GEOMETRICAL STRUCTURES OF CLASSICAL GENERAL RELATIVISTIC PHASE SPACE GIVEN BY GRAVITATIONAL AND ELECTROMAGNETIC FIELDS

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Abstract. We deal with basic geometric properties of the phase space of a classical general relativistic particle, regarded as the 1st jet space of motions, i.e. as the 1st jet space of timelike 1–dimensional submanifolds of spacetime. This setting allows us to skip constraints.

Our main goal is to determine the geometric conditions by which the Lorentz metric and a connection of the phase space yield contact and Jacobi structures. In particular, we specialise these conditions to the cases when the connection of the phase space is generated by the metric and an additional tensor. Indeed, the case generated by the metric and the electromagnetic field is included, as well.

Introduction

In [2, 4, 6] we studied geometrical structures on the phase space of a spacetime naturally induced (in the sense of [9]) by a metric and a phase connection. Some of these structures are well known and some are less standard. Actually, the geometric objects arising in the framework of the Galilei's phase space [2, 4], involve mainly the concepts of cosymplectic and (regular) coPoisson structures. On the other hand, the analogous geometric objects arising in the framework of the Einstein's phase space [6], involve mainly the concepts of almost–cosymplectic–contact and almost–coPoisson–Jacobi structures (eventually contact and Jacobi structures). In this paper we will deal with the Einstein's spacetime.

First, in Section 1, we recall some standard structures and some new structures, [5], namely almost–cosymplectic–contact, coPoisson and almost–coPoisson–Jacobi structures. In Section 2 we study geometrical structures on the phase space of the Einstein's spacetime given naturally by the metric field and a phase connection and finally, in Section 3, we study geometrical structures given by the gravitational and the electromagnetic fields.

In order to make our theory explicitly independent from units of measurement, we introduce the "spaces of scales" [7]. Actually, we assume the following basic spaces of scales: the space of *time intervals* $\mathbb{T}$, the space of *lengths* $\mathbb{L}$ and the space of *masses* $\mathbb{M}$. Moreover, we consider the following "universal scales": the *speed of light* $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$ and the *Planck constant* $\hbar \in \mathbb{T}^{*} \otimes \mathbb{L}^{2} \otimes \mathbb{M}$. Moreover, we will consider a *particle* of *mass* $m \in \mathbb{M}$ and *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$.

1. Geometrical structures

In this Section $\boldsymbol{M}$ is a $(2n+1)$–dimensional smooth manifold.

1.1. **Covariant and contravariant pairs.** We define a *covariant pair* to be a pair (ω, Ω) consisting of a 1–form ω and a 2–form Ω of constant rank $2r$, with $0 \leq r \leq n$, such that $\omega \wedge \Omega^r \not\equiv 0$, and a *contravariant pair* to be a pair (E, Λ) consisting of a vector field E and a 2–vector Λ of constant rank $2s$, with $0 \leq s \leq n$, such that $E \wedge \Lambda^s \not\equiv 0$. Thus, by definition, we have $\Omega^r \not\equiv 0$, $\Omega^{r+1} \equiv 0$ and $\Lambda^s \not\equiv 0$, $\Lambda^{s+1} \equiv 0$.

We say that the pairs (ω, Ω) and (E, Λ) are *regular* if, respectively, $\omega \wedge \Omega^n \not\equiv 0$ and $E \wedge \Lambda^n \not\equiv 0$.

As usual, we define the following linear maps

$$\Omega^\flat : T\boldsymbol{M} \to T^*\boldsymbol{M} : X \mapsto X^\flat =: i_X \Omega \,, \qquad \Lambda^\sharp : T^*\boldsymbol{M} \to T\boldsymbol{M} : \alpha \mapsto \alpha^\sharp =: i_\alpha \Lambda \,.$$

1.2. **Structures given by covariant and contravariant pairs.** According to [11], a *pre cosymplectic structure* on $\boldsymbol{M}$ is defined by a regular covariant pair (ω, Ω). Two distinguished types of pre cosymplectic structures appear in the literature. Namely, we recall that a *cosymplectic structure* [1] and a *contact structure* [10] are defined by a regular covariant pair (ω, Ω) such that, respectively, $d\omega = 0$, $d\Omega = 0$, and $\Omega = d\omega$. Thus, a contact structure is characterized just by a 1–form ω such that $\omega \wedge (d\omega)^n \not\equiv 0$.

We can easily generalize the above structures in the following way [5]. We define an *almost–cosymplectic–contact structure* to be a covariant pair (ω, Ω) such that

$$d\Omega = 0 \,, \qquad \omega \wedge \Omega^n \not\equiv 0 \,.$$

We define a *pre coPoisson structure* to be a contravariant pair (E, Λ). In particular, a *coPoisson structure* is defined by a contravariant pair (E, Λ) such that $[E, \Lambda] = 0$, $[\Lambda, \Lambda] = 0$, where $[\,,]$ denotes the Schouten bracket, and a *Jacobi structure* is defined by a contravariant pair (E, Λ) such that $[E, \Lambda] = 0$, $[\Lambda, \Lambda] = -2E \wedge \Lambda$. An *almost–coPoisson–Jacobi structure* [5] is defined by a 3–plet (E, Λ, ω), where (E, Λ) is a contravariant pair and ω a 1–form, such that

$$[E, \Lambda] = -E \wedge \Lambda^\sharp(L_E \omega) \,, \quad [\Lambda, \Lambda] = 2\, E \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega) \,, \qquad i_E \omega = 1 \,, \quad i_\omega \Lambda = 0 \,.$$

The 1-form ω is said to be the *fundamental 1-form* of the almost–coPoisson–Jacobi structure.

1.3. **Dual structures.** Let us consider a covariant pair (ω, Ω) and a contravariant pair (E, Λ). The pairs (ω, Ω) and (E, Λ) are said to be *mutually dual* if they are regular, the maps

$$\Omega^\flat{}_{|\mathrm{im}\,(\Lambda^\sharp)} : \mathrm{im}\,(\Lambda^\sharp) \to \mathrm{im}\,(\Omega^\flat) \subset T^*\boldsymbol{M} \qquad \text{and} \qquad \Lambda^\sharp{}_{|\mathrm{im}\,(\Omega^\flat)} : \mathrm{im}\,(\Omega^\flat) \to \mathrm{im}\,(\Lambda^\sharp) \subset T\boldsymbol{M}$$

are isomorphisms and

$$(\Omega^\flat{}_{|\mathrm{im}\,(\Lambda^\sharp)})^{-1} = \Lambda^\sharp{}_{|\mathrm{im}\,(\Omega^\flat)} \,, \;\; (\Lambda^\sharp{}_{|\mathrm{im}\,(\Omega^\flat)})^{-1} = \Omega^\flat{}_{|\mathrm{im}\,(\Lambda^\sharp)} \,, \;\; i_E \Omega = 0 \,, \;\; i_\omega \Lambda = 0 \,, \;\; i_E \omega = 1.$$

The relation of duality yields a bijection between regular covariant pairs (ω, Ω) and regular contravariant pairs (E, Λ) [11]. Thus, the geometric structures given by dual covariant and contravariant pairs are essentially the same.

1.4. **Relations between structures.** Now, let us consider dual pairs (ω, Ω) and (E, Λ). It is well known [8, 11] that if (ω, Ω) is contact, then (E, Λ) is Jacobi. Thus, the geometric structures given by dual contact and regular Jacobi pairs are essentially the same. But we obtain more general result.

1.1. **Theorem.** [5] *The following facts hold:*

(1) (ω, Ω) *is an almost–cosymplectic–contact pair if and only if* (E, Λ, ω) *is an almost–coPoisson–Jacobi 3–plet;*

(2) (ω, Ω) *is a cosymplectic pair if and only if* (E, Λ) *is a coPoisson pair;*

(3) (ω, Ω) *is a contact pair if and only if* (E, Λ) *is a Jacobi pair.* □

2. Einstein spacetime

We recall the geometrical structures arising on the phase space of an Einstein spacetime [6].

2.1. **Spacetime.** We assume *spacetime* to be an oriented 4–dimensional manifold $\boldsymbol{E}$ equipped with a scaled Lorentzian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$, with signature $(- + ++)$; we suppose spacetime to be time oriented. The contravariant metric is denoted by $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (T\boldsymbol{E} \otimes T\boldsymbol{E})$. Given a particle with mass m we will consider the rescaled metric field $G =: \frac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$ and its dual $\bar{G} =: \frac{\hbar}{m}\, \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \otimes T\boldsymbol{E}$.

2.2. **Phase space.** In view of the definition of the phase space we use the conceps of jets of submanifolds [14].

A *motion* is defined to be a 1–dimensional timelike submanifold $s : \boldsymbol{T} \hookrightarrow \boldsymbol{E}$. The *1st differential* of the motion s is defined to be the map $ds =: \frac{ds}{d\sigma} : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$. We assume as *phase space* the subspace $\mathcal{J}_1\boldsymbol{E} \subset J_1(\boldsymbol{E}, 1)$ consisting of all 1–jets of motions.

The *velocity* of a motion s is defined to be its 1–jet $j_1 s : \boldsymbol{T} \to \mathcal{J}_1(\boldsymbol{E}, 1)$. We define the *contact map* to be the unique fibred morphism $д : \mathcal{J}_1\boldsymbol{E} \to \bar{\mathbb{T}}^* \otimes T\boldsymbol{E}$ over $\boldsymbol{E}$, such that $д \circ j_1 s = ds$, for each motion s. We have $g\,(д, д) = -c^2$.

We define the *time form* to be the map $\tau =: -\frac{1}{c^2}\, g^\flat(д) : \mathcal{J}_1\boldsymbol{E} \to \bar{\mathbb{T}} \otimes T^*\boldsymbol{E}$. We have $\tau(д) = 1$ and $\bar{g}(\tau, \tau) = -\frac{1}{c^2}$. We define also the *complementary contact map* $\theta =: 1 - д \otimes \tau : \mathcal{J}_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$.

2.3. **Vertical bundle of the phase space.** Let $V\mathcal{J}_1\boldsymbol{E} \subset T\mathcal{J}_1\boldsymbol{E}$ be the vertical tangent subbundle over $\boldsymbol{E}$ and $V_\tau E$ be the τ-vertical subbundle. The vertical prolongation of the contact map yields the mutually inverse linear fibred isomorphisms

$$\nu_\tau : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T} \otimes V_\tau^*\boldsymbol{E} \otimes V\mathcal{J}_1\boldsymbol{E} \qquad \text{and} \qquad \nu_\tau^{-1} : \mathcal{J}_1\boldsymbol{E} \to V^*\mathcal{J}_1\boldsymbol{E} \otimes \mathbb{T}^* \otimes V_\tau\boldsymbol{E}\,.$$

2.4. **Spacetime and phase connections.** We define a *spacetime connection* to be a torsion free linear connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ of the bundle $T\boldsymbol{E} \to \boldsymbol{E}$. We denote by R the curvature tensor of K. We denote by $K^{\mathfrak{g}}$ the *Levi Civita connection*, i.e. the torsion free linear spacetime connection such that $\nabla g = 0$.

We define a *phase connection* to be a connection of the bundle $\mathcal{J}_1\boldsymbol{E} \to \boldsymbol{E}$. A phase connection can be represented, equivalently, by a tangent valued form $\Gamma : \mathcal{J}_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\mathcal{J}_1\boldsymbol{E}$, which is projectable over $\mathbf{1} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$, or by the complementary vertical valued form $\nu[\Gamma] : \mathcal{J}_1\boldsymbol{E} \to T^*\mathcal{J}_1\boldsymbol{E} \otimes V\mathcal{J}_1\boldsymbol{E}$, or by the vector valued form $\nu_\tau[\Gamma] =: \nu_\tau^{-1} \circ \nu[\Gamma] : \mathcal{J}_1\boldsymbol{E} \to T^*\mathcal{J}_1\boldsymbol{E} \otimes (\mathbb{T}^* \otimes V_\tau\boldsymbol{E})$.

We can prove [3] that there is a natural map $\chi : K \mapsto \Gamma$ between linear spacetime connections K and phase connections Γ,.

2.5. **Dynamical phase connection.** A *dynamical phase connection* is defined to be a 2nd–order connection, i.e. a section $\gamma : \mathcal{J}_1\boldsymbol{E} \to \mathcal{J}_2\boldsymbol{E}$, or, equivalently, a section $\gamma : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\mathcal{J}_1\boldsymbol{E}$, which projects on $д$. If γ is a dynamical phase connection, then we have $\gamma \lrcorner \tau = 1$.

The contact map $д$ and a phase connection Γ yield the section $\gamma \equiv \gamma[д, \Gamma] =: д \lrcorner \Gamma : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\mathcal{J}_1\boldsymbol{E}$, which turns out to be a dynamical phase connection. In particular, a linear spacetime connection K yields the dynamical phase connection $\gamma =: \gamma[д, K] =: д \lrcorner \chi(K)$.

2.6. **Phase 2–form and 2–vector.** The rescaled metric G and a phase connection Γ yield the 2–form Ω, called *phase 2–form*, and the vertical 2–vector Λ, called *phase 2–vector*,

$$\Omega =: \Omega[G, \Gamma] =: G \lrcorner \big(\nu_\tau[\Gamma] \wedge \theta\big) : \mathcal{J}_1\boldsymbol{E} \to \Lambda^2 T^*\mathcal{J}_1\boldsymbol{E}\,,$$

$$\Lambda =: \Lambda[G, \Gamma] =: \bar{G} \lrcorner (\Gamma \wedge \nu_\tau) \qquad : \mathcal{J}_1\boldsymbol{E} \to \Lambda^2 T\mathcal{J}_1\boldsymbol{E}\,.$$

We can easily see that $-\frac{m\,c^2}{\hbar}\;\tau\wedge\Omega^3\not\equiv 0$ and $-\frac{\hbar}{m\,c^2}\;\gamma\wedge\Lambda^3\not\equiv 0$, so the pairs $(-\frac{m\,c^2}{\hbar}\;\tau,\,\Omega)$ and $(-\frac{\hbar}{m\,c^2}\;\gamma,\,\Lambda)$ are regular. Moreover, it is easy to see that these pairs are mutually dual.

2.7. **Dynamical structures of the phase space.** Let us consider a phase connection Γ and the induced phase objects $\gamma =: \gamma[д,\Gamma]$, $\Omega =: \Omega[G,\Gamma]$, and $\Lambda =: \Lambda[G,\Gamma]$.

We define the Lie derivatives $L_\Gamma\tau = (i_\Gamma d - d i_\Gamma)\tau$ and $L_R\tau = (i_R d + d i_R)\tau$. Then, the following results holds [6].

2.1. **Theorem.** *The following assertions are equivalent.*

(1) $L_{\nu_\tau(X)}\,L_\Gamma\,\tau = 0\,,\ \forall\,X\in\sec(\boldsymbol{E},T\boldsymbol{E})$, *and* $L_R\,\tau = 0$.

(2) $d\Omega = 0$, *i.e.* $(-\frac{m\,c^2}{\hbar}\;\tau,\,\Omega)$ *is an almost–cosymplectic–contact pair.*

(3) $[-\frac{\hbar}{m\,c^2}\;\gamma,\,\Lambda] = \frac{\hbar}{m\,c^2}\;\gamma\wedge\Lambda^\sharp(L_\gamma\,\tau))$ *and* $[\Lambda,\,\Lambda] = 2\,\gamma\wedge(\Lambda^\sharp\otimes\Lambda^\sharp)(d\tau))$, *i.e. the 3–plet* $(-\frac{\hbar}{m\,c^2}\;\gamma,\,\Lambda,\,-\frac{m\,c^2}{\hbar}\;\tau)$ *is an almost–coPoisson–Jacobi 3–plet.* □

2.2. **Theorem.** *The following assertions are equivalent.*

(1) $L_\Gamma\,\tau = 0$.

(2) $\Omega = -\frac{m\,c^2}{\hbar}\;d\tau$, *i.e.* $(-\frac{m\,c^2}{\hbar}\;\tau,\Omega)$ *is a contact pair.*

(3) $[-\frac{\hbar}{m\,c^2}\;\gamma,\Lambda] = 0$ *and* $[\Lambda,\Lambda] = \frac{2\,\hbar}{m\,c^2}\;\gamma\wedge\Lambda$, *i.e.* $(-\frac{\hbar}{m\,c^2}\;\gamma,\Lambda)$ *is a Jacobi pair.* □

Next, let us consider a linear spacetime connection K and the induced phase objects $\Gamma =: \chi(K)$, $\gamma =: \gamma[д,\Gamma]$, $\Omega =: \Omega[G,\Gamma]$, and $\Lambda =: \Lambda[G,\Gamma]$.

2.3. **Theorem.** *The following assertions are equivalent.*

(1) $L_{\chi(K)}\,\tau = 0$.

(2)
$$g(Z,Z)\big((\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + g(T(X,Y),Z)\big) + \tfrac12\,g(Z,X)(\nabla_Y g)(Z,Z) - \tfrac12\,g(Z,Y)(\nabla_X g)(Z,Z) = 0\,,$$

for each $X,Y,Z\in\sec(\boldsymbol{E},T\boldsymbol{E})$, *where* T *is the torsion of* K.

(3) $\Omega = -\frac{m\,c^2}{\hbar}\;d\tau$, *i.e.* $(-\frac{m\,c^2}{\hbar}\;\tau,\Omega)$ *is a contact pair.*

(4) $[-\frac{\hbar}{m\,c^2}\;\gamma,\Lambda] = 0$ *and* $[\Lambda,\Lambda] = \frac{2\,\hbar}{m\,c^2}\;\gamma\wedge\Lambda$, *i.e.* $(-\frac{\hbar}{m\,c^2}\;\gamma,\Lambda)$ *is a Jacobi pair.* □

3. Gravitational and electromagnetic structures

3.1. **Gravitational objects and structures.** In what follows we will denote by $^{\mathfrak g}$ all objects induced by a gravitational rescaled metric field G and its dual $\bar G$. So we have the Levi Civita spacetime connection $K^{\mathfrak g}$ and the induced phase objects $\Gamma^{\mathfrak g}$, $\gamma^{\mathfrak g}$, $\Omega^{\mathfrak g}$, and $\Lambda^{\mathfrak g}$.

3.1. **Theorem.** *We have:*

(1) $\Omega^{\mathfrak g} = -\frac{m\,c^2}{\hbar}\;d\tau$, *i.e.* $(-\frac{m\,c^2}{\hbar}\;\tau,\Omega^{\mathfrak g})$ *is a contact pair.*

(2) $[-\frac{\hbar}{m\,c^2}\;\gamma^{\mathfrak g},\Lambda^{\mathfrak g}] = 0$ *and* $[\Lambda^{\mathfrak g},\Lambda^{\mathfrak g}] = \frac{2\,\hbar}{m\,c^2}\;\gamma^{\mathfrak g}\wedge\Lambda^{\mathfrak g}$, *i.e.* $(-\frac{\hbar}{m\,c^2}\;\gamma^{\mathfrak g},\Lambda^{\mathfrak g})$ *is a Jacobi pair.* □

3.2. **Electromagnetic objects and structures.** We assume spacetime to be equipped with a given *electromagnetic field*, which is a closed scaled 2–form $F:\boldsymbol{E}\to(\mathbb{L}^{1/2}\otimes\mathbb{M}^{1/2})\otimes\Lambda^2T^*\boldsymbol{E}$. With reference to a particle with mass m and charge q, we obtain the rescaled 2–form $\frac{q}{\hbar}\;F:\boldsymbol{E}\to\Lambda^2T^*\boldsymbol{E}$. Let us denote by A a local potential of F, i.e., $F = 2\,dA$.

The metric G, the phase connection $\Gamma^{\mathfrak{g}}$ and the electromagnetic field F induce on the phase space the "joint" phase connection $\Gamma = \Gamma^{\mathfrak{g}} + \Gamma^{\mathfrak{e}}$ such that

$$\Omega[G,\Gamma] =: G \lrcorner (\nu_\tau[\Gamma] \wedge \theta) = \Omega^{\mathfrak{g}} + \Omega^{\mathfrak{e}} = -\frac{m\,c^2}{\hbar}\; d\tau + \frac{q}{2\,\hbar}\; F\,.$$

Indeed the metric G and the rescaled electromagnetic field $\frac{q}{2\,\hbar}\,F$ give the phase π^1_0-vertical valued 1-form $\Gamma^{\mathfrak{e}}$. Then we have the "joint" dynamical connection $\gamma = \gamma^{\mathfrak{g}} + \gamma^{\mathfrak{e}}$, where $\gamma^{\mathfrak{e}} =: \gamma[\Gamma^{\mathfrak{e}}] = д \lrcorner \Gamma^{\mathfrak{e}}$. Further we have the "joint" phase 2-vector $\Lambda[G,\Gamma] = \Lambda^{\mathfrak{g}} + \Lambda^{\mathfrak{e}}$, where $\Lambda^{\mathfrak{e}} =: \Lambda[G,\Gamma^{\mathfrak{e}}] = \bar{G} \lrcorner (\Gamma^{\mathfrak{e}} \wedge \nu_\tau^{-1})$.

3.2. **Theorem.** *We have*

$$[-\frac{2\,\hbar}{m\,c^2}\;\gamma^{\mathfrak{e}}, \Lambda^{\mathfrak{e}}] = 0\,, \qquad [\Lambda^{\mathfrak{e}}, \Lambda^{\mathfrak{e}}] = \frac{4\,\hbar}{m\,c^2}\;\gamma^{\mathfrak{e}} \wedge \Lambda^{\mathfrak{e}}\,,$$

i.e. $(-\frac{2\,\hbar}{m\,c^2}\;\gamma^{\mathfrak{e}}, \Lambda^{\mathfrak{e}})$ *is a Jacobi structure of the phase space.* □

3.3. **Compatibility of gravitational and electromagnetic Jacobi structures.** We have two Jacobi structures on the phase space. The "gravitational" Jacobi structure $(-\frac{\hbar}{m\,c^2}\;\gamma^{\mathfrak{g}}, \Lambda^{\mathfrak{g}})$ is "regular" in the sense that $-\frac{\hbar}{m\,c^2}\;\gamma^{\mathfrak{g}} \wedge (\Lambda^{\mathfrak{g}})^3$ is a volume vector. The "electromagnetic" Jacobi structure $(-\frac{2\,\hbar}{m\,c^2}\;\gamma^{\mathfrak{e}}, \Lambda^{\mathfrak{e}})$ is "singular" since $-\frac{2\,\hbar}{m\,c^2}\;\gamma^{\mathfrak{e}} \wedge (\Lambda^{\mathfrak{e}})^3 = 0$.

Let us recall that, according to [12], two Jacobi structures (E_1, Λ_1) and (E_2, Λ_2) on a manifold are "compatible" if the sum $(E_1 + E_2, \Lambda_1 + \Lambda_2)$ is a Jacobi structure. Then it is easy to see that two Jacobi structures are compatible if and only if

$$[E_1, \Lambda_2] + [E_2, \Lambda_1] = 0\,, \quad [\Lambda_1, \Lambda_2] = E_1 \wedge \Lambda_2 + E_2 \wedge \Lambda_1\,.$$

It is easy to see that the "gravitational" and "electromagnetic" Jacobi structures $(-\frac{\hbar}{m\,c^2}\;\gamma^{\mathfrak{g}}, \Lambda^{\mathfrak{g}})$ and $(-\frac{2\,\hbar}{m\,c^2}\;\gamma^{\mathfrak{e}}, \Lambda^{\mathfrak{e}})$ we obtained on the phase space are not compatible in the above sense.

3.4. **Joint objects and structures.** We will study identities of the "joint" objects which are obtained as the sum

$$\gamma =: \gamma^{\mathfrak{g}} + \gamma^{\mathfrak{e}}\,, \qquad \Omega =: \Omega^{\mathfrak{g}} + \Omega^{\mathfrak{e}}\,, \qquad \Lambda =: \Lambda^{\mathfrak{g}} + \Lambda^{\mathfrak{e}}\,.$$

3.3. **Theorem.** *We have*

$$d\Omega = 0\,. \tag{3.1}$$

Moreover, $-\frac{m\,c^2}{\hbar}\;\tau \wedge \Omega^3 = -\frac{m\,c^2}{\hbar}\;\tau \wedge (\Omega^{\mathfrak{g}})^3$ *is a volume form. So the pair* $(-\frac{m\,c^2}{\hbar}\;\tau, \Omega)$ *is an almost–cosymplectic–contact structure of the phase space.* □

3.4. **Remark.** Let us remark that the pair $(-\frac{m\,c^2}{\hbar}\;\tau, \Omega)$ is not generally a contact structure. But if the electromagnetic field F has a potential A, i.e. $F = 2dA$, then

$$\Omega\big[G,\Gamma\big] = d(-\frac{m\,c^2}{\hbar}\;\tau + \frac{q}{\hbar}\;A)\,,$$

i.e., Ω has the potential $-\frac{m\,c^2}{\hbar}\;\tau + \frac{q}{\hbar}\;A$. Then $(-\frac{m\,c^2}{\hbar}\;\tau + \frac{q}{\hbar}\;A, \Omega)$ is a contact structure if and only if $(-\frac{m\,c^2}{\hbar}\;\tau + \frac{q}{\hbar}\;A) \wedge (\Omega^{\mathfrak{g}})^3$ is a volume form. The condition that $(-\frac{m\,c^2}{\hbar}\;\tau + \frac{q}{\hbar}\;A) \wedge (\Omega^{\mathfrak{g}})^3$ is a volume form is equivalent to $q\,(д \lrcorner A) \neq m\,c^2$. □

It is easy to see that the joint pairs $(-\frac{m\,c^2}{\hbar}\;\tau, \Omega)$ and $(-\frac{\hbar}{m\,c^2}\;\gamma, \Lambda)$ are regular and mutually dual.

3.5. **Theorem.** *We have*

$$[-\frac{\hbar}{m\,c^2}\;\gamma, \Lambda] = -\frac{\hbar^2}{m^2\,c^4}\;\gamma^{\mathfrak{g}} \wedge \gamma^{\mathfrak{e}}\,, \quad [\Lambda, \Lambda] = \frac{2\,\hbar}{m\,c^2}\;\big(\gamma \wedge \Lambda + \gamma \wedge \Lambda^{\mathfrak{e}}\big)\,,$$

i.e. the joint pair $(-\frac{\hbar}{m\,c^2}\;\gamma, \Lambda)$ *is not a Jacobi structure on the phase space.*

Proof. Let F be a closed 2–form. Then we have

$$[\gamma^{\mathfrak{g}}, \Lambda^{\mathfrak{e}}] + [\gamma^{\mathfrak{e}}, \Lambda^{\mathfrak{g}}] = \frac{\hbar}{m\,c^2}\,\gamma^{\mathfrak{g}} \wedge \gamma^{\mathfrak{e}}\,, \qquad [\Lambda^{\mathfrak{g}}, \Lambda^{\mathfrak{e}}] = \frac{\hbar}{m\,c^2}\,(\gamma^{\mathfrak{e}} \wedge \Lambda^{\mathfrak{g}} + 2\,\gamma^{\mathfrak{g}} \wedge \Lambda^{\mathfrak{e}})$$

which imply Theorem 3.5. □

3.6. **Theorem.** *The joint pair $(-\frac{\hbar}{m\,c^2}\,\gamma, \Lambda)$ is an almost–coPoisson–Jacobi structure along with the fundamental 1–form $-\frac{m\,c^2}{\hbar}\,\tau$.*

Proof. The joint phase 2–form $\Omega = \Omega^{\mathfrak{g}} + \frac{q}{2\hbar}\,F$ is closed so the pair $(-\frac{m\,c^2}{\hbar}\,\tau, \Omega)$ is an almost–cosymplectic–contac structure. Then, by Theorem 1.1, the dual pair $(-\frac{\hbar}{m\,c^2}\,\gamma, \Lambda)$ is an almost–coPoisson–Jacobi structure along with the fundamental 1–form $-\frac{m\,c^2}{\hbar}\,\tau$. □

Now, let us assume that F has a potential A and let us assume $(-\frac{m\,c^2}{\hbar}\,\tau + \frac{q}{\hbar}\,A, \Omega)$ to be a contact structure. We will study the dual Jacobi structure.

3.7. **Theorem.** *If $(-\frac{m\,c^2}{\hbar}\,\tau + \frac{q}{\hbar}\,A, \Omega)$ is a contact structure then $(\frac{\hbar}{q\,(д \lrcorner A) - m\,c^2}\,\gamma, \Lambda)$ is the dual Jacobi structure. Moreover, if $д \lrcorner A = 0$, then the dual Jacobi structure is $(-\frac{\hbar}{m\,c^2}\,\gamma, \Lambda)$.*

Proof. First, let us recall that the condition for $(-\frac{m\,c^2}{\hbar}\,\tau + \frac{q}{\hbar}\,A, \Omega)$ to be a contact structure is $q\,(д \lrcorner A) \neq m\,c^2$. Then we can prove Theorem 3.7 in coordinates. □

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