Comparison between Geometric Quantisation and Covariant Quantum Mechanics

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Abstract

We compare the covariant formulation of Quantum Mechanics on a curved spacetime fibred on absolute time with the standard Geometric Quantisation.

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Contents

Introduction

Some years ago A. Jadczyk and M. M. proposed a covariant formulation of Quantum Mechanics for a scalar particle on a curved spacetime with absolute time, based on non standard methods such as fibred manifolds, jet spaces, non–linear connections, systems of connections, cosymplectic structures and Froelicher smooth spaces [47, 48, 49]. This theory has been extended to spin particles in cooperation with D. Canarutto [5], further developed in cooperation with J. Janyška, D. Saller, C. Tejero Prieto and R. Vitolo [46, 50, 51, 52, 53, 55, 58, 81, 82, 83, 84, 86, 96, 97, 98, 99] and partially extended to a Lorentz manifold in cooperation with J. Janyška and R. Vitolo [54, 55, 56, 57, 100].

In the proceedings of the previous session of the meeting on Lie Theory, we have accounted for a summary of this theory [81]. In order to capture the non standard methods and results of this theory it would be useful to compare it with the more standard Geometric Quantisation. This is the goal of the present paper.

For the sake of simplicity, our theory in the Galileian case will be conventionally referred to as *Covariant Quantum Mechanics* (CQM). Moreover, we shall be concerned with the main thread of *Geometric Quantisation* (GQ) and omit to consider special approaches dealing with quantisation of cosymplectic structures [8] and so on. We have no pretension at all of analysing extensively the wide literature of Geometric Quantisation; such a task would require a much larger space than a short note. Here, we just try to discuss some basic items concerning the comparison of the above theories.

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1 Covariant Quantum Mechanics

Let us start with a very brief sketch of the skeleton of the theory. For further details the reader can refer, for instance, to [49, 46, 81].

The covariance of the theory includes also independence from the choice of units of measurements. For this reason, we need a rigorous treatment of this feature and assume the following "positive 1–dimensional semi–vector spaces" over $\mathbb{R}^+$ as fundamental unit spaces (roughly speaking they have the same algebraic structure of $\mathbb{R}^+$, but no distinguished generator over $\mathbb{R}^+$): the space $\mathbb{T}$ of *time intervals*, the space $\mathbb{L}$ of *lengths*, the space $\mathbb{M}$ of *masses*.

Moreover, we assume the *Planck constant* to be an element $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

We refer to a particle with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$.

1.1 Classical theory

The classical framework is described in the following way.

The *spacetime* is an oriented $(n+1)$–dimensional manifold $\boldsymbol{E}$ (in the standard case $n=3$), the *absolute time* is an affine space associated with the vector space $\mathrm{I\!R}\otimes\mathbb{T}$, the *absolute time map* is a fibring $t:\boldsymbol{E}\to\boldsymbol{T}$. We denote fibred charts of spacetime by $(x^\lambda)\equiv(x^0,x^i)$. The tangent space and the vertical space of $\boldsymbol{E}$ are denoted by $T\boldsymbol{E}$ and $V\boldsymbol{E}$.

A *motion* is a section $s:\boldsymbol{T}\to\boldsymbol{E}$. The *phase space* is the first jet space of motions $J_1\boldsymbol{E}$ [60, 75, 87]. We denote fibred charts of phase space by $(x^0,x^i;x^i_0)$. The *absolute velocity* of a motion s is its first jet prolongation $j_1s:\boldsymbol{T}\to J_1\boldsymbol{E}$. An *observer* is a section $o:\boldsymbol{E}\to J_1\boldsymbol{E}$ and the *observed velocity* of a motion s is the map $\nabla[o]s:=j_1s-o\circ s:\boldsymbol{T}\to\mathbb{T}^*\otimes V\boldsymbol{E}$.

The *spacelike metric* is a scaled Riemannian metric of the fibres of spacetime $g:\boldsymbol{E}\to\mathbb{L}^2\otimes(V^*\boldsymbol{E}\underset{\boldsymbol{E}}{\otimes}V^*\boldsymbol{E})$. Given a particle of mass m, it is convenient to consider the re–scaled spacelike metric $G:=\frac{m}{\hbar}\,g:\boldsymbol{E}\to\mathbb{T}\otimes(V^*\boldsymbol{E}\underset{\boldsymbol{E}}{\otimes}V^*\boldsymbol{E})$.

The *gravitational field* is a time preserving torsion free linear connection of the tangent space of spacetime $K^\natural:T\boldsymbol{E}\to T^*\boldsymbol{E}\underset{T\boldsymbol{E}}{\otimes}TT\boldsymbol{E}$, such that $\nabla[K^\natural]g=0$ and the curvature tensor $R[K^\natural]$ fulfills the condition $R^\natural{}_\lambda{}^i{}_\mu{}^j=R^\natural{}_\mu{}^j{}_\lambda{}^i$.

The *electromagnetic field* is a scaled 2–form $f:\boldsymbol{E}\to(\mathbb{L}^{1/2}\otimes\mathbb{M}^{1/2})\otimes\Lambda^2T^*\boldsymbol{E}$, such that $df=0$. Given a particle of charge q, it is convenient to consider the re–scaled electromagnetic field $F:=\frac{q}{\hbar}\,f:\boldsymbol{E}\to\Lambda^2T^*\boldsymbol{E}$.

The electromagnetic field F can be "added", in a covariant way, to the gravitational connection $K^\natural$ yielding a *(total) spacetime connection* K, with coordinate expression

$$K_i{}^h{}_j=K^\natural{}_i{}^h{}_j\,,\quad K_j{}^h{}_0=K_0{}^h{}_j=K^\natural{}_0{}^h{}_j+\tfrac{1}{2}\,F^h{}_j\,,\quad K_0{}^h{}_0=K^\natural{}_0{}^h{}_0+\tfrac{1}{2}\,F^h{}_0\,.$$

This turns out to be a time preserving torsion free linear connection of the tangent space of spacetime, which still fulfills the properties that we have assumed for $K^\natural$.

The fibring of spacetime, the total spacetime connection and the spacelike metric yield, in a covariant way, a 2–form $\Omega:J_1\boldsymbol{E}\to\Lambda^2T^*J_1\boldsymbol{E}$ of phase space, with coordinate expression

$$\Omega=G^0_{ij}\left(dx^i_0-(K_\lambda{}^i{}_0+K_\lambda{}^i{}_h\,x^h_0)\,dx^\lambda\right)\wedge\left(dx^j-x^j_0\,dx^0\right).$$

This is a *cosymplectic form* [6, 9, 10, 72, 73], i.e. it fulfills the following properties: 1) $d\Omega=0$, 2) $dt\wedge\Omega^n:J_1\boldsymbol{E}\to\mathbb{T}\otimes\Lambda^nT^*J_1\boldsymbol{E}$ is a scaled volume form of $J_1\boldsymbol{E}$. Conversely, the cosymplectic form Ω characterises the spacelike metric and the total spacetime connection. Moreover, the closure of Ω is equivalent to the conditions that we have assumed on K.

There is a unique second order connection [75] $\gamma:J_1\boldsymbol{E}\to\mathbb{T}^*\otimes TJ_1\boldsymbol{E}$, such that $i_\gamma\Omega=0$. We assume the generalised *Newton's equation* $\nabla[\gamma]j_1s=0$ as the equation of motion for classical dynamics [49, 78, 79, 95].

We can also obtain this equation by a Lagrangian formalism according to a cohomological procedure in the following way [62, 46, 84]. The cosymplectic form Ω admits locally potentials of the type $\Theta:J_1\boldsymbol{E}\to T^*\boldsymbol{E}$, defined up to a closed form of the type $\alpha:\boldsymbol{E}\to T^*\boldsymbol{E}$, which are called *Poincaré–Cartan forms* [31, 33, 84]. Each Poincaré–Cartan form Θ splits, according to the splitting of $T^*\boldsymbol{E}$ induced by $J_1\boldsymbol{E}$, into the horizontal component $\mathcal{L}:J_1\boldsymbol{E}\to T^*\boldsymbol{T}$, called *Lagrangian*, and the vertical component

$\mathcal{P} : J_1\boldsymbol{E} \to V^*\boldsymbol{E}$, called *momentum*. These components are observer independent, but depend on the chosen gauge of the starting Poincaré–Cartan form. On the other hand, given an observer o, each Poincaré–Cartan form Θ splits, according to the splitting of $T^*\boldsymbol{E}$ induced by o, into the horizontal component $-\mathcal{H}[o] : J_1\boldsymbol{E} \to T^*\boldsymbol{T}$, called observed *Hamiltonian*, and the vertical component $\mathcal{P}[o] : J_1\boldsymbol{E} \to V^*\boldsymbol{E}$, called observed *momentum*. Moreover, the horizontal component of Ω, according to the splitting of $T^*J_1\boldsymbol{E}$ induced by $J_2\boldsymbol{E}$, is the map $\mathcal{E} = G^\flat(\nabla[\gamma]) : J_2\boldsymbol{E} \to \mathbb{T}^* \otimes V^*\boldsymbol{E}$, which turns out to be the Euler–Lagrange operator associated with $\mathcal{L}$. We have the coordinate expressions

$$\mathcal{L} = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0 x^j_0 + A_i\, x^i_0 + A_0)\, dx^0, \quad \mathcal{P} = (G^0_{ij}\, x^j_0 + A_i)\,(dx^i - x^i_0\, dx^0)\,,$$

and, in a chart adapted to o,

$$\mathcal{H}[o] = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0 x^j_0 - A_0)\, dx^0, \quad \mathcal{P}[o] = (G^0_{ij}\, x^j_0 + A_i)\, dx^i\,,$$

where $A \equiv o^*\Theta$.

The cosymplectic form Ω yields in a covariant way the Hamiltonian lift of functions $f : J_1\boldsymbol{E} \to \mathbb{R}$ to vertical vector fields $H[f] : J_1\boldsymbol{E} \to VJ_1\boldsymbol{E}$; consequently, we obtain the Poisson bracket $\{f, g\}$ between functions of phase space. Given an observer, the law of motion can be expressed, in a non covariant way, in terms of the Poisson bracket and the Hamiltonian.

More generally, chosen a time scale $\tau : J_1\boldsymbol{E} \to T\boldsymbol{T}$, the cosymplectic form Ω yields, in a covariant way, the Hamiltonian lift of functions f of phase space to vector fields $H_\tau[f] : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$, whose time component is τ. In view of our developments in the Quantum Theory, we prove that $H_\tau[f]$ is projectable on a vector field $X[f] : \boldsymbol{E} \to T\boldsymbol{E}$ if and only if the following conditions hold: i) the function f is quadratic with respect to the affine fibres of $J_1\boldsymbol{E} \to \boldsymbol{E}$ with second fibre derivative $f'' \otimes G$, where $f'' : \boldsymbol{E} \to T\boldsymbol{T}$, ii) $\tau = f''$. A function of this type is called *special quadratic* and has coordinate expression of the type

$$f = \tfrac{1}{2} f^0\, G^0_{ij} x^i_0 x^j_0 + f^0_i\, x^i_0 + \overset{o}{f}\,, \qquad \text{with} \qquad f^0, f^0_i, \overset{o}{f} : \boldsymbol{E} \to \mathbb{R}\,.$$

The vector space of special quadratic functions is not closed under the Poisson bracket, but it turns out to be an $\mathbb{R}$–Lie algebra through the covariant *special bracket*

$$[\![\, f, g \,]\!] = \{f, g\} + \gamma(f'').g - \gamma(g'').f\,.$$

We have the subalgebra of *quantisable* functions whose time component factorises through $\boldsymbol{T}$, the subalgebra of functions whose time component is constant, the subalgebra of *affine* functions whose time component vanishes and the abelian subalgebra of *spacetime* functions which factorise through $\boldsymbol{E}$. In particular, the Hamiltonian is a quantisable function, the components of the momentum are affine functions and the spacetime coordinates are spacetime functions.

Moreover, the map $f \mapsto X[f]$ turns out to be a morphism of Lie algebras. The coordinate expression of the tangent lift of f is $X[f] = f^0\, \partial_0 - f^i\, \partial_i$.

1.2 Quantum theory

The quantum framework is described in the following way.

A *quantum bundle* is defined to be a 1–dimensional complex vector bundle over spacetime $\boldsymbol{Q} \to \boldsymbol{E}$ equipped with a Hermitian metric $h : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{C} \otimes \Lambda^n V^* \boldsymbol{E}$ with values in the complexified volume forms of the fibres of spacetime. We shall refer to normalised local bases b of $\boldsymbol{Q}$ and to the associated complex coordinates z; accordingly, the coordinate expression of a *quantum section* is of the type $\Psi = \psi\, b$, with $\psi : \boldsymbol{E} \to \mathbb{C}$.

We consider also the *extended quantum bundle* $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$, by taking the pullback of $\boldsymbol{Q} \to \boldsymbol{E}$, with respect to the map $J_1\boldsymbol{E} \to \boldsymbol{E}$. A system of connections of $\boldsymbol{Q}$ parametrised by the sections of $J_1\boldsymbol{E} \to \boldsymbol{E}$ induces, in a covariant way, a connection of $\boldsymbol{Q}^\uparrow$, which is called *universal* [30, 75, 46]. A characteristic property of the universal connection is that its contraction with any vertical vector field of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ vanishes.

A *quantum connection* is defined to be a connection ч of the extended quantum bundle, which is Hermitian, universal and whose curvature is $R[ч] = \mathfrak{i}\,\Omega$. We stress that $\frac{1}{\hbar}$ has been incorporated in Ω through the re–scaled metric G. In a chart adapted to the observer o, the coordinate expression of a quantum connection is locally of the type

$$ч_0 = -\mathcal{H}[o]\,, \qquad ч_i = \mathcal{P}[o]\,, \qquad ч_i^0 = 0\,,$$

where the choice of the potential $A[o]$ is locally determined by ч and by the quantum base b. A quantum connection exists if and only if the cohomology class of Ω is integer; the equivalence classes of quantum bundles equipped with a quantum connection are classified by the cohomology group $H^1\big(\boldsymbol{E}, U(1)\big)$ [99].

We stress the minimality of our quantum bundle and quantum connection.

Let us assume a quantum bundle equipped with a quantum connection.

Any other quantum object is obtained, in a covariant way, from this quantum structure. The quantum connection lives on the extended quantum bundle, while we are looking for further quantum objects living on the original quantum bundle. This goal is successfully achieved by a *method of projectability*: namely, we look for objects of the extended quantum bundle which are projectable to the quantum bundle and then we take their projections. Indeed, our method of projectability turns out to be our way of implementing the covariance of the theory; in fact, it allows us to get rid of the family of all observers, which is encoded in the quantum connection (through $J_1\boldsymbol{E}$).

J. Janyška [52, 53] has proved that all covariant quantum Lagrangians of the quantum bundle are proportional to

$$\mathsf{L}[\Psi] = dt \wedge \Big(h(\Psi, \mathfrak{i}\,\bar{\nabla}\Psi) + h(\mathfrak{i}\,\bar{\nabla}\Psi, \Psi) - (\bar{G} \otimes h)(\overset{\vee}{\nabla}\Psi, \overset{\vee}{\nabla}\Psi) + k\, r\, h(\Psi, \Psi) \Big)\,,$$

where k is an arbitrary real factor, $\bar{\nabla}$ denotes the covariant differential with respect to time induced by the phase space, $\overset{\vee}{\nabla}$ denotes the vertical covariant differential and $r : \boldsymbol{E} \to \mathbb{R} \otimes \mathbb{T}^*$ is the scalar curvature of the spacelike metric G. Thus, k remains undetermined in our scheme. Several authors have tried to determine this factor via Feynmann's path integral approach, but they found different results, according to different ways to perform the integral [7, 11, 29, 59].

The standard Lagrangian formalism yields, from the above covariant quantum Lagrangian, the covariant quantum $(n+1)$–momentum, the covariant Euler–Lagrange equation and the covariant conserved probability current. These objects can also be obtained directly in terms of covariant differentials through the quantum connection, by means of the projectability method. The coordinate expression of the Euler–Lagrange equation is

$$\mathsf{S}.\psi \equiv \big(\overset{o}{\nabla}_0 + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\big)\psi - \mathfrak{i}\,\tfrac{1}{2}\big(\overset{o}{\Delta}_0 - k\,r_0\big)\,\psi = 0\,,$$

where

$$\overset{o}{\Delta}_0 \;\equiv\; G_0^{hk}\,\overset{o}{\nabla}_h\,\overset{o}{\nabla}_k + K_h{}^{kh}\,\overset{o}{\nabla}_k\,, \qquad \overset{o}{\nabla}_\lambda \;\equiv\; \partial_\lambda - \mathfrak{i}\,A_\lambda\,,$$

denote the Laplacian and the covariant differential induced by the connections ч$, K$ and by the observer attached to the spacetime chart. J. Janyška [53] has proved that any covariant Schrödinger equation is of the above type, hence it is the Euler–Lagrange equation associated with a covariant Lagrangian. We assume the above equation as the *quantum dynamical equation*.

Next, we classify the Hermitian vector fields of the extended quantum bundle, which are projectable to the quantum bundle. We find that the projected vector fields $Y : \boldsymbol{Q} \to T\boldsymbol{Q}$ of the quantum bundle, called *quantum vector fields*, constitute a Lie algebra naturally isomorphic to the Lie algebra of quantisable functions. The coordinate expression of the quantum vector field associated with the quantisable function f is

$$Y[f] = f^0\,\partial_0 - f^j\,\partial_j + \Big(\mathfrak{i}\,(f^0\,A_0 - f^h\,A_h + \overset{o}{f}) - \tfrac{1}{2}\,\operatorname{div} X[f]\Big)\;z\,\partial z\,,$$

where $\operatorname{div} X[f] = f^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_j(f^j\sqrt{|g|})}{\sqrt{|g|}}$.

The quantum vector field $Y[f]$ acts on the sections Ψ of the quantum bundle via the associated Lie derivative $Z[f] := \mathfrak{i}\;Y[f]_\bullet$. In particular, we obtain

$$Z[\mathcal{H}_0](\Psi) = \mathfrak{i}\;(\partial_0 + \tfrac{1}{2}\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}})\,\psi\,b\,,\quad Z[\mathcal{P}_j](\Psi) = -\,\mathfrak{i}\;(\partial_j + \tfrac{1}{2}\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}})\,\psi\,b\,.$$

Next, we consider the pre–Hilbert *functional quantum bundle* $\boldsymbol{H} \to \boldsymbol{T}$ over time, whose infinite dimensional fibres are constituted by the sections of the quantum bundle at a given time and with compact support. The quantum dynamical operator S can be regarded as a covariant differential $\nabla[\chi]$ of the functional quantum bundle; hence, the quantum Lagrangian yields a lift of the quantum connection ч of the extended quantum bundle to a connection χ of the functional quantum bundle.

Moreover, we can see that, if f is a quantisable function, then

$$\hat{f} = \mathfrak{i}\,(Y[f]_{\bullet} - f'' \lrcorner \nabla[\chi])$$

is the unique combination of $Z[f]$ and $\nabla[\chi]$, which yields an operator acting on the fibres of the functional quantum bundle. We have the following coordinate expression

$$\hat{f}(\Psi) = \big(-\tfrac{1}{2}\, f^0 \overset{o}{\Delta}_0 - \mathfrak{i}\, f^j \overset{o}{\nabla}_j + \overset{o}{f} + \tfrac{1}{2}\, k\, f^0\, r_0 - \mathfrak{i}\, \tfrac{1}{2}\, \frac{\partial_j (f^j \sqrt{|g|})}{\sqrt{|g|}} \big)\, \psi\, b\,.$$

The map $f \mapsto \hat{f}$ is injective. Moreover, $\hat{f}$ is Hermitian.

We assume $\hat{f}$ to be the Hermitian *quantum operator* associated with the quantisable function f. This is our *correspondence principle*.

We define the commutator of Hermitian fibred operators h, k of the functional quantum bundle by $[h\,,\,k] := -\,\mathfrak{i}\,(hk - kh)$. Then, for each quantisable functions f, g, we obtain the formula

$$[\hat{f}\,,\,\hat{g}] = \widehat{[\![f\,,\,g]\!]} + \big[(g'' \otimes Y[f]_{\bullet} - f'' \otimes Y[g]_{\bullet})\,,\, S\big]\,.$$

The second term in the above formula is the obstruction for the map $\hat{}\,: f \mapsto \hat{f}$ to be a morphism of Lie algebras. There is any substantial physical reason by which the map $\hat{}$ should be a morphism of Lie algebras? On the other hand, the restriction of the map $\hat{}$ to the subalgebra of affine functions yields an injective morphism of Lie algebras.

The Feynmann path integral formulation of Quantum Mechanics [7, 29] can be naturally expressed in our formalism; in particular, the Feynmann amplitudes arise naturally via parallel transport with respect to the quantum connection [49]. So the Feynmann path integral can be regarded as a further way to lift the quantum connection ч to a functional quantum connection.

In the particular case when spacetime is flat, our quantum dynamical equations turns out to be the standard Schrödinger equation and our quantum operators associated with spacetime coordinates, momenta and energy coincide with the standard operators.

Therefore, all usual examples of standard Quantum Mechanics are automatically recovered in our covariant scheme.

The above procedure can be easily extended to classical and quantum multi–body systems.

The above covariant theory can be extended to particles with spin; in this way, we obtain a generalised Pauli equation and all that.

Several techniques of the above theory (including the Lie algebra of quantisable functions and the corresponding Lie algebra of quantum vector fields) can be reproduced on a Lorentz manifold in a covariant way in the sense of Einstein. However, we do not know so far how to achieve a Hilbert stuff in this contest. It is possible that this problem has no solution (out of the Quantum Field Theory), as it is commonly believed.

2 Comparison

Covariant Quantum Mechanics (CQM) has several points in common with Geometric Quantisation (GQ) [1, 32, 90, 101]. The differences between the two theories arise from the fact that their basic goals are different: quantisation procedure and covariant formulation, respectively.

2.1 Quantisation

GQ can be regarded as a general programme aimed at "quantising" a classical system. More precisely, GQ is aimed at establishing a procedure in order to represent an algebra of functions of a classic symplectic manifold into a Hilbert space, according to some reasonable rules.

Perhaps, the original notion of "canonical quantisation" goes back to P. M. Dirac [12]. The first rigorous mathematical formulation of the notion of "geometric pre-quantisation" was due to I. E. Segal [89]; later J. M. Soriau [91] and B. Kostant [61] founded the Geometric Quantisation. This theory has been refined by several authors, see [3, 4, 69] and [34]. For instance, see [1], a "full quantisation" of a symplectic manifold (M, ω) is defined to be a pair $(\mathcal{H}, \delta)$ where $\mathcal{H}$ is a separable complex Hilbert space and δ is a map taking functions $f \in C^\infty(M)$ to self adjoint operators δ_f of $\mathcal{H}$ such that

(1) δ is $\mathbb{R}$-linear,

(2) $\delta_1 = \mathrm{id}_{\mathcal{H}}$,

(3) $[\delta_f, \delta_g] = \mathfrak{i}\, \hbar\, \delta_{\{f,g\}}$,

(4) if $\mathcal{A} \in C^\infty(M)$ is a complete subalgebra, i.e. if its centraliser with respect to the Poisson bracket is $\mathbb{R}$, then $\delta_{\mathcal{A}}$ acts irreducibly on $\mathcal{H}$.

There are *no go* theorems stating that there are no such quantisations in several situations [1, 32, 34, 35, 36, 37, 41, 38, 39, 40, 90, 101]; among them we mention the famous Groenewold - Van Hove theorem for the symplectic manifold $(\mathbb{R}^{2n}, \omega)$. If there is no full quantisation, then one looks for a subalgebra $\mathcal{O} \in C^\infty(M)$ to be quantised.

Since the requirement of a quantisation is too restrictive, one defines, as first step, a *pre-quantisation* by requiring just properties $(1, 2, 3)$. A pre-quantisation, called the *Dirac problem*, exists for every symplectic manifold, whose symplectic form defines an integer cohomology class.

In some respects, the aim of CQM is not the quantisation of a classical system. More precisely, we are just looking for a covariant formulation of the standard Quantum Theory [12, 80]. On the other hand, any quantum measurement is eventually constituted by classical observations, so we need to consider a classical spacetime as background of the Quantum Theory. Then, this background structure plays an important role in the Quantum Theory. But our heuristic geometric techniques are partially different from those of representations of Lie algebras. We observe also that in our formulation the classical spacetime and its structures, rather than the classical dynamics, determine the Quantum Theory.

Eventually, we do obtain a correspondence principle, which is a consequence of a classification theorem and not a postulate. But, this can be regarded as a quantisation only partially.

2.2 Covariance

The standard literature on GQ is not concerned with the special or general relativistic covariance of the theory. Indeed, the language of GQ is geometric, hence coordinate free; but this is not sufficient for attaining the covariance. In fact, in the standard literature on GQ, a given frame of reference is implicitly assumed.

On the other hand, CQM looks for a formulation of standard Quantum Mechanics, which be manifestly covariant (with respect to all frames of references, including accelerated frames), in the spirit of General Relativity.

Indeed, it would be natural to take a curved Lorentz manifold as classical background spacetime for such a theory. However, it is well known that there are serious physical difficulties to formulate the Special or General Relativistic Quantum Mechanics in this framework; actually, these difficulties led to the Quantum Field Theory. On the other hand, we realised that it is possible to keep the framework of Quantum Mechanics (Schrödinger and Pauli equations and all that) and formulate our covariant theory in a curved spacetime with absolute time and spacelike Riemannian metric. This approach stands in between the standard non relativistic Quantum Mechanics and a possible general relativistic Quantum Mechanics. In fact, our classical spacetime supports accelerated frames and several features of General Relativity (including the geometric interpretation of the gravitational field), but misses all features strictly related to the Lorentz metric (including the finite speed of signals).

In the flat case, this setting allows us to recover the standard non relativistic Quantum Mechanics. In the curved case, it suggests several new interpretations and techniques which might be possibly useful for a "true" general relativistic theory.

Thus, the covariance is the leading principle of our theory. Indeed, it turns out to be a powerful heuristic guide, as in all general relativistic theories. All main differences between our theory and GQ are related to the covariance.

Our covariant approach can be compared with a large literature dealing with Galileian General Relativity [20, 22, 23, 24, 25, 26, 42, 44, 64, 65, 66, 67, 70, 71, 74, 92, 93] and covariant formulations of Quantum Mechanics [21, 27, 28, 43, 45, 63, 68, 85, 88, 94].

2.3 Generality

The starting programme of GQ is quite general and is based on weak assumptions. In fact, GQ deals just with a symplectic manifold without further structure. On the other hand, strong symmetries of the framework are usually assumed and specified case by case.

CQM starts with a fibred manifold equipped with a spacelike metric and a fibre preserving linear connection, which fulfill a natural condition. In our opinion, the above geometric structure well reflects the physical features occurring in all examples of interest for Quantum Mechanics and yields in a functorial way any further object which is needed for the development of the classical theory (including the cosymplectic structure).

Thus, the CQM deals with a type of model more specific than that of GQ. On the other hand, the large generality of GQ is a beautiful mathematical feature, which, in practice, cannot be physically implemented in full extent. Actually, perhaps all concrete examples of physical interest that can be treated in the framework of GQ can be regarded as particular cases of our model.

2.4 Role of time

In GQ, time is essentially an exterior parameter. This theory basically deals with classical and quantum systems which do not depend explicitly on time. So, the starting classical configuration space is a manifold $\boldsymbol{S}$ which does not "include" time. If the theory needs to consider time, then it refers to the product manifold $\boldsymbol{E} \equiv \boldsymbol{T} \times \boldsymbol{S}$; the fact that spacetime is a product manifold means that a global observer has been implicitly chosen.

In CQM, the requirement that the theory be observer independent imposes that spacetime "includes" time but be not naturally split into space and time.

In Einsteinian General Relativity, spacetime $\boldsymbol{E}$ yields no observer independent time $\boldsymbol{T}$ and space $\boldsymbol{S}$, hence we have no observer independent projections $\boldsymbol{E} \to \boldsymbol{T}$ and $\boldsymbol{E} \to \boldsymbol{S}$. In our Galileian General Relativity, spacetime $\boldsymbol{E}$ is equipped with an observer independent time $\boldsymbol{T}$ and projection $\boldsymbol{E} \to \boldsymbol{T}$, but we have no observer independent space $\boldsymbol{S}$ and projection $\boldsymbol{E} \to \boldsymbol{S}$. In non relativistic GQ, spacetime $\boldsymbol{E}$ is equipped with time $\boldsymbol{T}$, space $\boldsymbol{S}$, and projections $\boldsymbol{E} \to \boldsymbol{T}$ and $\boldsymbol{E} \to \boldsymbol{S}$.

The further developments of our theory respect the starting assumption on the existence of absolute time without a preferred splitting of spacetime. Thus, all peculiar features of our theory follow from the covariance through the role of time. In particular, the cosymplectic structure of our phase space, the universality of the quantum connection, the method of projectability, the absence of the problem of polarisations and the construction of classical and quantum Hamiltonians are related to the role of time.

2.5 Phase space

In GQ the phase space is, in principle, any manifold supporting a symplectic form. Usually, the cotangent manifold of a manifold plays the role of phase space in virtue of the fact that it carries a canonical symplectic form. Thus, phase space has even dimension.

In CQM phase space is constituted by the first jet of sections of the spacetime fibred over time [60, 75, 87]. This choice is essential for the covariance of the theory. Thus, the phase space has odd dimension. Actually, the techniques related to even and odd phase spaces, respectively, present important differences.

On the other hand, any observer induces an affine fibred isomorphism of the first jet bundle with the vertical tangent bundle of spacetime (up to a time–scale factor); moreover, the spacelike metric induces a linear fibred isomorphism of the vertical tangent bundle with the vertical cotangent bundle of spacetime. Thus, breaking the covariance, the choice of an observer and the reference to the spacelike metric allow us to compare our phase space with that of GQ.

2.6 Symplectic and cosymplectic structures

In GQ the basic geometric structure of Classical Mechanics is constituted by a symplectic form and a Hamiltonian function of phase space. In principle, nothing else is necessary; in practice, one adds the fibring of the phase space over the configuration space and a suitable group of symmetry.

In CQM the geometric structure of Classical Mechanics is constituted by the spacetime fibred over time, the spacelike metric and the spacetime connection. These objects yield a cosymplectic form in a covariant way.

Any observer yields, by pullback and vertical restriction, a Riemannian symplectic form of the fibres of the vertical tangent bundle of spacetime. In this way we recover the analogue of the symplectic form of GQ. However, we stress that this symplectic form is not covariant and carries less information than the original cosymplectic form.

Furthermore, the cosymplectic form and the choice of an observer yield a classical Hamiltonian function. Thus, in CQM the cosymplectic form encodes the Hamiltonian (but an observer is needed to extract it).

2.7 Classical Lie brackets

In the original programme of GQ the classical Lie algebra to be quantised is the Poisson Lie algebra of all functions of the phase space, whose bracket is associated with the symplectic form. Actually, we have mentioned before that some obstructions to this quantisation programme occur, but there is no subalgebra $\mathcal{O}$ that can be consistently considered for all cases.

In CQM we do define the Poisson Lie algebra of all functions of phase space, whose bracket is associated with the cosymplectic form. However, we are only partially interested in this algebra. Indeed, we exhibit a new Lie algebra of special quadratic functions, which is involved in our Quantum Theory. This Lie algebra includes all functions which are usually quantised in the standard approaches.

2.8 Symmetries

In GQ the conserved quantities associated with the group of symmetries of the classical system are not necessarily quantisable. For a system whose phase space is a co-adjoint orbit for some group, one possibly imposes that the generators of the group are quantisable and act irreducibly on the Hilbert space. This is done in order to establish a correspondence between "elementary systems" at the classical and quantum levels, and can be considered as a sort of irreducibility condition.

In CQM there is no need for any specific group of symmetries acting on the classical system. Actually, the procedure of quantisation does not depend on such a group. On the other hand, the possible classical symmetries yield interesting consequences, including the momentum map [1, 77, 86]. In particular, all symmetries of the classical structure yield conserved quantisable functions [86].

2.9 Quantum structure

In GQ the quantum bundle is assumed to be a Hermitian line bundle over phase space. Moreover, the quantum bundle is assumed to be equipped with a Hermitian connection whose curvature is proportional to the classical symplectic form.

In CQM the quantum bundle is assumed to be a Hermitian line bundle over spacetime. Moreover, the extended quantum bundle is assumed to be equipped with a Hermitian connection whose curvature is universal and proportional to the classical cosymplectic form.

Thus the novelties of CQM consist in the following minimal assumptions: the quantum bundle lives on spacetime and not on the phase space, the quantum connection is universal.

2.10 Polarisation and projection method

In GQ one realises that the base space of the quantum bundle is too big in order to obtain an irreducible representation of the classical Poisson Lie algebra and to fulfill the uncertainty principle. Then, one looks for a polarisation P, that is for a Lagrangian subbundle P of the complexified tangent bundle of the phase space M, such that $D_{\mathbb{C}} = P \cap \bar{P}$ has constant rank and P, $P + \bar{P}$ are closed under the Lie bracket; moreover, the polarisation is said to be reducible if the quotient of the phase space M by the distribution D exists and the canonical projection $\pi : M \to M/D$ is a submersion. Once a polarisation is chosen, we can consider the polarised sections of the quantum bundle, that is the sections whose covariant derivative with respect to every vector field of the polarisation vanish. The polarised sections should yield the Hilbert space with the correct size. Actually, the problem of finding polarisations is very hard in practice, should be faced case by case and leads to a lot of complications and ambiguities, where the beauty of the original programme misses over considerably.

In CQM we have, in a covariant way, an implicit natural polarisation, namely the vertical polarisation.

The quantum connection is the only source of all further quantum objects, such as quantum dynamical equation, probability current, quantum operators and so on.

On the other hand, the quantum connection lives on the extended quantum bundle, while we are looking for further quantum objects living on the original quantum bundle. This goal is successfully achieved by the method of projectability.

Thus, in simple words, the difficult search for the inclusion of a polarisation (which should be performed case by case) is substituted by the easy search for projectable objects (which is successful and can be performed in general).

2.11 Half–densities and half–forms

In GQ, once a polarisation P is chosen, we take the polarised sections of the quantum bundle. The problem is how to build a Hilbert space. For instance, if P is reducible, then the Hermitian product $\langle s_1, s_2 \rangle$ of two polarised sections can be understood as a function on the quotient space M/D; but a problem arises because M/D has no natural volume element, which is necessary to define the Hilbert space of L^2–polarised sections. One way to remedy this problem is to tensor out the sections by the half–densities associated to D and to use the natural partial flat connection of D to build the Hilbert space. Unfortunately, even for the simplest physical systems, the results do not agree with those found in Quantum Mechanics.

Therefore, a further modification of the theory is needed: here is where half-forms come into play. They are defined through a metaplectic structure for the bundle of frames of the polarisation P; this imposes new conditions, since metaplectic structures do not exist in general. Even further problems arise, since it may happen that the tensor product of the quantum bundle with the bundle of half–forms has no polarised section at all.

In CQM we do not see any trouble concerning half–densities and all that. It seems that this convenient feature depends on the fact that CQM gets rid of all problems concerning polarisations.

In the first version of CQM the theory assumed a $\mathbb{C}$–valued Hermitian metric of the quantum bundle. In this case the theory, in order to prove that quantum operators are Hermitian, needed to use half–densities.

In the recent version of CQM the theory assumes a $(\mathbb{C} \otimes \Lambda^n V^* \boldsymbol{E})$–valued Hermitian metric of the quantum bundle. In this case the theory uses just sections of the quantum bundle and does not need half–densities at all.

2.12 Schroedinger equation

In GQ there is no clear Schrödinger equation in an explicit formulation ready to be directly compared with the Schrödinger equation of standard Quantum Mechanics. The

Schrödinger equation in this framework is understood as the infinitesimal generator of the flow of the Hamiltonian acting on the Hilbert space. The problem is that in general the Hamiltonian does not preserve the chosen polarisation and therefore we are led to compare different polarisations, this is handled through the Blattner-Kostant- Sternberg kernel machinery, see [4], which is extremely cumbersome. Even here one encounters new problems since there are obstructions discovered by Blattner [4] in order to construct the BKS kernel.

In CQM we have an explicit, covariant and intrinsic Schrödinger equation, which is immediately comparable with the standard Schrödinger equation [12, 80] (see also [13, 14, 15, 16, 17, 18, 19]). Moreover, we prove that it comes from a quantum Lagrangian.

2.13 Hilbert space and Hilbert bundle

GQ and standard Quantum Mechanics deal with a Hilbert space, which usually consists of L^2 sections of the quantum bundle.

CQM deals with a Hilbert bundle over time. This formulation is unusual but does not seem to be really in contrast with standard Quantum Mechanics.
Once more, this novelty is related to the explicit role of time in the CQM.

2.14 Feynmann path integral

In CQM the Feynmann amplitudes appear very naturally in terms of parallel transport with respect to the quantum connection. In fact, the classical Lagrangian turns out to play the role of local symbol of the quantum connection of the extended quantum bundle over time.
The proof of the equivalence of the Feynmann path integral with the covariant Schrödinger equation has not yet been worked out in detail.

2.15 Energy

Most of the practical difficulties of GQ run around the quantisation of energy. Actually, the classical Hamiltonian function has to be explicitly postulated and is not encoded in the basic structure of spacetime. Moreover, the energy requires a treatment quite different from the simpler approach required by other observables such as spacetime coordinates and momentum.

Even in CQM the energy has a special role, but no hard problems arise in this respect. First of all, we stress that energy is encoded in the basic geometric structure of spacetime and that it appears explicitly, in a non covariant way, by means of the choice of an observer.
If one accepts the point of view of Covariant Quantum Mechanics, maybe one can understand the difficulties of GQ from this perspective. In fact, in practice what is done

in GQ is to take the vertical tangent (or cotangent) space of spacetime as phase space, instead of the first jet space; accordingly, GQ tries to formulate and quantise energy by methods related to vertical subspace. On the other hand, in CQM, it is very clear that energy is related to the horizontal aspect of phase space. We could roughly say that the vertical aspect of phase space is essentially related to the static geometry of spacetime, while the horizontal aspect is related to the dynamics. This observation can also be analysed in terms of the Lie algebra of quantisable functions and their tangent lift; actually, the quantisable functions dealing with the vertical aspects of spacetime constitute the subalgebra of affine functions, while energy is a quadratic function. In CQM, all quantisable functions can quantised on the same footing.

2.16 Examples

In GQ it is not granted that every reasonable physical example can be worked out. Actually, there are few examples that have been successfully solved.

CQM reduces to standard Quantum Mechanics in the flat case. Hence, in CQM, all standard physical examples can be formulated; moreover, the Schrödinger equation and quantum operators corresponding to the quantisable functions can be explicitly and immediately computed. Of course, the integration of the Schrödinger equation and the computation of the energy spectrum is an analytical question which should be faced case by case.

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