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**RELATIONS BETWEEN LINEAR CONNECTIONS ON THE
TANGENT BUNDLE AND CONNECTIONS ON THE JET
BUNDLE OF A FIBRED MANIFOLD**

JOSEF JANYŠKA AND MARCO MODUGNO

To Ivan Kolář, on the occasion of his 60th birthday.

ABSTRACT. All natural operations transforming linear connections on the tangent bundle of a fibred manifold to connections on the 1-jet bundle are classified. It is proved that such operators form a 2-parameter family (with real coefficients).

Introduction

This paper is motivated by the bijective relation between time-preserving linear connections on space-time with absolute time and affine connections on 1-jet bundle of space-time, [1], [2], [3]. We would like to know if similar relation holds also for a general fibred manifold and so we study all natural operations transforming linear connections on the tangent bundle of a fibred manifold to connections on the 1-jet bundle. We prove that such operators form a 2-parameter family (with real coefficients) and we give its coordinate and geometric expressions.

Our operator are natural in the sense of [4] and [5].

All manifolds and mappings are assumed to be smooth.

1. Linear connections

Let $p : Y \rightarrow X$ be a fibred manifold with a local fibred coordinate chart $(x^\lambda, x^i) = (x^A)$, $\lambda = 1, \dots, \dim X = n$, $i = 1, \dots, \dim Y - \dim X = m$, $A = 1, \dots, \dim Y = n + m$.

A linear connection Λ on the bundle $\pi_X : TX \rightarrow X$ and a linear connection K on the bundle $\pi_Y : TY \rightarrow Y$ can be expressed, respectively, by tangent valued

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forms

$$\begin{aligned} \Lambda &: TX \rightarrow T^*X \otimes_{TX} TTX, \\ K &: TY \rightarrow T^*Y \otimes_{TY} TTY \end{aligned}$$

with coordinate expressions, respectively,

$$(1.1) \quad \Lambda = d^\varphi \otimes (\partial_\varphi + \Lambda_\varphi^\lambda \dot{x}^\psi \dot{\partial}_\lambda), \quad \Lambda_\varphi^\lambda \psi \in C^\infty(X),$$

$$(1.2) \quad K = d^A \otimes (\partial_A + K_A^C \dot{x}^B \dot{\partial}_C), \quad K_A^C B \in C^\infty(Y),$$

where $(x^\varphi, \dot{x}^\varphi)$ and (x^A, \dot{x}^A) are the induced coordinate charts on TX and TY , respectively. The connections Λ and K are also characterised by the vertical projections $\nu_\Lambda : TTX \rightarrow TX$ and $\nu_K : TTY \rightarrow TY$, respectively, or equivalently by the forms $\nu_\Lambda : TX \rightarrow T^*TX \otimes_{TX} TX$ and $\nu_K : TY \rightarrow T^*TY \otimes_{TY} TY$ with coordinate expressions, respectively,

$$(1.3) \quad \nu_\Lambda = (d^\lambda - \Lambda_\varphi^\lambda \dot{x}^\psi d^\varphi) \otimes \partial_\lambda,$$

$$(1.4) \quad \nu_K = (d^A - K_B^A \dot{x}^C d^B) \otimes \partial_A.$$

Let us denote by $K \otimes_Y \Lambda^*$ the tensor product of the connection K and the pullback of the dual connection Λ^* with respect to p , i.e.

$$K \otimes_Y \Lambda^* : T^*X \otimes_Y TY \rightarrow T^*Y \otimes_{T^*X \otimes_Y TY} T(T^*X \otimes_Y TY)$$

with coordinate expression, in the induced fibred coordinate chart (x^A, x_λ^A) on $q : T^*X \otimes_Y TY \rightarrow Y$,

$$(1.5) \quad K \otimes_Y \Lambda^* = d^A \otimes (\partial_A + (K_A^C \delta_\lambda^\mu - \delta_B^C \Lambda_A^\mu \lambda) x_\mu^B \partial_C^A),$$

where we put $\Lambda_i^{\mu\lambda} = 0$. The connection $K \otimes_Y \Lambda^*$ can be defined by the vertical projection $\nu_{K \otimes_Y \Lambda^*} : T(T^*X \otimes_Y TY) \rightarrow T^*X \otimes_Y TY$. We have the coordinate expression

$$(1.6) \quad \nu_{K \otimes_Y \Lambda^*} = (d_\mu^A - (K_B^A \delta_C^\mu - \delta_C^A \Lambda_B^\mu \kappa_\mu) x_\kappa^C d^B) \otimes \partial_A^\mu.$$

A linear connection K on TY is said to be *projectable* on a linear connection Λ on TX if the following diagram commutes

$$\begin{array}{ccc} TTY & \xrightarrow{\nu_K} & TY \\ TTp \downarrow & & \downarrow Tp \\ TTX & \xrightarrow{\nu_\Lambda} & TX \end{array}$$

A pair of linear connections (K, Λ) is said to be *ibre preserving* if the covariant derivative of dp with respect to $K \otimes_Y \Lambda^*$ vanishes, i.e. $\nabla_{K \otimes_Y \Lambda^*}(dp) = 0$.

Lemma 1.1. *Let K be a linear connection on TY and Λ a linear connection on TX . The following three conditions are equivalent*

- i) K is projectable on Λ .
- ii) The pair (K, Λ) is fibre preserving.
- iii) In a fibred coordinate chart $K_A^\lambda{}_i = K_j^\lambda{}_B = 0$ and $K_\mu^\lambda{}_\nu = \Lambda_\mu^\lambda{}_\nu$.

PROOF. It can be proved by using (1.3), (1.4) and (1.6). □

2. Contact mappings

We deal with the natural complementary contact maps

$$\pi : J_1Y \times_Y TX \rightarrow TY, \quad \vartheta : J_1Y \times_Y TY \rightarrow VY,$$

or equivalently

$$\pi : J_1Y \rightarrow T^*X \otimes_Y TY, \quad \vartheta : J_1Y \rightarrow T^*Y \otimes_Y VY,$$

which split the natural exact sequence

$$(2.1) \quad 0 \rightarrow VY \rightarrow TY \xrightarrow{dp} TX \rightarrow 0,$$

through the exact sequence over J_1Y

$$(2.2) \quad 0 \rightarrow J_1Y \times_X TX \xrightarrow{\pi} J_1Y \times_Y TY \xrightarrow{\vartheta} J_1Y \times_Y VY \rightarrow 0.$$

We have the coordinate expressions

$$(2.3) \quad \pi = d^\lambda \otimes \pi_\lambda = d^\lambda \otimes (\partial_\lambda + x_\lambda^i \partial_i), \quad \vartheta = \vartheta^i \otimes \partial_i = (d^i - x_\lambda^i d^\lambda) \otimes \partial_i,$$

where $(x^\lambda, x^i, x_\lambda^i)$ is the induced coordinate chart on J_1Y .

We recall the canonical isomorphism

$$VJ_1Y \simeq J_1Y \times_Y (T^*X \otimes_Y VY)$$

given by

$$\partial_i^\lambda \mapsto d^\lambda \otimes \partial_i.$$

3. Induced connection

A connection Γ on the affine bundle $\pi_0^1 : J_1Y \rightarrow Y$ can be expressed by a tangent valued form

$$\Gamma : J_1Y \rightarrow T^*Y \otimes_{J_1Y} TJ_1Y$$

with coordinate expression

$$(3.1) \quad \Gamma = d^A \otimes (\partial_A + \Gamma_{A\lambda}^i \partial_i^\lambda), \quad \Gamma_{A\lambda}^i \in C^\infty(J_1Y).$$

Using the identification of VJ_1Y and $T^*X \otimes_Y VY$, the connection Γ can be characterised by the vertical projection $\nu_\Gamma : TJ_1Y \rightarrow T^*X \otimes_{J_1Y} VY$, or equivalently by the form $\nu_\Gamma : J_1Y \rightarrow T^*X \otimes_Y T^*J_1Y \otimes_{J_1Y} VY$. In coordinates we have

$$(3.2) \quad \nu_\Gamma = d^\lambda \otimes (d_\lambda^i - \Gamma_{A\lambda}^i d^A) \otimes \partial_i.$$

The connection Γ is affine if and only if its coordinate expression is of the type

$$\Gamma_{A\lambda}^i = \Gamma_{A\lambda_j^\mu}^i x_\mu^j + \Gamma_{A\lambda}^i, \quad \Gamma_{A\lambda_j^\mu}^i, \Gamma_{A\lambda}^i \in C^\infty(Y).$$

Theorem 3.1. *Let Λ be a linear connection on TX and K a linear connection on TY . The map*

$$\nu_\Gamma = \vartheta \circ \nu_{(K \otimes \Lambda^*)} \circ T\pi$$

given by the following diagram

$$\begin{array}{ccccc} TJ_1Y & \xrightarrow{\nu_\Gamma} & VJ_1Y & \xrightarrow{\cong} & J_1Y \times_Y (T^*X \otimes VY) \\ \left. \begin{array}{c} (\pi_{J_1Y}, T\pi) \\ \downarrow \end{array} \right\} & & & & \left. \begin{array}{c} \uparrow \\ (\text{id}_{J_1Y} \times (\text{id}_{T^*X} \otimes \vartheta)) \end{array} \right\} \\ J_1Y \times_Y T(T^*X \otimes TY) & \xrightarrow{(\text{id}_{J_1Y} \times \nu_{K \otimes \Lambda^*})} & J_1Y \times_Y (T^*X \otimes TY) & & \end{array}$$

turns out to be a connection on the bundle $\pi_0^1 : J_1Y \rightarrow Y$. Moreover, we have the coordinate expression

$$(3.3) \quad \Gamma_{A\lambda}^i = K_A^i x_\lambda^j + K_A^i \lambda - x_\mu^i (K_A^\mu x_\lambda^j + K_A^\mu \lambda),$$

i.e. the connection Γ is independent of Λ .

Thus, we have obtained a natural operator

$$\chi : K \mapsto \Gamma$$

transforming linear connections on TY into connections on J_1Y .

PROOF. It can be proved in coordinates by using (2.3), (1.6) and (3.2). □

Lemma 3.1. *If (K, Λ) are fibre preserving, then the induced connection $\chi(K)$ on J_1Y is affine.*

PROOF. From the coordinate expression (3.3), for a pair of fibre preserving connections K and Λ , we get

$$(3.4) \quad \Gamma_{A\lambda}^i = (\delta_\lambda^\mu K_A^i x_\mu^j - \delta_j^i K_A^\mu \lambda) x_\mu^j + K_A^i \lambda,$$

where we put $K_k^\mu \lambda = 0$ and $K_\nu^\mu \lambda = \Lambda_\nu^\mu \lambda$. □

Remark 3.1. In Galilei relativistic theory [1], [2], [3], the base manifold (time) is assumed to be 1-dimensional and affine. A linear connection on space-time is said to be *time-preserving* if it is projectable on the canonical flat connection on the base. (3.4) then implies that the relation between time-preserving linear connections on space-time and affine connections on its 1-jet bundle is bijective. But for $\dim X > 1$ and the flat connection on an affine base manifold this relation is not one-to-one.

4. Curvature

The curvatures of a linear connection K on TY and of a connection Γ on J_1Y are, respectively, the 2-forms

$$R_K = \frac{1}{2}[K, K] : TY \rightarrow \wedge^2 T^*Y \otimes_Y TY,$$

$$R_\Gamma = \frac{1}{2}[\Gamma, \Gamma] : J_1Y \rightarrow \wedge^2 T^*Y \otimes_Y (T^*X \otimes_Y VY),$$

with coordinate expressions

$$(4.1) \quad R_K = (R_K)_{AB}{}^C{}_D \dot{x}^D d^A \wedge d^B \otimes \partial_C =$$

$$= \left(\frac{\partial K_B{}^C{}_D}{\partial x^A} + K_A{}^E{}_D K_B{}^C{}_E \right) \dot{x}^D d^A \wedge d^B \otimes \partial_C$$

and

$$(4.2) \quad R_\Gamma = (R_\Gamma)_{AB}{}^i{}_\lambda d^A \wedge d^B \otimes d^\lambda \otimes \partial_i =$$

$$= \left(\frac{\partial \Gamma_B{}^i{}_\lambda}{\partial x^A} + \Gamma_A{}^j{}_\mu \frac{\partial \Gamma_B{}^i{}_\lambda}{\partial x_\mu^j} \right) d^A \wedge d^B \otimes d^\lambda \otimes \partial_i,$$

respectively.

Theorem 4.1. *If Γ is the connection on J_1Y induced by a linear connection K on TY , then we have*

$$R_\Gamma = \vartheta \circ R_K \circ \mathfrak{d},$$

according to the following commutative diagram

$$\begin{array}{ccc} J_1Y \times_Y T^*X \otimes_Y TY & \xrightarrow{\text{id}_{J_1Y} \times (\text{id}_{T^*X} \otimes R_K)} & J_1Y \times_Y (T^*X \otimes_Y \wedge^2 T^*Y \otimes_Y TY) \\ \uparrow (\text{id}_{J_1Y}, \mathfrak{d}) & & \downarrow \vartheta \\ J_1Y & \xrightarrow{R_\Gamma} & \wedge^2 T^*Y \otimes_Y (T^*X \otimes_Y VY) \end{array}$$

i.e. in coordinates

$$(R_\Gamma)_{AB}{}^i{}_\lambda = (R_K)_{AB}{}^i{}_j x_\lambda^j + (R_K)_{AB}{}^i{}_\lambda - x_\mu^i \left((R_K)_{AB}{}^\mu{}_j x_\lambda^j + (R_K)_{AB}{}^\mu{}_\lambda \right).$$

PROOF. It can be proved by using (3.3), (4.1) and (4.2). □

5. Main theorem

Let us denote by $G_{(n,m)}^k$, $k \geq 0$, the group of k -order jets of diffeomorphisms of \mathbb{R}^{n+m} which preserve the origin and the fibration $\mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$, i.e. $G_{(n,m)}^k$ is the subgroup in G_{n+m}^k given by $a_{iA_1 \dots A_r}^\lambda = 0$, $r = 0, \dots, k - 1$. We have the canonical group homomorphism $\pi_k^l : G_{(n,m)}^l \rightarrow G_{(n,m)}^k$, $l > k$, and we denote by $K_{(n,m)}^{(l,k)}$ its kernel.

Let us denote by $Q = \mathbb{R}^m \otimes \mathbb{R}^{n*} \times \mathbb{R}^{n+m} \otimes \otimes^2 \mathbb{R}^{(n+m)*}$ the $G_{(n,m)}^2$ -space with coordinates $(x_\lambda^i, K_B^A C)$ and the left action of the group $G_{(n,m)}^2$ given by

$$\begin{aligned} \bar{x}_\lambda^i &= a_j^i x_\mu^j \tilde{a}_\lambda^\mu + a_\mu^i \tilde{a}_\lambda^\mu, \\ \bar{K}_B^A C &= a_L^A K_M^L N \tilde{a}_B^M \tilde{a}_C^N + a_L^A \tilde{a}_{BC}^L. \end{aligned}$$

Let us denote by $\tilde{Q} = \mathbb{R}^m \otimes \mathbb{R}^{n*} \times \mathbb{R}^{n+m} \otimes \wedge^2 \mathbb{R}^{(n+m)*}$ the $G_{(n,m)}^1$ -space with coordinates $(x_\lambda^i, T_B^A C)$ and the tensor action of the group $G_{(n,m)}^1$. We denote by $tor : Q \rightarrow \tilde{Q}$ the $G_{(n,m)}^2$ -equivariant mapping given by the antisymmetrisation of subindices $K_B^A C$, i.e.

$$x_\lambda^i = \tilde{x}_\lambda^i, \quad T_B^A C = 1/2(K_B^A C - K_C^A B).$$

Let us consider the space $S = \mathbb{R}^{(n+m)*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^m$ with coordinates $(u_A^i \lambda)$ and the action of the group $G_{(n,m)}^1$ given by

$$\bar{u}_A^i \lambda = a_j^i u_B^j \mu \tilde{a}_A^B \tilde{a}_\lambda^\mu.$$

Lemma 5.1. *All $G_{(n,m)}^2$ -equivariant mappings from Q to S are of the form*

$$\begin{aligned} u_A^i \lambda &= k_1 (T_A^i j x_\lambda^j + T_A^i \lambda - x_\mu^i T_A^\mu j x_\lambda^j - x_\mu^i T_A^\mu \lambda) \\ (5.1) \quad &+ k_2 (\delta_A^i T_D^D j x_\lambda^j + \delta_A^i T_D^D \lambda - x_\mu^i \delta_A^\mu T_D^D j x_\lambda^j - x_\mu^i \delta_A^\mu T_D^D \lambda), \end{aligned}$$

where $T_B^A C = 1/2(K_B^A C - K_C^A B)$.

PROOF. The proof uses the standard techniques of computation of $G_{(n,m)}^2$ -equivariant mappings, [4], and we can divide it into three steps. We omit technical computations.

Step 1. Let $f : Q \rightarrow S$ be a $G_{(n,m)}^2$ -equivariant mapping. From the equivariancy of f with respect to $K_{(n,m)}^{(2,1)}$ we get that f is of the form $f = \tilde{f} \circ tor$, where $\tilde{f} : \tilde{Q} \rightarrow S$ is a $G_{(n,m)}^1$ -equivariant mapping, so it is sufficient to classify all mappings \tilde{f} .

Step 2. Let us denote by h_m the homotheties of \mathbb{R}^m . From the equivariancy of \tilde{f} with respect to $(h_n \times \text{id}_{\mathbb{R}^m})$ and $(\text{id}_{\mathbb{R}^n} \times h_m)$ we get that \tilde{f} is polynomial and any monomial is linear in $T_B^A C$ and of maximum degree 3 in x_λ^i . Coefficients are absolute invariant tensors and we have a polynomial with 33 coefficients.

Step 3. Finally, using equivariancy with respect to diffeomorphisms $(x^\lambda, x^i) \mapsto (x^\lambda, x^i + a_\mu^i x^\mu)$, we find relations between coefficients of \tilde{f} and we get (5.1). \square

Theorem 5.1. *All natural operations transforming a linear connection K on TY into connections on J_1Y form the following 2-parameter family*

$$(5.2) \quad \chi(K) + (\text{id} \otimes \mathcal{D}^* \otimes \vartheta)(k_1 T_K + k_2 I \otimes \hat{T}_K),$$

where $k_1, k_2 \in \mathbb{R}$, T_K is the torsion tensor of K , $\hat{\cdot}$ denotes the contraction and I is the identity tensor on TY .

PROOF. Any natural connection on J_1Y is of the form $\chi(K) + \Phi(K)$, where Φ is an operator (over J_1Y) transforming K into a section of $T^*Y \otimes_Y T^*X \otimes_Y VY$. So it is sufficient to classify all operators Φ . The generalized Peetre theorem implies that any operator Φ is of finite order, [4], [8].

Using homogeneity conditions, [4, Proposition 25.2], we get that all finite order operators Φ are of order 0 ($\Phi(K)$ depends only on coefficients of K and not on their derivatives).

All 0-order operators Φ are in a bijective correspondence with $G_{(n,m)}^2$ -equivariant mappings from Q to S and it is easy to see that the operator corresponding to the mapping of Lemma 5.1 is $(\text{id} \otimes \mathcal{D}^* \otimes \vartheta)(k_1 T_K + k_2 I \otimes \hat{T}_K)$. \square

Corollary 5.1. *For a torsion free connection K the connection $\chi(K)$ is the unique natural connection on J_1Y given by K .* \square

Another geometrical description of Theorem 5.1 is based on the following theorem, [4, Proposition 25.2].

Theorem 5.2. *All natural operations transforming a linear connection K on TY into linear connections on TY form the following 3-parameter family*

$$K + k_1 T_K + k_2 I \otimes \hat{T}_K + k_3 \hat{T}_K \otimes I,$$

where $k_1, k_2, k_3 \in \mathbb{R}$. \square

Theorem 5.1 now can be interpreted by applying the operator χ on the family of connections from Theorem 5.2. Then the resulting connection on J_1Y does not depend on k_3 and it is easy to see that

$$\chi(K + k_1 T_K + k_2 I \otimes \hat{T}_K + k_3 \hat{T}_K \otimes I) = \chi(K) + (\text{id} \otimes \mathcal{D}^* \otimes \vartheta)(k_1 T_K + k_2 I \otimes \hat{T}_K).$$

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