widehat\nabla\Psi)\Big)= \tag{1}$$

$$=\Big((\psi\,\partial_0\bar\psi-\bar\psi\,\partial_0\psi)+2\,\mathfrak{i}\,A_0\,\bar\psi\,\psi-x_0^i\,(\bar\psi\,\partial_i\psi-\psi\,\partial_i\bar\psi)+2\,\mathfrak{i}\,(\tfrac{1}{2}\,G^0_{ij}\,x_0^i\,x_0^j+x_0^i\,A_i)\,\bar\psi\,\psi\Big)\,\upsilon^0\,,$$

$$\overset{\vee}{\mathsf{L}}[\Psi] := \mathfrak{i}\,dt\wedge\Big((\bar G\otimes\mathsf{h})(\overset{\vee}{\nabla}\Psi,\overset{\vee}{\nabla}\Psi)\Big)= \tag{2}$$

$$\begin{aligned}&=\Big(\mathfrak{i}\,G_0^{ij}\,(\partial_i\bar\psi\,\partial_j\psi+A_i\,A_j\,\bar\psi\,\psi)-A_0^i\,(\bar\psi\,\partial_i\psi-\psi\,\partial_i\bar\psi)\\ &-x_0^i\,(\bar\psi\,\partial_i\psi-\psi\,\partial_i\bar\psi)+\mathfrak{i}\,(G^0_{ij}\,x_0^i\,x_0^j+2\,x_0^i\,A_i)\,\bar\psi\,\psi\Big)\,\upsilon^0\,.\end{aligned}$$

Then, we obtain the section on spacetime

$$\begin{aligned}\mathsf{L}[\Psi] &:= \tfrac{1}{4}\,dt\wedge\Big(\mathsf{h}(\Psi,\widehat\nabla\Psi)-\mathsf{h}(\widehat\nabla\Psi,\Psi)-\mathfrak{i}\,(\bar G\otimes\mathsf{h})(\overset{\vee}{\nabla}\Psi,\overset{\vee}{\nabla}\Psi)\Big)\\ &=\tfrac{1}{4}\Big((\psi\,\partial_0\bar\psi-\bar\psi\,\partial_0\psi)+2\,\mathfrak{i}\,A_0\,\bar\psi\,\psi-\mathfrak{i}\,G_0^{ij}\,(\partial_i\bar\psi\,\partial_j\psi+A_i\,A_j\,\bar\psi\,\psi)+A_0^i\,(\bar\psi\,\partial_i\psi-\psi\,\partial_i\bar\psi)\Big)\,\upsilon^0\,.\end{aligned}$$

Thus, we obtain, in a covariant way, a global, gauge independent and observer independent 1st order imaginary operator $\mathsf{L} : J_1\boldsymbol{Q} \to \mathfrak{i}\,\Lambda^4 T\boldsymbol{E}$, called *quantum Lagrangian*. □

The normalising coefficient $\frac{1}{4}$ has been chosen in view of next developments.

3.11 Proposition. The scalar curvature of G yields, in a covariant way, the time scale $r : \boldsymbol{E} \to \bar{\mathbb{T}}^*$. Hence, for each $k \in \mathbb{R}$ we obtain the extended map

$$\mathsf{L}_{(k)}[\Psi] := \mathsf{L}[\Psi] + \mathfrak{i}\,k\,r \wedge \mathfrak{h}(\Psi, \Psi)\,.$$

Thus, we obtain, in a covariant way, a global, gauge independent and observer independent 1st order operator, called *extended quantum Lagrangian*, $\mathsf{L}_{(k)} : J_1\boldsymbol{Q} \to \mathfrak{i}\,\Lambda^4 T^*\boldsymbol{E}$. □

3.12 Corollary. For each $k \in \mathbb{R}$, the momentum associated with the extended quantum Lagrangian turns out to be given by $(\mathrm{re}\,\mathfrak{h})^\sharp(V_0\mathsf{L}) = (\mathrm{re}\,\mathfrak{h})^\sharp(V_0\mathsf{L}_{(k)}) = \frac{1}{2}\,\mathsf{P}$. □

3.13 Corollary. The Euler–Lagrange operator associated with the extended quantum Lagrangian turns out to be given by $\mathsf{S}_{(k)} = (\mathrm{re}\,\mathfrak{h})^\sharp\big(\mathsf{E}[\mathsf{L}_{(k)}]\big)$. □

3.14 Corollary. The Poincaré–Cartan form associated with the extended quantum Lagrangian turns out to be the map

$$\begin{gathered}\Theta[\mathsf{L}_{(k)}] := \mathsf{L}_{(k)} + \theta^*(V_0\mathsf{L}_{(k)}) \\ = \Big(\tfrac{1}{8}\,G_0^{hk}\,(z_h\,z_k - \bar{z}_h\,\bar{z}_k) + (A_0 - \tfrac{1}{2}\,A_0^h\,A_h + \tfrac{1}{2}\,k\,r_0)\,z\,\bar{z}\Big) \wedge \upsilon^0 \\ +\tfrac{1}{2}\,(z\,d\bar{z} - \bar{z}\,dz) \wedge \upsilon_0^0 - \tfrac{1}{2}\,G_0^{hk}\,\Big(\mathfrak{i}\,(z_h\,d\bar{z} + \bar{z}_h\,dz) + A_h\,(z\,d\bar{z} - \bar{z}\,dz)\Big) \wedge \upsilon_k^0\,.\ \square\end{gathered}$$

3.15 Proposition. The extended quantum Lagrangian is equivariant with respect to the action of the group $U(1)$. Hence, the vertical vector field $\mathfrak{i}\,\mathbb{I} = \mathfrak{i}\,z\,\mathfrak{b}$ is an infinitesimal symmetry of the extended quantum Lagrangian. □

3.16 Corollary. For each $k \in \mathbb{R}$, the current

$$i(\mathfrak{i}\,\mathbb{I})\Theta[\mathsf{L}_{(k)}] = i(\mathfrak{i}\,\mathbb{I})\Theta[\mathsf{L}] = (z\,\bar{z})\,\upsilon_0^0 + G_0^{hk}\,\Big(\mathfrak{i}\,\tfrac{1}{2}\,(z\,\bar{z}_h - \bar{z}\,z_h) - A_h\,z\,\bar{z}\Big)\,\upsilon_k^0$$

is conserved along the solutions of the extended Schrödinger equation. Actually, we obtain $i(\mathfrak{i}\,\mathbb{I})\Theta[\mathsf{L}] = i(\mathfrak{j})\,\upsilon$. □

3.3 Uniqueness by covariance

We have found a Schrödinger operator and a quantum Lagrangian by a covariant geometric procedure and we have shown that the Schrödinger operator can be derived from the quantum Lagrangian. Then, the following natural questions arise:

- are there other possible covariant Schrödinger operators?
- are there other possible covariant quantum Lagrangians?
- are all covariant Schrödinger operators derived from a quantum Lagrangian?

We answer these questions by proving that, under reasonable weak conditions concerning the order of derivatives, our Schrödinger operator and our quantum Lagrangian are essentially unique, provided they are covariant with respect to the quantum covariance group and to the change of time scale.

3.3.1 Covariant operators

First of all, we recall the notion of covariance and formulate our general problem.

Let us consider two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p' : \boldsymbol{F}' \to \boldsymbol{B}$ equipped with fibred actions $\boldsymbol{G} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{F}$ and $\boldsymbol{G} \underset{\boldsymbol{B}}{\times} \boldsymbol{F}' \to \boldsymbol{F}'$ of a group bundle $q : \boldsymbol{G} \to \boldsymbol{B}$.

Moreover, let us consider a subsheaf $\widetilde{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{F}) \subset \mathrm{sec}(\boldsymbol{B}, \boldsymbol{F})$, which is invariant with respect to the action of $\boldsymbol{G}$.

3.17 Definition. We define an *r–order covariant fibred morphism* to be a fibred morphism $\phi : J_r\boldsymbol{F} \to \boldsymbol{F}'$, which is equivariant with respect to the fibred action of the group bundle $\boldsymbol{G} \to \boldsymbol{B}$. Moreover, we define an *r–order covariant operator* to be a sheaf morphism $\mathsf{O} : \widetilde{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{F}) \to \mathrm{sec}(\boldsymbol{B}, \boldsymbol{F}')$, which factorises as $\mathsf{O} = \phi \circ j_k$, where ϕ is an r–order covariant fibred morphism. □

In the previous sections, we have assumed a spacelike metric, a gravitational field, an electromagnetic field, a quantum metric and a quantum connection. Now, in order to prove the above uniqueness theorems, we need to consider the above objects as variables.

So, here we assume the spacetime fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ and the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ and consider:

- the bundle $\mathrm{Met}(\boldsymbol{E})$ of re–scaled spacelike metrics G,
- the bundle $\mathrm{Con}(\boldsymbol{E})$ of spacetime connections K,
- the bundle $\mathrm{Met}(\boldsymbol{Q})$ of quantum metrics $\mathfrak{h}$,
- the bundle $\mathrm{Uni}(\boldsymbol{Q}^{\uparrow})$ of universal connections $\Psi^{\uparrow}$.

We could define the covariant operators dealing directly with the universal connection $\Psi^{\uparrow}$, but we would meet cumbersome constructions, because this connection lives on the classical phase space. Then, it is more convenient to deal with the system of connections $\{\overset{o}{\Psi}\}$ and their transitions maps, as they live on spacetime.

Therefore, we introduce the following preliminary notions.

3.18 Definition. We define the sheaf of *observed fundamental fields* to be the sheaf $\overset{o}{\mathrm{fields}}(\boldsymbol{Q}) \subset \mathrm{sec}\Big(\boldsymbol{E},\ J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathrm{Con}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} \mathrm{Con}(\boldsymbol{Q})\Big)$, which is constituted by all $(o\,,\, G\,,\, K\,,\, \mathfrak{h}\,,\, \Psi)$ fulfilling the *fundamental identities*

$$\nabla[K]\, G = 0\,, \quad d\,\Omega[G, K] = 0\,, \quad \nabla[\Psi]\, \mathfrak{h} = 0\,, \quad R[\Psi] = -2\,\mathfrak{i}\, o^*\Omega[G, K] \otimes \mathbb{I}\,. \,\square$$

Moreover, for each integer $0 \leq r$, we set

$$\overset{o}{\text{fields}}_r(\boldsymbol{Q}) := \left\{(j_{r-1}o\,,\, j_rG\,,\, j_{r-1}K\,,\, j_r\mathfrak{h}\,,\, j_{r-1}\Upsilon) \quad | \quad (o\,,\, G\,,\, K\,,\, \mathfrak{h}\,,\, \Upsilon) \in \overset{o}{\text{fields}}(\boldsymbol{Q})\right\}$$

and denote the corresponding jet prolongation by $j_r : \overset{o}{\text{fields}}(\boldsymbol{Q}) \to \overset{o}{\text{fields}}_r(\boldsymbol{Q})$.

Now, let us consider a fibred manifold $\boldsymbol{F} \to \boldsymbol{E}$, with a fibred action of the group bundle $\mathcal{W}^{(1,0)}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$, and a sheaf morphism $\mathfrak{O} : \overset{o}{\text{fields}}(\boldsymbol{Q}) \times \text{sec}(\boldsymbol{E},\, \boldsymbol{Q}) \to \text{sec}(\boldsymbol{E},\, \boldsymbol{F})$.

3.19 Definition. We say that $\mathfrak{O}$ is:
- of *order* r, if it factorises as $\mathfrak{O} = \mathfrak{O}_r \circ j_r$ through the r–jet prolongation of sections;
- *covariant* if it is invariant with respect to the fibred action of the transition group bundle $\mathcal{T}(\boldsymbol{E})$ and, for each observer o, the induced sheaf morphism $\overset{o}{\mathfrak{O}} := \mathfrak{O}\,(o\,,\,\cdot\,)$ is covariant with respect to the fibred action of the group bundles $\mathcal{W}^{(s,r)}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$. □

Hence, if $\mathfrak{O}$ is covariant and $(G\,,\, K\,,\, \mathfrak{h}\,,\, \Upsilon^{\uparrow})$ are the postulated fundamental fields, then the sheaf morphism $\mathsf{O} := \mathfrak{O}\,(o\,, G\,,\, K\,,\, \mathfrak{h}\,,\, \overset{o}{\Upsilon}\,, \cdot) : \text{sec}(\boldsymbol{E},\, \boldsymbol{Q}) \to \text{sec}(\boldsymbol{E},\, \boldsymbol{F})$ does not depend on the choice of the observer o.

3.3.2 Schrödinger operator

We prove that all 2nd order Schrödinger operators are a complex linear combination of the Schrödinger operators exhibited in Propositions 3.3 and 3.6.

3.20 Definition. We define a *Schrödinger operator* to be a sheaf morphism

$$\mathfrak{O} : \overset{o}{\text{fields}}(\boldsymbol{Q}) \times \text{sec}(\boldsymbol{E},\, \boldsymbol{Q}) \to \text{sec}(\boldsymbol{E},\, \mathbb{T}^* \otimes \boldsymbol{Q})\,. \,\square$$

3.21 Theorem. *All 2nd order covariant Schrödinger operators*

$$\mathsf{O} : \text{sec}(\boldsymbol{E},\, \boldsymbol{Q}) \to \text{sec}(\boldsymbol{E},\, \mathbb{T}^* \otimes \boldsymbol{Q})$$

are of the type $\mathsf{O}(\Psi) = \alpha\, \mathsf{S}[\Psi] + \beta\, r\, \Psi$, *with* $\alpha,\, \beta \in \mathbb{C}$.

PROOF. Let $\mathfrak{O} : \overset{o}{\text{fields}}(\boldsymbol{Q}) \times \text{sec}(\boldsymbol{E},\, \boldsymbol{Q}) \to \text{sec}(\boldsymbol{E},\, \mathbb{T}^* \otimes \boldsymbol{Q})$ be a 2nd order covariant Schrödinger operator.

1) In virtue of the "orbit reduction theorem" [63], implemented by considering the fibred group epimorphism $\mathcal{W}^{(3,2)}(\boldsymbol{Q}) \to \mathcal{W}^{(1,0)}(\boldsymbol{Q})$, we can express $\mathfrak{O}$, through the covariant differentials and curvatures, as $\mathfrak{O}\left(o\,,\, G\,,\, K\,,\, \mathfrak{h}\,,\, \overset{o}{\Upsilon}\,,\, \Psi\right) = \mathfrak{O}'\left(\nabla^{(1)}\, o\,,\ \nabla^{(2)}\, G\,,\ R[K]\,,\ \overset{o}{\nabla}{}^{(2)}\, \mathfrak{h}\,,\ R[\overset{o}{\Upsilon}]\,,\ \overset{o}{\nabla}{}^{(2)}\, \Psi\right)$, where the covariant differentials are performed with respect to K and $\overset{o}{\Upsilon}$, as appropriate, we have set

$$\nabla^{(1)}o := (o,\, \nabla o)\,, \qquad \nabla^{(2)}G := (G,\, \nabla G,\, \nabla\nabla G)\,, \ldots, \qquad \overset{o}{\nabla}{}^{(2)}\Psi := (\Psi,\, \overset{o}{\nabla}\Psi,\, \overset{o}{\nabla}\overset{o}{\nabla}\Psi)\,,$$

the sheaf morphism $\mathfrak{O}'$ is invariant with respect to the fibred action of the group bundle $\mathcal{T}(\boldsymbol{E})$ and, for each observer o, the induced sheaf morphism $\overset{o}{\mathfrak{O}}{}'$ is covariant with respect to the fibred actions of the group bundles $\mathcal{W}^{(1,0)}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$.

2) By considering the fundamental identities, we can express $\mathfrak{O}$, as

$$\mathfrak{O}\left(o\,,\,G\,,\,K\,,\,\mathfrak{h}\,,\,\overset{o}{\mathfrak{Y}}\,,\,\Psi\right)=\mathfrak{O}''\left(\nabla^{(1)}o\,,\,G\,,\,R[K]\,,\,\mathfrak{h}\,,\,o^*\Omega[G,K]\,,\,\overset{o}{\nabla}{}^{(2)}\,\Psi\right),$$

where, for each observer o, the induced sheaf morphism $\overset{o}{\mathfrak{O}}{}''$ is covariant with respect to the fibred actions of the group bundles $\mathcal{W}^{(1,0)}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$.

3) Let us consider the Hermitian metric $\widetilde{\mathfrak{h}}:=\mathfrak{h}/\eta[G]$, where $\eta[G]:\boldsymbol{E}\to\mathbb{T}^{3/2}\otimes\Lambda^3V^*\boldsymbol{E}$ is the scaled volume form associated with G.

In virtue of the "homogeneous function theorem" [63], implemented by considering the subbundle $\boldsymbol{E}\times\mathbb{R}^+\subset\mathcal{W}^{(1,0)}(\boldsymbol{Q})$ of real homotheties of $\boldsymbol{Q}$, the sheaf $\mathfrak{O}$ can be expressed as a polynomial of the type

$$\mathfrak{O}\left(o\,,\,G\,,\,K\,,\,\mathfrak{h}\,,\,\overset{o}{\mathfrak{Y}}\,,\,\Psi\right)=\sum_{\substack{0\leq i,j\leq 2\\ 0\leq h\\ 0\leq k\leq 2}}A^{hk}_{ij}\left(\nabla^{(1)}o\,,\,G\,,\,R[K]\,,\,o^*\Omega[G,K]\right)\left(\widetilde{\mathfrak{h}}\,(\overset{o}{\nabla}{}^i\Psi,\,\overset{o}{\nabla}{}^j\Psi)\right)^h\,\overset{o}{\nabla}{}^k\Psi\,,$$

where the coefficients A^{hk}_{ij} are $\mathbb{T}^{*(1-3h/2)}$–valued tensors of $\boldsymbol{E}$, which are covariant with respect to $\mathcal{G}^1(\boldsymbol{E})$.

4) In virtue of the "homogeneous function theorem" and the "metric covariant function" [63], implemented by considering the covariance of $\mathfrak{O}$ with respect to the fibred action of the group bundle $\mathcal{G}(\mathbb{T})$, the covariance of A^{hk}_{ij} with respect to the fibred action of the group bundle $\mathcal{G}^1(\boldsymbol{E})$, and by counting the contravariant and covariant indices of the objects we are dealing with, we can express $\mathfrak{O}$ as

$$\mathfrak{O}\left(o\,,\,G\,,\,K\,,\,\mathfrak{h}\,,\,\overset{o}{\mathfrak{Y}}\,,\,\Psi\right)=a\,(C^1_1\nabla o)\,\Psi+b\,o\lrcorner\,\overset{o}{\nabla}\Psi+c\,\bar{G}\,(\overset{o}{\nabla}\overset{o}{\nabla}\Psi)+d\,r[G]\,\Psi\,,\qquad a,b,c,d\in\mathbb{C}\,.$$

5) In virtue of Lemma 2.37, the invariance of $\overset{o}{\mathfrak{O}}$ with respect to the fibred action of the group bundle $\mathcal{T}(\boldsymbol{E})$ yields $a=\mathfrak{i}\,c$ and $b=2\,\mathfrak{i}\,c$. Hence

$$\begin{aligned}\mathfrak{O}\left(o\,,\,G\,,\,K\,,\,\mathfrak{h}\,,\,\overset{o}{\mathfrak{Y}}\,,\,\Psi\right)&=c\left((\mathfrak{i}\,C^1_1\,\nabla o)\,\Psi+2\,\mathfrak{i}\;o\lrcorner\,\overset{o}{\nabla}\Psi+\bar{G}\,(\overset{o}{\nabla}\overset{o}{\nabla}\Psi)\right)+d\,r[G]\,\Psi\\&=2\,\mathfrak{i}\,c\,\mathrm{S}[\Psi]+d\,r[G]\,\Psi\,,\qquad c,\,d\in\mathbb{C}\,,\end{aligned}$$

and putting $\alpha=2\,\mathfrak{i}\,c\,,\ \ \beta=d$ we get the Theorem, by observing that $C^1_1\,\nabla o=\mathrm{div}_\eta\,o$. QED

3.22 Remark. If our quantum system would be involved with a "fundamental" distinguished time scale, then we could not require the covariance of the Schrödinger operator with respect to the change of time scale. In such a case, the step 4) of the above proof would be weaker and we would obtain many more solutions of our problem.

For instance, by postulating a distinguished fundamental time scale $\tau\in\mathbb{T}^*$, any additional term of the type $f\circ\left(\tau^{*3/2}\otimes\widetilde{\mathfrak{h}}(\Psi,\Psi)\right)\tau\otimes\Psi\in\mathrm{sec}(\boldsymbol{E},\,\mathbb{T}^*\otimes\boldsymbol{Q})$, where $f:\mathbb{C}\to\mathbb{C}$ is any function, would still yield a covariant Schrödinger operator, according to the weakened definition of covariance.

Equivalent considerations hold for a possible distinguished length scale, as time scales and length scales can be related by the fundamental constant $\hbar/m\in\mathbb{T}^*\otimes\mathbb{L}^2$. □

3.3.3 Quantum Lagrangian

We prove that all 2nd order quantum Lagrangians are essentially 1st order operators and are proportional to the Lagrangian exhibited in Proposition 3.11.

3.23 Definition. We define a *quantum Lagrangian* to be a sheaf morphism

$$\mathfrak{O} : \overset{o}{\text{fields}}(\boldsymbol{Q}) \times \sec(\boldsymbol{E},\, \boldsymbol{Q}) \to \sec(\boldsymbol{E},\, \mathfrak{i}\,\Lambda^4 T^*\boldsymbol{E})\,. \square$$

3.24 Theorem. *All 2nd order covariant quantum Lagrangians*

$$\mathsf{O} : \sec(\boldsymbol{E},\, \boldsymbol{Q}) \to \sec(\boldsymbol{E},\, \mathfrak{i}\,\Lambda^4 T^*\boldsymbol{E})$$

are of the type

$$\begin{aligned}\mathsf{O}(\Psi) = a\,\mathsf{L}_{(k)}[\Psi] \\ + \tfrac{1}{4} b\, dt \wedge \Big(\mathfrak{h}(\Psi, \mathsf{S}[\Psi]) - \mathfrak{h}(\mathsf{S}[\Psi], \Psi)\Big) + \tfrac{1}{4}\, c\,\mathfrak{i}\, dt \wedge \Big(\mathfrak{h}(\mathsf{S}[\Psi], \Psi) + \mathfrak{h}(\Psi, \mathsf{S}[\Psi])\Big)\,,\end{aligned}$$

with $a, b, c, k \in \mathbb{R}$.

PROOF. 1) By a procedure analogous to that of Theorem 3.21, we can prove that all 2nd order covariant Lagrangians can be expressed as a polynomial of the type

$$\mathfrak{O}\left(o\,,\, G\,,\, K\,,\, \mathfrak{h}\,,\, \overset{o}{\nabla}\,,\, \Psi\right) = \mathfrak{i} \sum_{\substack{0\le i,j\le 2\\ 0\le h}} A^h_{ij0}\left(\nabla^{(1)} o\,,\, G\,,\, R[K]\,,\, o^*\Omega[G,K]\right)\left(\widetilde{\mathfrak{h}}\,(\overset{o}{\nabla}{}^i\Psi,\, \overset{o}{\nabla}{}^j\Psi)\right)^h \upsilon^0\,,$$

where $\upsilon[G] : \boldsymbol{E} \to \mathbb{T}^{5/2} \otimes \Lambda^4 T^*\boldsymbol{E}$ is the scaled volume form induced by G, and where the coefficients A^h_{ij0} are $\mathbb{T}^{*q}$–valued tensors of $\boldsymbol{E}$, with an appropriate rational number q, and are covariant with respect to $\mathcal{G}^1(\boldsymbol{E})$.

2) In virtue of the "homogeneous function theorem" and the "metric covariant function" [63], implemented by considering the covariance of $\mathfrak{O}$ with respect to the fibred action of the group bundle $\mathcal{G}(\mathbb{T})$, the covariance of A^h_{ij0} with respect to the fibred action of the group bundle $\mathcal{G}(\boldsymbol{E})$, and by counting the contravariant and covariant indices of the objects we are dealing with, we can express $\mathfrak{O}$ as

$$\begin{aligned}\mathfrak{O}\left(o\,,\, G\,,\, K\,,\, \mathfrak{h}\,,\, \overset{o}{\nabla}\,,\, \Psi\right) = \\ = \mathfrak{i}\,\alpha\,(C^1_1 \nabla o)\, dt \wedge \mathfrak{h}\,(\Psi, \Psi) + \mathfrak{i}\,\gamma\, dt \wedge \bar{G}\,(\overset{o}{\nabla}\Psi,\, \overset{o}{\nabla}\Psi) + \mathfrak{i}\,\epsilon\, r[G]\, dt \wedge \mathfrak{h}\,(\Psi,\, \Psi) \\ + \beta\left(dt \wedge \mathfrak{h}\,(o \lrcorner \overset{o}{\nabla}\Psi,\, \Psi) - dt \wedge \mathfrak{h}\,(\Psi,\, o \lrcorner \overset{o}{\nabla}\Psi)\right) + \mathfrak{i}\,\beta'\left(dt \wedge \mathfrak{h}\,(o \lrcorner \overset{o}{\nabla}\Psi,\, \Psi) + dt \wedge \mathfrak{h}\,(\Psi,\, o \lrcorner \overset{o}{\nabla}\Psi)\right) \\ + \delta\left(dt \wedge \mathfrak{h}\,(\overset{o}{\Delta}\Psi,\, \Psi) - dt \wedge \mathfrak{h}\,(\Psi,\, \overset{o}{\Delta}\Psi)\right) + \mathfrak{i}\,\delta'\left(dt \wedge \mathfrak{h}\,(\overset{o}{\Delta}\Psi,\, \Psi) + dt \wedge \mathfrak{h}\,(\Psi,\, \overset{o}{\Delta}\Psi)\right), \\ \alpha, \beta, \beta', \gamma, \delta, \delta', \epsilon \in \mathbb{R}\,.\end{aligned}$$

3) In virtue of Lemma 2.37, the invariance of $\overset{o}{\mathfrak{O}}$ with respect to the fibred action of the group bundle

$\mathcal{T}(\boldsymbol{E})$ yields the following identities $\alpha = -2\,\delta$, $\beta = 2\,\delta' - \gamma$ and $\beta' = -2\,\delta$. Hence

$$\mathfrak{O}\left(o\,,\,G\,,\,K\,,\,\mathsf{h}\,,\,\overset{o}{\mathfrak{U}}\,,\,\Psi\right) = -4\,\gamma\,\mathsf{L}[\Psi] - \mathfrak{i}\,\epsilon\,r[G]\otimes dt\wedge\mathsf{h}\,(\Psi,\,\Psi) + 2\,\mathfrak{i}\,\delta\,dt\wedge\left(\mathsf{h}\,(\mathsf{S}[\Psi],\,\Psi) + \mathsf{h}\,(\Psi,\,\mathsf{S}[\Psi])\right) + 2\,\delta'\,dt\wedge\left(\mathsf{h}\,(\mathsf{S}[\Psi],\,\Psi) - \mathsf{h}\,(\Psi,\,\mathsf{S}[\Psi])\right)$$
$$\gamma,\delta,\delta',\epsilon\in\mathbb{R}\,.$$

Then, putting $a = -4\,\gamma$, $\epsilon = -k\,a$, $b = -8\,\delta'$, $c = 8\,\delta$, we get the Theorem. QED

3.25 Note. The Euler–Lagrange operator associated with the 2nd order covariant quantum Lagrangian exhibited in the above Theorem is $\mathsf{E}[\mathsf{L}] = (a+b)\,\mathsf{S}_{(k)}$.

Hence, the 2nd order terms in the above Lagrangian are not physically relevant. □

3.3.4 Schrödinger operator on the functional quantum bundle

We introduce briefly the functional quantum bundle and prove that all covariant Schrödinger operators which induce Hermitian operators on the functional quantum bundle coincide with our distinguished Schroedinger operator.

3.26 Definition. We define the (infinite dimensional) *functional quantum bundle* $\boldsymbol{H}\to\boldsymbol{T}$ to be the fibred set over $\boldsymbol{T}$, whose fibres are constituted by the compact support smooth sections, at fixed time, of the quantum bundle ("regular sections"). □

This bundle has no distinguished splittings into time and type fibre; such a splitting can be obtained by choosing a classical observer.

The quantum metric h equips the functional quantum bundle with a pre–Hilbert metric $\langle\,|\,\rangle$. Then, a true Hilbert bundle can be obtained by a completion procedure.

The functional quantum bundle is naturally endowed with a "smooth" structure in the sense of A. Frölicher [40], which allows us to introduce standard geometric notions such as tangent space, connections and so on [46].

Each regular section Ψ of $\boldsymbol{Q}$ can be regarded as a section $\widehat{\Psi}$ of $\boldsymbol{H}$. Accordingly, each "regular" operator $\mathsf{O}:\mathrm{sec}(\boldsymbol{E},\boldsymbol{Q})\to\mathrm{sec}(\boldsymbol{E},\boldsymbol{Q})$ can be regarded as a fibred morphism $\widehat{\mathsf{O}}:\boldsymbol{H}\to\boldsymbol{H}$ over $\boldsymbol{T}$.

In particular, the operator $\widehat{\mathsf{O}}$ associated with a Schrödinger operator can be regarded as the covariant differential of a linear connection of the functional quantum bundle.

3.27 Definition. A fibred morphism $\widehat{\mathsf{O}}:\boldsymbol{H}\to\boldsymbol{H}$ over $\boldsymbol{T}$ is said to be *Hermitian*, if for each $\hat{\Phi},\hat{\Psi}\in\mathrm{sec}(\boldsymbol{T},\boldsymbol{H})$, we have $d\,\langle\widehat{\Phi}\,|\,\widehat{\Psi}\rangle = \langle\widehat{\mathsf{O}}\,(\widehat{\Phi})\,|\,\widehat{\Psi}\rangle + \langle\widehat{\Phi}\,|\,\widehat{\mathsf{O}}\,(\widehat{\Psi})\rangle$. □

3.28 Theorem. *All 2nd order covariant Schrödinger operators* O*, such that* $\widehat{\mathsf{O}}$ *is Hermitian, are of the type* $\mathsf{O}(\Psi) = \mathsf{S}_{(k)}[\Psi]$*, with* $k\in\mathbb{R}$*.*

Proof. Let $\widehat{\Phi},\widehat{\Psi}\in\mathrm{Sec}(\boldsymbol{T},\boldsymbol{H})$.

By considering the trivialisation of $\boldsymbol{H}\to\boldsymbol{T}$ induced by any global observer o, we obtain

$$d\,\langle\Phi\,|\,\Psi\rangle = \int L_o\left(\mathsf{h}(\Phi,\,\Psi)\right) = \int\partial_0(\bar{\phi}\,\psi)\,\eta + \int\bar{\phi}\,\psi\,L_o\,\eta\,.$$

Moreover, we have $\langle \widehat{\overset{o}{\nabla}}\,(\widehat{\Phi}) \,|\, \widehat{\Psi}\rangle = \langle \widehat{\Phi} \,|\, \widehat{\overset{o}{\nabla}}\,(\widehat{\Psi})\rangle$.

According to Theorem 3.21, let us consider the most general 2nd order covariant Schrödinger operator $\mathsf{O}\,(\Psi) = \alpha\,\big(\overset{o}{\nabla}_0 + \frac{1}{2}\,(\mathrm{div}_\eta\, o)_0 - \mathfrak{i}\,\frac{1}{2}\,\overset{o}{\Delta}_0\big)\,\psi\,u^0 \otimes \mathfrak{b} + \beta\, r\,\Psi$, with $\alpha, \beta \in \mathbb{C}$. Then, we obtain

$$\begin{aligned}
\langle \widehat{\mathsf{O}}\,(\widehat{\Phi}) \mid \widehat{\Psi}\rangle + \langle \widehat{\Phi} \mid \widehat{\mathsf{O}}\,(\widehat{\Psi})\rangle &= \mathrm{re}\,(\alpha) \int \big(\partial_0(\bar{\phi}\,\psi) + \bar{\phi}\,\psi\,(\mathrm{div}_\eta\, o)_0\big)\,\eta + \mathrm{re}\,(\beta) \int \bar{\phi}\,\psi\, r\,\eta \\
&\quad + \mathrm{im}\,(\alpha) \int \big(2\,A_0\,\bar{\phi}\,\psi + \tfrac{1}{2}\,(\overline{\overset{o}{\Delta}_0\,\bar{\phi}})\,\psi + \tfrac{1}{2}\,\bar{\phi}\,(\overset{o}{\Delta}_0\,\psi)\big)\,\eta \\
&= \mathrm{re}\,(\alpha)\, d\,\langle \Phi \mid \Psi\rangle + \mathrm{re}\,(\beta) \int \bar{\phi}\,\psi\, r\,\eta \\
&\quad + \mathrm{im}\,(\alpha) \int \big(2\,A_0\,\bar{\phi}\,\psi + \tfrac{1}{2}\,(\overline{\overset{o}{\Delta}_0\,\bar{\phi}})\,\psi + \tfrac{1}{2}\,\bar{\phi}\,(\overset{o}{\Delta}_0\,\psi)\big)\,\eta\,.
\end{aligned}$$

Hence, $\widehat{\mathsf{O}}$ is Hermitian if and only if $\mathrm{re}\,(\alpha) = 1$, $\mathrm{im}\,(\alpha) = 0$ and $\mathrm{re}\,(\beta) = 0$. QED

References

[1] R. Abraham, J. E. Marsden: *Foundations of mechanics*, Second edition, The Benjamin, London, 1978.

[2] A. P. Balachandran, G. Marmo, A. Simoni, G. Sparano: *Quantum bundles and their symmetries*, Int. J. Mod. Phys. A, 7, 8 (1992), 1641–1667.

[3] V. Bargmann: Ann. Math., **59**, 1 (1954).

[4] I. Bialynicki–Birula, M. Kaliński, J. H. Eberly: *Lagrange Equilibrium Points in Celestial Mechanics and Nonspreading Wave Packets for Strongly Driven Rydberg Electrons*, Phys. Rev. Let. **73**, 13 (1994), 77–80.

[5] I. Bialynicki–Birula, Z. Bialynicki–Birula: *Rotational Frequency Shift*, Phys. Rev. Let. **78**, 13 (1997), 2539–2542.

[6] A. Cabras: *On the universal connection of a system of connections*, Note Math. **8** (1987), 173-209.

[7] A. Cabras, I. Kolář: *Connections on some functional bundles*, Czech. Math. J. **45** (1995), 529–548.

[8] A. Cabras, I. Kolář: *The universal connection of an arbitrary system*, Ann. Matematica pura ed applicata (IV), Vol. CLXXIV (1998), 1–11.

[9] A. Camacho: *On a quantum equivalence principle*, Mod. Phys. Lett. A, **14**, 4 (1999), 275–288.

[10] A. Camacho: *Aharanov–Bohm effect and coordinate transformations*, Mod. Phys. Lett. A, **14**, 21 (1999), 1445–1453.

[11] A. Camacho: *Flavor–oscillation clocks, continuous quaqntum measurements and a violation of Einstein equivalence principle*, Mod. Phys. Lett. A, **14**, 36 (1999), 2545–2556.

[12] A. Camacho: *The role of the quantum properties of gravitational radiation in the detection of gravitational waves*, Int. J. of Mod. Phys. A, **14**, 13 (1999), 1997–2012.

[13] A. Camacho: *Gravity induced interference and continuous quantum measurement*, Phys. Lett. A, **256**, (1999), 339–346.

[14] A. Camacho: *Measurement induced interference in an inhomogeneous gravitational field*, Phys. Lett. A, **262**, (1999), 110–120.

[15] A. Camacho, H. Dehnen: *Higgs field and localization problem*, Phys. Lett. A, **256**, (1999), 339–346.

[16] A. Camacho, A. Camacho–Galván: *Time emergence by self–measurement in a quantum anisotropic universe*, Il Nuovo Cimento, **114** B, 8 (1999), 923–938.

[17] D. Canarutto, A. Jadczyk, M. Modugno: *Quantum mechanics of a spin particle in a curved spacetime with absolute time*, Rep. on Math. Phys., **36**, 1 (1995), 95–140.

[18] E. Cartan: *Sur les variétés à connexion affine et la théorie de la rélativité généralisée*, Gauthier Villars, Paris, 1955.

[19] E. Cartan: *On manifolds with an affine connection and the theory of general relativity*, Bibliopolis, Napoli, 1986.

[20] K. S. Cheng: *Quantization of a General Dynamical system by Feynmann's Path Integration Formulation*, J. Math. Phys., **13**, 11 (1972), 1723–1726.

[21] M. de Leon, J. C. Marrero, E. Padron: *On the geometric quantization of Jacobi manifolds*, J. Math. Phys. **38**, 12, (1997), 6185–6213.

[22] G. DAUTCOURT: *On the Newtonian limit of general relativity*, Acta Phys. Pol. **B 21**, 10, (1990), 755–765.

[23] B. S. DEWITT: *Dynamical Theory in Curved Spaces. I. A Review of the Classical and Quantum Action Principles*, Rev. Modern Phys., **29**, 3 (1957), 377–397.

[24] H.-D. DÖBNER: *Remarks on on linear quantum theory*, preprint ASI-TPA, 02/98.

[25] H.-D. DÖBNER, G. A. GOLDIN: *On a general nonlinear Schrödinger equation admitting diffusion currents*, Phys. Lett. A, **162**, (1992), 397–401.

[26] H.-D. DÖBNER, G. A. GOLDIN: *Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations*, J. Phys. A: Math. Gen., **27**, (1994), 1771–1780.

[27] H.-D. DÖBNER, G. A. GOLDIN: *Introducing nonlinear gauge transformations in a family of nonlinear Schrödinger equations*, Phys. Rev. A, **54**, 5, (1996), 3764–3771.

[28] H.-D. DÖBNER, G. A. GOLDIN, P. NATTERMANN: *Gauge transformations in quantum mechanics and the unification of nonlinear Schrödinger equations*, J. Math. Phys., **40**, 1, (1999), 49-63.

[29] H.-D. DÖBNER, J.-D. HENNIG: *On a path to nonlinear quantum mechanics*, in "Symmetries in Science IX", Eds.: Gruber and Ramek, Plenum Press, New York, 1997, 81–90.

[30] H.-D. DÖBNER, P. NATTERMANN: *Borel quantization: kinematics and dynamics*, Acta Phys. Pol. B, **27**, (1996), 2327–2339.

[31] H. D. DOMBROWSKI, K. HORNEFFER: *Die Differentialgeometrie des Galileischen Relativitätsprinzips*, Math. Z. 86 (1964), 291–311.

[32] C. DUVAL: *The Dirac & Levy-Leblond equations and geometric quantization*, in "Diff. Geom. Meth. in Math. Phys.", P. L. García and A. Pérez-Rendón Eds., L.N.M. **1251** (1985), Springer-Verlag, 205–221.

[33] C. DUVAL: *On Galilean isometries*, Clas. Quant. Grav. **10** (1993), 2217–2221.

[34] C. DUVAL, G. BURDET, H. P. KÜNZLE, M. PERRIN: *Bargmann structures and Newton-Cartan theory*, Phys. Rev. D, **31**, N.8 (1985), 1841–1853.

[35] C. DUVAL, G. GIBBONS, P. HORVATY: *Celestial mechanics, conformal structures, and gravitational waves*, Phys. Rev. D **43**, 12 (1991), 3907–3921.

[36] C. DUVAL, H. P. KÜNZLE: *Minimal gravitational coupling in the Newtonian theory and the covariant Schrödinger equation*, G.R.G., **16**, 4 (1984), 333–347.

[37] J. EHLERS: *The Newtonian limit of general relativity*, in "Fisica Matematica Classica e Relatività", Elba 9-13 giugno 1989, 95–106.

[38] J. R. FANCHI: *Review of invariant time formulations of relativistic quantum theories*, Found. Phys. **23** (1993), 487–548.

[39] R. P. FEYNMANN: *Space–time approach to non–relativistic quantum mechanics*, Reviews of Modern Physics, Vol. 20 (1948), 267–287.

[40] A. FRÖLICHER: *Smooth structures*, L.N.M. **962**, Springer-Verlag, 1982, 69-81.

[41] P. L. GARCÍA: *Connections and 1-jet fibre bundle*, Rendic. Sem. Mat. Univ. Padova, **47** (1972), 227–242.

[42] P. L. GARCÍA: *The Poincaré-Cartan invariant in the calculus of variations*, Symposia Mathematica **14** (1974), 219–246.

[43] C. C. GHIRARDI, A, RIMINI, T. WEBER: *Unified dynamics for microscopic and macroscopic systems*, Phys. Rew. D, **34**, 2, (1986), 470–491.

[44] M. J. Gotay: *Constraints, reduction and quantization*, J. Math. Phys. **27** n. 8 (1986), 2051–2066.

[45] P. Havas: *Four-dimensional formulation of Newtonian mechanics and their relation to the special and general theory of relativity*, Rev. Modern Phys. **36** (1964), 938–965.

[46] A. Jadczyk, J. Janyška, M. Modugno: *Galilei general relativistic quantum mechanics revisited*, in "Geometria, Física-Matemática e outros Ensaios", Homenagem a António Ribeiro Gomes, A. S. Alves, F. J. Craveiro de Carvalho and J. A. Pereira da Silva Eds., Coimbra 1998, 253–313.

[47] A. Jadczyk, M. Modugno: *An outline of a new geometric approach to Galilei general relativistic quantum mechanics*, in "Differential geometric methods in theoretical physics", C. N. Yang, M. L. Ge and X. W. Zhou Eds., World Scientific, Singapore, 1992, 543–556.

[48] A. Jadczyk, M. Modugno: *A scheme for Galilei general relativistic quantum mechanics*, in "General Relativity and Gravitational Physics", M. Cerdonio, R. D'Auria, M. Francaviglia and G. Magnano Eds., World Scientific, Singapore, 1994, 319–337.

[49] A. Jadczyk, M. Modugno: *Galilei general relativistic quantum mechanics*, report, Dept. of Appl. Math., Univ. of Florence, 1994.

[50] J. Janyška: *Remarks on symplectic and contact 2–forms in relativistic theories*, Bollettino U.M.I. (7) 9–B (1995), 587–616.

[51] J. Janyška: *Natural quantum Lagrangians in Galilei quantum mechanics*, Rendiconti di Matematica, S. VII, Vol. 15, Roma (1995), 457–468.

[52] J. Janyška: *Natural Lagrangians for quantum structures over 4–dimensional spaces*, Rend. di Mat., S VII, Vol 18, Roma (1998), 623–648.

[53] J. Janyška: *A remark on natural quantum Lagrangians and natural generalized Schrödinger operators in Galilei quantum mechanics*, The Proceedings of the Winter School "Geometry and Topology" (Srní, 2000), Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II, No. 66 (2001), 117–128.

[54] J. Janyška, M. Modugno: *Classical particle phase space in general relativity*, Proc. Conf. Diff. Geom. and its Appl., Brno 1995, Masaryk University 1996, 573–602, electronic edition: `http://www.emis.de/proceedings/`.

[55] J. Janyška, M. Modugno: *Relations between linear connections on the tangent bundle and connections on the jet bundle of a fibred manifold*, Arch. Math. (Brno), **32** (1996), 281–288, electronic edition: `http://www.emis.de/journals/`.

[56] J Janyška, M. Modugno : *On quantum vector fields in general relativistic quantum mechanics*, General Mathematics **5** (1997), Proc. 3rd Internat. Workshop Diff. Geom. and its Appl., Sibiu (Romania) 1997, 199–217.

[57] J. Janyška, M. Modugno: *On the graded Lie algebra of quantisable forms*, in "Differential Geometry and Applications", Eds.: I. Kolář, O. Kowalski, D. Krupka J. Slovak , Proc. of the 7th Intern. Conf., Brno, 10-14 August 1998, Masaryk University, 1999, 601–620, electronic edition: `http://www.emis.de/journals/`.

[58] J. Janyška, M. Modugno: *Quantisable functions in general relativity*, in "Opérateurs différentiels et Physique Mathématique", Eds.: J. Vaillant, J. Carvalho e Silva, Textos Mat. Ser. B, 24, 2000, 161–181.

[59] J. Janyška, M. Modugno: *Covariant pre-quantum operators*, to appear in Proceedings 'Differential Geometry and its Applications", Conf. Opava (Czech Republic), 27–31 August 2001, Silesian University, Opava, 2002, pages 24.

[60] J. Janyška, M. Modugno: *Covariant Quantum Mechanics*, in preparation 2001.

[61] J. JANYŠKA, M. MODUGNO, D. SALLER: *Covariant quantum mechanics and quantum symmetries*, to appear on Proc. "Recent Developments in General Relativity", Genova, September 2000, Springer–Verlag.

[62] M. KALINSKI, J. H. EBERLY: *Trojan wave packets; Mathieu theory and generation from circular states*, Phys. Rev. A **53**, 3 (1996), 1715–1724.

[63] I. KOLÁŘ, P. MICHOR, J. SLOVÁK: *Natural operators in differential geometry*, Springer-Verlag, Berlin, 1993.

[64] D. KRUPKA, J. JANYŠKA: *Lectures on Differential Invariants*, Folia Fac. Sci. Nat. Univ. Purkynianae Brunensis, Brno, 1990.

[65] K. KUCHAŘ: *Gravitation, geometry and nonrelativistic quantum theory*, Phys. Rev. D, **22**, 6 (1980), 1285–1299.

[66] H. P. KÜNZLE: *Galilei and Lorentz structures on space-time: comparison of the corresponding geometry and physics*, Ann. Inst. H. Poinc. **17**, 4 (1972), 337–362.

[67] H. P. KÜNZLE: *Galilei and Lorentz invariance of classical particle interaction*, Symposia Mathematica **14** (1974), 53–84.

[68] H. P. KÜNZLE: *Covariant Newtonian limit of Lorentz space–times*, G.R.G. **7**, 5 (1976), 445–457.

[69] H. P. KÜNZLE: *General covariance and minimal gravitational coupling in Newtonian space-time*, in "Geometrodynamics", A. Prastaro Ed., Tecnoprint, Bologna 1984, 37–48.

[70] H. P. KÜNZLE, C. DUVAL: *Dirac field on Newtonian space–time*, Ann. Inst. H. Poinc., **41**, 4 (1984), 363–384.

[71] A. KYPRIANIDIS: *Scalar time parametrization of relativistic quantum mechanics: the covariant Schrödinger formalism*, Phys. Rep **155** (1987), 1–27.

[72] C. LÄMMERZAHL: *On the Equivalence Principle in Quantum Theory*, Gen. Rel. and Grav. **28**, 9, (1996), 1043-1070.

[73] M. LE BELLAC, J. M. LEVY-LEBLOND: *Galilean electromagnetism*, Nuovo Cim. **14** B, 2 (1973), 217–233.

[74] J. M. LEVY-LEBLOND: *Galilei group and Galilean invariance*, in "Group theory and its applications", E. M. Loebl Ed., Vol. 2, Academic, New York, 1971, 221–299.

[75] P. LIBERMANN, CH. M. MARLE: *Symplectic Geometry and Analytical Mechanics*, Reidel Publ., Dordrecht, 1987.

[76] L. MANGIAROTTI: *Mechanics on a Galilean manifold*, Riv. Mat. Univ. Parma (4) **5** (1979), 1-14.

[77] B. MASHHOON: *Neutron interferometry in a Rotating Frame of Reference*, Phys. Rev. Let. **61**, 23 (1988), 2639–2642.

[78] E. MASSA, E. PAGANI: *Classical dynamics of non-holonomic systems: a geometric approach*, Ann. Inst. H. Poinc. **55**, 1 (1991), 511-544.

[79] A. MESSIAH: Quantum mechanics, Vol. I, II, III, North-Holland, Amsterdam, 1961.

[80] M. MODUGNO: *On a covariant formulation of quantum mechanics*, in Proceedings of the II International Workshop "Lie Theory and Its Applications in Physics", August 17 – 20, 1997, Clausthal, Eds. H.-D. Doebner, V. K. Dobrev and J. Hilgert, World Scientific, 1998, 183-203.

[81] M. MODUGNO, C. TEJERO PRIETO, R. VITOLO: *Comparison between Geometric Quantisation and Covariant Quantum Mechanics*, in Proceed. "Lie Theory and Its Applications in Physics - Lie III", 11 - 14 July 1999, Clausthal, Germany, H.-D. Doebner, V.K. Dobrev and J. Hilgert Eds., World Scientific, Singapore, 2000, 155–175.

[82] M. Modugno, R. Vitolo: *Quantum connection and Poincaré–Cartan form*, in "Gravitation, electromagnetism and geometrical structures", G. Ferrarese Ed., Pitagora Editrice Bologna, 1996, 237–279.

[83] W. Pauli: in "Handbuch der Physik", S. Flügge Ed., Vol. V, 18–19, Springer, Berlin, 1958.

[84] E. Prugovečki: *Quantum Geometry: A Framework for Quantum General Relativity*, Kluwer, Dordrecht, 1992.

[85] E. Prugovečki: *Principles of Quantum General Relativity*, World Scientific, Singapore, 1995.

[86] F. Riehle, Th. Kisters, A. Witte, J. Helmecke, Ch. Bordé: *Optical Ramsey Spectroscopy in a Rotating Frame: Sagnac Effect in a Matter–Wave Interferometer*, Phys. Rev. Lett., **67**, 2 (1991), 177–180.

[87] D. Saller, R. Vitolo: *Symmetries in covariant classical mechanics*, J. Math. Phys., **41**, 10, (2000), 6824–6842.

[88] D. J. Saunders: *The geometry of jet bundles*, Cambridge Univ. Press, 1989.

[89] E. Schmutzer, J. Plebanski: *Quantum mechanics in non inertial frames of reference*, Fortschritte der Physik **25** (1977), 37–82.

[90] D. J. Simms: *Lie groups and Quantum mechanics*, in Lect. Notes Math. Vol. **52** (Berlin: Springer 1968).

[91] J. Sniatycki: *Geometric quantization and quantum mechanics*, Springer, New York, 1980.

[92] A. Trautman: *Sur la théorie Newtonienne de la gravitation*, C. R. Acad. Sc. Paris, t. **257** (1963), 617–620.

[93] A. Trautman: *Comparison of Newtonian and relativistic theories of space-time*, In "Perspectives in geometry and relativity", N. **42**, Indiana Univ. Press, 1966, 413–425.

[94] W. M. Tulczyjew: *An intrinsic formulation of nonrelativistic analytical mechanics and wave mechanics*, J. Geom. Phys., **2**, 3 (1985), 93–105.

[95] G. M. Tuynman, W. A. J. J. Wigerinck: *Central extensions and physics*, J. Geom. Phys., **4**, n.3 (1987), 207–258.

[96] R. Vitolo: *Spherical symmetry in classical and quantum Galilei general relativity*, Annales de l'Institut Henri Poincaré, **64**, n. 2 (1996), 177–203.

[97] R. Vitolo: *Quantum structures in general relativistic theories*, Proc. of the XII It. Conf. on G.R. and Grav. Phys., Roma, 1996; World Scientific, Singapore.

[98] R. Vitolo: *Quantising the rigid body*, Proc. of the VII Conf. on Diff. Geom and Appl., Brno 1998, 653–664.

[99] R. Vitolo: *Quantum structures in Galilei general relativity*, Ann. Inst. 'H. Poinc. **70**, n.3, 1999, 239–257.

[100] N. Woodhouse: *Geometric quantization*, Clarendon Press, Oxford, 2nd Edit. 1992.