Quantisable functions in general relativity

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This a reprint of a published paper: 26.11.2000

The original paper has been written on 27.07.1996.
The original paper has been published in
"Opérateurs différentiels et Physique Mathématique",
J. Vaillant, J. Carvalho e Silva Eds., Textos Mat. Ser. B, **24**, 2000, 161–181.

Abstract

A Lie algebra of functions on the phase space of a classical general relativistic particle is introduced by means of a projectability criterion for the Hamiltonian lift. This algebra includes the coordinate, the momentum and the Hamiltonian functions. The bracket is a natural deformation of the Poisson bracket. Moreover, a homomorphism of this Lie algebra to the Lie algebra of spacetime vector fields is shown.

These constructions have been inspired by analogous results in the Galilei framework, which led to a new quantisation scheme. The present results are expected to lead to an analogous covariant quantisation procedure in the Einstein framework.

Key words: Lie algebra of functions, Hamiltonian lift of functions, tangent lift of functions, quantisation.
1991 MSC: 17B66, 70H40, 58F05, 58F06.

This research has been supported by grant No. 201/96/0079 of GA CR, GNFM of CNR, MURST, University of Florence and Contract ERB CHRXCT 930096 of EEC.

Introduction

A new geometric approach to quantum mechanics on a curved spacetime with absolute time has been proposed in [1, 2, 3]. This general relativistically covariant formulation of quantum mechanics reduces to the standard quantum mechanics in the flat case. The procedure has evident analogies with geometric quantisation but also relevant novelties. The main differences are related to the essential role of time, which is not just a parameter. Accordingly, spacetime is assumed to be a curved manifold fibred over time and equipped with a vertical Riemannian metric, a gravitational spacetime connection and an electromagnetic field. The jet bundle of spacetime sections is taken as classical phase space. The spacetime structures induce a connection, a second order connection and a cosymplectic form on the phase space. The gravitational and electromagnetic objects are naturally coupled, yielding a total version of the above objects. The interaction of metric and gravitational spacetime connection is expressed by the closure of the cosymplectic form. Then, a quantum bundle over classical spacetime equipped with a quantum connection is considered. A quantum connection is assumed on the pullback bundle of the quantum bundle over the classical phase space; it is supposed to be Hermitian, with curvature proportional to the cosymplectic form and universal. The cohomological problem of existence is discussed in [8]. The universality property allows us to skip the troubles of polarisation. The quantum connection is the only assumption for the quantum theory. The fact that the quantum connection lives on the bullbacked quantum bundle depends on its relation to the classical cosymplectic form. On the other hand, the wave sections and related equations and operators must live on the original quantum bundle. Such reductions are systematically achieved by a projectability criterion, which yields the physically significant objects in a covariant way. This principle yields a distinguished Lagrangian, hence the generalised Schrödinger equation on the curved background. Moreover, this principle allows us to select a Lie algebra of classical functions (which admit a Hamiltonian lift projectable over spacetime) and a Lie algebra of vector fields over the pullbacked quantum bundle (which preserve the quantum structures and are projectable over the quantum bundle). These Lie algebras turn out to be naturally isomorphic. This is the proposed quantisation procedure, which eventually leads to the covariant quantum operators and commutators. The role played by time throughout the theory leads us to the quantum energy operator in a direct way.

Obviously, the starting hypothesis that spacetime is fibred over absolute time is physically unsatisfactory and produces a theory able to fit physical phenomena only in a certain approximation. Actually, this approximation stands in between that of the standard non relativistic and true general relativistic theories. On the other hand the interest of the above approach depends on the fact that the geometric procedures which have been developed are suitable for extension to the true general relativistic case.

The present paper is aimed at developing a programme of quantisation according to the above scheme in the Einstein's general relativistic context. In [5] we have analysed the geometric structure of the classical general relativistic phase space; here, the jet bundle of sections is replaced by the jet bundle of time–like 1–dimensional submanifolds and the Lorentz metric allows us to recover the analogous of the contact structure. In this paper we study the Lie algebra of quantisable functions. Actually, we show a Lie algebra of functions on the phase space which admit a lift to vector fields over spacetime. These functions and the corresponding vector fields are candidate to be interpreted as quantisable functions and pre–quantum operators.

We denote the sheaf of local smooth function on a manifold M by $C^\infty(M)$.

We assume the following fundamental unit spaces [2]:
(1) the oriented 1–dimensional vector space $\mathbb{T}$ over $I\!R$ of *time intervals*,
(2) the positive 1–dimensional semi–vector space $\mathbb{L}$ over $I\!R^+$ of *lengths*,
(3) the positive 1–dimensional semi–vector space $\mathbb{M}$ over $I\!R^+$ of *masses*.

A *time unit of measurement* is defined to be an oriented base of $\mathbb{T}$ or its dual

$$u_0 \in \mathbb{T}\,, \qquad u^0 \in \mathbb{T}^*\,.$$

Moreover, we refer to the *light velocity* and the *Planck constant*

$$c \in \mathbb{T}^{+*} \otimes \mathbb{L}\,, \qquad \hbar \in \mathbb{T}^{+*} \otimes \mathbb{L}^2 \otimes \mathbb{M}\,,$$

and consider a classical particle with *mass* and charge

$$m \in \mathbb{M}\,, \qquad q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}\,.$$

1 Geometry of phase space

In this Section we recall basic geometric properties of phase space in the Einstein general relativistic framework. For further details and proofs see [5].

1.1 Phase space

We start with the basic assumptions and definitions.

We assume *space–time* to be a 4–dimensional oriented and time–oriented manifold M equipped with a scaled Lorentzian metric of signature $(+---)$

$$g : M \to \mathbb{L}^2 \otimes T^*M \underset{M}{\otimes} T^*M\,.$$

Local coordinate charts on M will be denoted by (x^λ), $\lambda = 0,1,2,3$. We stress that we find it convenient to assume the generic coordinate function $x^\lambda : M \to I\!R$ to be adimensional.

The coordinate expression of g is then

$$g = g_{\lambda\mu} d^\lambda \otimes d^\mu\,, \quad g_{\lambda\mu} : M \to \mathbb{L}^2 \otimes I\!R\,.$$

In what follows we shall use local coordinate charts such that the vector ∂_0 is time–like and time oriented and $\partial_1, \partial_2, \partial_3$ are space–like; hence $g_{00} > 0, g_{11}, g_{22}, g_{33} < 0$.

Latin indices $i, j, p, \dots$ will span space–like coordinates, while Greek indices λ, μ, ϕ, $\dots$ will span space–time coordinates.

A k–jet of 1–dimensional submanifolds of M at $x \in M$ is defined to be an equivalence class of 1–dimensional submanifolds touching each other at x with a contact of order k, [7]. The k–jet of a 1-dimensional submanifold $l \subset M$ at $x \in l$ is denoted by $j_k l(x)$. The set of all k–jets of 1–dimensional submanifolds of M can be equipped, in a natural way, with a smooth structure. The corresponding manifold is denoted by $J_k(M, 1)$ and its projection on M is denoted by $\pi^k_0 : J_k(M, 1) \to M$.

We have the canonical fibred isomorphism over M of the first jet bundle with the Grassmannian bundle of dimension 1

$$J(M, 1) \equiv J_1(M, 1) \to \mathrm{Grass}(M, 1) : \phi \mapsto L_\phi,$$

where $\phi \in J(M, 1)$ and $L_\phi \subset T_{\underline{\phi}}M$ is the tangent space at $\underline{\phi} = \pi^1_0(\phi)$ of 1–dimensional submanifolds generating ϕ.

A local chart on M is said to be *divided* if the set of its coordinate functions is divided into two subsets of 1 and $(\dim M - 1)$ elements. Our typical notation for a divided chart will be

$$(x^0, x^i)\,, \qquad 1 \le i \le \dim M - 1\,.$$

A divided chart and a 1–dimensional submanifold $l \subset M$ are said to be *related* if the submanifold l can be expressed locally by formulas of the type

$$x^i = l^i(x^0)\,;$$

i.e. , more precisely $x^i|l = l^i \circ x^0|l$, with $l^i : I\!\!R \to I\!\!R$.

Every divided chart on M determines canonically a local fibred chart

$$(x^0, x^i; x^i_0)$$

on $J(M, 1)$. We shall always refer to such charts.

Thus we can write

$$x^i_0 \circ j_1 l = \partial_0 l^i \equiv (Dl^i) \circ (x^0|l)\,.$$

A *motion* in M is defined to be a 1–dimensional time–like submanifold $l \subset M$.

We define the *phase space*

$$UM \equiv U_1 M \subset J(M, 1)$$

to be the subspace of all 1–jets of motions. Hence, $\phi = j_1 l(\underline{\phi}) \in J(M, 1)$ belongs to UM if and only if $L_\phi = T_{\underline{\phi}} l$ lies inside the light cone.

1.2 Contact structure

The geometric structure of the phase space and the Lorentz metric allow us to recover the contact structure.

We have the following *contact maps*

$$Д : UM \to \mathbb{T}^* \otimes TM \,, \qquad \tau := \frac{g^\flat}{c^2} \circ Д : UM \to \mathbb{T} \otimes T^*M \,,$$

with coordinate expressions

$$Д = c\,\alpha\, Д_0 = c\,\alpha\,(\partial_0 + x_0^i \partial_i)\,, \qquad \tau \equiv \tau_\lambda\, d^\lambda = \frac{\alpha}{c}\,(g_{0\lambda} + g_{i\lambda} x_0^i) d^\lambda \,,$$

where

$$\alpha = 1/\|Д_0\| = 1/\sqrt{g_{00} + 2 g_{0j} x_0^j + g_{ij} x_0^i x_0^j} \in \mathbb{L}^* \,.$$

We have

$$Д \lrcorner \tau = 1 \,,$$

i.e. , in coordinates,

$$c\,\alpha\,(\tau_0 + \tau_h x_0^h) = 1 \,.$$

In order to interpret the above maps let us consider a motion $l \subset M$, an observer $o : M \to UM$ and any chart (x^0, x^i) adapted to o.

Then, the 4–velocity of l with respect to its *proper time* (expressed by means of length units of measurement) and with respect to the time function of the observer o (expressed by means of time unit of measurement) are, respectively, the vector fields with coordinate expressions

$$Д \circ jl = c\,\alpha\,(\partial_0 + x_0^i \partial_i)|jl \,, \qquad v := \partial_0 + x_0^i \partial_i \,.$$

Hence, the ratio between the proper time and the observer time along the motion is given by

$$\frac{ds}{dt} := c\,\alpha|jl \,.$$

The above formula corresponds to the following standard formula of special relativity, expressed by classical notation,

$$\frac{ds}{dt} = \frac{c}{\sqrt{1 - \beta^2}} \,.$$

1.3 Splitting of the tangent space of spacetime

The contact structure induces a natural splitting of the tangent space of spacetime into two orthogonal components over the phase space.

We define the vector bundles over UM

$$T^{\|}M := \{(\phi, X) \in UM \underset{M}{\times} TM \mid X \in L_\phi\}\,,$$
$$T^{\perp}M := \{(\phi, X) \in UM \underset{M}{\times} TM \mid X \in L^{\perp}_\phi\}\,,$$

which yield the splitting

$$UM \underset{M}{\times} TM = T^{\|}M \underset{UM}{\oplus} T^{\perp}M\,.$$

We observe that

$$T^{\|}M = \{(\phi, X) \in UM \underset{M}{\times} TM \mid X \in Д(\phi)(\mathbb{T})\}\,,$$
$$T^{\perp}M = \{(\phi, X) \in UM \underset{M}{\times} TM \mid X \,\lrcorner\, \tau(\phi) = 0\}\,.$$

The following mutually dual local bases of vector fields and forms are adapted to the above splitting

$$Д_0 := \partial_0 + x^i_0 \partial_i\,, \qquad b_i := \partial_i - c\alpha\tau_i Д_0\,,$$
$$\lambda^0 := d^0 + c\alpha\tau_i\theta^i = c\alpha\tau\,, \qquad \theta^i := d^i - x^i_0 d^0\,.$$

It is convenient to introduce the matrices, with respect to a natural base,

$$h_{i\mu} := g_{i\mu} - c^2\tau_i\tau_\mu\,, \qquad h^{i\mu} := g^{i\mu} - x^i_0 g^{0\mu}\,,$$

and the mutually inverse matrices, with respect to an adapted base,

$$g^{\perp}{}_{ij} := g_{ij} - c^2\tau_i\tau_j\,, \qquad g_{\perp}{}^{ij} := g^{ij} - g^{i0}x^j_0 - g^{j0}x^i_0 + g^{00}x^i_0 x^j_0\,.$$

Then, we obtain the following coordinate expressions

$$g^{\flat} \circ b_i = g^{\perp}{}_{ij}\theta^j = h_{i\mu} d^\mu\,, \qquad g^{\sharp} \circ \theta^i = g_{\perp}{}^{ij} b_j = h^{i\mu}\partial_\mu\,.$$

Moreover, the parallel and orthogonal projections

$$\lambda : UM \underset{M}{\times} TM \to T^{\|}M\,, \qquad \theta = 1_M - \lambda : UM \underset{M}{\times} TM \to T^{\perp}M$$

have the coordinate expressions

$$\lambda = \lambda^0 \otimes Д_0\,, \qquad \theta = h^{i\mu}h_{i\nu} d^\nu \otimes \partial_\mu\,.$$

Additionally, we have a natural linear fibred isomorphism over UM

$$v^{\perp} : VUM \to \mathbb{T}^* \otimes T^{\perp}M \,,$$

with the coordinate expression

$$v^{\perp} = c\,\alpha\, d^i_0 \otimes b_i \,, \qquad v^{\perp -1} = \frac{1}{c\,\alpha} \theta^i \otimes \partial^0_i \,.$$

1.4 Connections, 2–forms and 2–vectors

A linear connection of spacetime induces naturally on the phase space a connection, a second order connection, a 2–form and a 2–vector, which fulfill certain relations.

A linear connection K on the vector bundle $\pi_M : TM \to M$ can be expressed, equivalently, by a tangent valued form, or by a vertical valued form

$$K : TM \to T^*M \underset{TM}{\otimes} TTM \,, \qquad \nu_K : TM \to T^*TM \underset{TM}{\otimes} TM \,,$$

respectively, with coordinate expressions

$$K = d^\phi \otimes (\partial_\phi + K_\phi{}^\mu{}_\psi \, \dot{x}^\psi \, \dot{\partial}_\mu) \,, \qquad \nu_K = (\dot{d}^\mu - K_\phi{}^\mu{}_\psi \, \dot{x}^\psi \, d^\phi) \otimes \partial_\mu \,,$$

where $K_\phi{}^\mu{}_\psi \in C^\infty(M)$ and $(x^\phi, \dot{x}^\phi)$ is the induced coordinate chart on TM.

We observe that a linear connection ν_K on $TM \to M$ induces a linear connection $\nu' : T(\mathbb{T}^* \otimes TM) \to \mathbb{T}^* \otimes TM$ on the vector bundle $\mathbb{T}^* \otimes TM \to M$, with coordinate expression

$$\nu'_K = u^0 \otimes (\dot{d}^\mu_0 - K_\phi{}^\mu{}_\psi \, \dot{x}^\psi_0 \, d^\phi) \otimes \partial_\mu \,,$$

where $(x^\phi, \dot{x}^\phi_0)$ denotes the induced chart on $\mathbb{T}^* \otimes TM$.

A connection Γ on UM can be expressed, equivalently, by a tangent valued form, or by a vector valued form

$$\Gamma : UM \to T^*M \underset{UM}{\otimes} TUM \,, \qquad v^{\perp} \circ \nu_\Gamma : UM \to T^*UM \underset{UM}{\otimes} (\mathbb{T}^* \otimes T^{\perp}M) \,,$$

with coordinate expressions

$$\Gamma = d^\phi \otimes (\partial_\phi + \Gamma_\phi{}^i_0 \, \partial^0_i) \,, \qquad v^{\perp} \circ \nu_\Gamma = c\,\alpha (d^i_0 - \Gamma_\phi{}^i_0 \, d^\phi) \otimes b_i \,,$$

where $\Gamma_\phi{}^i_0 \in C^\infty(UM)$.

For any linear connection K on TM the map

$$\nu_\Gamma = v^{\perp -1} \circ \theta \circ \nu'_K \circ T Д$$

turns out to be a connection on the bundle $UM \to M$ with coordinate expression

$$\Gamma_{\phi 0}^{\ i} = K_{\phi}{}^{i}{}_{j}\, x_0^j + K_{\phi}{}^{i}{}_{0} - x_0^i (K_{\phi}{}^{0}{}_{j}\, x_0^j + K_{\phi}{}^{0}{}_{0})\,.$$

A connection Γ on UM and the metric g yield the scaled 2–form on UM

$$\Omega(g,\Gamma) := (v^{\perp} \circ \nu_{\Gamma}) \bar{\wedge} \theta : UM \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^* UM\,,$$

where $\bar{\wedge}$ denotes the wedge product and the contraction via the metric g.

We have the coordinate expression

$$\Omega(g,\Gamma) = c\,\alpha\, h_{i\mu}\,(d_0^i - \Gamma_{\phi 0}^{\ i}\, d^{\phi}) \wedge d^{\mu}\,.$$

Moreover, a connection Γ on UM and the metric g yield the scaled 2–vector on UM

$$\tilde{\Omega}(g,\Gamma) := \Gamma\, \tilde{\bar{\wedge}}\, v^{\perp -1} : UM \to \mathbb{T} \otimes \mathbb{L}^{*2} \otimes \wedge^2 TUM\,,$$

where $\tilde{\bar{\wedge}}$ denotes the wedge product and the contraction via the dual metric

$$\tilde{g} : M \to \mathbb{L}^{*2} \otimes TM \underset{M}{\otimes} TM\,.$$

We have the coordinate expression

$$\tilde{\Omega}(g,\Gamma) = \frac{1}{c\,\alpha}\, h^{i\lambda} (\partial_{\lambda} + \Gamma_{\lambda 0}^{\ j}\, \partial_j^0) \wedge \partial_i^0\,.$$

A *second order connection* on UM is defined to be a scaled vector field

$$\gamma : UM \to \mathbb{T}^* \otimes TUM\,,$$

such that

$$T\pi_0^1 \circ \gamma = Д : UM \to \mathbb{T}^* \otimes TM\,, \qquad \tau \lrcorner \gamma = 1 : UM \to \mathbb{T}^* \otimes \mathbb{T}\,.$$

The coordinate expression of γ is of the type

$$\gamma = c\,\alpha\,(\partial_0 + x_0^i\, \partial_i + \gamma_{00}^{\ i}\, \partial_i^0)\,, \qquad \gamma_{00}^{\ i} \in C^{\infty}(UM)\,.$$

There is a unique second order connection γ such that

$$\gamma \lrcorner \Omega(g,\Gamma) = 0\,.$$

Namely,

$$\gamma = Д \lrcorner \Gamma : UM \to \mathbb{T}^* \otimes TUM$$

with coordinate expression

$$\gamma^i_{00} = \Gamma_k{}^i_0 \, x^k_0 + \Gamma_0{}^i_0 \,.$$

1.5 Gravitational objects

First of all, we consider the objects introduced in the above section, that come from the Lorentz metric.

The metric g yields the *gravitational* connection on TM, the *gravitational* connection on UM, the *gravitational* second order connection on UM

$$K^\natural := \varkappa \,, \qquad \nu_{\Gamma^\natural} := v^{\perp -1} \circ \theta \circ \nu'_{K^\natural} \circ T\text{Д} \,, \qquad \gamma^\natural := \text{Д} \lrcorner \Gamma^\natural \,,$$

respectively, where $\varkappa$ is the Levi–Civita connection with the Christoffel symbols

$$\varkappa^\sigma_{\phi\psi} = -\frac{g^{\sigma\tau}}{2}(\partial_\phi g_{\tau\psi} + \partial_\psi g_{\tau\phi} - \partial_\tau g_{\phi\psi}) \,.$$

Moreover, the *gravitational* 2–form

$$\Omega^\natural := \Omega(g, \Gamma^\natural)$$

turns out to be the contact 2–form generated by $c^2\tau$; namely, we obtain the equality

$$\Omega^\natural = c^2 d\tau$$

and the volume form

$$c^2\tau \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural : UM \to \mathbb{T}^{*4} \otimes \mathbb{L}^8 \otimes \wedge^7 T^*UM \,,$$

with coordinate expression

$$c^2\tau \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural = 6\, c^4 \alpha^4 \,|g|\, d^0 \wedge d^1 \wedge d^2 \wedge d^3 \wedge d^1_0 \wedge d^2_0 \wedge d^3_0 \,.$$

Eventually, we have the *gravitational* 2–vector

$$\tilde{\Omega}^\natural := \tilde{\Omega}(g, \Gamma^\natural) := \Gamma^\natural \,\tilde{\bar{\wedge}}\, v^{\perp -1} : UM \to \mathbb{T} \otimes \mathbb{L}^{*2} \otimes \wedge^2 TUM \,.$$

1.6 Total connections and 2–forms

Next, we deform the above geometric structures by a suitable coupling with electromagnetic field.

Thus, we assume the *electromagnetic field* to be a closed scaled 2–form on M

$$F : M \to (\mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes \wedge^2 T^*M \,.$$

We define the *'universal' electric field* to be the scaled 1–form on UM

$$E := -Д \lrcorner F : UM \to (\mathbb{T}^{*2} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes T^*_{\perp}M \,.$$

Hence, the *electric field* associated with an observer $o : M \to UM$ is just the pullback form

$$o^*E : M \to (\mathbb{T}^{*2} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes T^*M \,.$$

We stress that the universal electric field carries the full information of the electromagnetic field.

We have the coordinate expressions

$$E = E_i \theta^i = E_j d^j + E_0 d^0 \,,$$

where

$$E_j = -c\alpha(F_{0j} + F_{hj}\, x^h_0) \qquad E_0 = -c\alpha\, F_{h0}\, x^h_0 \,.$$

Next, we show that the electromagnetic field can be naturally incorporated into the gravitational structures of the phase space. Namely, we obtain *total* objects obtained correcting the gravitational objects by an electromagnetic term, in such a way to preserve their original relations.

For this purpose we need a suitable coupling constant. So, we consider a particle with a given mass and charge

$$m \in \mathbb{M} \,, \qquad q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \,,$$

and refer to the coupling constant

$$\frac{q}{m} \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{*1/2} \,.$$

We start from the obvious coupling of the electromagnetic field with the gravitational contact 2–form on UM. Accordingly, we define the *total 2–form* to be

$$\Omega := \Omega^\natural + \frac{q}{2mc} F : UM \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^*UM \,.$$

Of course, we obtain

$$d\Omega = 0 \,.$$

Moreover,

$$c^2 \tau \wedge \Omega \wedge \Omega \wedge \Omega = c^2 \tau \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural$$

is a scaled volume form on UM.

There is a unique second order connection γ such that [5]

$$\gamma \lrcorner \Omega = 0\,.$$

Namely, γ is given by

$$\gamma = \gamma^{\natural} + \gamma^{e}\,,$$

where

$$\gamma^{e} = \frac{q}{mc} v^{\perp -1} \circ g^{\sharp} \circ E\,,$$

is just the *Lorentz force* with coordinate expression

$$\gamma^{e} = -\frac{q}{mc}\, h^{i\mu}(F_{0\mu} + F_{j\mu}\, x^{j}_{0})\partial^{0}_{i} = \frac{q}{mc^{2}\alpha}\, g^{ij}_{\perp} E_{j}\, \partial^{0}_{i}\,.$$

Given a section

$$H : UM \to T^{*}M \underset{UM}{\otimes} VUM\,,$$

we define the map

$$A(v^{\perp} \circ H) : UM \to \mathbb{T}^{*} \otimes \wedge^{2} T_{\perp}{}^{*}M$$

by means of the composition

$$UM \xrightarrow{H} T^{*}M \underset{UM}{\otimes} VUM \xrightarrow{\theta^{*}\otimes(g^{\flat}\circ v^{\perp})} \mathbb{T}^{*} \otimes T_{\perp}{}^{*}M \underset{UM}{\otimes} T_{\perp}{}^{*}M \xrightarrow{\mathrm{Alt}} \mathbb{T}^{*} \otimes \wedge^{2} T_{\perp}{}^{*}M\,,$$

where $\theta^{*} : UM \times_{M} T^{*}M \to T_{\perp}{}^{*}M$ is the transpose of $\theta : UM \times_{M} TM \to T^{\perp}M$.

There is a unique connection $\Gamma : UM \to T^{*}M \underset{UM}{\otimes} VUM$, such that [5]

$$Д \lrcorner \Gamma = \gamma \qquad (v^{\perp} \circ \nu_{\Gamma}) \bar{\wedge} \theta = \Omega\,, \qquad A(v^{\perp} \circ (\Gamma - \Gamma^{\natural})) = 0\,.$$

Namely,

$$\Gamma = \Gamma^{\natural} + \Gamma^{e}\,,$$

where

$$\Gamma^{e} = -\frac{q}{2mc} v^{\perp -1} \circ g^{\sharp 2} \circ (F + 4\tau \wedge E)\,,$$

i.e. , in coordinates

$$\Gamma^{e} = \frac{q}{2mc^{2}\alpha} g_{\perp}{}^{ij} \Big(E_{j}\tau - (F_{hj} + \tau_{h} E_{j} - \tau_{j} E_{h})\theta^{h} \Big) \otimes \partial^{0}_{i}\,.$$

The total phase connection Γ yields the total 2–vector

$$\tilde{\Omega} := \Gamma \,\bar{\tilde{\Lambda}}\, v^{\perp -1} : UM \to \mathbb{T} \otimes \mathbb{L}^{*2} \otimes \wedge^2 TUM\,.$$

We can write

$$\tilde{\Omega} = \tilde{\Omega}^\natural + \Gamma^e \,\bar{\tilde{\Lambda}}\, v^{\perp -1}$$

and the coordinate expression of the electromagnetic term is

$$\Gamma^e \,\bar{\tilde{\Lambda}}\, v^{\perp -1} = -\frac{q}{2mc^3\alpha^2}\, {g_\perp}^{ih}\, {g_\perp}^{jk} (F_{kh} + \tau_k E_h - \tau_h E_k)\, \partial^0_i \wedge \partial^0_j\,.$$

We can normalise the scaled total 2–form Ω and 2–vector $\tilde{\Omega}$ and obtain the true total 2–form and 2–vector

$$\frac{m}{\hbar}\Omega : UM \to \wedge^2 T^*UM\,, \qquad \frac{\hbar}{m}\tilde{\Omega} : UM \to \wedge^2 TUM\,.$$

From now on we shall refer to the total objects

$$\Gamma\,, \qquad \gamma\,, \qquad \Omega\,, \qquad \tilde{\Omega}\,.$$

2 Tangent space of phase space

In this section we analyse further geometric structures of the phase space, such as the splitting of its tangent space and the musical maps induced by the cosymplectic form.

2.1 Splitting of the tangent space of phase space

We start with the splittings of the sheaves $\mathcal{T}(UM)$ of vector fields and $\mathcal{T}^*(UM)$ of 1–forms on UM induced by the 1–form τ and the second order connection γ. We observe that this splitting can be introduced on the tangent and cotangent bundles of the phase space as well.

We introduce the subsheaves

$$\mathcal{T}^\perp_\tau \subset \mathcal{T}(UM)\,, \qquad \mathcal{T}^\gamma_\perp \subset \mathcal{T}^*(UM)\,,$$

defined by

$$\mathcal{T}^\perp_\tau := \{X \in \mathcal{T}(UM) \mid X \lrcorner\, \tau = 0\}\,, \qquad \mathcal{T}^\gamma_\perp := \{\phi \in \mathcal{T}^*(UM) \mid \gamma \lrcorner\, \phi = 0\}\,,$$

and the subsheaves

$$\langle\gamma\rangle \subset \mathcal{T}(UM)\,, \qquad \langle\tau\rangle \subset \mathcal{T}^*(UM)$$

generated by γ and τ by means of $\mathbb{T}-$ and $\mathbb{T}^*-$ valued functions, respectively.

2.1. Proposition. We have the splittings

$$\mathcal{T}(UM) = \langle\gamma\rangle \oplus \mathcal{T}^{\perp}_{\tau}\,, \qquad \mathcal{T}^*(UM) = \langle\tau\rangle \oplus \mathcal{T}^{\gamma}_{\perp}\,,$$

given by

$$X = \gamma(X \lrcorner \tau) + \Big(X - \gamma(X \lrcorner \tau)\Big)\,, \qquad \phi = \tau(\gamma \lrcorner \phi) + \Big(\phi - \tau(\gamma \lrcorner \phi)\Big)\,,$$

and the natural sheaf isomorphisms

$$\langle\gamma\rangle^* \simeq \langle\tau\rangle\,, \qquad \mathcal{T}^{\perp *}_{\tau}(UM) \simeq \mathcal{T}^{\gamma}_{\perp}(UM)\,.\square$$

We have the coordinate expressions

$$\gamma(X \lrcorner \tau) = c\,\alpha\,\tau_\lambda X^\lambda(\partial_0 + x^i_0\,\partial_i + \gamma^i_{00}\,\partial^0_i)\,, \quad \tau(\gamma \lrcorner \phi) = c\,\alpha(\phi_0 + \phi_p\,x^p_0 + \phi^0_p\,\gamma^p_{00})\tau_\lambda d^\lambda\,.$$

2.2. Remark. Obviously, we have

$$\mathcal{V}(UM) \subset \mathcal{T}^{\perp}_{\tau}(UM)\,, \qquad T\pi^1_0(\mathcal{T}^{\perp}_{\tau}(UM)) = \mathcal{T}^{\perp}M\,,$$

where $\mathcal{V}(UM)$ denotes the sheaf of vertical vector fields of the bundle $UM \to M$. $\square$

2.2 Adapted bases

We can introduce new local bases of vector fields and 1–forms on UM, which are more suitable for coordinate calculations.

2.3. Lemma. The following mutually dual local bases of vector fields and 1–forms on UM are adapted to the splittings induced by γ and τ

$$B_0 = \partial_0 + x^j_0\partial_j + \gamma^i_{00}\partial^0_i\,, \qquad B_i = \partial_i - c\,\alpha\,\tau_i(\partial_0 + x^j_0\,\partial_j + \gamma^j_{00}\,\partial^0_j)\,, \qquad B^0_i = \partial^0_i\,,$$
$$\lambda^0 = d^0 - c\,\alpha\,\tau_i\theta^i\,, \qquad \theta^i = d^i - x^i_0\,d^0\,, \qquad \theta^i_0 = d^i_0 - \gamma^i_{00}\,d^0\,.\square$$

2.4. Remark. If $X = X^0B_0 + X^iB_i + X^i_0B^0_i \in \mathcal{T}(UM)$, then

$$\gamma(X \lrcorner \tau) = X^0B_0 \in \langle\gamma\rangle \qquad X - \gamma(X \lrcorner \tau) = X^iB_i + X^i_0B^0_i \in \mathcal{T}^{\perp}_{\tau}(UM)\,.$$

If $\phi = \phi_0\theta^0 + \phi_i\theta^i + \phi^0_i\theta^i_0 \in \mathcal{T}^*(UM)$, then

$$\tau(\gamma \lrcorner \phi) = \phi_0\theta^0 \in \langle\tau\rangle \qquad \phi - \tau(\gamma \lrcorner \phi) = \phi_i\theta^i + \phi^0_i\theta^i_0 \in \mathcal{T}^{\gamma}_{\perp}(UM)\,.\square$$

2.5. Remark. The coordinate expressions of the form Ω and the 2-vector $\tilde{\Omega}$ in an adapted base are

$$\Omega = c\,\alpha\,g^{\perp}{}_{ij}\,(\theta^i_0 - \Gamma_h{}^i{}_0\theta^h) \wedge \theta^j\,,$$
$$\tilde{\Omega} = \frac{1}{c\,\alpha}\,g_{\perp}{}^{ij}\,(B_j + \Gamma_j{}^h{}_0 B^0_h) \wedge B^0_i\,.\square$$

2.3 Musical mappings

The cosymplectic form yields an isomorphism between the contravariant and covariant orthogonal components of the above splitting.

2.6. Lemma. The total sections

$$\frac{m}{\hbar}\Omega : UM \to \wedge^2 T^* UM \,, \qquad \frac{\hbar}{m}\tilde{\Omega} : UM \to \wedge^2 TUM$$

yield the linear sheaf epimorphisms

$$\mathcal{T}(UM) \to \mathcal{T}^{\gamma}_{\perp}(UM) : X \mapsto \frac{m}{\hbar} i_X \Omega \,, \qquad \mathcal{T}^*(UM) \to \mathcal{T}^{\perp}_{\tau}(UM) : \phi \mapsto \frac{\hbar}{m} i_{\phi}\tilde{\Omega} \,,$$

with coordinate expressions, in a canonical base,

$$\begin{aligned} \frac{m}{\hbar} i_X \Omega &= \frac{m\,c\,\alpha}{\hbar}\Big((h_{i\lambda} X^i_0 + h_{i\mu}\, X^{\mu}\, \Gamma_{\lambda 0}{}^{i} - h_{i\lambda}\Gamma_{\phi 0}{}^{i}\, X^{\phi})\, d^{\lambda} - h_{i\mu}\, X^{\mu}\, d^i_0\Big) \,, \\ \frac{\hbar}{m} i_{\phi}\tilde{\Omega} &= \frac{\hbar}{m\,c\,\alpha}\Big(- h^{j\lambda}\, \phi^0_j\, \partial_{\lambda} + (h^{i\lambda}\, \phi_{\lambda} + h^{i\lambda}\, \Gamma_{\lambda 0}{}^{j}\, \phi^0_j - h^{j\lambda}\, \Gamma_{\lambda 0}{}^{i}\, \phi^0_j)\, \partial^0_i\Big) \,. \square \end{aligned}$$

2.7. Theorem. *The above maps restrict to the mutually inverse linear sheaf isomorphisms*

$$\frac{m}{\hbar}\Omega^{\flat} : \mathcal{T}^{\perp}_{\tau}(UM) \to \mathcal{T}^{\gamma}_{\perp}(UM) \,, \qquad \frac{\hbar}{m}\Omega^{\sharp} : \mathcal{T}^{\gamma}_{\perp}(UM) \to \mathcal{T}^{\perp}_{\tau}(UM) \,,$$

with coordinate expressions, in an adapted base,

$$\begin{aligned} \frac{m}{\hbar}\Omega^{\flat}(X) &= \frac{m\,c\,\alpha}{\hbar}\Big((g^{\perp}{}_{ij}\, X^j_0 + g^{\perp}{}_{mj}\, \Gamma_{i0}{}^{m}\, X^j - g^{\perp}{}_{mi}\, \Gamma_{j0}{}^{m}\, X^j)\theta^i - g^{\perp}{}_{ij}\, X^j\, \theta^i_0\Big) \,, \\ \frac{\hbar}{m}\Omega^{\sharp}(\phi) &= \frac{\hbar}{m\,c\,\alpha}\Big(- g_{\perp}{}^{ij}\, \phi^0_j\, B_i + (g_{\perp}{}^{ij}\, \phi_j + g_{\perp}{}^{mi}\, \Gamma_{m0}{}^{j}\, \phi^0_j - g_{\perp}{}^{jm}\, \Gamma_{m0}{}^{i}\, \phi^0_j) B^0_i\Big) \,. \square \end{aligned}$$

2.4 Time scale

Here, we introduce the notion of time scale, which will play a role in view of Hamiltonian lift of quantisable functions.

We define a *time scale* to be a smooth map $t : UM \to \mathbb{T}$. Given a time scale t, we introduce the subsheaf

$$\mathcal{T}_t(UM) \subset \mathcal{T}(UM) := \{X \in \mathcal{T}(UM) \mid X \lrcorner \tau = t\} \,.$$

Thus, in coordinates, $X \in \mathcal{T}_t(UM)$ if and only if $X^{\alpha}\tau_{\alpha} = t$.

2.8. Remark. Each fibred morphism $\phi : UM \to \mathbb{L}^2 \otimes T^*M$ over M yields the time scale

$$t(\phi) := \frac{Л \lrcorner \phi}{c^2}$$

with coordinate expression

$$t(\phi) = \frac{\alpha}{c}(\phi_p\, x_0^p + \phi_0)\,. \square$$

2.9. Lemma. For each $X \in \mathcal{T}(UM)$, we can write

$$X \lrcorner \tau = \frac{Л}{c^2} \lrcorner (g^\flat \circ \underline{X}) = t(g^\flat \circ \underline{X})\,,$$

where we set

$$\underline{X} := T\pi_0^1 \circ X : UM \to TM\,. \square$$

2.10. Proposition. Let $t : UM \to \mathbb{T}$ be a time scale constant on fibres of $UM \to M$. A vector field $X \in \mathcal{T}_t(UM)$ is projectable on a vector field $\underline{X} \in \mathcal{T}(M)$ if and only if $t = 0$ and X is vertical. $\square$

3 Quantisable functions

In this Section we describe the Lie algebra of quantisable functions on UM and its lift to the tangent space of spacetime.

3.1 Quantisable functions

In the section we exhibits a distinguished sheaf of functions on the phase space which admit a tangent lift. Our construction is based on a projectability criterion as in the Galilei case [1, 2, 3]. This sheaf of functions includes spacetime coordinates, momentum and energy functions for a classical particle. Their lifts are vector fields which are expected to play a role in the covariant quantisation procedure analogous to that developed in the Galilei case [1, 2, 3]. For this reason, we call these functions quantisable.

Let f be a local function on UM.

We define the γ*–covariant differential* of f to be the section

$$d_\gamma f := df - \tau(\gamma \lrcorner df) \in \mathcal{T}_\perp^{\gamma}(UM)\,,$$

with coordinate expression

$$d_\gamma f = (\partial_\lambda f - c\,\alpha\,\tau_\lambda\,(\partial_0 f + x_0^p\,\partial_p f + \gamma_{00}^p\,\partial_p^0 f))\,d^\lambda + (\partial_i^0 f)\,d_0^i\,.$$

Moreover, we define the sheaf morphism

$$C^\infty(UM) \to \mathcal{T}_\tau^\perp(UM) : f \mapsto f^\sharp := \frac{\hbar}{m}\Omega^\sharp(d_\gamma f)\,,$$

with coordinate expression

$$f^\sharp = \frac{\hbar}{m\,c\,\alpha}\Big(- h^{\lambda j}\,\partial^0_j f\,\partial_\lambda + (h^{\lambda i}\,\partial_\lambda f + (h^{\lambda i}\,\Gamma_\lambda{}^j_0 - h^{\lambda j}\,\Gamma_\lambda{}^i_0)\,\partial^0_j f)\partial^0_i\Big)\,.$$

Furthermore, if $t : UM \to \mathbb{T}$ is a time scale, then we define the sheaf morphism

$$C^\infty(UM) \to \mathcal{T}^t_\tau(UM) : f \mapsto f^\sharp{}_t := \gamma(t) + f^\sharp\,,$$

with coordinate expression

$$f^\sharp{}_t = \Big(- \frac{\hbar}{m\,c\,\alpha} h^{\lambda j}\,\partial^0_j f + c\,\alpha\,t(\delta^\lambda_0 + \delta^\lambda_i\,x^i_0)\Big)\,\partial_\lambda + \Big(\frac{\hbar}{m\,c\,\alpha}(h^{\lambda i}\,\partial_\lambda f + (h^{\lambda i}\,\Gamma_\lambda{}^j_0 - h^{\lambda j}\,\Gamma_\lambda{}^i_0)\,\partial^0_j f) + c\,\alpha\,t\,\gamma^i_{00}\Big)\,\partial^0_i$$

Obviously, we have

$$f^\sharp{}_0 = f^\sharp\,.$$

3.1. Remark. Let $f_0, f'_0 : M \to I\!\!R$ and $\phi, \phi' : M \to \mathbb{L}^2 \otimes T^*M$ be such that

$$-\frac{m}{\hbar} Д \lrcorner \phi + f_0 = -\frac{m}{\hbar} Д \lrcorner \phi' + f'_0\,.$$

Then, we obtain

$$f_0 = f'_0\,, \qquad \phi = \phi'\,.$$

Accordingly, if

$$f := -\frac{m}{\hbar} Д \lrcorner \phi + f_0\,,$$

then we set

$$l(f) := \phi\,, \qquad t(f) := \frac{Д}{c^2} \lrcorner \phi\,. \square$$

3.2. Theorem. *Let $f \in C^\infty(UM)$ and t be a time scale. Then, the vector field $f^\sharp{}_t$ is projectable on the vector field*

$$X[f] := T\pi^1_0 \circ f^\sharp{}_{t(f)} : M \to TM$$

if and only if the following conditions are fulfilled

$$f = -\frac{m}{\hbar} Д \lrcorner \phi + f_0\,, \qquad \textit{with } f_0 \in C^\infty(M)\,, \quad \phi : M \to \mathbb{L}^2 \otimes T^*M\,,$$
$$t = t(f) := \frac{Д}{c^2} \lrcorner l(f) := \frac{Д}{c^2} \lrcorner \phi\,.$$

Moreover, if the above conditions are fulfilled, then we obtain

$$X[f] = g^\sharp \circ \phi \,.$$

PROOF. From the coordinate expression of $f^\sharp{}_t$ we get

$$-\frac{\hbar}{m\, c\, \alpha} g_{\mu\lambda}\, h^{\lambda j}\, \partial^0_j f + c\, \alpha (g_{0\mu} + g_{\mu p}\, x^p_0)\, t = \phi_\mu \,.$$

Then,

$$x^j_0\, \partial^0_j f = \frac{m\, c\, \alpha}{\hbar} (\phi_0 - c^2 \tau_0\, t)\,, \qquad \partial^0_i f = -\frac{m\, c\, \alpha}{\hbar} (\phi_i - c^2 \tau_i\, t)$$

imply the Theorem. QED

3.3. Corollary. Let $f \in C^\infty(UM)$. Then, the following conditions are equivalent:
i) $f^\sharp$ is projectable on on M,
ii) $f \in C^\infty(M)$,
iii) $f^\sharp$ is vertical,
iv) $X[f] = 0$.

PROOF. $f^\sharp$ corresponds to the zero time scale, then the implication i) $\Rightarrow$ ii) is a consequence of the above Theorem.
The implications ii) $\Rightarrow$ iii), iii) $\Rightarrow$ i) and iii) $\Leftrightarrow$ iv) are evident. QED

The functions $f \in C^\infty(UM)$ of the type

$$f = -\frac{m}{\hbar} \mathit{\Pi} \lrcorner\, \phi + f_0\,, \qquad \text{with } f_0 \in C^\infty(M)\,, \quad \phi : M \to \mathbb{L}^2 \otimes T^*M\,,$$

are said to be *quantisable*. The subsheaf of quantisable functions is denoted by

$$\mathcal{Q}(UM) \subset C^\infty(UM)\,.$$

The vector prolongations

$$\mathcal{Q}(UM) \to \mathcal{T}(UM) : f \mapsto f^\sharp{}_{t(f)}\,, \qquad \mathcal{Q}(UM) \to \mathcal{T}(M) : f \mapsto X[f]\,,$$

and are said to be the *phase prolongation* and the *tangent prolongation* of f, respectively.

3.4. Corollary. The coordinate expression of a quantisable function is of the type

$$f = -\frac{m\, c}{\hbar} \frac{\phi_j x^j_0 + \phi_0}{\sqrt{g_{00} + 2 g_{j0} x^j_0 + g_{ji} x^j_0 x^i_0}} + f_0\,.$$

The corresponding vector fields have the coordinate expressions

$$f^{\sharp}{}_{t(f)} = g^{\lambda\mu}\,\phi_\mu\,\partial_\lambda + \Big(-\frac{\hbar}{m\,c\,\alpha} h^{\lambda i}\,\partial_\lambda f_0 + h^{\lambda i}(x_0^k\,\partial_\lambda\phi_k + \partial_\lambda\,\phi_0) + (\partial_\rho g_{0\lambda} + x_0^k\,\partial_\rho g_{k\lambda})(g^{\mu\lambda}\,g^{i\rho} - g^{\mu\rho}\,g^{i\lambda})\,\phi_\mu + (\partial_\rho g_{0\lambda} + x_0^k\,\partial_\rho g_{k\lambda})(g^{0\lambda}\,g^{i\rho} - g^{0\rho}\,g^{j\lambda})\,x_0^i\,\phi_j\Big)\,\partial_i^0\,,$$

$$X[f] = g^{\lambda\mu}\,\phi_\lambda\,\partial_\mu\,.\square$$

3.5. Example. Let us consider an observer o and any adapted chart (x^0, x^i). We have the following examples of quantisable functions and corresponding vector fields:
- the time and position coordinate functions and corresponding vector fields

$$f := x^\alpha\,, \qquad X[x^\alpha] = 0\,;$$

- the components of the momentum observed by o and corresponding vector fields

$$f := \frac{p_i}{\hbar} = \frac{m\,c}{\hbar}\tau_i = \frac{m\,c}{\hbar}\frac{g_{i0} + g_{ij}x_0^j}{\sqrt{g_{00} + 2g_{j0}x_0^j + g_{ji}x_0^j x_0^i}}\,, \qquad X[\frac{p_i}{\hbar}] = -\partial_i\,;$$

- the energy observed by o and corresponding vector field

$$f := \frac{p_0}{\hbar} = \frac{m\,c}{\hbar}\tau_0 = \frac{m\,c}{\hbar}\frac{g_{00} + g_{0j}x_0^j}{\sqrt{g_{00} + 2g_{j0}x_0^j + g_{ji}x_0^j x_0^i}}\,, \qquad X[\frac{p_0}{\hbar}] = -\partial_0\,.\square$$

3.2 Lie algebra of quantisable functions

Eventually, we exhibit a Lie bracket for quantisable functions and show that the tangent lift is a Lie algebra homomorphism.

We define the Poisson–Lie bracket

$$\{f, g\} = \frac{m}{\hbar} i_{f^\sharp} i_{g^\sharp}\Omega = i_{f^\sharp} d_\gamma g$$

with coordinate expression, in a canonical base,

$$\{f, g\} = \frac{\hbar}{m\,c\,\alpha}\Big(h^{\lambda p}(\partial_p^0 f\,\partial_\lambda g - \partial_\lambda f\,\partial_p^0 g) + (h^{\lambda p}\,\Gamma_{\lambda 0}{}^{q} - h^{\lambda q}\,\Gamma_{\lambda 0}{}^{p})\partial_p^0 f\,\partial_q^0 g\Big)\,.$$

If $f, g \in \mathcal{Q}(UM)$, then $\{f, g\} \in \mathcal{Q}(UM)$ is not true, in general.
In order to make the sheaf of quantisable functions a sheaf of Lie algebras, we define the bracket

$$[f, g] := \{f, g\} + \gamma(t(f)).g - \gamma(t(g)).f\,.$$

3.6. Lemma. We have the Lie bracket of 1–forms $\phi, \psi : M \to \mathbb{L}^2 \otimes T^*M$

$$\{\phi, \psi\}_g = g^\flat([g^\sharp(\phi),\, g^\sharp(\psi)])\,,$$

with coordinate expression

$$\{\phi, \psi\}_g = \Big(\partial_\sigma g_{\lambda\rho}\,(g^{\rho\mu}\,g^{\sigma\nu} - g^{\nu\rho}\,g^{\sigma\mu})\,\phi_\mu\,\psi_\nu + g^{\sigma\mu}\,(\phi_\mu\,\partial_\sigma\psi_\lambda - \psi_\mu\,\partial_\sigma\phi_\lambda)\Big)\, d^\lambda\,.\square$$

3.7. Theorem. *If $f, g \in \mathcal{Q}(UM)$, then*

$$[f, g] = -\frac{m}{\hbar}\, \text{Л} \lrcorner \{l(f), l(g)\}_g + g^\sharp(l(f)).g_0 - g^\sharp(l(g)).f_0\,.$$

PROOF. The proof consists in difficult calculations in coordinates, by taking into account the formula

$$\begin{aligned}[f, g] = -\frac{m\,c\,\alpha}{\hbar}\Big(&(\partial_\sigma g_{k\rho} x_0^k + \partial_\sigma g_{0\rho})(g^{\lambda\sigma} g^{\rho\mu} - g^{\lambda\rho} g^{\sigma\mu})\phi_\lambda\psi_\mu \\ &+ g^{\mu\lambda}(\phi_\lambda\partial_\mu\psi_k - \psi_\lambda\partial_\mu\phi_k)x_0^k + g^{\mu\lambda}(\phi_\lambda\partial_\mu\psi_0 - \psi_\lambda\partial_\mu\phi_0)\Big) \\ &+ g^{\mu\lambda}(\phi_\lambda\partial_\mu g_0 - \psi_\lambda\partial_\mu f_0)\,.\ \text{QED}\end{aligned}$$

3.8. Example. Let ϕ, ψ be two $\mathbb{L}^2$-valued 1–forms and $f_0, g_0 \in C^\infty(M)$. Then

$$[f_0\,, g_0] = 0\,, \qquad [-\frac{m}{\hbar}\text{Л} \lrcorner \phi\,, g_0] = g^\sharp(\phi).g_0\,,$$
$$[-\frac{m}{\hbar}\text{Л} \lrcorner \phi\,, -\frac{m}{\hbar}\text{Л} \lrcorner \psi] = -\frac{m}{\hbar}\text{Л} \lrcorner \{\phi, \psi\}_g\,.\square$$

3.9. Theorem. *$\mathcal{Q}(UM)$ with the bracket $[\,,]$ is a sheaf of $\mathbb{R}$–Lie algebras.*

PROOF. Let $f, g, h \in \mathcal{Q}(UM)$. Then, Theorem 3.7 and Example 3.8 yield Let $f, g, h \in \mathcal{Q}(UM)$. Then, Theorem 3.7 and Example 3.8 yield

$$\begin{aligned}[f\,, [g, h]] = [-\frac{m}{\hbar}\text{Л} \lrcorner l(f) + f_0\,, -\frac{m}{\hbar}\text{Л} \lrcorner \{l(g), l(h)\}_g + g^\sharp(l(g)).h_0 - g^\sharp(l(h)).g_0] \\ = -\frac{m}{\hbar}\text{Л} \lrcorner \{l(f), \{l(g), l(h)\}_g\}_g \\ + g^\sharp(l(f)).g^\sharp(l(g)).h_0 - g^\sharp(l(f)).g^\sharp(l(h)).g_0 \\ - g^\sharp(l(g)).g^\sharp(l(h)).f_0 + g^\sharp(l(h)).g^\sharp(l(g)).f_0\,.\end{aligned}$$

The Jacobi identity now follows from the Jacobi identity for the bracket $\{\,,\}_g$. QED

3.10. Corollary. The tangent prolongation of quantisable functions is an $\mathbb{R}$–Lie algebra homomorphism, i.e.

$$X[[f, g]] = [X[f]\,, X[g]]\,.\square$$

References

[1] D. Canarutto, A. Jadczyk, M. Modugno: *Quantum mechanics of a spin particle in a curved spacetime with absolute time*, Rep. on Math. Phys., **36**, 1 (1995), 95–140.

[2] A. Jadczyk, M. Modugno: *An outline of a new geometric approach to Galilei general relativistic quantum mechanics*, in C.N. Yang, M.L. Ge and X. W. Zhou editors, Proc. XXI Int. Conf. on Differential geometric methods in theoretical physics, Tianjin 5-9 June 1992, 543-556, Singapore, 1992, World Scientific.

[3] A. Jadczyk, M. Modugno: Galilei general relativistic quantum mechanics, book manuscript, 1994.

[4] J. Janyška: *Lie algebra structures on $\Omega^1(M)$ and $\Omega^1(TM)$ for a Riemannian manifold,* Rendiconti di Matematica, Serie VII, Vol. 13, Roma (1993), 573–593.

[5] J. Janyška, M. Modugno: *Classical particle phase space in general relativity,* Proc. Conf. Diff. Geom. Appl., Brno 1995, Masaryk University 1996, 573–602, electronic edition: `http://www.emis.de/proceedings/`.

[6] P. Libermann, Ch. M. Marle: Symplectic Geometry and Analytical Mechanics, Reidel Publ., Dordrecht, 1987.

[7] M. Modugno, A. M. Vinogradov: *Some variations on the notions of connection,* Annali di Matematica pura ed Applicata (IV), Vol. CLXVII (1994), 33–71.

[8] M. Modugno, R. Vitolo: *Quantum connection and Poincaré–Cartan form*, to appear in Proc. Meeting in honour of A. Lichnerowicz (Frascati 1995), 1996.

[9] N. M. J. Woodhouse: *Geometric Quantization*, second edition, Clarendon Press - Oxford, 1992.