GALILEI GENERAL RELATIVISTIC QUANTUM MECHANICS

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Abstract

We present a general relativistic approach to quantum mechanics of a spinless charged particle, subject to external classical gravitational and electromagnetic fields in a curved space-time with absolute time. The scheme is also extended in order to treat the n-body quantum mechanics.

First, we study the Galilei general relativistic space-time, as classical background; then, we develop the quantum theory.

The formulation is fully based on geometrical ideas and methods and is explicitly covariant.
In the special relativistic case, our theory agrees with the standard one referred to a given frame of reference.

Our approach takes into account several classical ideas and results of Galilei general relativity and geometric quantisation (see E. Cartan, C. Duval, K. Kuchař, H. P. Künzle, E. Prugovecki, A. Trautman, N. Woodhouse and several others). However, we present original ideas and results as well.

Key words: Galilei general relativity, curved space-time, classical field theory, classical mechanics, quantum mechanics on a curved back-ground; fibred manifolds, jets, connections; tangent valued forms, systems.

1991 MSC: 83D05, 81P05, 81S10, 81S40, 81T20, 70G50; 53B05, 55R05, 55R10, 58A20.

Preface

The book is addressed to a double audience: to mathematicians who at some point were attracted by the subject of quantum mechanics, but soon were repelled because they could not find a geometrical door to this magic palace. It is also addressed to those physicists who knowing all kinds of culinary recipes of how to compute quantum mechanical effects are still unsatisfied and thirsty of knowing some solid and primary mathematical principles that can be used to derive or to justify some of their successful formulae.

In fact the book is more than just a compendium building from scratch geometrical foundation for Galileian relativity, wave functions, quantisation and Schrödinger equation. It is also an invitation to a further research.

The greatest unsolved problem of the XX-th century physics is as old as the famous Einstein-Bohr debate. There were two main revolutions in physics witnessed by this century: relativity and quantum theory. Both were radical enough to change not only physics but also our entire Weltanschauung. After the great drama of Einstein's failure to reduce quantum theory to a unified non-linear classical field theory, after so many and so spectacular successes both of relativity and quantum theory, we are tempted to believe that what we need is a union of the two opposites rather than a reduction of one to the other. It is with this in mind that we have undertaken the research whose fruits we want to share in this book. We hope that perhaps our way of approaching quantum mechanics geometrically will trigger new ideas in some readers, and clearly new ideas are necessary to catalyse the fruitful chemical reaction between so different components.

We would like to point out the difference between our approach and that of geometric quantisation. Geometrical quantisation method is a powerful machine feeding itself on symplectic manifolds and their polarisations. So, it has a different scope than our approach because we are concerned with structures related to space-time. On the other hand "time" is merely a parameter in geometrical quantisation; it is never treated fully geometrically. Thus it is difficult or impossible to discuss in that framework changes of states corresponding to accelerated observers. Our approach stresses the full covariance from the very beginning, and covariance proves to be a powerful guiding principle.

We thank our colleagues for their interest, questions, comments and criticism; they allowed us to shape our research domain.

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Florence, 31 December 1993

Arkadiusz Jadczyk, Marco Modugno

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> La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi agli occhi (io dico l'universo), ma non si puó intendere se prima non si impara a intender la lingua, e conoscer i caratteri, ne quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi ed altre figure geometriche, senza i quali mezzi è impossibile intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro labirinto.
>
> G. Galilei, VI, 232, Il Saggiatore, 1623.

0 - INTRODUCTION

0.1. Aims

The standard quantum mechanics (see, for instance, [Me], [Sk], [Sc]) is quite well established and tested, so that it must be taken as touchstone for any further development.

The supporting framework of this theory is the standard flat Galilei space-time. Moreover, an inertial frame of reference is usually assumed and the implicit covariance of the theory is achieved by imposing a suitable transformation under the action of the Galilei group.

As it is well known, the standard quantum mechanics conflicts with the classical theory of curved space-time and gravitational field. This great problem is still open, in spite of several important attempts. We share the opinion that the first step aimed at approaching the solution should be a general relativistic formulation of quantum mechanics interacting with a given classical gravitational field in a curved space-time.

In the physical literature, the principle of covariance is mostly formulated in terms of representations. This viewpoint is very powerful and has been largely successful. Moreover, it is related to the view of geometry based on the famous Klein's programme, hence to the theories of representation of groups and Lie algebras. However, we think that the modern developments of geometry cannot be exhausted by this approach. Indeed, we think that a direct approach to geometrical structures in terms of intrinsic algebraic structures, operators and functors is quite interesting and might deserve a primitive consideration. Then, the groups of automorphisms of such structures arise subsequently. Moreover, if the developments of the primitive structures are derived intrinsically, through functorial methods, then the invariance of the theory under the action of the automorphism groups is automatic. So, we think that, when we know the basic structures of our physical model, it is worthwhile following an intrinsic, i.e. manifestly covariant, approach. Indeed, in the cases when we know both an intrinsic formulation of a physical theory and a formulation via representations, the first one appears to be simpler and neater. Just to fix the ideas, consider a very simple example and refer to the formulations of electromagnetic field through the modern intrinsic language of exterior differential calculus and the older language based on components and the action of the Lorentz group. Actually, it is a pity that an intrinsic geometrical language has not yet been achieved for all domains that occur in physics. On the other hand, the method of representations remains essential for the study of classifications.

So, the goal of our paper is a general relativistic quantum mechanics. As usual, by 'general relativistic' we mean 'covariant' with respect to the change of frames of reference (observers and units of measurement) and charts. Even more, we look for an explicitly covariant formulation based on intrinsic structures. The reader will judge if such an approach is neat and heuristically valuable.

A general relativistic quantum mechanics demands a general relativistic classical space-time as necessary support. Certainly, the most natural and interesting programme would be to study quantum mechanics on an Einstein general relativistic back-ground, hence on a curved space-time equipped with a Lorentz metric. On the other hand, it is possible to develop a general relativistic classical theory, based on a space-time fibred over absolute time and equipped with a vertical Riemannian metric. This theory - which will be referred to as Galileian - is mathematically rigorous and self-contained and provides a description of physical phenomena with a good approximation (with respect to the corresponding Einstein theory) in presence of low velocities and weak gravitational field. The Galilei classical mechanics has been studied by several authors (for instance, see [Ca], [Dv1], [Dv2], [DBKP], [DGH],

[DH], [Eh], [Ha], [Ku], [Ke1], [Ke2], [Ku], [Kü1], [Kü2], [Kü3], [Kü4], [KD], [LBL], [Le], [Ma], [Mo1], [Pl], [Pr1], [Pr2], [SP], [Tr1], [Tr2], [Tu]); nevertheless, it is not common belief that many features, which are usually attached exclusively to Einstein general relativity, are also present in the Galilei theory. Then, in order to avoid confusion, we stress the difference between the general validity of notions such as general relativity, curved space-time manifold, accelerated observers, equivalence principle and so on and their possible specifications into an Einstein or a Galilei theory. In spite of its weaker physical validity, the Galilei theory has some advantages due to its simplicity. Hence, we found worth starting our approach to quantum mechanics from the Galilei case. Later we shall apply to the Einstein case what we have learned in the Galilei case. On the other hand, this study can be considered not only as an useful exercise in view of further developments, but also physically interesting by itself.

0.2. Summary

In order to help the reader to get a quick synthesis of our approach, we sketch the main ideas and steps.

0.2.1. Classical theory

We assume the classical *space-time* to be a 4-dimensional oriented fibred manifold (see § III.1)

$$t : \boldsymbol{E} \to \boldsymbol{T}$$

over a 1-dimensional oriented affine space associated with the vector space $\mathbb{T}$. The typical space-time chart is denoted by (x^0, y^i) and the corresponding time unit of measurement by $u_0 \in \mathbb{T}$ or $u^0 \in \mathbb{T}^*$.

We obtain the scaled *time form*

$$dt : \boldsymbol{E} \to \mathbb{T} \otimes T^* \boldsymbol{E},$$

with coordinate expression

$$dt = u_0 \otimes d^0.$$

We deal with the jet space (see § III.3)

$$J_1\boldsymbol{E} \to \boldsymbol{E}$$

and the natural complementary *contact maps*

$$\text{д} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \qquad \vartheta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E},$$

with coordinate expressions

$$\text{д} = u^0 \otimes \text{д}_0 = u^0 \otimes (\partial_0 + y_0^i \, \partial_i) \qquad \vartheta = \vartheta^i \otimes \partial_i = (d^i - y_0^i \, d^0) \otimes \partial_i .$$

A classical *(absolute) motion* is defined to be a section

$$s : \boldsymbol{T} \to \boldsymbol{E}$$

and its *(absolute) velocity* is the jet prolongation

$$j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E},$$

with coordinate expression

$$j_1 s = u^0 \otimes \big((\partial_0 \circ s + \partial_0 s^i \, (\partial_i \circ s) \big).$$

An *observer* is defined to be a section

$$o : \boldsymbol{E} \to J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$$

and the *observed velocity* of the motion s is the vertical section

$$\nabla_o s := j_1 s - o \circ s : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E},$$

with coordinate expression in an adapted chart

$$\nabla_o s = \partial_0 s^i \, u^0 \otimes (\partial_i \circ s).$$

Vertical restrictions are denoted by "ˇ".

We assume space-time to be equipped with a scaled *vertical Riemannian metric*

$$g : \boldsymbol{E} \to \mathbb{A} \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}),$$

with coordinate expression

$$g = g_{ij} \, \check{d}^i \otimes \check{d}^j \qquad g_{ij} \in \mathcal{M}(\boldsymbol{E}, \mathbb{A} \otimes \mathbb{R}).$$

The metric and the time form, along with a choice of the orientation, yield the scaled *space-time* and *space-like volume forms*

$$\upsilon : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{A}^{3/2}) \otimes \overset{4}{\wedge} T^* \boldsymbol{E} \qquad \eta : \boldsymbol{E} \to \mathbb{A}^{3/2} \otimes \overset{3}{\wedge} V^* \boldsymbol{E},$$

with coordinate expressions

$$\upsilon = \sqrt{|g|}\; u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 \qquad \eta = \sqrt{|g|}\; \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3.$$

Moreover, the metric yields the *vertical Riemannian connection*

$$\varkappa : V\boldsymbol{E} \to V^* \boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} VV\boldsymbol{E}$$

on the fibres of space time.

There is a natural bijection between the dt-preserving torsion free linear connections

$$K : T\boldsymbol{E} \to T^* \boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E}$$

on the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$ and the torsion free affine connections

$$\Gamma : J_1\boldsymbol{E} \to T^* \boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E}$$

on the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, with coordinate expressions

$$K = d^\lambda \otimes (\partial_\lambda + (K_\lambda{}^i{}_h\, \dot{y}^h + K_\lambda{}^i{}_0\, \dot{x}^0)\, \dot{\partial}_i) \qquad \Gamma = d^\lambda \otimes (\partial_\lambda + (\Gamma_\lambda{}^i{}_h\, y^h_0 + \Gamma_\lambda{}^i{}_\circ)\, \partial^0_i)$$

$$K_\mu{}^i{}_\lambda = K_\lambda{}^i{}_\mu = \Gamma_\lambda{}^i{}_\mu = \Gamma_\mu{}^i{}_\lambda.$$

Then, each of such equivalent connections, will be called a *space-time connection*.

A space-time connection K yields, by vertical restriction, the space-time *vertical connection*

$$\check{K} : V\boldsymbol{E} \to V^* \boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} VV\boldsymbol{E}$$

on the fibres of space-time, with coordinate expression

$$\check{K} = \check{d}^j \otimes (\partial_j + K_j{}^i{}_h\, \dot{y}^h\, \dot{\partial}_i).$$

If Γ is a space-time connection and o an observer, then we obtain the covariant differential

$$\nabla o : \boldsymbol{E} \to \mathbb{T}^* \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}).$$

Then, the vertical metric and the observer itself yield the splitting of ∇o into its symmetrical and anti-symmetrical components

$$\Sigma : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{}{\underset{2}{\vee}} T^*\boldsymbol{E} \qquad \Phi : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^*\boldsymbol{E},$$

with coordinate expressions

$$\Sigma := -2\, u^0 \otimes (\Gamma_{0j\circ}\, d^0 \vee d^j + \Gamma_{ij\circ}\, d^i \vee d^j) \qquad \Phi := -2\, u^0 \otimes (\Gamma_{0j\circ}\, d^0 \wedge d^j + \Gamma_{ij\circ}\, d^i \wedge d^j)$$

The connection Γ is characterised by $\check{K}$, $\check{\Sigma}$ and Φ.

A space-time connection K is said to be *metrical* if it preserves the contravariant vertical metric, i.e. if

$$\nabla_K \bar{g} = 0.$$

We cannot fully apply the methods of Riemannian geometry, because the metric g is degenerate.

A space-time connection Γ yields the connection

$$\gamma := д \lrcorner \Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T J_1\boldsymbol{E}$$

on the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$ and the scaled 2-form

$$\Omega := \nu_\Gamma \barwedge \vartheta : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^* J_1\boldsymbol{E}$$

on the manifold $J_1\boldsymbol{E}$, with coordinate expressions

$$\gamma = u^0 \otimes \big(\partial_0 + y_0^i\, \partial_i + (\Gamma_{h\,k}{}^{i}\, y_0^h\, y_0^k + 2\,\Gamma_{h\,\circ}{}^{i}\, y_0^h + \Gamma_{0\,\circ}{}^{i})\, \partial_i^0\big)$$

$$\Omega = g_{ij}\, u^0 \otimes (d_0^i - \gamma^i\, d^0 - \Gamma_h{}^i\, \vartheta^h) \wedge \vartheta^j.$$

They are said to be, respectively, the *second order connection* and the *contact 2-form* associated with Γ.

These objects fulfill the equality

$$\gamma \lrcorner \Omega = 0;$$

moreover

$$dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to (\mathbb{T}^{*2} \otimes \mathbb{A}^3) \otimes \overset{7}{\wedge} T^* J_1\boldsymbol{E}$$

is a scaled volume form on $J_1\boldsymbol{E}$; furthermore, for each observer o, we obtain

$$\Phi = 2\, o^* \Omega,$$

hence, we say that Φ is the *observed contact 2-form*.

On the other hand, γ and Ω characterise Γ itself.

We assume space-time to be equipped with a space-time connection

$$\Gamma^\natural : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} T J_1\boldsymbol{E}$$

and a scaled 2-form

$$F : \boldsymbol{E} \to \mathbb{B} \otimes \overset{2}{\wedge} T^*\boldsymbol{E},$$

representing the *gravitational connection* and the *electromagnetic field*.

The gravitational connection and the electromagnetic field can be coupled in a natural way through a constant $\mathbf{c}$, which can be either the square root $\sqrt{\mathbf{k}}$ of the gravitational constant, or the ratio $\boldsymbol{q}/m$ of a *mass* $m \in \mathbb{M}$ and a *charge* $\boldsymbol{q} \in \mathbb{Q}$ of a given particle: the coupled objects will be called *total*. In practice, we are concerned with $\boldsymbol{c} = \sqrt{\mathbf{k}}$ only in the context of the second gravitational field equation and in all other cases we consider $\boldsymbol{c} = \boldsymbol{q}/m$. So, we obtain the *total contact 2-form*

$$\Omega := \Omega^\natural + \Omega^e := \Omega^\natural + \frac{1}{2}\, \boldsymbol{c}\, F,$$

the *total second order connection*

$$\gamma := \gamma^\natural + \gamma^e$$

and the *total space-time connection*

$$\Gamma := \Gamma^\natural + \Gamma^e,$$

where

$$\gamma^e : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes (\mathbb{T}^* \otimes V\boldsymbol{E})$$

turns out to be the *Lorentz force* and

$$\Gamma^e : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} (\mathbb{T}^* \otimes V\boldsymbol{E})$$

a certain *soldering form* associated canonically with F, with coordinate expressions

$$\gamma^e = -c\,(F_0{}^i + F_h{}^i\, y_0^h)\, u^0 \otimes \partial_i^0 \qquad\qquad \Gamma^e = \tfrac{1}{2}\,c\,\big((F^i{}_h\, y_0^h + 2\,F^i{}_0)\, d^0 + F^i{}_j\, d^j\big) \otimes \partial_i^0.$$

This splitting will be reflected in all other objects derived from the total connection.

In order to couple the total connection with the vertical metric, we postulate the *first field equation*: for each charge and mass, the total contact 2-form is closed, i.e.

$$d\Omega = 0.$$

This equation expresses, in a compact way, a large number of important conditions. Namely, the first field equation is equivalent to the fact that the total connection is metrical and the total curvature tensor fulfills the standard symmetry properties. Moreover, the first field equation is equivalent to the fact that the vertical total connection $\check{K}$ coincides with the vertical Riemannian connection $\varkappa$ and, for each observer o, $\check{\Sigma}$ is given by the time derivative of the metric and Φ is closed. Moreover, in virtue of the arbitrariness of the mass and the charge, the first field equation implies the first Maxwell equation.

In order to couple the total connection with the matter source, we postulate the *second gravitational and electromagnetic field equations*:

$$r^{\natural} = \mathrm{T}^{\natural} \qquad\qquad \operatorname{div}^{\natural} F = j.$$

We restrict ourselves to consider a charged incoherent fluid, just as an example; these equations yield an Einstein type equation for the total connection

$$r = \mathrm{T}.$$

The only observer independent way of expressing the generalised *Newton law* of motion of a classical particle, under the action of the gravitational and electromagnetic fields, is to assume that the covariant differential of the motion with respect to the second order total connection γ vanishes

$$\nabla_{\gamma} j_1 s = 0.$$

We stress that the standard *Hamiltonian* and *Lagrangian approaches* to classical dynamics depend on the choice of an observer in an essential way. Hence, they are not suitable for a general relativistic formulation of classical mechanics.

Under reasonable hypothesis, there exist background affine structures on space-time, which allow us to re-interpret the second field equation as the Newton law of gravitation.

In particular, the *special relativistic case* is obtained by considering an affine space-time with vanishing energy tensor of matter.

By means of a slight modification of the above scheme we can formulate the n-body field theory and mechanics on a curved Galilei space-time. In particular, the standard results for the two-body classical mechanics can be recovered as a special solution of our equations.

0.2.2. Quantum theory

We assume the *quantum bundle* to be a line-bundle

$$\pi : \boldsymbol{Q} \to \boldsymbol{E}$$

over space-time. The quantum histories are described by the *quantum sections*

$$\Psi : \boldsymbol{E} \to \boldsymbol{Q}.$$

In some respects, it is useful to regard a quantum section Ψ as a *quantum density*

$$\Psi^{\eta} := \Psi \otimes \sqrt{\eta} : \boldsymbol{E} \to \boldsymbol{Q}^{\eta}.$$

The typical normal chart of $\boldsymbol{Q}$ will be denoted by (z) and the corresponding base by (b); accordingly, we write

$$\Psi = \psi\, b, \qquad \psi := z \circ \Psi.$$

Then, we assume the *quantum connection* to be a Hermitian universal connection

$$\text{ч} : \boldsymbol{Q}^{\uparrow} \to T^{*} J_1 \boldsymbol{E} \underset{J_1 \boldsymbol{E}}{\otimes} T \boldsymbol{Q}^{\uparrow}$$

on the *pullback quantum bundle*

$$\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E},$$

whose curvature is proportional to the classical total contact 2-form, according to the formula

$$R_{\text{ч}} = i\,\frac{m}{\hbar}\,\Omega \otimes \text{и} : \boldsymbol{Q}^{\uparrow} \to \overset{2}{\wedge} T^{*} J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} \boldsymbol{Q}^{\uparrow} .$$

The universal connection ч can be naturally regarded as a *system of Hermitian connections*

$$\xi : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T^{*}\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{Q}$$

on the bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, whose curvature is proportional to the observed total contact 2-form Φ, for each observer o, according to the formula

$$R_{\xi_o} = \frac{1}{2}\, i\,\frac{m}{\hbar}\,\Phi \otimes \text{и} : \boldsymbol{Q} \to \overset{2}{\wedge} T^{*}\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}$$

The quantum connection is essentially our unique structure postulated for quantum mechanics; all other structures and objects will be derived from this in a natural way.

We prove that the coordinate expression of the quantum connection, is of the type

$$\text{ч}_0 = -H/\hbar \qquad \text{ч}_j = p_j/\hbar \qquad \text{ч}^0_j = 0,$$

where H and p are the classical Hamiltonian and momentum associated with the frame of reference attached to the chosen chart, with a suitable gauge of the total potential

$$a : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{A}) \otimes T^{*}\boldsymbol{E}$$

of the closed 2-form Φ, which refers both to the gravitational and electromagnetic fields.

The composition

$$\gamma \lrcorner\, \text{ч} : \boldsymbol{Q}^{\uparrow} \to \mathbb{T}^{*} \otimes T\boldsymbol{Q}^{\uparrow}$$

turns out to be a connection on the fibred manifold $\boldsymbol{Q}^{\uparrow} \to \boldsymbol{T}$, whose coordinate expression is of the type

$$\gamma \lrcorner \text{ч} = u^0 \otimes (\partial_0 + y^i_0\, \partial_i + \gamma^i\, \partial^0_i + i\, L/\hbar\ \text{и}),$$

where L is the classical Lagrangian associated with the frame of reference attached to the chosen chart.

If $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ is a quantum section, then we obtain the *quantum covariant differential*

$$\nabla_{\text{ч}} \Psi : J_1 \boldsymbol{E} \to T^* \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q},$$

with coordinate expression

$$\nabla \Psi = \big((\partial_0 \psi + i\, H/\hbar\ \psi)\, d^0 + (\partial_j \psi - i\, p_j/\hbar\ \psi)\, d^j \big) \otimes b.$$

The quantum connection lives on the pull-back quantum bundle $\boldsymbol{Q}^{\uparrow} \to J_1 \boldsymbol{E}$ (i.e. is parametrised by all observers), but we wish to derive further physical objects living on the quantum bundle $\boldsymbol{Q} \to \boldsymbol{E}$ (i.e. observer independent objects). We shall achieve them by means of a *principle of projectability*, which turns out to be our way to implement the principle of general relativity in the framework of quantum mechanics.

By means of the principle of projectability, we can exhibit a distinguished *quantum Lagrangian*

$$\mathscr{L} : J_1 \boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{4}{\wedge} T^* \boldsymbol{E},$$

with coordinate expression

$$\mathscr{L}_\Psi = \tfrac{1}{2} \Big(-\tfrac{h}{m}\, g^{ij}\, \partial_i \bar\psi\ \partial_j \psi - i\, (\partial_0 . \bar\psi\ \psi - \bar\psi\ \partial_0 . \psi) + i\, a^i\, (\partial_i \bar\psi\ \psi - \bar\psi\ \partial_i \psi) +$$

$$+ \tfrac{m}{\hbar}\, (2\, a_0 - a_i\, a^i)\ \bar\psi\ \psi \Big)\, u^0 \otimes \upsilon.$$

The quantum Lagrangian yields the *quantum 4-momentum*

$$\mathfrak{p} : J_1 \boldsymbol{Q} \to \mathbb{T}^* \otimes T \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q},$$

with coordinate expression

$$\mathfrak{p}_\Psi = u^0 \otimes \big(\psi\ \partial_0 - i\, \tfrac{\hbar}{m}\, g^{ij}\, (\partial_j \psi - i\, \tfrac{m}{\hbar}\, a_j\ \psi)\ \partial_i \big) \otimes b.$$

This can also be obtained directly from the contact structure of spacetime and the vertical quantum covariant differential, by means of the principle of projectability.

Then, the Euler-Lagrange equation associated with the quantum Lagrangian turns out to be the *generalised Schrödinger equation*

$$*\check{\mathcal{E}}^{\neq} : J_2\boldsymbol{Q} \to \mathbb{T}^* \otimes \boldsymbol{Q},$$

with coordinate expression

$$*\check{\mathcal{E}}^{\neq}{}_{\Psi} = 2\Big(i\,\partial_0\psi + \frac{m}{\hbar}\,a_0\,\psi + \frac{1}{2}\,i\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi +$$

$$+ \frac{\hbar}{2m}\,(g^{ij}\,(\partial_{ij}\psi - 2\,i\,\frac{m}{\hbar}\,a_i\,\partial_j\psi - (i\,\frac{m}{\hbar}\,\partial_i a_j + \frac{m^2}{\hbar^2}\,a_i\,a_j)\,\psi) +$$

$$+ \frac{\partial_i(g^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_j\psi - i\,\frac{m}{\hbar}\,a_j\,\psi))\Big)\,b.$$

This can also be obtained directly from the time-like quantum covariant differential of the quantum section and the quantum covariant codifferential of the quantum 4-momentum, by means of the principle of projectability.

The invariance of the quantum Lagrangian with respect to the group $U(1)$ yields a conserved *probability 4-current*

$$j : J_1\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{3}{\wedge} T^*\boldsymbol{E},$$

with coordinate expression

$$j_{\Psi} = \sqrt{|g|}\,\Big(\bar{\psi}\,\psi\,d^1 \wedge d^2 \wedge d^3 +$$

$$+ (-1)^h\,(-\,i\,\frac{\hbar}{2m}\,g^{hk}\,(\bar{\psi}\,\partial_k\psi - \partial_k\bar{\psi}\,\psi) - a^h\,\bar{\psi}\,\psi)\,d^0 \wedge d^1 \dots \wedge \hat{d}^h \dots \wedge d^3\Big).$$

In view of quantum operators, we need further preliminary results on classical mechanics.

The contact 2-form

$$\frac{m}{\hbar}\,\Omega : J_1\boldsymbol{E} \to \overset{2}{\wedge} T^* J_1\boldsymbol{E}$$

yields a natural *Hamiltonian lift* of the *classical functions* $f : J_1\boldsymbol{E} \to \mathbb{R}$ into vector fields

$$f^{\#}_{\tau} : J_1\boldsymbol{E} \to T J_1\boldsymbol{E}$$

with a given time-component $\boldsymbol{\tau} : J_1\boldsymbol{E} \to \mathbb{T}$. We have the coordinate expression

$$f^{\#}_{\tau} = \tau\,(\partial_0 + y^i_0\,\partial_i + \gamma^i\,\partial^0_i) + \frac{\hbar}{m}\,g^{ij}\left(-\,\partial^0_j f\,\partial_i + (\partial_j f + (\Gamma_j{}^k - \Gamma^k{}_j)\,\partial^0_k f)\,\partial^0_i\right).$$

Moreover, such a vector field is projectable over a vector field

$$f^H : \boldsymbol{E} \to T\boldsymbol{E}$$

if and only if the function f is quadratic with respect to the fibre of $J_1\boldsymbol{E} \to \boldsymbol{E}$ and its second fibre derivative is proportional to the metric g through the coefficient $\boldsymbol{\tau}$. The coordinate expression of such a function is of the type

$$f = f''\,\frac{m}{2\hbar}\,g_{ij}\,y^i_0\,y^j_0 + f_i\,y^i_0 + f_\circ, \qquad f'', f_\circ, f_i \in \mathcal{F}(\boldsymbol{E}).$$

These functions are called *quantisable functions* and we prove that they constitute naturally a Lie algebra. The coordinate expression of the bracket is quite long.

The classical time, position, momentum, Hamiltonian and Lagrangian functions are quantisable functions.

Then, we consider the vector fields

$$X^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T\boldsymbol{Q}^{\uparrow}$$

on the pull-back quantum bundle, with a given time-component $\boldsymbol{\tau} : J_1\boldsymbol{E} \to \mathbb{T}$, which preserve the quantum structures. We prove that they are of the type

$$X^{\uparrow}{}_{f,\tau} := \text{ч}(f^{\#}_{\tau}) + i\,\text{и}\,f,$$

where $f : J_1\boldsymbol{E} \to \mathbb{R}$ is a function.

Then, we prove that the vector field $X^{\uparrow}{}_{f,\tau}$ is projectable over a vector field

$$X_f : \boldsymbol{Q} \to T\boldsymbol{Q}$$

if and only if the corresponding function f is a quantisable function and the time-component of the vector field coincides with the time-component of the quantisable function. The coordinate expression of such a vector field is given by the important formula

$$X_f = f'' \, \partial_0 - \frac{\hbar}{m} f^i \, \partial_i + i \, (\frac{m}{\hbar} f'' \, a_0 - f^i \, a_i + f_\circ) \, \text{и}.$$

These projected vector fields X_f are called *quantum vector fields*. Moreover, we prove that they constitute a Lie algebra and that the map

$$f \mapsto X_f$$

is a Lie algebra isomorphism.

The quantum vector fields act naturally on the quantum densities Ψ^η as *quantum Lie operators*

$$Y_f := i \, X_f \bullet,$$

according to the coordinate expression

$$Y_f(\Psi^\eta) =$$

$$= i \, \Big(f'' \, \nabla^o{}_0 \psi^\eta - \frac{\hbar}{m} f^i \, \nabla^o{}_i \psi^\eta - i \, f_\circ \, \psi^\eta + \frac{1}{2} \, (\partial_0 f'' - \frac{\hbar}{m} \, \partial_i f^i) \, \psi^\eta \Big) \, b \otimes \sqrt{\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3}.$$

Therefore, we obtain a Lie algebra isomorphism

$$f \mapsto Y_f$$

between the quantisable functions and the quantum Lie operators.

In particular, in the special Galilei case, the classical Hamiltonian corresponds to the time-derivative and the affine quantisable functions correspond to the standard quantum operators.

So far, the quantum theory has been developed on the finite dimensional bundle $\boldsymbol{Q} \to \boldsymbol{E}$ over space-time. Next, in order to achieve the Hilbert structure in the quantum framework, we derive in a natural way an infinite dimensional Hilbert bundle $H\boldsymbol{Q}^\eta \to \boldsymbol{T}$ over time.

Namely, we consider the infinite dimensional *fibred set*

$$\sigma : S\boldsymbol{Q}^\eta \to \boldsymbol{T}$$

constituted by the tube sections of the double fibred manifold $\boldsymbol{Q}^\eta \to \boldsymbol{E} \to \boldsymbol{T}$ and obtain a natural bijection

$$\Psi \mapsto \hat{\Psi}$$

between the sections $\Psi^\eta : \boldsymbol{E} \to \boldsymbol{Q}^\eta$ and $\hat{\Psi}^\eta : \boldsymbol{T} \to \boldsymbol{SQ}^\eta$.

Then, we define a smooth structure on $\boldsymbol{SQ}^\eta \to \boldsymbol{T}$, according to the Frölicher's definition of *smoothness*; hence, we are able to construct the tangent space and define the connections on the fibred set $\boldsymbol{SQ}^\eta \to \boldsymbol{T}$.

The above constructions are compatible with any subsheaf of tube sections of the double fibred manifold $\boldsymbol{Q}^\eta \to \boldsymbol{E} \to \boldsymbol{T}$; in particular, we are interested to the tube sections with space-like compact support. They yield the fibred set

$$\sigma^c : \boldsymbol{S}^c\boldsymbol{Q}^\eta \to \boldsymbol{T}.$$

Moreover, we prove that the Schrödinger equation can be regarded as the equation

$$\nabla_{\hat{k}} \hat{\Psi}^\eta = 0,$$

where

$$\hat{k} : S^c\boldsymbol{Q}^\eta \to \mathbb{T}^* \otimes T S^c \boldsymbol{Q}^\eta$$

is a symmetric connection on the infinite dimensional bundle $\boldsymbol{S}^c\boldsymbol{Q}^\eta \to \boldsymbol{T}$, which is called the *Schrödinger connection*, and has the coordinate expression

$$k_0(\Psi^\eta) = i\,\left(\frac{\hbar}{2m}\, \check{\Delta}^o \psi^\eta + \frac{m}{\hbar}\, a_0\, \psi^\eta\right) u^0 \otimes b \otimes \sqrt{\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3}.$$

There is a unique natural way to obtain a fibred morphism $\boldsymbol{S}^c\boldsymbol{Q}^\eta \to \boldsymbol{S}^c\boldsymbol{Q}^\eta$ over $\boldsymbol{T}$ (and not only a differential operator acting on the sections $\hat{\Psi}^\eta : \boldsymbol{T} \to \boldsymbol{Q}^\eta$) from any quantisable function. Namely, the *quantum operator* associated with the quantisable function f is defined to be the symmetric fibred morphism

$$\hat{\Xi}_f : \boldsymbol{S}^c\boldsymbol{Q}^\eta \to \boldsymbol{S}^c\boldsymbol{Q}^\eta$$

induced by the sheaf morphism

$$\hat{\Xi}_f := \hat{Y}_f - i\, \boldsymbol{f}'' \lrcorner \nabla_{\hat{k}},$$

with coordinate expression

$$\Xi_f(\Psi^\eta) =$$

$$= \big(f_o\, \psi^\eta + i\, \tfrac{1}{2}\, (\partial_0 f^{\shortparallel} - \tfrac{\hbar}{m}\, \partial_i f^i)\, \psi^\eta - i\, \tfrac{\hbar}{m}\, f^i\, \nabla^o{}_i \psi^\eta - f^{\shortparallel}\, \tfrac{\hbar}{2m}\, \check{\Delta}^o \psi^\eta\big)\, b \otimes \sqrt{\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3}.$$

Thus, the above formula is our implementation of the principle of correspondence, achieved in a pure geometrical way.

In particular, in the special Galilei case, these operators and their commutators correspond to the standard ones.

Eventually, the fibred set $S^c\boldsymbol{Q}^\eta \to \boldsymbol{T}$ yields the *quantum Hilbert bundle*

$$H\boldsymbol{Q}^\eta \to \boldsymbol{T}\,,$$

by the completion procedure. This bundle will carry the standard probabilistic interpretation of quantum mechanics. We stress that we do not have a unique Hilbert space, but a Hilbert bundle over time. Indeed, a unique Hilbert space would be in conflict with the principle of relativity. On the other hand, a global observer yields an isometry between the fibres of the quantum Hilbert bundle.

The *Feynmann amplitudes* arise in a natural and nice way in our framework.

By means of a slight modification of the above scheme we can formulate the n-body quantum mechanics on a curved Galilei space-time. In particular, the standard results for the two-body quantum mechanics can be recovered as a special solution of our equations.

0.3. Main features

The literature concerning classical Galilei general relativity and geometric approaches to quantum mechanics is very extended.

We have been mainly influenced by the ideas due to E. Cartan (see [Ca]), C. Duval (see [DBKP]), H. P. Künzle (see [DBKP], [Kü1], [Kü2], [Kü3]), E. Prugovecki (see [Pr]) and A. Trautman (see [Tr1], [Tr2]) and by the scheme of geometrical quantisation due to B. Kostant and J. M. Souriau (see [St], [Wo]). Also the papers by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer (see BFFLS]), W. Pauli (see [Pl]), H. D. Dombrowski and K. Horneffer (see [DH]), P. Havas (see [Ha]), X. Kenec (see [Ke1], [Ke2]), C. Kiefer and T.P. Singh (see [KS]), K. Kuchař (see [Ku]), M. Le Bellac (see [LBL]), J. M. Levy-Leblond (see [LBL], [Le]), L. Mangiarotti (see [Ma]), M.

Modugno (see [Mo1]), E. Schmutzer and J. Plebanski (see [SP]) and W. M. Tulczyjew (see [Tu]) have been considered.

We omit a detailed comparison between the above literature and our paper, because it would take too much space. Indeed, sometimes, such a comparison turns out to be very hard because it is not possible to recover our intrinsic and well defined spaces in other papers. We just say that our approach and results seem to be original in several respects.

Our touchstone for quantum mechanics is the standard theory. Actually, even if our scheme is quite far from the usual one, we stress that we do not touch the standard probabilistic interpretation and, eventually, our concrete results agree with the standard ones in the special Galilei case. So, our theory can be regarded both as a generalisation of the standard theory (in order to fulfill the principle of general relativity and to include the interaction with a gravitational field on a curved space-time) and as a new heuristic language (in view of further interpretations and developments).

All ideas and developments are achieved in a fully geometrical way. All formulas are expressed intrinsically and their coordinate or observer dependent expressions are given as well.

As we have already largely discussed, groups have no direct role. On the other hand we define carefully the geometrical structures of the fundamental spaces and derive the physical theory from them. Of course, the transformation laws of the derived objects can also be checked directly.

We stress that the representations arising from our intrinsic methods are not trivial and could not be guessed as consequence of standard procedures. In particular, our implementation of the covariant principle of correspondence seems to be a miracle due to these specific geometrical structures.

The principle of relativity is basically implemented by the fact that space-time is a fibred manifold, without any distinguished trivialisation. Each splitting of space-time is associated with an observer and no distinguished observers are assumed. All subsequent structures must respect this original feature. So, time cannot be just a trivial parameter, but the fibring over time yields structures playing an important role in the theory.

The most usual (mostly Lagrangian or Hamiltonian) formulations of classical mechanics are based on the vertical tangent or cotangent spaces, or on the cotangent space of space-time. These approaches are related to a philosophy, which is very far from ours; in fact, the final physical interpretation of these theories cannot be expressed in an observer independent way; so, we disregard this viewpoint. Conversely, the approach based on the tangent space of space-time is manifestly observer independent, but it depends on the choice of a time unit of measurement. Our approach is based on the jet space, because this is the only way to obtain a formulation which is independent of observers and units of measurement of time. Our choice yields important consequences both for the classical and quantum theories.

Another, typical feature of our formulation depends on the fundamental role played by connections both in the classical and quantum theories. So, the classical field theory and mechanics is based on the space-time connection; moreover, we derive the quantum dynamics and operators from the quantum connection.

We mostly deal with linear or affine connections, but we are also concerned with notions and methods related to general connections (see [MM], [Mo2], [Mo3]). As it is well known, a general connection on a fibred manifold is defined as a section of the jet bundle; such a section can also be regarded, equivalently, as a horizontal valued 1-form on the base space, or a vertical valued 1-form on the total space. The first viewpoint is more suitable for its relation with jets and the Frölicher-Nijenhuis bracket, while the second one is more directly related to the covariant differential of forms. We refer to the first viewpoint as the primitive one; for this reason, our coefficients of the connections turn out to be the negatives of the standard ones.

It is well known that the differential calculus associated with a general connection can be derived from the Frölicher-Nijenhuis graded Lie algebra of tangent valued forms (see [MM], [Mo2], [Mo3]); this calculus is simple and more powerful than the standard one, even in the case of linear connections. Therefore, we find it convenient to refer always to this general method. Indeed, some steps of our theory require specifically notions of this general calculus (see, for instance, the upper quantum vector fields, § II.3.2).

Classical mechanics cannot be formulated by Hamiltonian or Lagrangian approaches in an observer independent way. On the other hand, a classical Hamiltonian (contact) formalism can be developed; however, it has no covariant role in classical mechanics, but yields a fundamental link between classical and quantum structures, with respect to the quantum connection and quantum operators.

Some analogies between our approach and the geometrical quantisation (see, for instance, [St], [Wo]) are evident; but also several important differences arise. The main source of differences is again due to our requirement of general relativistic covariance, hence to the role of time. In particular, we are led to base the quantisation procedure on a contact 2-form; indeed the symplectic structure of geometric quantisation is essentially vertical, hence cannot have a relativistically covariant total role.

As the vertical metric is degenerate, the standard methods of Riemannian geometry cannot be applied fully. However, the first field equation, based on the closure of the contact 2-form, provides a compact way of expressing the coupling between the vertical metric and the space-time connection, and several other important equations as well.

The gravitational and electromagnetic coupling works well and consistently in all respect, in the classical and quantum theories. This seems to be an original aspect of our theory.

The Lie algebra of quantisable functions is new, as far as we know. It is one of the key points of the covariant principle of correspondence.

The coordinate expression of the generalised Schrödinger equation is similar to the standard one in the flat case. For short, it replaces the wave function with a wave density; but the difference is more subtle than it could seem at a first insight.

We stress that, in the quantum theory, the total potential associated with an observer cannot be split into its gravitational and electromagnetic components.

The reader is only requested to have a standard knowledge of differential geometry, general relativity and quantum mechanics. Besides that, the work is rather self-contained.

An appendix provides a quick outline of the basic notions on fibred manifolds, tangent and jet spaces, general connections and tangent valued forms, which are traceable only in a specialised literature. These notions are necessary for a full understanding of all details of our treatment. However, we stress that the reader, who does not like to spend too much time on an abstract geometrical language, does not need to go thoroughly through this subject: a glance will be sufficient for understanding the greatest part of the paper.

0.4. Units of measurement

A further original feature of our formulation concerns the way we treat the units of measurement, in order to emphasise, in a clear and rigorous way, the independence of the theory from any choice of scales.

In fact, some physical objects (mass, charge, and so on) can be described by elements of one dimensional vector spaces. Moreover, some other physical objects (metric, electromagnetic field, and so on) can be described by sections of vector bundles, which can be identified with geometrical bundles up to a scale factor. Furthermore, each frame of reference involves a time scale.

Only ratios of two vectors of such a 1 dimensional vector space or of two scale factors are numbers. Then, we are led to consider "semi-vector" spaces over the "semi-field" $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ and define the dual of a semi-vector space and the tensor products over $\mathbb{R}^+$ of semi-vector spaces. In particular, a vector space is also a semi-vector space and the tensor product of a semi-vector space with a vector space turns out to be a vector space. A *positive semi-vector space* is defined to be a semi-vector space whose scalar multiplication cannot be extended neither to $\mathbb{R}^+ \cup \{0\}$ nor to $\mathbb{R}$.

When we are concerned with a 1-dimensional positive semi-vector space, we often denote the duals of its elements as inverses and the tensor products of its elements with vectors as scalar products; in this way, we can treat elements of 1-dimensional positive semi-vector spaces as they were numbers. So, our practical formulas look like the standard ones in the physical literature.

We can also define the roots of 1-dimensional positive semi-vector spaces.

The half-densities can be obtained as a by-product of the above algebraic scheme.

Thus, in our theory we obtain vector fields, forms, tensors and so on, which are tensorialised with some scale factor belonging to a 1-dimensional vector or positive semi-vector space. We stress that the usual differential operations, such as Lie derivative, exterior differential, covariant differential, and so on, can be naturally extended to the above scaled objects. We shall perform these operations without any further warning.

0.5. Further developments

In the special relativistic Galilei case, our practical results agree with the corresponding ones of standard quantum mechanics. For example, in this case, the concrete computations concerning harmonic oscillator, hydrogen atom and so on agree with the standard ones. Thus, unlike some other geometrical approaches to quantum mechanics, nothing needs to be checked in this direction. Nevertheless, a possible theoretical interest of our scheme might be maintained also in the special relativistic case.

Therefore, in order to provide some new concrete quantum examples on an effectively curved space-time, one has first to find non-trivial solutions of the classical fields.

Eventually, we observe that our theory can be also considered from an experimental viewpoint. In fact, some results could be checked in principle by experiments. But a detailed analysis of this aspect is beyond the purpose of the present work.

In a forthcoming paper we shall extend our approach, preserving the spirit of the present work, in order to include spin. Moreover, we expect that our methods be suitable for further extension to Einstein general relativistic space-time.

I - THE CLASSICAL THEORY

The general relativistic quantum theory requires a general relativistic classical space-time as support.

Therefore, the first part of the paper is devoted to a model of Galilei curved space-time with absolute time. In this framework, we formulate the dynamics of classical gravitational and electromagnetic fields and of a classical test particle.

I.1 - Space-time

First, we introduce the space-time fibred manifold and its space-like metrical structure.

I.1.1. Space-time fibred manifold

In this section, we introduce the space-time fibred manifold and study its tangent and jet prolongations. Moreover, we state our conventions about coordinates.

ASSUMPTION C1. We assume *space-time* to be a 4-dimensional orientable fibred manifold (see § III.1)

$$t : \boldsymbol{E} \to \boldsymbol{T}$$

over a 1-dimensional oriented affine space $\mathbf{T}$, associated with the vector space $\mathbb{T}$. ◇

REMARK I.1.1.1. Thus, we assume the *absolute time* $\boldsymbol{T}$ and the *absolute time function* t. But we do not mention any "absolute space", as we do not assume any distinguished splitting of the space-time fibred manifold into a product of time and space. Later, any choice of such a local splitting will be associated with an observer; no distinguished observer is assumed. ❑

REMARK I.1.1.2. The differential of the time function is the $\mathbb{T}$-valued form

$$dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}.$$

We shall be involved with the tangent space $T\boldsymbol{E}$ and the vertical subspace $V\boldsymbol{E}$; we recall the exact sequence of vector bundles over $\boldsymbol{E}$ (see § III.2)

$$0 \longrightarrow \longrightarrow V \boldsymbol{E} \longrightarrow T \boldsymbol{E}$$

The 1-jet space $J_1\mathbf{E}$ (see § III.3) plays an important role in the classical and quantum theories. We recall that $J_1\boldsymbol{E} \to \boldsymbol{E}$ is an affine bundle associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E} \to \boldsymbol{E}$.

We shall be involved with the canonical fibred morphisms over $\boldsymbol{E}$ (see [Mo2])[1]

$$д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \qquad \vartheta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E},$$

which provide a natural splitting of the above exact sequence over $J_1\boldsymbol{E}$ (they are quoted as the *contact structure* of jets). ❑

DEFINITION I.1.1.1. An *(absolute) motion* is defined to be a section

$$s : \boldsymbol{T} \to \boldsymbol{E}$$

and its *(absolute) velocity* is defined to be its first jet prolongation

$$j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E}. \qquad ♣$$

DEFINITION I.1.1.2. An *observer* is defined to be a section

$$o : \boldsymbol{E} \to J_1\boldsymbol{E}. \qquad ♣$$

---

[1] д is the Cyrillic character corresponding to "d".

Thus, $J_1\boldsymbol{E}$ can be considered as the target space both of the velocity of particles and of observers.

REMARK I.1.1.3. Global observers exist, because of the affine structure of $J_1\boldsymbol{E}\to\boldsymbol{E}$.

Each observer o is nothing but a connection on the fibred manifold $t:\boldsymbol{E}\to\boldsymbol{T}$ (see § III.5). Hence, an observer o yields a splitting of the exact tangent sequence

$$0 \to \boldsymbol{E}\times\mathbb{T} \xrightarrow{o} T\boldsymbol{E} \xrightarrow{\nu_o} V\boldsymbol{E} \to 0,$$

where

$$\nu_o : T\boldsymbol{E} \to V\boldsymbol{E} : X \mapsto X - dt(X) \lrcorner o.$$

In other words, an observer o yields a splitting of the tangent bundle of space-time into its "observed" time-like and space-like components

$$T\boldsymbol{E} \simeq (\boldsymbol{E}\times\mathbb{T}) \underset{\boldsymbol{E}}{\times} V\boldsymbol{E},$$

given by the two linear fibred projections over $\boldsymbol{E}$

$$dt : T\boldsymbol{E}\to\boldsymbol{E}\times\mathbb{T} \qquad \nu_o : T\boldsymbol{E}\to V\boldsymbol{E}.$$

Moreover, an observer o yields the translation fibred morphism over $\boldsymbol{E}$

$$\nabla_o : J_1\boldsymbol{E} \to \mathbb{T}^*\otimes V\boldsymbol{E} : \sigma_1 - o\circ\sigma.$$

So, if $s:\boldsymbol{T}\to\boldsymbol{E}$ is a motion and o is an observer, then we define the *observed velocity* to be the fibred morphism over s

$$\nabla_o s := \nabla_o \circ j_1 s : \boldsymbol{T} \to \mathbb{T}^*\otimes V\boldsymbol{E}$$

and we obtain

$$j_1 s = \nabla_o s + o\circ s. \qquad \square$$

A *time unit of measurement* is defined to be an oriented basis or its dual (see § III.1)

$$u_0\in\mathbb{T}^+ \qquad u^0\in\mathbb{T}^{+*}.$$

A *frame of reference* is defined to be a pair (u_0, o), where u_0 is a time unit of measurement and o an observer.

We denote the typical chart of $\boldsymbol{E}$, adapted to the fibring, to a time unit of measurement $u_0 \in \mathbb{T}^+$ and to the space-time orientation, by

$$(x^0, y^i).$$

The induced charts of $T\boldsymbol{E}$, $J_1\mathbf{E}$ and $TJ_1\boldsymbol{E}$ are denoted by

$$(x^0, y^i; \dot{x}^0, \dot{y}^i), \quad (x^0, y^i; y^i_0), \quad (x^0, y^i, y^i_0; \dot{x}^0, \dot{y}^i, \dot{y}^i_0).$$

Moreover, the corresponding local bases of vector fields and 1-forms of $\boldsymbol{E}$, $T\boldsymbol{E}$ and $J_1\boldsymbol{E}$ are denoted by

$$(\partial_0, \partial_i), \quad (\partial_0, \partial_i, \dot{\partial}_0, \dot{\partial}_i), \quad (\partial_0, \partial_i, \partial^0_i) \qquad (d^0, d^i), \quad (d^0, d^i, \dot{d}^0, \dot{d}^i), \quad (d^0, d^i, d^i_0),$$

Thus, by construction, we have

$$dt \circ \partial_0 = u_0 \qquad t^* u^0 = dx^0.$$

Moreover, we can write

$$\partial^0_i = u^0 \otimes \partial_i.$$

In general, vertical restrictions will be denoted by "ˇ". In particular, the local base of vertical 1-forms of $\boldsymbol{E}$ will be denoted by $(\check{d}^i)$.

Greek indices λ, μ, ... run from 0 to 3, Latin indices i, j, h, k, ... run from 1 to 3 and capital Latin indices A, B, ... span the values $0, 1, 2, 3, {}^1_0, {}^2_0, {}^3_0$.

The coordinate expressions of dt, д and ϑ are

$$dt = u_0 \otimes dx^0$$

$$д = u^0 \otimes д_0 \qquad \vartheta = \vartheta^i \otimes \partial_i,$$

with

$$д_0 := \partial_0 + y^i_0\, \partial_i \qquad \vartheta^i := d^i - y^i_0\, d^0.$$

The coordinate expression of the absolute velocity of a motion s is

$$j_1 s = u^0 \otimes \big((\partial_0 \circ s + \partial_0 s^i \, (\partial_i \circ s) \big).$$

Each chart (x^0, y^i) determines the local observer

$$o := u^0 \otimes \partial_0 : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E},$$

with coordinate expression (in the same chart) $o_0^i = 0$. This chart is said to be *adapted* to o. Conversely, each observer admits many adapted charts.

Let o be an observer and let us refer to adapted coordinates. Then, we obtain the following coordinate expressions

$$\nu_o = d^i \otimes \partial_i \qquad \nabla_o = y_0^i \, u^0 \otimes \partial_i;$$

moreover if $s : \boldsymbol{T} \to \boldsymbol{E}$ is a motion, then the coordinate expression of the observed velocity is

$$\nabla_o s = \partial_0 s^i \, u^0 \otimes \partial_i.$$

I.1.2. Vertical metric

In this section, we introduce the space-time metric and study the main related structures.

Assumption C2. We assume space-time to be equipped with a *scaled*[2] *vertical Riemannian metric*

$$g : \boldsymbol{E} \to \mathbb{A} \otimes (V^* \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^* \boldsymbol{E}),$$

where $\mathbb{A}$ is a 1-dimensional positive semi-vector space (see § III.1.3). ◇

Thus, $\mathbb{A}$ represents the space of area units.

We can also regard g as a degenerate 4-dimensional metric of signature (0,3) by considering the associated contravariant tensor

$$\bar{g} : \boldsymbol{E} \to \mathbb{A}^* \otimes (V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}) \subset \mathbb{A}^* \otimes (T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{E}).$$

The metrical linear fibred isomorphism and its inverse will be denoted by

---

[2] Here and later, "scaled" means "defined up to a scale factor".

$$g^\flat : V\boldsymbol{E} \to \mathbb{A} \otimes V^*\boldsymbol{E} \qquad g^\sharp : V^*\boldsymbol{E} \to \mathbb{A}^* \otimes V\boldsymbol{E}.$$

The coordinate expressions of g and $\bar{g}$ are

$$g = g_{ij}\, \check{d}^i \otimes \check{d}^j \qquad \bar{g} = g^{ij}\, \partial_i \otimes \partial_j$$

with

$$g_{ij} \in \mathcal{M}(\boldsymbol{E}, \mathbb{A} \otimes \mathbb{R}) \qquad g^{ij} \in \mathcal{M}(\boldsymbol{E}, \mathbb{A}^* \otimes \mathbb{R}).$$

REMARK I.1.2.1. The metric, the time-fibring and the choice of an orientation of the manifold $\boldsymbol{E}$ yield a *space-time* and a *space-like* scaled *volume form*

$$\upsilon : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{A}^{3/2}) \otimes \overset{4}{\wedge} T^*\boldsymbol{E} \qquad \eta : \boldsymbol{E} \to \mathbb{A}^{3/2} \otimes \overset{3}{\wedge} V^*\boldsymbol{E}.$$

Then, we obtain the dual elements

$$\bar{\upsilon} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}^{*3/2}) \otimes \overset{4}{\wedge} T\boldsymbol{E} \qquad \bar{\eta} : \boldsymbol{E} \to \mathbb{A}^{*3/2} \otimes \overset{3}{\wedge} V\boldsymbol{E}.$$

We have the coordinate expressions

$$\upsilon = \sqrt{|g|}\; u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 \qquad \eta = \sqrt{|g|}\; \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3$$

$$\bar{\upsilon} = \frac{1}{\sqrt{|g|}}\, u^0 \otimes \partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3 \qquad \bar{\eta} = \frac{1}{\sqrt{|g|}}\, \partial_1 \wedge \partial_2 \wedge \partial_3,$$

where

$$|g| := det\,(g_{ij}) \in \mathcal{M}(\boldsymbol{E}, \mathbb{A}^3).$$ ❑

REMARK I.1.2.2. The metric g yields the Riemannian connection on the fibres of $t : \boldsymbol{E} \to \boldsymbol{T}$, which can be regarded as the section

$$\varkappa : V\boldsymbol{E} \to V^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} VV\boldsymbol{E},$$

with coordinate expression

$$\varkappa = \check{d}^j \otimes (\partial_j + \varkappa_j{}^i{}_h\, \dot{y}^h\, \dot{\partial}_i),$$

where

$$\varkappa_{h\,k}{}^{i} = -\tfrac{1}{2}\, g^{ij}\, (\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk}). \qquad \square$$

Thus, the differences between the Einstein and Galilei general relativistic space-times can be summarised as follows:

- in the Einstein case, we have a Lorentz metric and no fibring over absolute time;

- in the Galilei case, we have a "space-like" metric $\bar{g}$ and a "time-like" 1-form $dt: \boldsymbol{E} \to \mathbb{T} \otimes T^* \boldsymbol{E}$, which determines the fibring over absolute time.

In brief words, we can say that the essential difference between the two theories consists in the replacement of the light cones with the vertical subspaces.

I.1.3. Units of measurement

We have already introduced the 1-dimensional oriented vector space of units of measurement of time $\mathbb{T}$ (Ass. C1 in I.1.1) and 1-dimensional positive vector space of units of measurement of area (Ass.C2 in I.1.2). Now, we complete our assumptions of fundamental spaces of units of measurement by introducing the 1-dimensional positive vector space of masses.

These three spaces generate all other spaces of units of measurement.

In the classical theory we assume a distinguished element in one of these spaces, namely the universal gravitational coupling constant.

In the quantum theory we shall assume another distinguished element in one of these spaces, namely the Plank constant (see Ass. Q2 in § II.1.4).

ASSUMPTION C3. We assume the *space of masses* to be a 1-dimensional positive semi-vector space (see § III.1.3) $\mathbb{M}$. ◇

The *mass* of a classical or quantum particle is defined to be an element

$$m \in \mathbb{M}.$$

The mass plays the role of coupling constants for the metric.

The three fundamental 1-dimensional semi-vector spaces (see § III.1.3) $\mathbb{T}$, $\mathbb{A}$ and $\mathbb{M}$ generate all spaces of units of measurement.

DEFINITION I.1.3.1. A *space of units of measurement* is defined to be a

1-dimensional semi-vector space of the type

$$\mathbb{U} := \mathbb{T}^p \otimes \mathbb{A}^q \otimes \mathbb{M}^r,$$

where p, q, and r are rational numbers (see § III.1.3).

We say that $\mathbb{U}$ *has dimensions*

$$(p, q, r). \qquad ♣$$

In the classical theory we assume just one universal unit of measurement.

ASSUMPTION C4. We assume the *gravitational coupling constant* to be an element

$$\boldsymbol{\kappa} \in \mathbb{T}^{*2} \otimes \mathbb{A}^{3/2} \otimes \mathbb{M}^{*}. \qquad \diamond$$

I.2 - Space-time connections

In view of further development of our model, we need a preliminary study of connections which preserve the fibred and metrical structure of space-time.

I.2.1. Space-time connections

In this section, we introduce the notion of space-time connection by referring to the tangent or to the jet space, equivalently.

REMARK I.2.1.1. Let K be a linear connection on the vector bundle $T\boldsymbol{E}\to\boldsymbol{E}$ (see § III.5).

The following conditions are equivalent:

i) K is dt-preserving, i.e.

$$\nabla dt = 0;$$

ii) the coefficients $K_\lambda{}^0{}_\mu \in \mathcal{F}(\boldsymbol{E})$ of K with time-like superscript vanish, i.e.

$$K_\lambda{}^0{}_\mu = 0.$$

Moreover, the following conditions are equivalent:

iii) the fibres of $t:\boldsymbol{E}\to\boldsymbol{T}$ are auto parallel with respect to K, i.e.

$$\nabla_X Y \in \mathcal{S}(V\boldsymbol{E}\to\boldsymbol{E}) \qquad \forall X,Y\in\mathcal{S}(V\boldsymbol{E}\to\boldsymbol{E}):$$

iv) K can be restricted to the fibres of $t:\boldsymbol{E}\to\boldsymbol{T}$;

v) the coefficients $K_i{}^0{}_j \in \mathcal{F}(\boldsymbol{E})$ of K with time-like superscript vanish, i.e.

$$K_i{}^0{}_j = 0.$$

Furthermore, i) implies iii). ❑

Let us consider a dt-preserving torsion free linear connection

$$K:T\boldsymbol{E}\to T^*\boldsymbol{E}\underset{T\boldsymbol{E}}{\otimes}TT\boldsymbol{E}$$

on the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$ and a torsion free[3] affine connection

$$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E},$$

on the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$.

Their coordinate expressions are of the type

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i\, \dot{\partial}_i) \qquad\qquad \Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^i\, \partial_i^0)$$

where

$$K_\lambda{}^i := K_\lambda{}^i{}_h\, \dot{y}^h + K_\lambda{}^i{}_0\, \dot{x}^0 \qquad\qquad \Gamma_\lambda{}^i := \Gamma_\lambda{}^i{}_h\, y_0^h + \Gamma_\lambda{}^i{}_\circ$$

$$\Gamma_\lambda{}^i{}_\mu = \Gamma_\mu{}^i{}_\lambda \qquad K_\lambda{}^i{}_\mu = K_\mu{}^i{}_\lambda,$$

with

$$\Gamma_\lambda{}^i{}_\mu,\ K_\lambda{}^i{}_\mu \in \mathcal{F}(\boldsymbol{E}).$$

PROPOSITION I.2.1.1. There is a natural bijection

$$K \mapsto \Gamma$$

between such connections; its coordinate expression is given by

$$\Gamma_\lambda{}^i{}_\mu = K_\lambda{}^i{}_\mu.$$

PROOF. It follows by considering the following commutative diagram

$$\begin{array}{ccc} J_1\boldsymbol{E} & \xrightarrow{\ \Gamma\ } & T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E} \\ \downarrow & & \downarrow \\ \mathbb{T}^* \otimes T\boldsymbol{E} & \xrightarrow{\ K\ } & T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E} \simeq T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T(\mathbb{T}^* \otimes T\boldsymbol{E}) \end{array}$$

□

DEFINITION I.2.1.1. A *space-time connection* is defined to be, equivalently, a connection K, or Γ, of the above type. ♣

---

[3] The torsion for such an affine connection is defined through the $\mathbb{T}$-valued soldering form ϑ, via the Frölicher-Nijenhuis bracket (see § III.4, § III.5).

The first viewpoint is more suitable for field theory (where we have to take covariant derivatives of space-time tensors), the second one for classical and quantum particle mechanics (where the jet space plays the role of kinematical space).

We shall be involved with the vertical valued 1-forms associated with the space-time connection (see § III.5)

$$\nu_K : T\boldsymbol{E} \to T^*T\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} T\boldsymbol{E} \qquad \nu_\Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes (T^*J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} V\boldsymbol{E})$$

with coordinate expressions

$$\nu_K = \dot{d}^0 \otimes \partial_0 + (\dot{d}^i - K_\lambda{}^i\, d^\lambda) \otimes \dot{\partial}_i \qquad \nu_\Gamma = (d^i_0 - \Gamma_\lambda{}^i\, d^\lambda) \otimes \partial^0_i .$$

REMARK I.2.1.2. The space-time connection K restricts to the linear connection

$$K' : V\boldsymbol{E} \to T^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} TV\boldsymbol{E}$$

of the vertical bundle $V\boldsymbol{E} \to \boldsymbol{E}$, with coordinate expression

$$K' = d^\lambda \otimes (\partial_j + K_\lambda{}^i{}_h\, \dot{y}^h\, \dot{\partial}_i). \qquad \square$$

REMARK I.2.1.3. The natural linear fibred epimorphism $T^*\boldsymbol{E} \to V^*\boldsymbol{E}$ over $\boldsymbol{E}$ yields the further restriction

$$\check{K} : V\boldsymbol{E} \to V^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} VV\boldsymbol{E},$$

which can be regarded as a smooth family of linear connections on the fibres of $t : \boldsymbol{E} \to \boldsymbol{T}$, and has coordinate expression

$$\check{K} = \check{d}^j \otimes (\partial_j + K_j{}^i{}_h\, \dot{y}^h\, \dot{\partial}_i). \qquad \square$$

REMARK I.2.1.4. Given a space-time connection K, we can define, as usual, its curvature[4] (see § III.5)

[4] [,] is the Frölicher Nijenhuis bracket.

$$R := \frac{1}{2}[K,K] : T\boldsymbol{E} \to \overset{2}{\wedge} T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}$$

its Ricci tensor

$$r := 2\, C^1_1 R : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E}$$

and its scalar curvature

$$s := \langle \bar{g},\, r \rangle : \boldsymbol{E} \to \mathbb{R},$$

with coordinate expressions

$$R := R_{\lambda\mu}{}^{i}{}_{\nu}\, d^\lambda \wedge d^\mu \otimes \partial_i \otimes d^\nu = (\partial_\lambda K_\mu{}^i{}_\nu + K_\lambda{}^j{}_\nu\, K_\mu{}^i{}_j)\, d^\lambda \wedge d^\mu \otimes \partial_i \otimes d^\nu$$

$$r = (\partial_i K_\lambda{}^i{}_\mu - \partial_\lambda K_i{}^i{}_\mu + K_i{}^j{}_\mu\, K_\lambda{}^i{}_j - K_\lambda{}^j{}_\mu\, K_i{}^i{}_j)\, d^\lambda \otimes d^\mu$$

$$s = g^{hk}\, (\partial_i K_h{}^i{}_k - \partial_h K_i{}^i{}_k + K_i{}^j{}_h\, K_k{}^i{}_j - K_h{}^j{}_k\, K_i{}^i{}_j).$$

Then, the scalar curvature of K coincides with the scalar curvature of $\check{K}$

$$s = \check{s}. \qquad \square$$

I.2.2. Space-time connections and observers

In this section, we consider a space-time connection Γ, an observer o along with an adapted space-time chart (x^0, y^i) and describe the connection through the observer.

Remark I.2.2.1. The covariant differential of the observer is the section

$$\nabla o : \boldsymbol{E} \to \mathbb{T}^* \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}),$$

with coordinate expression in adapted coordinates

$$\nabla o = -\, u^0 \otimes (\Gamma_0{}^i{}_\circ\, d^0 + \Gamma_j{}^i{}_\circ\, d^j) \otimes \partial_i. \qquad \square$$

Remark I.2.2.2. The metric g and the inclusion $V^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ induced by the observer allow us to regard ∇o as a section

$$(\nabla o)^\flat : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E}),$$

whose expression in adapted coordinates is

$$(\nabla o)^\flat = -\, u^0 \otimes (\Gamma_{0i\circ}\, d^0 + \Gamma_{ij\circ}\, d^i) \otimes d^j.$$

We can split the above tensor into its symmetrical and anti symmetrical components

$$(\nabla o)^\flat = \frac{1}{2}\, \Sigma + \frac{1}{2}\, \Phi,$$

where

$$\Sigma : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \underset{2}{\vee} T^*\boldsymbol{E} \qquad \Phi : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^*\boldsymbol{E}.$$

Then, we obtain the coordinate expressions

$$\Sigma := u^0 \otimes \big(2\, \Sigma_{0j}\, d^0 \vee d^j + \Sigma_{ij}\, d^i \vee d^j \big) = -\, 2\, u^0 \otimes (\Gamma_{0j\circ}\, d^0 \vee d^j + \Gamma_{ij\circ}\, d^i \vee d^j)$$

$$\Phi := u^0 \otimes \big(2\, \Phi_{0j}\, d^0 \wedge d^j + \Phi_{ij}\, d^i \wedge d^j \big) = -\, 2\, u^0 \otimes (\Gamma_{0j\circ}\, d^0 \wedge d^j + \Gamma_{ij\circ}\, d^i \wedge d^j),$$

where

$$\Sigma_{ij} = -\, (\Gamma_{ij\circ} + \Gamma_{ji\circ}) \qquad \Sigma_{0j} = -\, \Gamma_{0j\circ} = \Phi_{0j} \qquad \Phi_{ij} = -\, (\Gamma_{ij\circ} - \Gamma_{ji\circ}),$$

with

$$\Sigma_{\lambda j},\ \Phi_{\lambda j} \in \mathcal{M}(\boldsymbol{E}, \mathbb{A} \otimes \mathbb{R}).$$ ❑

PROPOSITION I.2.2.1. The maps

$$K \mapsto (\check{K},\ \nabla o) \mapsto (\check{K},\ \check{\Sigma},\ \Phi)$$

are bijections. ❑

In other words, $\check{K}$ and ∇o carry independent information on K; $\check{\Sigma}$ and Φ carry independent information on o; moreover, $\check{\Sigma}$ and Φ characterise ∇o and the pair $(\check{K},\ \nabla o)$ characterises K itself.

An observer o is said to be *inertial* with respect to the space-time connection K if $\nabla o = 0$.

Of course, inertial observers on a curved space-time might not exist at all.

I.2.3. Metrical space-time connections

This section is devoted to study space-time connections which preserve the metric. This subject needs a little care because of the degeneracy of the metric.

PROPOSITION I.2.3.1. Let K be a space-time connection. Then, the following four conditions i), ii), iii), iv) are equivalent:

i) $$\nabla \bar{g} = 0;$$

ii) in a space-time chart

$$\partial_\lambda g^{ij} - K_\lambda{}^i{}_h \, g^{hj} - K_\lambda{}^j{}_h \, g^{ih} = 0;$$

iii) iii)′ $$\check{K} = \varkappa,$$

iii)″ for an observer $o : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$

$$\check{\Sigma} = g^\flat \, L_o \bar{g};$$

iv) in a space-time chart

iv)′ $$K_{ihj} = -\tfrac{1}{2}(\partial_i g_{hj} + \partial_j g_{hi} - \partial_h g_{ij}),$$

iv)″ $$K_{0ij} + K_{0ji} = -\partial_0 g_{ij}.$$

PROOF. We can easily see that i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iv) and iii)′ $\Leftrightarrow$ iv)′.

Moreover, we can easily see that, with reference to an observer o and any adapted chart, iii)′ $\Leftrightarrow$ iv)″. Then, we can conclude the proof by closing the circle of equivalencies (recalling that each space-time chart is adapted to an observer). ❑

Then, we give the following definition.

DEFINITION I.2.3.1. A space-time connection K is said to be *metrical* if

$$\nabla \bar{g} = 0.$$ ♣

COROLLARY I.2.3.1. Let K be a metrical space-time connection and let o be a global observer. Then, the following conditions are equivalent:

- for each $\tau, \tau' \in T$ the diffeomorphism

$$\boldsymbol{E}_\tau \to \boldsymbol{E}_{\tau'}$$

induced by o is an isometry;

- $$\check{\Sigma} = 0.$$

PROOF. It follows immediately from iii)" of Prop. I.2.3.1. ❑

REMARK I.2.3.1. If K is a metrical space-time connection, then

$$*) \qquad K_j{}^i{}_i = -\tfrac{1}{2} g^{ih} \partial_j g_{ih}.$$

PROOF. Formula ∗) follows immediately from iv)'. ❑

COROLLARY I.2.3.2. If K is a metrical space-time connection, then

$$**) \qquad \nabla \upsilon = 0;$$

$$**)' \qquad K_\lambda{}^i{}_i = -\frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}};$$

$$***) \qquad g^{hk} K_h{}^i{}_k = \frac{\partial_j (g^{ij} \sqrt{|g|})}{\sqrt{|g|}}.$$

PROOF. Formula ∗∗)' follows from ∗) of Rem. I.2.3.1 and from the algebraic identity

$$\langle A^{-1}, DA \rangle = \frac{D(\det A)}{\det A},$$

which holds for any map

$$A : \mathbb{R} \to \mathrm{Auto}\,(V) \subset V^* \otimes V,$$

where V is a vector space[5].

Moreover, formula $**)'$ is the coordinate expression of $**)$.

Furthermore, we have, in virtue of iv)' of Prop. I.2.3.1,

$$g^{hk}\, K_{h\,k}^{\ \,i} = \partial_j g^{ij} + \tfrac{1}{2}\, g^{ij}\, g^{hk}\, \partial_j g_{hk},$$

hence, in virtue of $*)$,

$$g^{hk}\, K_{h\,k}^{\ \,i} = \partial_j g^{ij} - g^{ij}\, K_{j\ h}^{\ h},$$

hence, in virtue of $**)'$,

$$g^{hk}\, K_{h\,k}^{\ \,i} = \partial_j g^{ji} + g^{ij}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}},$$

which yields $***)$. ❑

COROLLARY I.2.3.4. If K is a metrical space-time connection, then

$$\nabla \eta = 0,$$

where the covariant differential is performed through the induced linear connection K' on the vector bundle $V\boldsymbol{E} \to \boldsymbol{E}$ (see Rem. I.2.1.2). ❑

I.2.4. Divergence and codifferential operators

This section is devoted to the study of different kinds of divergence operators. This subject needs a little care because of the degeneracy of the metric.

Let us start by considering just the vertical metric g.

If

$$X : \boldsymbol{E} \to T\boldsymbol{E}$$

is a vector field, then we define the *codifferential* of X to be the function

[5] In a (pseudo-)Riemannian manifold, we can define the volume form υ by means of the condition $g(\upsilon,\upsilon) = \pm 1$; then, the identity $\nabla \upsilon = 0$ can be deduced directly from the metricity of the connection. But this direct argument does not hold for our degenerate metric.

$$\delta X := \langle \bar{\upsilon}, d i_X \upsilon \rangle = \langle \bar{\upsilon}, L_X \upsilon \rangle : \boldsymbol{E} \to \mathbb{R},$$

with coordinate expression

$$\delta X = \frac{\partial_\lambda (X^\lambda \sqrt{|g|})}{\sqrt{|g|}}.$$

We have no corresponding codifferential for forms, because the metric g is degenerate. However, we can define the vertical codifferential of the vertical restrictions of forms, as usual.

Next, let us assume a space-time connection K.

If

$$X : \boldsymbol{E} \to T\boldsymbol{E}$$

is a vector field, then we define the *divergence* of X to be the function

$$\operatorname{div} X := tr\, \nabla X : \boldsymbol{E} \to \mathbb{R},$$

with coordinate expression

$$\operatorname{div} X = \partial_\lambda X^\lambda - K_\lambda{}^i{}_i X^\lambda.$$

If

$$\omega : \boldsymbol{E} \to T^*\boldsymbol{E}$$

is a 1-form, then we define the *divergence* of ω to be the function

$$\operatorname{div} \omega := \langle \bar{g}, \nabla \omega \rangle : \boldsymbol{E} \to \mathbb{A}^* \otimes \mathbb{R},$$

with coordinate expression

$$\operatorname{div} \omega = g^{ij} (\partial_i \omega_j + K_i{}^h{}_j\, \omega_h).$$

We remark that this divergence depends only on the vertical restrictions of ω and K, i.e. we can write

$$\operatorname{div} \omega = \check{\operatorname{div}}\, \check{\omega} := \langle \bar{g}, \check{\nabla} \check{\omega} \rangle.$$

The above two divergence operators can be extended to the tensor algebra by means of the Leibnitz rule.

Eventually, let us assume that the space-time connection K be metrical.

Then we obtain the following results.

If

$$X : \boldsymbol{E} \to T\boldsymbol{E}$$

is a vector field, then

$$\delta X = \operatorname{div} X,$$

because the coordinate expressions of the two hand sides coincide.

If

$$\omega : \boldsymbol{E} \to \overset{2}{\wedge} T^* \boldsymbol{E}$$

is a 2-form, then we obtain

$$\operatorname{div}^2 \omega = 0;$$

in fact $\operatorname{div}^2 \omega = \check{\operatorname{div}}^2 \check{\omega}$ and we can apply the classical Riemannian identity.

I.2.5. Second order connection and contact 2-form

Next, we associate two further objects with a space-time connection: a second order connection and a contact 2-form. These objects will play a fundamental role in the classical and quantum theories.

First, let us show how a space-time connection yields naturally these two objects.

PROPOSITION I.2.5.1. If Γ is a space-time connection, then we obtain the connection on the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$

$$\gamma := д \lrcorner \Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T J_1\boldsymbol{E},$$

with coordinate expression

$$\gamma = u^0 \otimes (\partial_0 + y_0^{\ i} \, \partial_i + \gamma^i \, \partial_i^0),$$

where

$$\gamma^i = \Gamma_{h\,k}^{\ i} \, y_0^h \, y_0^k + 2\, \Gamma_{h\,\circ}^{\ i} \, y_0^h + \Gamma_{0\,\circ}^{\ i}. \qquad \square$$

PROPOSITION I.2.5.2. If Γ is a space-time connection, then we obtain the scaled 2-form on the manifold $J_1\boldsymbol{E}$[6]

$$\Omega := \nu_\Gamma \bar{\wedge} \vartheta : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^* J_1\boldsymbol{E},$$

with coordinate expression

$$\Omega = g_{ij}\, u^0 \otimes (d_0^i - \gamma^i\, d^0 - \Gamma_h{}^i\, \vartheta^h) \wedge \vartheta^j. \qquad \square$$

It can be proved that Ω is the unique scaled 2-form on $J_1\boldsymbol{E}$ naturally induced by g and Γ (see [Ja]).

Next, let us study the main properties of these two objects.

Let us recall that a *second order connection* on the fibred manifold $t:\boldsymbol{E}\to\boldsymbol{B}$ is defined to be a section (see § III.3, [MM1])

$$c : J_1\boldsymbol{E} \to J_2\boldsymbol{E}.$$

Moreover, we have natural fibred mono-morphisms over J_1E

$$J_1 J_1\boldsymbol{E} \hookrightarrow \mathbb{T}^* \otimes T J_1\boldsymbol{E} \qquad J_2\boldsymbol{E} \hookrightarrow J_1 J_1\boldsymbol{E}.$$

Actually, $J_2\boldsymbol{E}$ turns out to be the fibred submanifold

$$J_2\boldsymbol{E} \hookrightarrow \mathbb{T}^* \otimes T J_1\boldsymbol{E}$$

over $J_1\boldsymbol{E}$, which projects over $\vartheta : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$.

Hence, a (first order) connection

$$c : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T J_1\boldsymbol{E}$$

of the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$ is a second order connection of the fibred manifold $t:\boldsymbol{E}\to\boldsymbol{T}$ if and only if c is projectable over $\vartheta : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$, i.e. if and only if the coordinate expression of c is of the type

$$\gamma = u^0 \otimes \left(\partial_0 + y_0^i\, \partial_i + c^i\, \partial_i^0\right) \qquad c^i \in \mathcal{F}(J_1\boldsymbol{E}).$$

---

[6] The symbol "$\bar{\wedge}$" denotes wedge product $\wedge$ followed by scalar product g.

Moreover, by considering the algebraic structure of the bundle $J_2\boldsymbol{E} \to \boldsymbol{E}$, we can define the homogeneous second order connections: they are characterised in coordinates by the fact that the coefficients c^i are second order polynomials in the coordinates y^j_0.

REMARK I.2.5.1. If Γ is a space-time connection, then γ is a homogeneous second order connection[7]. ❑

PROPOSITION I.2.5.1. If Γ is a space-time connection and o is an observer, then we obtain the following important equality (see § I.2.2)

$$\Phi := 2\, o^*\Omega. \qquad ❑$$

PROPOSITION I.2.5.2. If Γ is a space-time connection, then the 2-form Ω is non-degenerate in the sense that it yields the non singular scaled volume form[8] on the manifold $J_1\boldsymbol{E}$

$$dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to (\mathbb{T}^{*2} \otimes \mathbb{A}^3) \otimes \overset{7}{\wedge} T^* J_1\boldsymbol{E}. \qquad ❑$$

PROPOSITION I.2.5.3. If Γ is a space-time connection, then the 2-form Ω is characterised by the following property:

- for each second order connection γ', we obtain the formula

$$i_{\gamma'}\Omega = \vartheta \lrcorner (\gamma' - \gamma),$$

with coordinate expression

$$i_{\gamma'}\Omega = g_{ij}\, (\gamma'^i - \gamma^i)\, u^0 \otimes \vartheta^j.$$

---

[7] If we choose a time unit of measurement u^0 and replace $\mathbb{T}$ with $\mathbb{R}$, then the theory of second order connections on the jet space reduces to the more usual theory of second order differential equations on the tangent space. Our approach based on jets is required by the explicit independence from time units of measurements of our theory.

[8] This form can be taken as the “Liouville volume form” of our model and can be used for developing a Galilei general relativistic statistics.

PROOF. It follows from a computation in coordinates, by considering the base of forms $\big(d^0,\ \vartheta^i,\ (d^i_0 - \gamma^i\, d^0)\big)$. ❑

COROLLARY I.2.5.1. If Γ is a space-time connection, then γ and Ω fulfill the property

$$\gamma \lrcorner \Omega = 0. \qquad ❑$$

So, we introduce the following definition.

DEFINITION I.2.5.1. If Γ is a space-time connection, then

$$\gamma := д \lrcorner \Gamma \qquad\qquad \Omega := \nu_\Gamma \wedge \vartheta$$

are said to be, respectively, the *second order connection* and the scaled *contact 2-form* [9] associated with Γ. ♣

Eventually, let us see how γ and Ω characterise Γ.

PROPOSITION I.2.5.4. If γ is a homogeneous second order connection on $\boldsymbol{E}\to\boldsymbol{T}$, then there is a unique torsion free affine connection Γ on $J_1\mathbf{E}\to\boldsymbol{E}$, such that

$$\gamma = д \lrcorner \Gamma.$$

PROOF. It follows from a comparison of the coordinate expressions of γ and Γ. ❑

COROLLARY I.2.5.2. If Ω is the contact 2-form associated with the space-time connection Γ, then there is a unique connection γ' on the fibred manifold $J_1\boldsymbol{E}\to\boldsymbol{T}$, such that

$$i_{\gamma'}\Omega = 0.$$

Namely, we have

[9] In the literature, the term "contact form" is devoted to a 1-form of class $2n+1$ on a manifold of dimension $2n+1$. So, there is no conflict between the usual terminology and ours.

$$\gamma' = \gamma.$$

PROOF. It follows from Prop. I.2.5.3. ❑

COROLLARY I.2.5.3. If Ω is the contact 2-form associated with the space-time connection Γ, then there is a unique space-time connection Γ' such that

$$\Omega = \nu_{\Gamma'} \wedge \vartheta.$$

Namely, we have

$$\Gamma' = \Gamma.$$

PROOF. It follows from Prop. I.2.5.4 and Cor. I.2.5.2. ❑

We observe that, if the space-time connection is metrical, then the vertical restriction of the contact 2-form Ω turns out to be the vertical symplectic 2-form

$$\check{\Omega} = \nu_{\varkappa} \wedge V\pi_{\boldsymbol{E}} : V\boldsymbol{E} \to \mathbb{A} \otimes \overset{2}{\wedge} V^* V\boldsymbol{E}$$

associated with the vertical metric g. But $\check{\Omega}$ cannot have a total role in our theory, as a consequence of the principle of relativity. The replacement of $\check{\Omega}$ with Ω makes an essential difference between our approach and the viewpoint of geometrical quantisation.

I.2.6. Space-time connections and acceleration

We conclude this section with the study of the acceleration of a motion with respect to a given space-time connection. This subject is necessary for a full understanding of the meaning of the space-time second order connections. The results below will be used in the expression of the law of motion of classical particles (see § I.5.1).

Let us consider the second jet space $J_2\boldsymbol{E}$ (see § III.3). We recall that the natural map $J_2\boldsymbol{E} \to J_1\boldsymbol{E} : \sigma_2 \mapsto \sigma_1$ is an affine bundle associated with the vector bundle $(\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$.

Now, let us consider a space-time connection Γ and the associated second order connection γ (see § I.2.5).

We obtain the translation fibred morphism over $J_1\boldsymbol{E} \to \boldsymbol{E}$

$$\nabla_\gamma : J_2\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E} : \sigma_2 \mapsto \sigma_2 - \gamma(\sigma_1).$$

Then, let us consider a motion $s : \boldsymbol{T} \to \boldsymbol{E}$.

We define the *(absolute) acceleration* of s to be the second order covariant differential of s, i.e. the section

$$\nabla_\gamma \, j_1 s := j_2 s - \gamma \circ j_1 s : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$

with coordinate expression

$$\nabla_\gamma \, j_1 s = (\partial_{00} s^i - \gamma^i \circ j_1 s) \, u^0 \otimes u^0 \otimes (\partial_i \circ s) =$$

$$= (\partial_{00} s^i - \Gamma_{h\,k}^{\,i} \circ s \; \partial_0 s^h \, \partial_0 s^k - 2 \, \Gamma_{h\,\circ}^{\,i} \circ s \; \partial_0 s^h - \Gamma_{0\,\circ}^{\,i} \circ s) \, u^0 \otimes u^0 \otimes (\partial_i \circ s).$$

I.3 - Gravitational and electromagnetic fields

So far, the space-time fibred manifold has been equipped only with the vertical metric. Now, we complete the structure of space-time by adding the gravitational and electromagnetic fields.

I.3.1. The fields

In this section, we introduce the gravitational and electromagnetic fields.

Assumption C5. We assume space-time to be equipped with a space-time connection (see § I.2.1)

$$\Gamma^{\natural} : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} T J_1\boldsymbol{E},$$

and a scaled 2-form

$$F : \boldsymbol{E} \to \mathbb{B} \otimes \overset{2}{\wedge} T^*\boldsymbol{E},$$

where $\mathbb{B}$ is the 1-dimensional vector space

$$\mathbb{B} = \mathbb{A}^{1/4} \otimes \mathbb{M}^{1/2}. \qquad \diamond$$

We say that $\Gamma^{\natural}$ is the *gravitational field* and F the *electromagnetic field*.

The superscript "$\natural$" will label objects related to the gravitational connection $\Gamma^{\natural}$. In particular, we have (see § I.2.1, § I.2.5):

$$\nabla^{\natural} := \nabla_{\Gamma^{\natural}} \qquad \gamma^{\natural} := д \lrcorner \Gamma^{\natural} \qquad \Omega^{\natural} := \nu_{\Gamma^{\natural}} \bar{\wedge} \vartheta.$$

We stress that, as the metric is degenerate, it cannot determine fully the gravitational connection. Hence, we must assume that the gravitational field is described just by the gravitational connection $\Gamma^{\natural}$.

The sections $s : \boldsymbol{T} \to \boldsymbol{E}$, which fulfill the equality

$$\nabla_{\gamma^{\natural}} j_1 s = 0$$

will be interpreted as *free falling motions*, according to the Newton law of motion (see § I.5.1).

The coordinate expression of F is

$$F = 2\, F_{0j}\, d^0 \wedge d^j + F_{ij}\, d^i \wedge d^j, \qquad F_{\lambda j} \in \mathcal{M}(\boldsymbol{E}, \mathbb{B}\otimes\mathbb{R}).$$

The (observer independent) *magnetic field* and *electric field* related to an observer o are defined to be the vertical 1-forms

$$B := \tfrac{1}{2}\, \check{*}\, \check{F} : \boldsymbol{E} \to (\mathbb{B}\otimes\mathbb{A}^{1/2*})\otimes V^*\boldsymbol{E} \qquad E := -\,(o \check{\lrcorner} F) : \boldsymbol{E} \to (\mathbb{T}^*\otimes\mathbb{B})\otimes V^*\boldsymbol{E}$$

with coordinate expressions

$$B = \sqrt{|g|}\,\left(F^{12}\, \check{d}^3 + F^{31}\, \check{d}^2 + F^{23}\, \check{d}^1\right) \qquad E = F_{i0}\, \check{d}^i$$

and we can write

$$F = 2\, dt \wedge o^*(E) + 2\, o^*(\check{*}\, B).$$

I.3.2. Gravitational and electromagnetic coupling

Next, we exhibit a natural way to incorporate the electromagnetic field and the gravitational connection into the geometric structures of space-time. This coupling is parametrised either by the ratio of a charge and a mass or by the square root of the gravitational constant.

Such a procedure works very well both in classical and quantum theories.

DEFINITION I.3.2.1. We define the *space of charges* to be the oriented 1-dimensional vector space

$$\mathbb{Q} = \mathbb{T}^*\otimes\mathbb{A}^{3/4}\otimes\mathbb{M}^{1/2}. \qquad \square$$

The *charge* of a classical or quantum particle is defined to be an element

$$\boldsymbol{q} \in \mathbb{Q}.$$

Moreover, given $u_0 \in \mathbb{T}^+$, we set

$$q := \boldsymbol{q}(u_0) \in \mathbb{A}^{3/4}\otimes\mathbb{M}^{1/2}.$$

The charge plays the role of coupling constants for the electromagnetic field.

REMARK I.3.2.1. The square root of the gravitational coupling constant (see Ass. C4 in §I.1.3) and the ratio $\boldsymbol{q}/m$ of any charge $\boldsymbol{q}$ and mass m have the same dimensions:

$$\sqrt{\boldsymbol{\kappa}} \in \mathbb{T}^* \otimes \mathbb{A}^{3/4} \otimes \mathbb{M}^{*\,1/2} \qquad \frac{\boldsymbol{q}}{m} \in \mathbb{T}^* \otimes \mathbb{A}^{3/4} \otimes \mathbb{M}^{*\,1/2}.$$

Hence, the following objects have the same dimensions

$$\Omega^\natural : J_1 \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^* J_1 \boldsymbol{E}$$

$$\sqrt{\boldsymbol{\kappa}}\, F : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^* \boldsymbol{E} \qquad \frac{\boldsymbol{q}}{m}\, F : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^* \boldsymbol{E}.$$

Therefore, there are two distinguished ways to couple the gravitational contact 2-form $\Omega^\natural$ and the electromagnetic field F in a way independent of the choice of any unit of measurement. Namely, we can use as coupling constant both the square root of the universal gravitational coupling constant $\sqrt{\boldsymbol{\kappa}}$ and the ratio $\boldsymbol{q}/m$ of a given charge $\boldsymbol{q}$ and a mass m. In the first case we obtain a universal coupling, in the second case the coupling depends on the choice of a particular particle. ❑

In practice, we are concerned with $\boldsymbol{c} = \sqrt{\boldsymbol{\kappa}}$ only in the context of the second gravitational field equation (see § I.4.5, § I.4.7) and in all other cases we consider $\boldsymbol{c} = \boldsymbol{q}/m$.

Thus, let us consider an element

$$\boldsymbol{c} \in \mathbb{T}^* \otimes \mathbb{A}^{3/4} \otimes \mathbb{M}^{*\,1/2},$$

which might be either $\sqrt{\boldsymbol{\kappa}}$, or $\frac{\boldsymbol{q}}{m}$ (for a certain given charge $\boldsymbol{q}$ and mass m).

Moreover, given $u_0 \in \mathbb{T}^+$, we set

$$c := \boldsymbol{c}(u_0) \in \mathbb{A}^{3/4} \otimes \mathbb{M}^{*\,1/2}.$$

We shall exhibit a natural way to deform the gravitational space-time connection $\Gamma^\natural$, the related second order connection $\gamma^\natural$ and contact 2-form $\Omega^\natural$

into corresponding "*total objects*" Γ, γ and Ω, through the electromagnetic field F.

Let us start with $\Omega^\natural$. In fact, it is natural to consider the deformed "total" 2-form

$$\Omega := \Omega^\natural + \Omega^e := \Omega^\natural + \tfrac{1}{2}\boldsymbol{c}\, F,$$

obtained by adding the electromagnetic 2-form to the gravitational contact 2-form.

Here, the coupling constant $\boldsymbol{c}$ is necessary for the equality of the left and right hand sides (i.e. the correct dimensionality of the above formula). On the other hand, we remark that we could have multiplied F by any other non-zero scalar factor; the factor 1/2 has been chosen just in order to obtain the standard normalisation in the classical and quantum equations.

Then, we can prove the following result.

THEOREM I.3.2.1. Let us consider the total contact 2-form

$$\Omega := \Omega^\natural + \Omega^e := \Omega^\natural + \tfrac{1}{2}\boldsymbol{c}\, F.$$

Then, there exist a unique torsion free affine connection Γ on the bundle $J_1\boldsymbol{E}\to\boldsymbol{E}$ and a unique connection γ on the fibred manifold $J_1\boldsymbol{E}\to\boldsymbol{T}$

$$\Gamma = \Gamma^\natural + \Gamma^e \qquad\qquad \gamma = \gamma^\natural + \gamma^e$$

such that[10]

$$\gamma := д \lrcorner \Gamma \qquad \Omega := \boldsymbol{\nu}_\Gamma \barwedge \vartheta \qquad \gamma \lrcorner \Omega = 0.$$

Namely, γ^e turns out to be the *Lorentz force*

$$\gamma^e := -\,\boldsymbol{c}\; g^{\sharp}(д \check{\lrcorner} F) : J_1\boldsymbol{E} \to \mathbb{T}^*\otimes(\mathbb{T}^*\otimes V\boldsymbol{E})$$

and Γ^e the *electromagnetic soldering form*

$$\Gamma^e := \tfrac{1}{4}\boldsymbol{c}\; g^{\sharp}\big((\vartheta + 3\,д)\check{\barwedge} F\big) : J_1\boldsymbol{E} \to T^*\boldsymbol{E}\underset{\boldsymbol{E}}{\otimes}(\mathbb{T}^*\otimes V\boldsymbol{E}).$$

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[10] These are just the same equalities valid for the corresponding pure gravitational objects (see § I.2.5).

Moreover, the above objects fulfill the following equalities

$$\gamma^e = d \lrcorner \Gamma^e \qquad - \Gamma^e \barwedge \vartheta = \Omega^e .$$

We have the coordinate expressions

$$\gamma^e = - c\, (F_0{}^i + F_h{}^i\, y_0^h)\, u^0 \otimes \partial_i^0 \qquad \Gamma^e = \tfrac{1}{2}\, c\, \big((F^i{}_h\, y_0^h + 2\, F^i{}_0)\, d^0 + F^i{}_j\, d^j\big) \otimes \partial_i^0,$$

hence

$$\Gamma_{h\,k}{}^i = \Gamma^{\natural}{}_{h\,k}{}^i \qquad \Gamma_{0\,k}{}^i = \Gamma^{\natural}{}_{0\,k}{}^i + \tfrac{1}{2}\, c\, F^i{}_k \qquad \Gamma_{0\,\circ}{}^i = \Gamma^{\natural}{}_{0\,\circ}{}^i + c\, F^i{}_0 .$$

PROOF. The proof can be obtained by a computation in coordinates. ❑

COROLLARY I.3.2.1. If o is an observer, then the gravitational and electromagnetic coupling can be read in the following way

$$\check{K} = \check{K}^{\natural} \qquad \check{\Sigma} = \check{\Sigma}^{\natural} \qquad \Phi = \Phi^{\natural} + \boldsymbol{c}\, F .$$

PROOF. It follows from the Theor. I.3.2.1 and Prop. I.2.2.1. ❑

We stress that the Lorentz force has been derived and not postulated. The coupling of the gravitational connection and the electromagnetic field does not affect the torsion: both $K^{\natural}$ and $K = K^{\natural} + K^e$ are torsion free.

This coupling of the gravitational connection with the electromagnetic field seems to be a non standard result.

REMARK I.3.2.1. Suppose that the forms Φ, $\Phi^{\natural}$ and F be closed and a be a local potential of Φ. Then, the above splitting of Φ into its gravitational and electromagnetic components does not yield an analogous distinguished splitting of a. In fact, the potentials of Φ, $\Phi^{\natural}$ and F are defined up to a gauge and it is not possible to split naturally the gauge of a into gravitational and electromagnetic components. ❑

COROLLARY I.3.2.2. The curvature R of the total connection K splits as follows

$$R = R^{\natural} + R^{\natural e} + R^e ,$$

where

$$R^\natural := \tfrac{1}{2}\,[K^\natural, K^\natural]$$

is the standard gravitational curvature and the other two terms are given by

$$R^{\natural e} := d^\natural K^e := [K^\natural, K^e] \qquad R^e := [K^e, K^e].$$

We have the following coordinate expressions

$$R^\natural := R^\natural{}_{\lambda\mu}{}^i{}_\nu \, d^\lambda \wedge d^\mu \otimes \partial_i \otimes d^\nu = (\partial_\lambda K^\natural{}_\mu{}^i{}_\nu + K^\natural{}_\lambda{}^j{}_\nu \, K^\natural{}_\mu{}^i{}_j)\, d^\lambda \wedge d^\mu \otimes \partial_i \otimes d^\nu$$

$$R^{\natural e} = \tfrac{1}{2}\, c \,\Big(\big((\nabla^\natural{}_0 F^k{}_j - 2\,\nabla^\natural{}_j F^k{}_0)\, \dot{x}^0 - \nabla^\natural{}_j F^k{}_h \, \dot{y}^h\big)\, d^0 \wedge d^j + \nabla^\natural{}_i F^k{}_j \, \dot{x}^0 \, d^i \wedge d^j\Big) \otimes \partial_k$$

$$R^e = \tfrac{1}{4}\, c^2 \, F^h{}_j \, F^i{}_h \, d^j \wedge d^0 \otimes \partial_i \otimes d^0.$$

PROOF. We have (see § III.5 and § III.4)

$$R = \tfrac{1}{2}\,[K, K] = \tfrac{1}{2}\,[K^\natural, K^\natural] + \tfrac{1}{2}\,[K^\natural, K^e] + \tfrac{1}{2}\,[K^e, K^\natural] + \tfrac{1}{2}\,[K^e, K^e].$$

Moreover, we can write (see § III.4)

$$[K^\natural, K^e] = [K^e, K^\natural].$$ ❑

COROLLARY I.3.2.4. The Ricci tensor r of the total connection K splits as follows

$$r = r^\natural + r^{\natural e} + r^e,$$

where

$$r^\natural := 2\; C^1_1 R^\natural$$

is the standard gravitational Ricci tensor and the other two terms are given by

$$r^{\natural e} := \tfrac{1}{2}\, \boldsymbol{c}\, (dt \otimes \mathrm{div}^\natural F + \mathrm{div}^\natural F \otimes dt) \qquad r^e := -\tfrac{1}{4}\, \check{F}^2\, \boldsymbol{c}^2\, dt \otimes dt,$$

where

$$\check{F}^2 := (g^{\#} \otimes g^{\#}) \circ \check{F} \lrcorner \check{F} : \boldsymbol{E} \to \mathbb{A}^{*3/2} \otimes \mathbb{M}.$$

We have the following coordinate expressions

$$r^{\natural e} = \frac{1}{2} c \ \nabla_k F^k{}_\lambda (d^0 \otimes d^\lambda + d^\lambda \otimes d^0) \qquad r^e = - \frac{1}{4} c^2 \ F_{ij} \ F^{ij} \ d^0 \otimes d^0 . \qquad \square$$

COROLLARY I.3.2.5. The scalar curvature s of the total connection K is just the scalar curvature of the gravitational connection

$$s = s^\natural = \check{s}^\natural . \qquad \square$$

I.4 - Field equations

Now, we introduce the gravitational and electromagnetic field equations. We consider two equations: the first one couples just the gravitational and electromagnetic fields, while the second one couples the gravitational and electromagnetic fields with the charged matter sources.

I.4.1. First field equation

The first field equation is expressed through the closure of the total contact 2-form.

This equation turns out to be a compact way to express several important conditions involving the classical fields. Moreover, it could be used to formulate the inverse Lagrangian problem of dynamics (for the trivial case, see, for instance, [Cr], [CPT]). Furthermore, it will occur as an essential integrability condition in the quantum theory (see § II.1.4).

ASSUMPTION C6. *(First field equation)* We assume that, for any coupling constant $\boldsymbol{c}$, the total contact 2-form Ω is closed, i.e.

$$d\Omega = 0 \qquad \forall\, \boldsymbol{c} \in \mathbb{T}^* \otimes \mathbb{A}^{3/4} \otimes \mathbb{M}^{*\,1/2}. \ \diamond$$

We stress that there is no canonical local potential of Ω even in the case when the space-time connection is flat[11].

We can interpret the above equation in several interesting ways.

I.4.2. Geometrical interpretation of the first field equation

The first field equation says that, for each coupling constant $\boldsymbol{c}$, the to-

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[11] In a sense, in the quantum framework, we shall postulate a distinguished potential of Ω, after adding a complex dimension to space-time through the quantum bundle $\boldsymbol{Q} \to \boldsymbol{E}$.

tal connection is metrical and the standard algebraic Riemannian identities for the total curvature hold.

We stress that we cannot apply fully the standard procedures of Riemannian geometry in our context, because the metric is degenerate.

First, we prove a technical lemma.

LEMMA I.4.2.1. The equation

i) $$d\Omega = 0$$

is locally equivalent to the system

ii)$^{\mathrm{I}}$ $$\partial_\lambda g_{ij} = - \Gamma_{\lambda ij} - \Gamma_{\lambda ji}$$

ii)$^{\mathrm{II}}$ $$\partial_\lambda \Gamma_{\mu ji} - \partial_\lambda \Gamma_{\mu ij} + \partial_\mu \Gamma_{\lambda ji} - \partial_\mu \Gamma_{\lambda ij} = 2\, \partial_i \Gamma_{\lambda j\mu} - 2\, \partial_j \Gamma_{\lambda i\mu}$$

ii)$^{\mathrm{III}}$ $$\partial_i \Gamma_{jh\lambda} + \partial_h \Gamma_{ij\lambda} + \partial_j \Gamma_{hi\lambda} - \partial_j \Gamma_{ih\lambda} - \partial_i \Gamma_{hj\lambda} - \partial_h \Gamma_{ji\lambda} = 0.$$

PROOF. The coordinate expression of $d\Omega$ is

$$d\Omega = - (\text{д}_0.g_{ij} + \Gamma_{ji} + \Gamma_{ij})\, d^0 \wedge \vartheta^i \wedge d^j_0 + (- \text{д}_0.\Gamma_{ij} + \partial_i \gamma_j)\, d^0 \wedge \vartheta^i \wedge \vartheta^j -$$

$$- \partial_i \Gamma_{jh}\, \vartheta^i \wedge \vartheta^j \wedge \vartheta^h - (\partial_i g_{hj} + \Gamma_{ijh})\, d^h_0 \wedge \vartheta^i \wedge \vartheta^j.$$

Therefore, i) is equivalent to the following system

$$\text{д}_0.g_{ij} = - \Gamma_{ji} - \Gamma_{ij}$$

$$\text{д}_0.(\Gamma_{ij} - \Gamma_{ji}) = \partial_i \gamma_j - \partial_j \gamma_i$$

$$\partial_i \Gamma_{jh} + \partial_h \Gamma_{ij} + \partial_j \Gamma_{hi} - \partial_i \Gamma_{hj} - \partial_h \Gamma_{ji} - \partial_j \Gamma_{ih} = 0$$

$$\partial_i g_{hj} - \partial_j g_{hi} = \Gamma_{jih} - \Gamma_{ijh},$$

which is equivalent to the following system

$$\partial_0 g_{ij} = - \Gamma_{0ij} - \Gamma_{0ji}$$

$$\partial_h g_{ij} = -\Gamma_{hij} - \Gamma_{hji}$$

$$\partial_k \Gamma_{ijh} - \partial_k \Gamma_{jih} + \partial_h \Gamma_{kji} - \partial_h \Gamma_{kij} = 2\,\partial_i \Gamma_{hjk} - 2\,\partial_j \Gamma_{hik}$$

$$\partial_0 \Gamma_{ijh} - \partial_0 \Gamma_{jih} + \partial_h \Gamma_{0ji} - \partial_h \Gamma_{0ij} = 2\,\partial_i \Gamma_{0jh} - 2\,\partial_j \Gamma_{0ih}$$

$$\partial_0 \Gamma_{0ji} - \partial_0 \Gamma_{0ij} = \partial_i \Gamma_{0j\circ} - \partial_j \Gamma_{0i\circ}$$

$$\partial_i \Gamma_{jhk} + \partial_h \Gamma_{ijk} + \partial_j \Gamma_{hik} - \partial_j \Gamma_{ihk} - \partial_i \Gamma_{hjk} - \partial_h \Gamma_{jik} = 0$$

$$\partial_i \Gamma_{jh\circ} + \partial_h \Gamma_{ij\circ} + \partial_j \Gamma_{hi\circ} - \partial_j \Gamma_{ih\circ} - \partial_i \Gamma_{hj\circ} - \partial_h \Gamma_{ji\circ} = 0,$$

which is equivalent to ii). ❑

Then, we can state a first result.

PROPOSITION I.4.2.1. The first field equation implies that the space-time connection is metrical, i.e.

$$\nabla \bar{g} = 0.$$

PROOF. It follows from i) ⇒ ii)[I]. ❑

In order to complete the geometrical interpretation of the first field equation in terms of the curvature R of K, we need further technical lemmas.

Let K be any metrical space-time connection.

LEMMA I.4.2.2. The coordinate expression of

$$R^\flat := g^\flat(R) = R_{\lambda\mu i\nu}\, d^\lambda \wedge d^\mu \otimes \check{d}^i \wedge d^\nu$$

can be written as

a) $$2\,R_{\lambda\mu i\nu} = \partial_\lambda \Gamma_{\mu i\nu} - \partial_\mu \Gamma_{\lambda i\nu} + \Gamma_{\lambda hi}\,\Gamma_{\mu}{}^{h}{}_{\nu} - \Gamma_{\mu hi}\,\Gamma_{\lambda}{}^{h}{}_{\nu}.$$

PROOF. We have

$$2\,R_{\lambda\mu i\nu} = \partial_\lambda \Gamma_{\mu i\nu} - \partial_\mu \Gamma_{\lambda i\nu} - \partial_\lambda g_{ij}\,\Gamma_{\mu}{}^{j}{}_{\nu} + \partial_\mu g_{ij}\,\Gamma_{\lambda}{}^{j}{}_{\nu} + \Gamma_{\lambda}{}^{h}{}_{\nu}\,\Gamma_{\mu ih} - \Gamma_{\mu}{}^{h}{}_{\nu}\,\Gamma_{\lambda ih}.$$

Then, the condition $\nabla \bar{g} = 0$ yields the result. ❑

REMARK I.4.2.1. The fact that K is torsion free yields the standard first Bianchi identity

$b)$
$$R_{\lambda\mu i\nu} + R_{\nu\lambda i\mu} + R_{\mu\nu i\lambda} = 0. \quad ❑$$

LEMMA I.4.2.3. The following algebraic identity holds

$c)$
$$R_{\lambda\mu ij} = - R_{\lambda\mu ji}.$$

PROOF. Formula a) gives

$$2\,(R_{\lambda\mu ij} + R_{\lambda\mu ji}) = \partial_\lambda \Gamma_{\mu ij} - \partial_\mu \Gamma_{\lambda ij} + \partial_\lambda \Gamma_{\mu ji} - \partial_\mu \Gamma_{\lambda ji}.$$

Hence, the condition $\nabla \bar{g} = 0$ implies

$$2\,(R_{\lambda\mu ij} + R_{\lambda\mu ji}) = - \partial_{\lambda\mu} g_{ij} + \partial_{\lambda\mu} g_{ji} = 0. \quad ❑$$

LEMMA I.4.2.4. The following conditions are equivalent:

$\mathrm{ii})^{\mathrm{II}}$
$$\partial_\lambda \Gamma_{ij\mu} - \partial_\lambda \Gamma_{ji\mu} + \partial_\mu \Gamma_{\lambda ji} - \partial_\mu \Gamma_{\lambda ij} = 2\,\partial_i \Gamma_{\mu j\lambda} - 2\,\partial_j \Gamma_{\mu i\lambda}$$

$\mathrm{iii})^{\mathrm{I}}$
$$(R_{i\lambda j\mu} - R_{j\mu i\lambda}) + (R_{i\mu j\lambda} - R_{j\lambda i\mu}) = 0.$$

PROOF. The result follows from the above expression a) of $R_{\lambda\mu i\nu}$. ❑

We observe that the vertical restriction (for $\lambda = h$, $\mu = k$) of condition $\mathrm{iii})^{\mathrm{I}}$, turns out to be a consequence of the metricity condition only, according to a standard argument of Riemannian geometry.

LEMMA I.4.2.5. The following conditions are equivalent:

$\mathrm{ii})^{\mathrm{III}}$
$$\partial_i \Gamma_{jh\lambda} + \partial_h \Gamma_{ij\lambda} + \partial_j \Gamma_{hi\lambda} - \partial_i \Gamma_{hj\lambda} - \partial_h \Gamma_{ji\lambda} - \partial_j \Gamma_{ih\lambda} = 0$$

$\mathrm{iii})^{\mathrm{II}}$
$$R_{ijh\lambda} + R_{hij\lambda} + R_{jhi\lambda} = 0.$$

PROOF. The result follows from the above expression a) of $R_{\lambda\mu i\nu}$. ❑

We observe that the vertical restriction (for $\lambda = h$) of condition iii)$^{\mathrm{II}}$, turns out to be a consequence of the metricity condition only, according to a standard argument of Riemannian geometry.

The above conditions iii)$^{\mathrm{I}}$ and iii)$^{\mathrm{II}}$ on the curvature tensor can be expressed together in the following compact way.

For this purpose we need some preliminary results.

REMARK I.4.2.2. The condition

iii)$^{\mathrm{II}}$ $$R_{ijh\lambda} + R_{hij\lambda} + R_{jhi\lambda} = 0$$

implies the condition

iii)$^{\mathrm{II}'}$ $$R_{\lambda ijh} + R_{\lambda hij} + R_{\lambda jhi} = 0.$$

PROOF. Condition iii)$^{\mathrm{II}}$ yields

d) $$R_{ijh\lambda} = - R_{hij\lambda} - R_{jhi\lambda}.$$

Then, d), c) and iii)$^{\mathrm{I}}$, respectively, yield

$$R_{ijh\lambda} + R_{\lambda ihj} + R_{\lambda jih} = - R_{hij\lambda} - R_{jhi\lambda} + R_{\lambda ihj} + R_{\lambda jih} = - R_{hij\lambda} - R_{jhi\lambda} + R_{i\lambda jh} + R_{j\lambda hi} = 0.$$

Eventually, by adding term by term the circular permutation of the space-like indices in the above result

$$R_{ijh\lambda} + R_{\lambda ihj} + R_{\lambda jih} = 0 \qquad R_{hij\lambda} + R_{\lambda jih} + R_{\lambda hji} = 0 \qquad R_{jhi\lambda} + R_{\lambda hji} + R_{\lambda ihj} = 0$$

we obtain, in virtue of iii)$^{\mathrm{II}}$,

$$2\,(R_{\lambda ihj} + R_{\lambda jih} + R_{\lambda hji}) = 0.$$ ❑

LEMMA I.4.2.6. The conditions

iii)$^{\mathrm{I}}$ $$R_{ijhk} + R_{ikhj} - R_{hjik} - R_{hkij} = 0$$

iii)$^{\mathrm{II}}$ $$R_{ijh\lambda} + R_{hij\lambda} + R_{jhi\lambda} = 0$$

are equivalent to

iii)$^{\mathrm{R}}$
$$R_{i\lambda j\mu} = R_{j\mu i\lambda}.$$

PROOF. Let us prove that iii)$^{\mathrm{I}}$, iii)$^{\mathrm{II}}$ $\Rightarrow$ iii)$^{\mathrm{R}}$. Identities iii)$^{\mathrm{II}'}$, c) and b), respectively, yield

$$R_{i\lambda jh} = - R_{\lambda ijh} = R_{\lambda hij} + R_{\lambda jhi} = R_{\lambda hij} + R_{j\lambda ih} = - R_{hji\lambda} = R_{jhi\lambda},$$

hence

$$R_{i\lambda jh} = R_{jhi\lambda}.$$

Moreover, iii)$^{\mathrm{I}}$ implies

$$R_{i0j0} = R_{j0i0}.$$

Now, let us prove that iii)$^{\mathrm{R}}$ $\Rightarrow$ iii)$^{\mathrm{I}}$, iii)$^{\mathrm{II}}$. We can see immediately that iii)$^{\mathrm{R}}$ $\Rightarrow$ iii)$^{\mathrm{I}}$. Next, condition iii)$^{\mathrm{R}}$ and identities c) and b), respectively, yield

$$R_{ijh\lambda} + R_{hij\lambda} + R_{jhi\lambda} = R_{h\lambda ij} + R_{j\lambda hi} + R_{jhi\lambda} = - R_{\lambda hij} - R_{j\lambda ih} + R_{hji\lambda} = 0. \qquad \square$$

Then, we can state a second result.

PROPOSITION I.4.2.2. The first field equation implies the identity

$$R^{i}{}_{\lambda}{}^{j}{}_{\mu} = R^{j}{}_{\mu}{}^{i}{}_{\lambda}.$$

PROOF. It follows from i) $\Rightarrow$ iii)$^{\mathrm{II}}$, iii)$^{\mathrm{III}}$. $\square$

The two above results can be joined to provide a first geometrical interpretation of the first field equation.

THEOREM I.4.2.1. The first field equation is equivalent to the system

$$\nabla \bar{g} = 0 \qquad R^{i}{}_{\lambda}{}^{j}{}_{\mu} = R^{j}{}_{\mu}{}^{i}{}_{\lambda}.$$

PROOF. It follows from i) $\Leftrightarrow$ ii)$^{\mathrm{II}}$, ii)$^{\mathrm{III}}$. $\square$

I.4.3. First field equation interpreted through an observer

We can exhibit another interesting interpretation of the first field equation in terms of the vertical restriction of the total connection and the total symmetric 2-tensor and 2-form associated with an observer.

We have a first immediate result.

PROPOSITION I.4.3.1. The first field equation implies that the vertical restriction of K is just the vertical Riemannian connection (see Rem. I.1.2.2)

$$\check{K} = \varkappa \, .$$

PROOF. The condition ii)$^{\mathrm{I}}$ yields

$$\text{iv)}^{\mathrm{I}} \qquad \check{K}_{hik} := K_{hik} = -\tfrac{1}{2}\,(\partial_h g_{ik} + \partial_k g_{ih} - \partial_i g_{hk}) = \varkappa_{hik}. \qquad \square$$

Now, we refer to the tensors Σ and Φ related to a given observer o and to adapted coordinates (see § I.2.2).

Then, we have a second immediate result.

PROPOSITION I.4.3.2. The first field equation implies that the vertical restriction of the tensor Σ is determined by the metric, according to the equation (see formula iii)" in Prop. I.2.3.1)

$$\check{\Sigma} = g^{\flat}\, L_o \bar{g},$$

where L_0 is the Lie derivative with respect to the observer o, i.e. in adapted coordinates

$$\text{iv)}^{\mathrm{II}} \qquad \Sigma_{ij} = \partial_0 g_{ij}.$$

PROOF. It follows from ii)$^{\mathrm{I}}$. $\square$

In order to complete our interpretation, we need some technical lemmas.

LEMMA I.4.3.1. If iv)$^{\mathrm{I}}$ and iv)$^{\mathrm{II}}$ hold, then, the two following conditions are equivalent:

iii)^{IV}
$$\partial_0 \Gamma_{ijh} - \partial_0 \Gamma_{jih} + \partial_h \Gamma_{0ji} - \partial_h \Gamma_{0ij} = 2\, \partial_i \Gamma_{0jh} - 2\, \partial_j \Gamma_{0ih}$$

iv)^{III}
$$\partial_h \Phi_{ij} + \partial_j \Phi_{hi} + \partial_i \Phi_{jh} = 0. \qquad \square$$

LEMMA I.4.3.2. If iv)^{II} holds, then the two following conditions are equivalent:

iii)^{VII}
$$\partial_i \Gamma_{0jh} + \partial_h \Gamma_{0ij} + \partial_j \Gamma_{0hi} - \partial_i \Gamma_{0hj} - \partial_h \Gamma_{0ji} - \partial_j \Gamma_{0ih} = 0$$

iv)^{III}
$$\partial_h \Phi_{ij} + \partial_j \Phi_{hi} + \partial_i \Phi_{jh} = 0. \qquad \square$$

LEMMA I.4.3.3. If iv)^{II} holds, then the two following conditions are equivalent:

iii)^{V}
$$\partial_0 \Gamma_{0ji} - \partial_0 \Gamma_{0ij} = \partial_i \Gamma_{0j\circ} - \partial_j \Gamma_{0i\circ}$$

iv)^{IV}
$$\partial_0 \Phi_{ij} = \partial_i \Phi_{0j} - \partial_j \Phi_{0i}. \qquad \square$$

Then we have a third result.

PROPOSITION I.4.3.3. The first field equation implies that the form Φ is closed

$$d\Phi = 0.$$

PROOF. It follows from i) $\Rightarrow$ iv)^{I}, iv)^{III}, iv)^{IV}. $\square$

The three above results can be joined to provide a second geometrical interpretation of the first field equation.

THEOREM I.4.3.1. Given an observer o, the first field equation is equivalent to the following conditions:

– the (observer independent) vertical restriction of K coincides with the vertical Riemannian connection

$$\check{K} = \varkappa,$$

i.e. in coordinates

$$K_{hik} = -\tfrac{1}{2}\,(\partial_h g_{ik} + \partial_k g_{ih} - \partial_i g_{hk});$$

– the vertical restriction of the tensor Σ is determined by the metric, according to the equation (see formula iii)" in Prop. I.2.3.1)

$$\check{\Sigma} = g^{\flat}\, L_o \bar{g},$$

where L_0 is the Lie derivative with respect to the observer, i.e. in adapted coordinates

$$K_{0ij} + K_{0ji} = -\,\partial_0 g_{ij};$$

– the form Φ is closed

$$d\Phi = 0,$$

i.e. in adapted coordinates

$$(\partial_0 \Phi_{ij} - 2\,\partial_i \Phi_{0j})\, d^0 \wedge d^i \wedge d^j + \partial_h \Phi_{ij}\, d^h \wedge d^i \wedge d^j =$$

$$= 2\,\big((\partial_0 \Gamma_{0ij} + \partial_i \Gamma_{0j0})\, d^0 \wedge d^i \wedge d^j + \partial_h \Phi_{ij}\, d^h \wedge d^i \wedge d^j\big) = 0.$$

PROOF. It follows from i) $\Leftrightarrow$ iv)$^{\mathrm{I}}$, iv)$^{\mathrm{II}}$, iv)$^{\mathrm{III}}$, iv)$^{\mathrm{IV}}$. ❑

Of course, we might have deduced immediately $d\Phi = 0$ from $d\Omega = 0$ because the pullback o^* commutes with d; but in order to prove the rest, it is necessary to consider the above lemmas dealing with coordinate formulas.

We recall that any space-time connection K is characterised by its vertical restriction $\check{K}$ and the tensors $\check{\Sigma}$ and Φ related to a given observer (see Prop. I.2.2.1). Hence, we have proved that K is determined by the first jet of the metric g and by an observer dependent closed 2-form Φ.

The above result can be re-formulated as follows.

THEOREM I.4.3.2. Let

$$a : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes T^*\boldsymbol{E}$$

be a local potential of the closed 2-form Φ, i.e. a local solution of the equation

$$\Phi = 2\, da,$$

with coordinate expression

$$\Phi_{\lambda\mu} = \partial_\lambda a_\mu - \partial_\mu a_\lambda,$$

where

$$a := a_\lambda\, u^0 \otimes d^\lambda.$$

Then, the coordinate expression of the total connection can be written as

$$K_{\lambda i \mu} = -\tfrac{1}{2}\,(\partial_\lambda g_{i\mu} + \partial_\mu g_{i\lambda} - \partial_i g_{\lambda\mu}),$$

where we have set

$$g_{i0} := g_{0i} := a_i \qquad g_{00} := 2\, a_0. \qquad \square$$

Thus, as in the Einstein theory, the space-time connection is obtained locally from 10 scalar potentials; in the Galilei theory, only 6 potentials are the components of the metric g and we have 4 additional potentials. This difference between the Galilei and the Einstein theories is related to the fact that, in the Galilei case, g is degenerate.

We stress that the total potential a involves both the gravitational and electromagnetic fields.

I.4.4. First field equation interpreted through the fields

On the other hand, in virtue of the arbitrariness of the coupling constant $\boldsymbol{c}$, the above results split into their gravitational and electromagnetic components, respectively.

PROPOSITION I.4.4.1. The first field equation is equivalent to the system

$$d\Omega^\natural = 0 \qquad dF = 0.$$

PROOF. It follows from the arbitrariness of the coupling constant $\boldsymbol{c}$. $\square$

REMARK I.4.4.1. The equation $dF = 0$ is nothing but the first Maxwell equation.

If o is an observer, then, the first Maxwell equation reads

$$\check{d}B = 0 \qquad \check{\mathrm{curl}}\, E = \partial_0 B,$$

where ∂_0 denotes the time derivative with respect to the observer. ❑

PROPOSITION I.4.4.2. The first field equation is equivalent to the system

$$\nabla^{\natural}\bar{g} = 0 \qquad R^{\natural i}{}_{\lambda}{}^{j}{}_{\mu} - R^{\natural j}{}_{\lambda}{}^{i}{}_{\mu} = 0$$

$$F^{ij} + F^{ji} = 0 \qquad dF = 0,$$

where $R^{\natural}$ is the curvature of $K^{\natural}$.

PROOF. It follows from Theor. I.4.2.1, in virtue of the arbitrariness of the mass and the charge. ❑

PROPOSITION I.4.4.3. Given an observer o, the first field equation is equivalent to the following conditions:

- the (observer independent) vertical restriction of $K^{\natural}$ is just the vertical Riemannian connection (see Rem. I.1.2.2)

$$\check{K}^{\natural} = \check{K} = \varkappa,$$

i.e. in coordinates

$$K^{\natural}{}_{hik} = K_{hik} = -\tfrac{1}{2}(\partial_h g_{ik} + \partial_k g_{ih} - \partial_i g_{hk});$$

- the vertical restriction of the tensor $\Sigma^{\natural}$ is determined by the metric, according to the equation (see formula iii)" in Prop. I.2.3.1)

$$\check{\Sigma}^{\natural} = \check{\Sigma} = g^{\flat}\, L_o \bar{g},$$

where L_0 is the Lie derivative with respect to the observer, i.e. in adapted coordinates

$$K^{\natural}{}_{0ij} + K^{\natural}{}_{0ji} = K_{0ij} + K_{0ji} = -\,\partial_0 g_{ij};$$

- the forms $\Phi^{\natural}$ and F are closed

$$d\Phi^\natural = 0 \qquad dF = 0.$$

PROOF. It follows from Theor. I.4.3.1, in virtue of the arbitrariness of the coupling constant $\boldsymbol{c}$. ❑

Thus, $\check{K}$ and $\check{\Sigma}$ are completely determined by the first jet of the metric, while the closed 2-form Φ splits into a gravitational component, which is not related to the metric, and an electromagnetic component, which is just the electromagnetic field F.

I.4.5. Second gravitational equation

The second gravitational equation expresses the coupling of the gravitational field with the matter and electromagnetic sources, by comparing the gravitational Ricci tensor with the energy tensor of the matter and the electromagnetic field. We restrict our study to the case of the matter source constituted by an incoherent fluid, just as an example.

We define the *energy tensor* of the electromagnetic field F to be the section[12]

$$\text{т}^e : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E},$$

given by

$$\text{т}^e := \frac{1}{4}\,\boldsymbol{\kappa}\,\check{F}^2\, dt \otimes dt.$$

We have the coordinate expression

$$\text{т}^e = \frac{1}{4}\,\kappa\, F_{ij}\, F^{ij}\, d^0 \otimes d^0.$$

We define a *mass density* to be a section

$$\mu : \boldsymbol{E} \to \mathbb{A}^{*3/2} \otimes \mathbb{M}.$$

We define the *energy tensor* of the mass density μ to be the section

[12] т is the Cyrillic character corresponding to "t".

$$\tau^{\mu} : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E},$$

given by

$$\tau^{\mu} := \boldsymbol{\kappa}\, \mu\, dt \otimes dt.$$

We have the coordinate expression

$$\tau^{\mu} := \kappa\, \mu\, d^0 \otimes d^0.$$

DEFINITION I.4.5.1. The *energy tensor* of the electromagnetic field F and of the mass density μ is defined to be the section

$$\tau^{\natural} := \tau^{e} + \tau^{\mu} : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E}. \qquad \clubsuit$$

Just as example, we assume the source of the gravitational field to be constituted by the electromagnetic field F and a mass density μ. Accordingly, we make the following assumption.

ASSUMPTION C7. We assume the gravitational field $K^{\natural}$ to fulfill the *second gravitational equation*

$$r^{\natural} = \tau^{\natural}. \qquad \diamond$$

PROPOSITION I.4.5.1. The coordinate expression of the second gravitational equation is

$$r^{\natural}{}_{00} = \kappa\, \mu \qquad r^{\natural}{}_{0h} = r^{\natural}{}_{h0} = 0 \qquad r^{\natural}{}_{hk} = 0;$$

i.e.

$$\partial_i K^{\natural}{}_0{}^i{}_0 - \partial_0 K^{\natural}{}_i{}^i{}_0 + K^{\natural}{}_i{}^j{}_0\, K^{\natural}{}_0{}^i{}_j - K^{\natural}{}_0{}^j{}_0\, K^{\natural}{}_i{}^i{}_j = \kappa\, \mu$$

$$\partial_i K^{\natural}{}_0{}^i{}_h - \partial_0 K^{\natural}{}_i{}^i{}_h + K^{\natural}{}_i{}^j{}_h\, K^{\natural}{}_0{}^i{}_j - K^{\natural}{}_0{}^j{}_h\, K^{\natural}{}_i{}^i{}_j = 0$$

$$\partial_i K^{\natural}{}_h{}^i{}_k - \partial_h K^{\natural}{}_i{}^i{}_k + K^{\natural}{}_i{}^j{}_k\, K^{\natural}{}_h{}^i{}_j - K^{\natural}{}_h{}^j{}_k\, K^{\natural}{}_i{}^i{}_j = 0. \qquad \square$$

The particular form of the source of the gravitational field implies that the fibres of $t : \boldsymbol{E} \to \boldsymbol{T}$ are flat.

PROPOSITION I.4.5.2. We have (see § I.2.4)

$$\check{R}^{\natural} = 0 .$$

PROOF. The fibres of $t{:}\boldsymbol{E}\to\boldsymbol{T}$ are Ricci flat because $\check{\mathrm{T}}^{\natural} = 0$.
Hence they are flat, because they are 3-dimensional (see [GHL]). ❑

REMARK I.4.5.1. If we take into account the Bianchi identities and try to obtain a compatibility condition for the second gravitational equation, by repeating a standard procedure in a way appropriate to our Galileian case, we obtain just a trivial identity. So, in this way, we do not obtain any information of the state equation of the source. ❑

REMARK I.4.5.2. As far as we know, our second gravitational equation seems to be the most reasonable equation appropriate to the Galileian framework and inspired by the Einstein equation. The problem of finding the appropriate coupling between the gravitational field and the matter source has been investigated by several authors, but we do not know a really definite answer. In fact, we are not able to derive such an equation from a fully satisfactory unifying principle (for instance a variational principle). On the other hand, we could guess several further proposals of stress tensors associated with the source, but we cannot couple them appropriately with the Ricci tensor because we do not dispose of a non degenerate metric and of appropriate coupling constants. ❑

REMARK I.4.5.3. Our second gravitational equation does not imply any direct effect of the electric field and of the movement of masses on the gravitational field. This is a weaker feature of the Galilei theory with respect to the Einstein one.

In principle, we could partially overcome this deficiency of the above Galilei theory, by introducing an energy-momentum tensor, with non-vanishing vertical component and coupling it with the Ricci tensor. However, such an energy-momentum tensor would be a primitive object, which could not be related to the movement of matter. We do not develop such a theory in detail. ❑

I.4.6. Second electromagnetic equation

The second electromagnetic equation expresses the coupling of the electromagnetic field with the charge source.

We define a *charge density* to be a section

$$\boldsymbol{\varrho} : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A}^{*3/4} \otimes \mathbb{M}^{1/2}.$$

Moreover, given $u_0 \in \mathbb{T}^+$, we set

$$\varrho := \boldsymbol{\varrho}(u_0) \in \mathbb{A}^{*3/4} \otimes \mathbb{M}^{1/2}.$$

We define the *time-like current* of the charge density $\boldsymbol{\varrho}$ to be the section

$$j : \boldsymbol{E} \to (\mathbb{A}^{*3/4} \otimes \mathbb{M}^{1/2}) \otimes T^* \boldsymbol{E},$$

given by

$$j := \boldsymbol{\varrho}\, dt.$$

We have the coordinate expression

$$j = \varrho\, d^0.$$

Just as example, we assume the source of the electromagnetic field to be constituted by the charge density $\boldsymbol{\varrho}$. Accordingly, we make the following assumption.

ASSUMPTION C8. We assume the electromagnetic field F to fulfill the *second electromagnetic equation*

$$\mathrm{div}^{\natural} F = j. \qquad \diamond$$

PROPOSITION I.4.6.1. The coordinate expression of the second electromagnetic equation is

$$g^{ij}\,(\partial_i F_{j0} + K^{\natural}{}_i{}^h{}_j\, F_{h0}) = \varrho \qquad g^{ij}\,(\partial_i F_{jk} + K^{\natural}{}_i{}^h{}_j\, F_{hk}) = 0. \qquad \square$$

PROPOSITION I.4.6.2. We have

$$\mathrm{div}^{\natural} \check{F} = 0. \qquad \square$$

REMARK I.4.6.1. Both hand sides of the second electromagnetic equation are identically divergence free. So, application of the divergence operator does not yield any information of the state equation of the source. ❑

REMARK I.4.5.2. We can assume that the mass and charge densities are carried by the same charged incoherent fluid, which can be described in the following way.

We assume the charge density to be proportional to the mass density

$$\varrho = \varepsilon\, \mu \qquad \text{with } \varepsilon \in \mathbb{Q}\otimes\mathbb{M}^*.$$

We assume a velocity field v in the domain where μ is non vanishing and define the *contravariant momentum*, *current* and *stress tensors* and the *Lorentz force density*

$$p := \mu\, v \qquad c := \varrho\, v \qquad e := \mu\, v\otimes v$$

$$f := -\varrho\, g^{\sharp}(\check{v}\lrcorner F).$$

Then, we assume the equation

$$\operatorname{div}^{\natural} e = f,$$

which splits into the mass continuity equation and the Newton's equation of motion

$$\operatorname{div}^{\natural} p = 0 \qquad \mu\, \nabla^{\natural}{}_{v} v = f;$$

as a consequence of the proportionality between mass and charge densities, we obtain also the charge continuity equation

$$\operatorname{div}^{\natural} c = 0$$ ❑

REMARK I.4.5.3. As far as we know, our second electromagnetic equation seems to be the most reasonable equation appropriate to the Galileian framework and inspired by the Maxwell equation. We can define the contravariant current involving the charge density and velocity, but we cannot couple it with the divergence of the electromagnetic field because we do not dispose of a non degenerate metric. ❑

REMARK I.4.5.4. Our second electromagnetic equation does not imply any direct effect of the space-like current on the electromagnetic field. This is a weaker feature of the Galilei theory with respect to the Maxwell one.

In principle, we could partially overcome this deficiency of the above Galilei theory, by introducing a current, with non-vanishing vertical component and coupling it with the divergence of the electromagnetic field. However, such a current would be a primitive object, which could not be related to the movement of charges. We do not develop such a theory in detail.❑

I.4.7. Second field equation

The second gravitational and electromagnetic equations can be coupled in a natural way through the gravitational coupling constant. In this way we obtain the second field equation for the total space-time connection associated with the gravitational coupling constant. The corresponding matter source turns out to be constituted just by the mass and charge densities, as the contribution of the electromagnetic field is incorporated in the total Ricci tensor.

We define the *energy tensor* of the charge density ϱ to be the section

$$\mathrm{T}^{\varrho} : \boldsymbol{E} \to T^{*}\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^{*}\boldsymbol{E},$$

given by

$$\mathrm{T}^{\varrho} := \sqrt{\boldsymbol{\kappa}}\, \varrho\; dt \otimes dt.$$

We have the coordinate expression

$$\mathrm{T}^{\varrho} := \sqrt{\kappa}\, \varrho\; d^{0} \otimes d^{0}.$$

DEFINITION I.4.7.1. The *energy tensor* of the mass density μ and charge density ϱ is defined to be the section

$$\mathrm{T} := \mathrm{T}^{\mu} + \mathrm{T}^{\varrho} : \boldsymbol{E} \to T^{*}\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^{*}\boldsymbol{E}. \qquad ♣$$

THEOREM I.4.7.1. The second gravitational and electromagnetic equations imply the following *second field equation*

$$r = \mathrm{T},$$

where r is the Ricci tensor of the total space-time connection K induced by the coupling of the gravitational connection $K^{\natural}$ and the electromagnetic field F, through the coupling constant $\sqrt{\mathbf{k}}$ (see § I.3.2).

PROOF. In fact, we have (see Cor. I.3.2.4, Ass. C7, Ass. C8)

$$r = r^{\natural} + r^{\natural e} + r^{e} = r^{\natural} + \frac{1}{2} \sqrt{\mathbf{k}} \, (dt \otimes \operatorname{div}^{\natural} F + \operatorname{div}^{\natural} F \otimes dt) - \frac{1}{4} \check{F}^2 \, \mathbf{k} \, dt \otimes dt =$$

$$= \frac{1}{4} \mathbf{k} \, \check{F}^2 \, dt \otimes dt + \mathbf{k} \, \mu \, dt \otimes dt + \sqrt{\mathbf{k}} \, \varrho \, dt \otimes dt - \frac{1}{4} \check{F}^2 \, \mathbf{k} \, dt \otimes dt =$$

$$= \tau . \qquad \square$$

I.5 - Particle mechanics

Now, we study the dynamics of classical charged particles in the given gravitational and electromagnetic field.

I.5.1. The equation of motion

The only observer independent approach to classical mechanics can be achieved in terms of the total connection γ.

Then, we introduce the fundamental law of particle dynamics.

ASSUMPTION C9. *(Generalised Newton law of motion)* We assume the *law of motion* for a particle, with mass $m \in \mathbb{M}$ and charge $\boldsymbol{q} \in \mathbb{Q}$, whose motion is $s : \boldsymbol{T} \to \boldsymbol{E}$, to be the equation

$$\nabla_\gamma \, j_1 s = 0. \qquad \diamond$$

REMARK I.5.1.1. We obtain

$$\nabla_\gamma \, j_1 s = \nabla_{\gamma^\natural} \, j_1 s - \gamma^e \circ j_1 s.$$

Hence, the Newton law of motion can be written as

$$\nabla_{\gamma^\natural} \, j_1 s = \gamma^e \circ j_1 s,$$

i.e., in coordinates,

$$\partial_{00} s^i - (\Gamma^\natural{}_h{}^i{}_k \circ s) \, \partial_0 s^h \, \partial_0 s^k - 2 \, (\Gamma^\natural{}_0{}^i{}_h \circ s) \, \partial_0 s^h - (\Gamma^\natural{}_0{}^i{}_\circ \circ s) =$$

$$= \frac{q}{m} \, (F^i{}_0 \circ s + F^i{}_h \circ s \, \partial_0 s^h). \qquad \square$$

The Newton law of motion can be expressed in a dual way, in terms of the differential of functions.

REMARK I.5.1.2. If $f \in \mathcal{F}(J_1 \boldsymbol{E})$, then, for each solution of the Newton law of motion $s \in \mathcal{S}(\boldsymbol{E} \to \boldsymbol{T})$, we can write

$$d(f \circ j_1 s) = (\gamma . f) \circ j_1 s .$$

In particular, f turns out to be a *constant of motion* if and only if

$$\gamma . f = 0 .$$

PROOF. We have

$$d(f \circ j_1 s) = \langle (df) \circ j_1 s, T j_1 s \rangle = \langle (df) \circ j_1 s, j_2 s \rangle = \langle (df) \circ j_1 s, \gamma \circ j_1 s \rangle =$$

$$= (\gamma . f) \circ j_1 s . \qquad \square$$

REMARK I.5.1.3. In a sense, the Newton law of motion can be expressed by a Hamiltonian approach (in terms of a contact structure). In fact, we recall (see Cor. I.2.5.2) that γ is the unique second order connection which fulfills the equation

$$i_\gamma \Omega = 0 .$$

Hence, the flow of solutions of the Newton law of motion preserves the total contact 2-form and the induced volume form, i.e.

$$L_\gamma \Omega = 0 \qquad L_\gamma (dt \wedge \Omega \wedge \Omega \wedge \Omega) = 0 . \qquad \square$$

I.5.2. Observer dependent formulations of the Newton law of motion

By choosing an observer, we can re-formulate the Newton law of motion in terms of Euler-Lagrange, or Hamilton, or Poisson equations.

However, the choice of the observer turns out to be essential. So, explicitly general relativistic Lagrangian, Hamiltonian and Poissonian formulations of classical mechanics do not exist.

The basic maps of classical particle mechanics related to an observer will be extensively used in quantum mechanics (see Theor. II.1.4.1 and Cor. II.1.4.1).

First, we recall the following maps.

Let o be an observer and consider the induced translation fibred morphisms over $\boldsymbol{E}$ (see Rem. I.1.1.3)

$$\nabla_o : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$$

and a local potential (which is defined up to a gauge)

$$a : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes T^*\boldsymbol{E}$$

of the 2-form $\Phi := 2\, o^*\Omega$ (see Rem. I.2.2.2 and Prop. I.2.5.1).

Then, we define the following *classical* objects in a coordinate free way.

Definition I.5.2.1. We define the *kinetic energy, kinetic momentum, Lagrangian, momentum* and *Hamiltonian* to be, respectively, the maps

$$\boldsymbol{G} := \tfrac{1}{2}\, m\; g \circ (\nabla_o,\ \nabla_o) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M}$$

$$\check{\boldsymbol{P}} := m\; g^\flat \circ \nabla_o : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes V^*\boldsymbol{E}$$

$$\boldsymbol{L} := \boldsymbol{G} + m\; \text{д} \lrcorner a : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M}$$

$$\check{\boldsymbol{p}} := m\; g^\flat \circ \nabla_o + m\; \text{д} \lrcorner a : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes V^*\boldsymbol{E}$$

$$\boldsymbol{H} := \langle \nabla_o, \check{\boldsymbol{p}} \rangle - \boldsymbol{L} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M}. \qquad \clubsuit$$

We can write

$$\check{\boldsymbol{P}} = V_{\boldsymbol{E}}\boldsymbol{G} \qquad \check{\boldsymbol{p}} = V_{\boldsymbol{E}}\boldsymbol{L}.$$

Remark I.5.2.1. By considering the natural inclusion $\mathbf{E} \times \mathbb{T}^* \subset T^*\boldsymbol{E}$, we can regard the kinetic energy, Lagrangian and Hamiltonian maps as time-like forms

$$\boldsymbol{G} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes T^*\boldsymbol{E} \qquad \boldsymbol{L} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes T^*\boldsymbol{E} \qquad \boldsymbol{H} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes T^*\boldsymbol{E}.$$

By considering the natural fibred inclusion $\vartheta^* : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ (see Rem. I.1.1.2), we obtain the forms

$$\boldsymbol{P} := \vartheta^* \lrcorner \check{\boldsymbol{P}} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes T^*\boldsymbol{E} \qquad \boldsymbol{p} := \vartheta^* \lrcorner \check{\boldsymbol{p}} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes T^*\boldsymbol{E},$$

which will be said to be the *kinetic momentum form* and *momentum form*, respectively. ❑

Moreover, if $u_0 \in \mathbb{T}^+$, then we set

$$G := \boldsymbol{G}(u_0, u_0) : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes \mathbb{R} \qquad L := \boldsymbol{L}(u_0, u_0) : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes \mathbb{R}$$

$$H := \boldsymbol{H}(u_0, u_0) : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes \mathbb{R}$$

$$\check{P} := \check{\boldsymbol{P}}(u_0) : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes V^*\boldsymbol{E} \qquad \check{p} := \check{\boldsymbol{p}}(u_0) : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes V^*\boldsymbol{E}$$

$$P := \boldsymbol{P}(u_0) : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes T^*\boldsymbol{E} \qquad p := \boldsymbol{p}(u_0) : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes T^*\boldsymbol{E}.$$

REMARK I.5.2.2. With reference to a space-time chart adapted to the frame of reference (u_0, o), we have the following coordinate expressions

$$G = \tfrac{1}{2} m\, g_{ij}\, y_0^i\, y_0^j \qquad L = \tfrac{1}{2} m\, g_{ij}\, y_0^i\, y_0^j + m\,(a_i\, y_0^i + a_0)$$

$$H = \tfrac{1}{2} m\, g_{ij}\, y_0^i\, y_0^j - m\, a_0$$

$$\check{P} = m\, g_{ij}\, y_0^j\, \check{d}^i \qquad \check{p} = (m\, g_{ij}\, y_0^j + m\, a_i)\, \check{d}^i$$

$$P = -m\, g_{ij}\, y_0^i\, y_0^j\, d^0 + m\, g_{ij}\, y_0^j\, d^i \qquad p = -m\, g_{ij}\, y_0^i\, y_0^j\, d^0 + (m\, g_{ij}\, y_0^j + m\, a_i)\, d^i.$$

REMARK I.5.2.3. Let o and o' be two observers and set

$$v := o' - o : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}.$$

Then, we obtain the following relation between the respective kinetic forms

$$\boldsymbol{G}' = \boldsymbol{G} + \tfrac{1}{2} m\, g \circ (v, v) - m\, \nabla_o \lrcorner v^\flat \qquad \boldsymbol{P}' = \boldsymbol{P} - m\, \vartheta^* \lrcorner v^\flat,$$

hence (see Remark I.1.1.3)

$$\boldsymbol{G}' + \boldsymbol{P}' = \boldsymbol{G} + \boldsymbol{P} + \frac{1}{2} m\, g \circ (v, v) - m\, \nu_0{}^* \lrcorner v^\flat ,$$

where

$$v^\flat := g^\flat \circ v : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes V^* \boldsymbol{E}.$$

PROOF. It follows from

$$\nabla_{o'} = \nabla_o - v \qquad \nabla_o + \vartheta^* = \nu_o . \qquad \square$$

REMARK I.5.2.4. We can achieve a Hamiltonian formulation of the Newton law of motion in three different ways, by considering the symplectic structures naturally induced by the total contact 2-form Ω on the even dimensional vector bundles

$$VJ_1\boldsymbol{E} \to J_1\boldsymbol{E} \qquad VV\boldsymbol{E} \to V\boldsymbol{E} \qquad VV^*\boldsymbol{E} \to V^*\boldsymbol{E}$$

and the observer dependent Hamiltonian and momentum functions.

Moreover, the Newton law of motion, in the version quoted in Rem. I.5.1.2, can be expressed in terms of the Poisson bracket (see, later, Prop. II.3.2.1) between the function $f \in \mathcal{F}(J_1\boldsymbol{E})$ and the observer dependent Hamiltonian function.

Furthermore, the Newton law of motion turns out to be the Euler-Lagrange equation associated with the observer dependent Lagrangian function.

We omit here the details. $\square$

I.6 - Special relativistic case

Under reasonable hypothesis, there exist distinguished metrical space-time connections, which yield an affine structure on space-time. In such a case, we avail of this structure to simplify several formulas; in particular, we prove that the Einstein equation turns out to be just the Newton law of gravitation.

In the special case when the energy tensor vanishes, possibly the same gravitational connection yields an affine structure on space-time. Such a distinguished solution of the field equations is referred to as special relativistic and constitutes the background for the standard classical Newtonian mechanics.

I.6.1. Affine structure of space-time

Let us start with preliminary observations on a possible background connection inducing an affine structure on space-time.

REMARK I.6.1.1. Let A be an affine space associated with the vector space $\bar{A}$ and the translation map $f : A \times \bar{A} \to A$.

Then, the affine structure yields a linear connection on A

$$\nu_{K''} : TTA = (A\times\bar{A})\times(\bar{A}\times\bar{A}) \to VTA = (A\times\bar{A})\times(0\times\bar{A}) : (a,u;v,w) \mapsto (a,u;0,w),$$

which is geodesically complete and whose curvature vanishes.

Conversely, the above connection K'' characterises naturally the affine structure $(\bar{A},\, f)$ on A in a unique way. ❑

DEFINITION I.6.1.1. A *background connection* is defined to be a space-time connection K'', such that

- K'' is metrical (see § I.2.3), i.e. $\nabla''\bar{g} = 0$;
- K'' induces an affine structure on the space-time manifold $\boldsymbol{E}$, such that the time function t is an affine map. ♣

LEMMA I.6.1.1. Let K'' be a background connection.

Then, $t : \boldsymbol{E} \to \boldsymbol{T}$ turns out to be an affine bundle associated with a trivial vector bundle

$$\bar{t} : \bar{\boldsymbol{E}} = \boldsymbol{T} \times \boldsymbol{S} \to \boldsymbol{T},$$

where $\boldsymbol{S}$ is a three dimensional vector space. Thus, $t : \boldsymbol{E} \to \boldsymbol{T}$ turns out to be a principal bundle.

Moreover, the vertical metric can be regarded as a constant element

$$g \in \mathbb{A} \otimes \boldsymbol{S}^* \otimes \boldsymbol{S}^*.$$

PROOF. Set

$$\boldsymbol{S} := \ker Dt \subset DE,$$

where DE denotes the vector space associated with the affine space E.

Then, for each $\tau \in \boldsymbol{T}$, we obtain

$$D(t^{-1}(\tau)) = \boldsymbol{S},$$

in virtue of the fact that t is affine. ❑

REMARK I.6.1.2. Not all space-times $t : \boldsymbol{E} \to \boldsymbol{T}$ admit background connections K''.

If the space-time $t : \boldsymbol{E} \to \boldsymbol{T}$ admits a background connection K'', then we can easily see that it admits many background connections. ❑

REMARK I.6.1.3. Let K'' be a background connection. If o and o' are global observers such that

$$\nabla_{K''} o = 0 = \nabla_{K''} o',$$

then we have

$$o' = o + v \qquad v \in \mathbb{T}^* \otimes \boldsymbol{S}.$$ ❑

REMARK I.6.1.4. Let K'' be a background connection. Then, each global space-time chart (x^0, y^i) adapted to the induced affine structure of space-time yields

$$K''{}_{\lambda}{}^{i}{}_{\mu} = 0 \qquad g_{ij} \in \mathbb{A}.$$

Of course, the observer o associated with such a chart is of the above type. ❑

I.6.2. Newtonian connections and the Newton law of gravitation

Next, we consider the interesting case of a background connection which is “tangent” to the gravitational connections on the fibres of space-time.

By considering such a connection, we can write the first and second gravitational equations in an interesting way.

If the source of the gravitational connection $K^{\natural}$ is an incoherent fluid, then the gravitational Einstein equation implies that the vertical gravitational connection $\check{K}^{\natural}$ is flat (see Prop. I.4.5.2). Then, under further reasonable hypothesis, $\check{K}^{\natural}$ may induce an affine structure on the fibres of space-time. Of course, if the mass density of the source does not vanish, $K^{\natural}$ cannot induce an affine structure on the whole space-time. However, under further reasonable hypothesis, $\check{K}^{\natural}$ may be extended to a background connection $K^{''}$ (in a non unique way). So, we introduce the following notion.

DEFINITION I.6.2.1. A *Newtonian connection* is defined to be a space-time connection $K^{''}$ which fulfills the following properties:

i) the restrictions of $K^{''}$ and $K^{\natural}$ to the vertical subspace coincide

$$K^{''}{}_{|VE} = K^{\natural}{}_{|VE} : V\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TV\boldsymbol{E};$$

ii) $K^{''}$ induces an affine structure on the space-time manifold $\boldsymbol{E}$, such that the time function t is an affine map. ♣

If $K^{''}$ is a Newtonian connection, then a *Newtonian observer* is defined to be a global observer o such that $\nabla_{K^{''}} o = 0$ and a *Newtonian chart* is defined to be a global space-time chart (x^0, y^i) adapted to the induced affine structure of space-time.

Now on, in this section, we assume that a Newtonian connection $K^{''}$ exist and we refer to such a connection $K^{''}$ and to Newtonian observers and charts.

Accordingly, some formulas of the classical field theory and particle mechanics assume a simplified expression.

REMARK I.6.2.1. Condition i) implies

$$\check{K}^{''} = \check{K}^{\natural}.$$

But we stress that condition i) is stronger than the above equality. ❑

LEMMA I.6.2.1. The connection $K^{\shortparallel}$ is metrical, hence it is a background connection.

PROOF. Condition i) and the first gravitational equation (see Prop. I.4.4.2) yield

$$\nabla^{\shortparallel}\bar{g} = \nabla^{\natural}\bar{g} = 0. \qquad \square$$

LEMMA I.6.2.2. Condition i) reads in coordinates as

$$K^{\natural}{}_{\lambda}{}^{i}{}_{j} = 0. \qquad \square$$

LEMMA I.6.2.3. The gravitational connection can be uniquely written as

$$K^{\natural} = K^{\shortparallel} + dt \otimes dt \otimes N^{\natural}$$

where N is a section

$$N^{\natural} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$$

whose coordinate expression is of the type

$$N^{\natural} = N^{\natural}{}^{i}_{0\,0}\, u^0 \otimes u^0 \otimes \partial_i, \qquad N^{\natural}{}^{i}_{0\,0} \in \mathcal{F}(\boldsymbol{E}). \square$$

We say that $N^{\natural}$ is the *Newton vector field* associated with the Newtonian connection $K^{\shortparallel}$.

Let us discuss at which extent the Newtonian connection is uniquely determined by the gravitational connection (provided at least one Newtonian connection exists).

PROPOSITION I.6.2.1. Let us suppose that $K^{\shortparallel}$ and $K^{\shortparallel\prime}$ be two Newtonian connections.

Then, the two affine structures induced by $K^{\shortparallel}$ and $K^{\shortparallel\prime}$ on the space-time manifold $\boldsymbol{E}$ yield the same affine structure on the bundle $t : \boldsymbol{E} \to \boldsymbol{T}$. Hence, we obtain the same vector bundle

$$\bar{t} : \bar{\boldsymbol{E}} = \boldsymbol{T} \times \boldsymbol{S} \to \boldsymbol{T},$$

with respect to the two Newtonian connections.

Moreover, the Newtonian vector fields associated with $K^{\shortparallel}$ and $K^{\shortparallel\prime}$ differ by a time-dependent vector field:

$$(*) \qquad N^{\natural\prime} - N^{\natural} : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes \boldsymbol{S}.$$

Furthermore, let o and o' be two Newtonian observers with respect to $K^{\shortparallel}$ and $K^{\shortparallel\prime}$, respectively. Then, the two observers perform mutually a rigid translation[13]:

$$(**) \qquad o' - o : \boldsymbol{T} \to \mathbb{T}^* \otimes \boldsymbol{S}.$$

PROOF. The affine structures induced on the fibres of space-time by $K^{\shortparallel}$ and $K^{\shortparallel\prime}$ coincide because

$$\check{K}^{\shortparallel} = \check{K}^{\natural} = \check{K}^{\shortparallel\prime}.$$

In other words, $K^{\shortparallel}$ and $K^{\shortparallel\prime}$ induce on the space-time fibred manifold the same structure of affine bundle, associated with the vector bundle

$$\boldsymbol{V} \to \boldsymbol{T}.$$

Hence, $K^{\shortparallel}$ and $K^{\shortparallel\prime}$ induce the same vertical parallelisation

$$V\boldsymbol{E} = \boldsymbol{E} \underset{\boldsymbol{B}}{\times} \boldsymbol{V}.$$

Moreover, we have

$$\nabla^{\natural}(o' - o) = \nabla^{\natural} o' - \nabla^{\natural} o = (\nabla^{\shortparallel\prime} o' - dt \otimes dt \otimes N^{\natural\prime}) - (\nabla^{\shortparallel} o - dt \otimes dt \otimes N^{\natural}) =$$

$$= dt \otimes dt \otimes (N^{\natural} - N^{\natural\prime}),$$

which yields

$$(\mathrm{i}) \qquad \nabla^{\natural}(o' - o) = dt \otimes dt \otimes (N^{\natural} - N^{\natural\prime}).$$

The vertical restriction of (i) gives

$$\check{\nabla}^{\natural}(o' - o) = 0,$$

i.e. $o' - o$ can be regarded as a section

$$(**) \qquad o' - o : \boldsymbol{T} \to \mathbb{T}^* \otimes \boldsymbol{V}.$$

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[13] As usual, the difference $o' - o$ is taken with respect to the affine structure of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$.

Furthermore, formula $(*)$ implies that $\nabla^{\natural}(o' - o)$ can be regarded as a section

(ii) $$\nabla^{\natural}(o' - o) : \boldsymbol{T} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \boldsymbol{V}.$$

Then, formulas (i) and (ii) yield $(*)$.

Additionally, formula $(**)$ implies that o and o' yield the same linear splitting over $\boldsymbol{T}$

$$\boldsymbol{V} \to \boldsymbol{T} \times \boldsymbol{S}.$$ ❑

Next, let us show that the first and second gravitational equation imply that N fulfills the Newton law of gravitation.

REMARK I.6.2.2. The 2-form $\Omega^{\natural}$ becomes

$$\Omega^{\natural} = \Omega'' - \langle dt, N^{\natural} \rangle \bar{\wedge} \vartheta.$$

Let o be a Newtonian observer. Then the 2-form $\Phi^{\natural}$ becomes

$$\Phi^{\natural} = 2\, N^{\natural\flat} \wedge dt.$$ ❑

PROPOSITION I.6.2.2. The first gravitational field equation reduces to

$$\check{d} N^{\natural\flat} = 0.$$

PROOF. Let o be a Newtonian observer. Then, we obtain

$$\check{K}^{\natural} = \check{K}'' = \varkappa \qquad \check{\Sigma}^{\natural} = \check{\Sigma}'' = 0 \qquad \check{\Phi}'' = 0.$$

Hence, the first field equation (see Theorem I.4.3.1) reduces to $\check{d} N^{\natural\flat} = 0$. ❑

COROLLARY I.6.2.1. The first gravitational field equation reduces to

$$N^{\natural} = \check{\mathrm{grad}}\, U,$$

where U is a map

$$U^{\natural} : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{A}.$$ ❑

LEMMA I.6.2.4. The Ricci tensor of the gravitational connection turns out to be

$$r^{\natural} = \check{\mathrm{div}}^{\natural}\, N^{\natural}\, dt \otimes dt : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E}. \qquad \square$$

Next, let us assume that the source of the gravitational connection $K^{\natural}$ is an incoherent fluid with density mass μ, as in § I.4.5. This hypothesis does not conflict with the above lemma, as both $r^{\natural}$ and $\tau^{\natural}$ turn out to be time-like tensors.

Then, we can re-interpret the second gravitational field equation as follows.

THEOREM I.6.2.1. The second gravitational field equation reduces to

$$\check{\mathrm{div}}^{\natural}\, N^{\natural} = \boldsymbol{\kappa} \otimes \mu,$$

i.e. to

$$\check{\Delta}''\, U = \boldsymbol{\kappa} \otimes \mu. \qquad \square$$

REMARK I.6.2.2. Let o be a Newtonian observer. The potential of $\Phi^{\natural}$ becomes

$$a^{\natural} = U \otimes dt. \qquad \square$$

We can express the curvature through the gravitational potential.

REMARK I.6.2.3. Let us regard $dt \otimes dt \otimes N^{\natural}$ as a vertical valued 1-form on the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$

$$dt \otimes dt \otimes N^{\natural} : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}.$$

Then, the gravitational curvature turns out to be the covariant exterior differential (see § III.5) of $N^{\natural}$, with respect to the background connection K'',

$$R^{\natural} = d_{K''}(dt \otimes dt \otimes N^{\natural}),$$

with coordinate expression

$$R^{\natural} = u_0 \otimes u_0 \otimes g^{ih}\, \partial_{hj} U\, d^j \wedge d^0 \otimes \partial_i \otimes d^0.$$

PROOF. We have (see Rem. I.1.1.3)

$$R^{\natural} = \frac{1}{2}\,[K^{\natural}, K^{\natural}] = \frac{1}{2}\,[K'' + dt\otimes dt\otimes N^{\natural}, K'' + dt\otimes dt\otimes N^{\natural}] = [K'', dt\otimes dt\otimes N^{\natural}] :=$$

$$:= d_{K''}(dt\otimes dt\otimes N^{\natural})\ . \qquad \square$$

Finally, we study the law of motion of a classical test particle in the and show that it reduces exactly to the classical Newton law of gravitation.

Lemma I.6.2.5. The gravitational second order connection can be written as

$$\gamma^{\natural} = \gamma'' + N^{\natural},$$

where γ'' is the second order connection associated with Γ''. $\square$

Proposition I.6.2.3. The Newton law of motion reads as

$$\nabla_{\gamma''} s = N^{\natural} \circ js\,. \qquad \square$$

Therefore, $N^{\natural}$ can be interpreted as the gravitational force with respect to the background connection K''.

Remark I.6.2.4. Two different Newtonian connections K'' and K''' yield two different accelerations and gravitational forces

$$\nabla_{\gamma''} s \qquad \nabla_{\gamma'''} s \qquad\qquad N^{\natural} \circ js \qquad N^{\natural\prime} \circ js\,. \qquad \square$$

Remark I.6.2.5. Neither the two fields equations, nor the law of motion yield any distinguished choice among Newtonian connections admitted by $K^{\natural}$ (if they exist). $\square$

I.6.3. The special relativistic space-time

Eventually, we consider the case when the source of the gravitational field vanishes. In this case, the gravitational connection is Ricci flat; hence it is possibly flat and, under reasonable hypothesis, the gravitational connection itself is a Newtonian connection. So, in this case, we have a distinguished choice of the Newtonian background connection.

Therefore, we are led to introduce the following notion.

DEFINITION I.6.3.1. A space-time is said to be *special* if the gravitational connection $K^{\natural}$ induces an affine structure on the space-time manifold $\boldsymbol{E}$, such that the time function t is an affine map. ♣

Of course, if the space-time is special, then we can apply all above results with the identifications

$$K'' = K^{\natural} \qquad N^{\natural} = 0 .$$

This case will be referred to as the *special relativistic Galilei case*. The corresponding field theory and mechanics are just the standard classical theories.

I.7 - Classical two-body mechanics

So far, space-time has been regarded as the manifold of possible events "touched" by a classical particle. Moreover, the source of gravitational and electromagnetic fields has been a given external incoherent charged fluid (as an example). Furthermore, we have described the mechanics of a classical particle.

Now, we modify slightly our model in order to describe a closed system constituted by n classical particles interacting through the gravitational and electromagnetic fields. Thus, we no longer consider an external source of the fields, but the source is constituted by the particles themselves. Then, we are led to consider the fibred product over time of n identical copies of the "pattern" space-time as the framework of the system; each component is referred to one of the n particles. In order to describe the singularities corresponding to the possible collisions, we could describe the fields in terms of distributions. However, we follow a simpler way: we just cut the multi-events corresponding to possible collisions and consider smooth fields. It is a striking fact that the fields and the mechanics of particles can be formulated in terms of this multi-space-time in strong analogy with the case of a single particle. Actually, in the one particle theory, the dimension 3 of the space-time fibre has never played an essential role. So, the basic change from the case $n = 1$ to the case $n > 1$ consists in the increase of the dimension of the fibre of space-time (along with the addition of n projections on the components).

This scheme can be developed for any $n \geq 1$. However, we do it explicitly only for the case $n = 2$. The reader can generalise it without any difficulty.

We do not find the most general solution of field equations, but we just exhibit the simplest solution whose symmetries and boundary conditions are physically sensible. Then, the classical dynamics follows easily. This solution is nothing but a Galilei general relativistic formulation of the well known standard classical two body problem. We stress that the field equations and solutions can be formulated independently of the explicit motion of the particles.

Many concepts and results of this chapter are quite simple and standard.

However, it is necessary to spend some words for fixing the notation and showing how standard notions arise very well in our general formalism.

I.7.1. Two body space-time and equations

We start by introducing the multi-space-time associated with two classical particles. We consider the associated basic multi-objects such as the metric, gravitational and electromagnetic fields and we write the multi-field equations.

We label the objects related to the two particles by the indices 1 and 2, respectively.

ASSUMPTION CTB1. We assume a space-time fibred manifold

$$t : \boldsymbol{E} \to \boldsymbol{T}$$

as in Assumption C1, which will be referred to as the *pattern space-time*. ◇

More generally, the adjective *pattern* will be used for all objects associated with the pattern space-time.

Then, we define the *multi-space-time* to be the fibred manifold

$$\mathbb{t} : \mathbb{E} := (\boldsymbol{E}_1) \underset{\boldsymbol{T}}{\times} (\boldsymbol{E}_2) \to \boldsymbol{T}$$

where $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are two identical copies of the pattern space-time $\boldsymbol{E}$:

$$\boldsymbol{E}_1 \equiv \boldsymbol{E} \equiv \boldsymbol{E}_2 .$$

We denote the projections on the two components by

$$p_1 : \mathbb{E} \to \boldsymbol{E}_1 \qquad p_2 : \mathbb{E} \to \boldsymbol{E}_2 .$$

Of course, we can write

$$T\mathbb{E} = (T\boldsymbol{E}_1) \underset{T\boldsymbol{T}}{\times} (T\boldsymbol{E}_2) \qquad V\mathbb{E} = (V\boldsymbol{E}_1) \underset{\boldsymbol{T}}{\times} (V\boldsymbol{E}_2) \qquad J_1\mathbb{E} = (J_1\boldsymbol{E}_1) \underset{\boldsymbol{T}}{\times} (J_1\boldsymbol{E}_2).$$

Moreover, we consider the *reduced multi-space-time* obtained by subtracting the diagonal from the multi-space-time:

$$\mathbb{t}' : \mathbb{E}' := \{(y_1, y_2) \in \mathbb{E} \mid y_1 \neq y_2\} \to \boldsymbol{T} .$$

Then, we can define multi-objects related to multi-space-time analogously to the one body theory. Let us analyse a few concepts as an example.

A *multi-motion* is defined to be a section

$$\mathbb{s} : \boldsymbol{T} \hookrightarrow \mathbb{E}$$

and its *velocity* is defined to be its first jet prolongation

$$j_1\mathbb{s} : \boldsymbol{T} \to J_1\mathbb{E}.$$

Of course, we can write

$$\mathbb{s} = (s_1, s_2) := (\mathbb{s} \circ pr_1, \mathbb{s} \circ pr_2) \qquad j_1\mathbb{s} = (j_1 s_1, j_1 s_2) := \big(j_1(\mathbb{s} \circ pr_1), j_1(\mathbb{s} \circ pr_2)\big),$$

where

$$s_1 : \boldsymbol{T} \to \boldsymbol{E}_1 \qquad s_2 : \boldsymbol{T} \to \boldsymbol{E}_2 \qquad\qquad j_1 s_1 : \boldsymbol{T} \to J_1\boldsymbol{E}_1 \qquad j_1 s_2 : \boldsymbol{T} \to J_1\boldsymbol{E}_2.$$

A *multi-observer* is defined to be a section

$$\mathbb{o} : \mathbb{E} \to J_1\mathbb{E}.$$

Of course, we can write

$$\mathbb{o} = (o_1, o_2) := (\mathbb{o} \circ J_1 pr_1, \mathbb{o} \circ J_1 pr_2),$$

where

$$o_1 : \mathbb{E} \to J_1\boldsymbol{E}_1 \qquad o_2 : \mathbb{E} \to J_1\boldsymbol{E}_2,$$

need not to be pattern observers in the standard sense because they may depend on the two-body events. In particular, let

$$o : \boldsymbol{E} \to J_1\boldsymbol{E}$$

be a pattern observer and let o_1 and o_2 be two identical copies of o on $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, respectively; then, o yields the multi-observer

$$\mathbb{o} := (o_1, o_2).$$

A pattern space-time chart (x^0, y^i) yields the *multi-space-time chart*

$$(x^0, y_1{}^i, y_2{}^i) := (x^0, y^i \circ pr_1, y^i \circ pr_1).$$

Next, we pursue by assuming further structures on the multi-space-time analogously to the one body case.

ASSUMPTION TB2. We assume the reduced multi-space-time to be equipped with a *multi-metric*

$$\mathbb{g} : \mathbb{E}' \to \mathbb{A} \otimes (V^*\mathbb{E} \underset{\mathbb{E}}{\otimes} V^*\mathbb{E}),$$

a *multi-gravitational connection*

$$\mathbb{K}^\natural : T\mathbb{E}' \to T^*\mathbb{E}' \underset{\mathbb{E}'}{\otimes} TT\mathbb{E}'$$

and a *multi-electromagnetic field*

$$\mathbb{F} : \mathbb{E}' \to \mathbb{B} \otimes \overset{2}{\wedge} T^*\mathbb{E}'. \qquad \diamond$$

Analogously to the one body case, the above fields yield several objects, which fulfill several relations. In particular, we obtain the *multi-gravitational connection* (see Prop. I.2.1.1)

$$\Gamma^\natural : J_1\mathbb{E}' \to T^*\mathbb{E}' \underset{J_1\mathbb{E}'}{\otimes} TJ_1\mathbb{E}',$$

the *multi-second order gravitational connection* (see Prop. I.2.5.1)

$$\gamma^\natural : J_1\mathbb{E}' \to \mathbb{T}^* \underset{J_1\mathbb{E}'}{\otimes} TJ_1\mathbb{E}',$$

the *multi-gravitational contact 2-form* (see Prop. I.2.5.2)

$$\Omega^\natural : J_1\mathbb{E}' \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^*J_1\mathbb{E}'$$

the *multi-gravitational curvature tensor* (see Rem. I.2.1.4)

$$\mathbb{R}^\natural : \mathbb{E}' \to \overset{2}{\wedge} T^*\mathbb{E}' \underset{\mathbb{E}'}{\otimes} V^*\mathbb{E}'.$$

and the *multi-gravitational Ricci tensor* (see Rem. I.2.1.4)

$$\mathbb{r}^\natural : \mathbb{E}' \to T^*\mathbb{E}' \underset{\mathbb{E}'}{\otimes} T^*\mathbb{E}'.$$

ASSUMPTION TB3. We assume two masses and charges

$$m_1,\ m_2 \in \mathbb{M} \qquad \boldsymbol{q}_1,\ \boldsymbol{q}_2 \in \mathbb{Q}. \qquad \diamond$$

We set

$$\mu_1 := \frac{m_1}{m} \qquad m := m_1 + m_2 \qquad \mu_2 := \frac{m_2}{m}$$

$$\chi_1 := \frac{\boldsymbol{q}_1}{\boldsymbol{q}} \qquad \boldsymbol{q} := \boldsymbol{q}_1 + \boldsymbol{q}_2 \qquad \chi_2 := \frac{\boldsymbol{q}_2}{\boldsymbol{q}}\ .$$

Then, we obtain the *multi-total* objects by coupling the gravitational and electromagnetic fields (see § I.3.2), analogously to the case of one particle, both with reference to the above mass $m \in \mathbb{M}$ and charge $\boldsymbol{q} \in \mathbb{Q}$ and to the square root of the universal gravitational constant $\boldsymbol{\kappa}$.

In particular, we have the following total objects (with respect to both couplings)

$$\Gamma = \Gamma^{\natural} + \Gamma^{e} \qquad \gamma = \gamma^{\natural} + \gamma^{e} \qquad \Omega = \Omega^{\natural} + \Omega^{e}.$$

Moreover, we obtain the following splittings

$$\mathbb{R} = \mathbb{R}^{\natural} + \mathbb{R}^{\natural e} + \mathbb{R}^{e} \qquad r = r^{\natural} + r^{\natural e} + r^{e}.$$

The one body field theory suggests the following field equations.

ASSUMPTION TB4. We assume the metric, gravitational and electromagnetic field to fulfill the following field equations on the reduced multi-spacetime

$$d\,\Omega^{\natural} = 0 \qquad d\mathbb{F} = 0$$

$$r^{\natural} = 0 \qquad \operatorname{div}^{\natural}\mathbb{F} = 0 \qquad \diamond$$

The one body mechanics suggests the following dynamical equation.

ASSUMPTION TB5. *(Generalised Newton law of motion)* We assume the *law of motion* for two particles, with masses $m_1, m_2 \in \mathbb{M}$ and charges $\boldsymbol{q}_1$,

$\mathbf{q}_2 \in \mathbb{Q}$, whose multi-motion is $\mathbb{s} : \boldsymbol{T} \to \mathbb{E}$, to be the equation

$$\nabla_{\mathbb{\gamma}} j_1 \mathbb{s} = 0. \qquad \diamond$$

Later, in the two body quantum mechanics (see § II.7.2), we shall be involved with the following classical objects.

Let us consider a multi-observer $\mathbb{o}$ and the associated map

$$\nabla_{\mathbb{o}} : J_1\mathbb{E} \to \mathbb{T}^* \otimes V\mathbb{E}.$$

Then, we define the *multi-kinetic energy* and *multi-kinetic momentum* to be the maps (see Def. I.5.2.1)

$$\mathbb{G} := \frac{1}{2} m\, \mathbb{g} \circ (\nabla_{\mathbb{o}}, \nabla_{\mathbb{o}}) : J_1\mathbb{E} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M}$$

$$\check{\mathbb{P}} := m\, \mathbb{g}^\flat \circ \nabla_{\mathbb{o}} : J_1\mathbb{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes V^*\mathbb{E}.$$

Moreover, we obtain the *kinetic energy form* and *kinetic momentum form* (see Rem. I.5.2.1)

$$\mathbb{G} : J_1\mathbb{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes T^*\mathbb{E}$$

$$\mathbb{P} := \mathbb{\vartheta}^* \lrcorner \check{\mathbb{P}} : J_1\mathbb{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M} \otimes T^*\mathbb{E}.$$

I.7.2. The standard two-body solution

Next, we exhibit a distinguished solution of the field equations whose symmetries and boundary values are physically appropriate.

The background space-time structure

First, we introduce a multi-metric and a background affine structure of the multi-space-time in the following way.

We consider a vertical metric (see § I.1.2)

$$g : \boldsymbol{E} \to \mathbb{A} \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E})$$

and a background connection (see Def. I.6.1.1)

$$K^{\natural} : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} TT\boldsymbol{E}$$

on the pattern space-time.

Then, we denote by g_1 and g_2 two identical copies of g on $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, respectively, and consider the *multi-metric*

$$\mathbb{g} : \mathbb{E} \to \mathbb{A} \otimes (V^*\mathbb{E} \underset{\mathbb{E}}{\otimes} V^*\mathbb{E})$$

given by

$$\mathbb{g} : V\mathbb{E} \underset{\mathbb{E}}{\otimes} V\mathbb{E} \to \mathbb{A} : (X, Y) \mapsto \mu_1 g_1(X_1, Y_1) + \mu_2 g_2(X_2, Y_2).$$

Analogously, we denote by $K^{\natural}{}_1$ and $K^{\natural}{}_2$ two identical copies of $K^{\natural}$ on $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, respectively, and consider the *multi-background connection*

$$\mathbb{K}^{\natural} := K^{\natural}{}_1 \times K^{\natural}{}_2 : T\mathbb{E} \to T^*\mathbb{E} \underset{\mathbb{E}}{\otimes} TT\mathbb{E}.$$

Hence, the background affine structure on the multi-space-time yields the splittings (see Lemma I.6.1.1)

$$\bar{\mathbb{E}} = \boldsymbol{T} \times \mathbb{S} \qquad V\mathbb{E} = \mathbb{E} \times \mathbb{S},$$

where $\mathbb{S}$ is the 6-dimensional vector space

$$\mathbb{S} = \boldsymbol{S}_1 \times \boldsymbol{S}_2,$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ denote two identical copies of $\boldsymbol{S}$ related to $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, respectively.

We consider also the *reduced* subset

$$\boldsymbol{S}' := \boldsymbol{S} - \{0\}.$$

Moreover, the multi-metric turns out to be an element

$$\mathbb{g} \in \mathbb{A} \otimes \big((\boldsymbol{S}_1{}^* \otimes \boldsymbol{S}_1{}^*) \times (\boldsymbol{S}_2{}^* \otimes \boldsymbol{S}_2{}^*)\big).$$

Before pursuing with the construction of the standard solution of the field equations, we analyse some important consequences of the above structures on the multi-space-time.

We define the *space of the centre of mass* to be the diagonal fibred submanifold over $\boldsymbol{T}$

$$\mathbb{E}_c := \{(e_1,e_2) \in \mathbb{E} \mid e_1 = e_2\} \subset \mathbb{E} \to \boldsymbol{T}$$

and the *centre of mass projection* to be the fibred morphism over $\boldsymbol{T}$

$$pr_c : \mathbb{E} \to \mathbb{E}_c : \mathbb{e} \equiv (e_1,e_2) \mapsto \mathbb{e}_c \equiv (e_c,e_c),$$

where $e_c \in \boldsymbol{E}$ is the unique element such that

$$t(e_1) = t(e_c) = t(e_2) \qquad \mu_1 (e_1 - e_c) + \mu_2 (e_2 - e_c) = 0.$$

Moreover, we define the *relative space* to be the vector subspace

$$\mathbb{S}_s := \{(X_1,X_2) \in \mathbb{S} \mid \mu_1 X_1 + \mu_2 X_2 = 0\} \subset \mathbb{S}$$

and the *c-relative projection* to be the map

$$pr_s : \mathbb{E} \to \mathbb{S}_s : \mathbb{e} \equiv (e_1,e_2) \mapsto \mathbb{s} \equiv (\boldsymbol{s}_1,\boldsymbol{s}_2) := (e_1 - e_c, e_2 - e_c).$$

Furthermore, we define the *relative projections* to be the map[14]

$$pr_r : \mathbb{E}\to\mathbb{S} : \mathbb{e} \equiv (e_1,e_2) \mapsto (\boldsymbol{r}_1,\boldsymbol{r}_2) := (e_1 - e_2, e_2 - e_1).$$

We set also

$$\rho : \boldsymbol{S} \to \mathbb{A}^{1/2}\otimes\mathbb{R} : v \mapsto \|v\| := \sqrt{g(v,v)}$$

$$r := \rho\circ\boldsymbol{r}_1 = \rho\circ\boldsymbol{r}_2 : \mathbb{E} \to \mathbb{A}^{1/2}\otimes\mathbb{R}$$

$$s := \|\mathbb{s}\| := \sqrt{\mathbb{g}(\mathbb{s},\mathbb{s})}\,.$$

Hence, we can write

$$\boldsymbol{r}_1 = \frac{1}{\mu_2}\boldsymbol{s}_1 \qquad r^2 = \frac{1}{\mu_1\mu_2}s^2 \qquad \boldsymbol{r}_2 = \frac{1}{\mu_1}\boldsymbol{s}_2.$$

Therefore, besides the natural fibred splitting over $\boldsymbol{T}$

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[14] Of course, these maps should not be confused with the Ricci tensor.

$$(pr_1, pr_2) : \mathbb{E} \to \boldsymbol{E}_1 \underset{T}{\times} \boldsymbol{E}_2 : e \mapsto (e_1, e_2),$$

we obtain the further splittings induced by the affine background structure

$$(pr_c, pr_s) : \mathbb{E} \to \mathbb{E}_c \times \mathbb{S}_s : e \mapsto \big(e_c, (\boldsymbol{s}_1, \boldsymbol{s}_2)\big) := \big(e_c, (e_1 - e_c, e_2 - e_c)\big)$$

$$(pr_c, pr_{r_1}) : \mathbb{E} \to \mathbb{E}_c \times \boldsymbol{S} : e \mapsto (e_c, \boldsymbol{r}_1) := (e_c, e_1 - e_2)$$

$$(pr_c, pr_{r_2}) : \mathbb{E} \to \mathbb{E}_c \times \boldsymbol{S} : e \mapsto (e_c, \boldsymbol{r}_2) := (e_c, e_2 - e_1).$$

The above splitting is naturally prolonged by the tangent and jet functors, hence it yields analogous splittings of structures and equations.

If o is a pattern Newtonian observer and (x^0, y^i) an adapted Cartesian chart, then we obtain the two distinguished multi-charts

$$(x^0, y_1{}^i, y_1{}^i) := (x^0, y^i \circ pr_1, y^i \circ pr_2) \qquad (x^0, y_c{}^i, y_r{}^i) := (x^0, y^i \circ pr_c, y_2{}^i - y_1{}^i).$$

The gravitational and electromagnetic fields

Next, we exhibit the multi-gravitational connection.

We consider the map

$$U^\natural := -\frac{\mathrm{k}}{m} \frac{m_1 m_2}{\varrho} : \boldsymbol{S}' \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{M},$$

which yields

$$\mathbb{U}^\natural := U^\natural \circ \boldsymbol{r}_1 = U^\natural \circ \boldsymbol{r}_2 : \mathbb{E}' \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{A}.$$

Then, we obtain the section

$$\mathbb{N}^\natural := \check{\operatorname{grad}}\, \mathbb{U}^\natural : \mathbb{E}' \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\mathbb{E},$$

which can be expressed as

$$\mathbb{N}^\natural = (N^\natural{}_1, N^\natural{}_2) = -\frac{\mathrm{k}}{m} \frac{m_1 m_2}{r^3} \left(\frac{1}{\mu_1} \boldsymbol{r}_1, \frac{1}{\mu_2} \boldsymbol{r}_2\right).$$

Eventually, we exhibit the multi-electromagnetic field.

We consider the map

$$U^e := \frac{1}{\boldsymbol{q}} \frac{\boldsymbol{q}_1 \boldsymbol{q}_2}{\varrho} : \boldsymbol{S}' \to \mathbb{T}^* \otimes \mathbb{A}^{1/4} \otimes \mathbb{M}^{1/2}$$

which yields

$$\mathbb{U}^e := U^e \circ \boldsymbol{r}_1 = U^e \circ \boldsymbol{r}_2 : \mathbb{E}' \to \mathbb{T}^* \otimes \mathbb{A}^{1/4} \otimes \mathbb{M}^{1/2}.$$

Then, we obtain the section

$$\mathbb{F} := -2\, dt \wedge d\, \mathbb{U}^e : \mathbb{E}' \to \mathbb{B} \otimes \overset{2}{\wedge} T^* \mathbb{E}',$$

which can be expressed as

$$\mathbb{F} := (\chi_1 F_1, \chi_2 F_2) : \mathbb{E}' \to \mathbb{B} \otimes (\overset{2}{\wedge} T^* \boldsymbol{E}_1) \underset{\boldsymbol{T}}{\times} (\overset{2}{\wedge} T^* \boldsymbol{E}_2) \subset \mathbb{B} \otimes \overset{2}{\wedge} T^* \mathbb{E}'.$$

Moreover, we obtain the section

$$\mathbb{N}^e := \frac{\boldsymbol{q}}{m} \check{\operatorname{grad}}\, \mathbb{U}^e : \mathbb{E}' \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V \mathbb{E}',$$

which can be expressed as

$$\mathbb{N}^e = (N^e{}_1, N^e{}_2) = \frac{1}{m} \frac{\boldsymbol{q}_1 \boldsymbol{q}_2}{r^3} \left(\frac{1}{\mu_1} \boldsymbol{r}_1, \frac{1}{\mu_2} \boldsymbol{r}_2\right).$$

PROPOSITION I.7.2.1. The sections

$$\mathbb{K}^\natural = \mathbb{K}'' + dt \otimes dt \otimes \mathbb{N}^\natural : T\mathbb{E}' \to T^* \mathbb{E}' \underset{\mathbb{E}'}{\otimes} T\, T\mathbb{E}'$$

and

$$\mathbb{F} := -2\, dt \wedge d\, \mathbb{U}^e : \mathbb{E}' \to \mathbb{B} \otimes \overset{2}{\wedge} T^* \mathbb{E}'$$

are a multi-gravitational connection and a multi-electromagnetic field and fulfill the two-body field equations (see Ass. TB4). ❑

REMARK I.7.2.1. The Newtonian vector field and the potential factorise through the following commutative diagrams

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\mathbb{N}"] \arrow[d] & (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes \mathbb{S} \\ \mathbb{S}_r \arrow[r] & (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes \mathbb{S}_r \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[rr, "\mathbb{U}"] \arrow[dr] & & \mathbb{T}^* \mathbb{E} \\ & \mathbb{S}_r \arrow[ur] & \end{tikzcd}$$ ❑

Remark I.7.2.2. We obtain

$$\Omega^\natural = \Omega'' - 2\, dt \wedge d\, \mathbb{U} \qquad \Omega^e = -\, 2\, \frac{\boldsymbol{q}}{m}\, dt \wedge d\, \mathbb{U}^e$$

$$\gamma^\natural = \gamma'' + dt \otimes dt \otimes \mathbb{N}^\natural \qquad \gamma^e = dt \otimes dt \otimes \mathbb{N}^e .$$

Hence, with reference to any Newtonian multi-observer (see § I.6.2), the form (see Theor. I.4.3.2))

$$\mathbb{a} = \mathbb{U}\, dt := (\mathbb{U}^\natural + \frac{\boldsymbol{q}}{m}\, \mathbb{U}^e)\, dt = -\, \frac{1}{m}\, \frac{\boldsymbol{\kappa} m_1 m_2 - \boldsymbol{q}_1 \boldsymbol{q}_2}{r}\, dt$$

is a potential of the 2-form $\Phi := 2\, \mathbb{o}^* \Omega$. ❑

Remark I.7.2.3. With reference to any multi-space-time chart, we have the following coordinate expression

$$\mathbb{a} := \mathbb{a}_0\, u^0 \otimes d^0 = -\, \frac{1}{m}\, \frac{\kappa m_1 m_2 - q_1 q_2}{r}\, u^0 \otimes d^0 .$$ ❑

The two body mechanics

The dynamics of the two classical particles follows from the above solution of the field equations analogously to the one body dynamics, without any problem, in full accordance with well known standard results.

We just give explicitly a few further details about objects we shall be involved with later in the quantum theory (see § II.7.2).

Remark I.7.2.4. Let us consider a multi-observer $\mathbb{o}$ of the standard type $\mathbb{o} = (o, o)$, where $o : \boldsymbol{E} \to J_1 \boldsymbol{E}$ is a pattern observer.

Then, we obtain the standard formulas

$$\mathbb{G} = \boldsymbol{G}_1 + \boldsymbol{G}_2 := \tfrac{1}{2}\, m_1\, g \circ (\nabla_o, \nabla_o) \circ J_1 pr_1 + \tfrac{1}{2}\, m_2\, g \circ (\nabla_o, \nabla_o) \circ J_1 pr_2$$

$$= \boldsymbol{G}_c + \boldsymbol{G}_r := \tfrac{1}{2} m\; g \circ (\nabla_o, \nabla_o) \circ J_1 pr_c + \tfrac{1}{2} m\; \mu_1\, \mu_2\, g \circ (\nabla_o, \nabla_o) \circ J_1 pr_r$$

$$\check{\mathbb{P}} = \check{\boldsymbol{P}}_1 + \check{\boldsymbol{P}}_2 := (m_1\, g^\flat \circ \nabla_o) \circ J_1 pr_1 + (m_2\, g^\flat \circ \nabla_o) \circ J_1 pr_2$$

$$= \check{\boldsymbol{P}}_c + \check{\boldsymbol{P}}_r := (m\; g^\flat \circ \nabla_o) \circ J_1 pr_c + (m\; \mu_1\, \mu_2\, g^\flat \circ \nabla_o) \circ J_1 pr_r$$

$$\mathbb{P} = \boldsymbol{P}_1 + \boldsymbol{P}_2 := \vartheta^* \lrcorner \check{\boldsymbol{P}}_1 + \vartheta^* \lrcorner \check{\boldsymbol{P}}_2$$

$$= \boldsymbol{P}_c + \boldsymbol{P}_r := \vartheta^* \lrcorner \check{\boldsymbol{P}}_c + \vartheta^* \lrcorner \check{\boldsymbol{P}}_r.$$

With reference to a Cartesian chart (x^0, y^i) adapted to o, we obtain the following coordinate expressions

$$\mathbb{G} = \tfrac{1}{2}\, \delta_{ij}\, (m_1\, y^i_{10}\, y^j_{10} + m_2\, y^i_{20}\, y^j_{20})$$

$$= \tfrac{1}{2}\, \delta_{ij}\, m\, (y^i_{c0}\, y^j_{c0} + \mu_1\, \mu_2\; y^i_{r0}\, y^j_{r0})$$

$$\mathbb{P} = -\, \delta_{ij} \big((m_1\, y^i_{10}\, y^j_{10} + m_2\, y^i_{20}\, y^j_{20})\, d^0 + (m_1\, y^j_{10}\, d_1{}^i + m_2\, y^j_{20}\, d_2{}^i) \big)$$

$$= -\, \delta_{ij}\, m \big((y^i_{c0}\, y^j_{c0} + \mu_1\, \mu_2\; y^i_{r0}\, y^j_{r0})\, d^0 + (y^j_{c0}\, d_c{}^i + \mu_1\, \mu_2\, y^j_{r0}\, d_r{}^i) \big). \qquad \square$$

We conclude this chapter with the observation that, in most respects, the system of two particles with masses (m_1, m_2) and charges $(\boldsymbol{q}_1, \boldsymbol{q}_2)$ can be treated just as a system consisting of one particle with mass m and charge $\boldsymbol{q}$ moving in the multi-space-time.

II - THE QUANTUM THEORY

The second part of the paper is devoted to a model of general relativistic quantum mechanics for a spin-less charged particle, interacting with given classical gravitational and electromagnetic fields in a given classical Galilei general relativistic curved space-time with absolute time.

So, we consider a classical curved space-time $t : \boldsymbol{E} \to \boldsymbol{T}$ (§ I.1.1) equipped with a vertical metric g (§ I.1.2), a gravitational connection $\Gamma^{\natural}$ and an electromagnetic field F (§ I.3.1) fulfilling the field equations (§ I.4.1, § I.4.5). Moreover, we consider a mass $m \in \mathbb{M}$, a charge $\boldsymbol{q} \in \mathbb{Q}$ and the related total objects, which involve both the gravitational and electromagnetic fields (§ I.3.2); in particular, we are involved with the total contact 2-form Ω (§ I.2.5).

Next, we introduce the quantum framework: it is constituted by a line bundle, over the classical space-time, equipped with a system of connections, parametrised by the observers, whose universal curvature is proportional (through the Planck constant) to the classical contact 2-form. Then, a criterion of projectability expressing the principle of general relativity, yields the dynamics for the sections of the quantum bundle. Moreover, pure geometrical constructions produce the quantum operators.

The above theory is referred to the quantum bundle, which is based on space-time. We can re-formulate this theory, in terms of infinite dimensional systems, by taking time as base space. In this context, we achieve the Hilbert bundle and related quantum operators.

The standard probabilistic interpretation of quantum mechanics is assumed without any essential change.

II.1 - The quantum framework

This chapter is devoted to the study of the quantum bundle and the quantum connection.

II.1.1. The quantum bundle

We start by introducing the quantum bundle, which is the fundamental space of the quantum theory.

ASSUMPTION Q1. We assume the *quantum bundle* to be a Hermitian line bundle over space-time

$$\pi : \boldsymbol{Q} \to \boldsymbol{E}. \qquad \diamond$$

Thus, we assume the quantum bundle to be a 2-dimensional (real) vector bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ equipped with a linear fibred morphism over $\boldsymbol{E}$

$$\iota : \boldsymbol{Q} \to \boldsymbol{Q}$$

such that $\iota^2 = -1$ and a Hermitian fibred product

$$h : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{C}.$$

As usual, $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ becomes a 1-dimensional complex vector bundle, by setting

$$i\, q := \iota(q), \qquad \forall q \in \boldsymbol{Q}.$$

We shall be involved with the real and imaginary *Liouville vector fields*[15]

$$\text{и} : \boldsymbol{Q} \to V\boldsymbol{Q} = \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q, q) \qquad i\,\text{и} : \boldsymbol{Q} \to V\boldsymbol{Q} = \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q, i\, q)$$

and we shall often make the identifications

$$1 = \mathrm{id}_{\boldsymbol{Q}} \simeq \text{и} : \boldsymbol{E} \to (\boldsymbol{Q}^* \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}) \qquad i = i\, \mathrm{id}_{\boldsymbol{Q}} \simeq i\, \text{и} : \boldsymbol{E} \to (\boldsymbol{Q}^* \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}).$$

We denote the tube-like (real) linear charts and dual bases, respectively,

[15] и is the Cyrillic character corresponding to “i”.

by

$$(w^a) : \boldsymbol{Q} \to \mathbb{R}^2 \qquad (b_a) : \boldsymbol{E} \to \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \qquad 1 \le a \le 2.$$

Moreover, we denote the associated (real) linear bundle trivialisation and (real) linear complex valued coordinate, respectively, by

$$\varphi := (\pi,\ w^1 + i\ w^2) : \boldsymbol{Q} \to \boldsymbol{E} \times \mathbb{C} \qquad z := w^1 + i\ w^2 : \boldsymbol{Q} \to \mathbb{C}.$$

The above four objects (w^a), (b_a), φ and z characterise each other in an obvious way.

Of course, the tube-like vector field

$$b := b_1 : \boldsymbol{E} \to \boldsymbol{Q}$$

turns out to be a complex basis.

The above four objects (w^a), (b_a), φ and z are said to be *normal* if

$$\mathrm{b}_2 = \mathrm{i}\ \mathrm{b}_1 \qquad h(b_1, b_1) = 1.$$

Normality is characterised by each of the following equivalent conditions:

i) $$z : \boldsymbol{Q} \to \mathbb{C}$$

is a complex linear coordinate,

$$b : \boldsymbol{E} \to \boldsymbol{Q}$$

is the complex dual basis and

$$h(b, b) = 1;$$

ii) $\iota = w^1 \otimes b_2 - w^2 \otimes b_1 \qquad h = (w^1 \otimes w^1 + w^2 \otimes w^2) + i\ (w^1 \otimes w^2 - w^2 \otimes w^1);$

iii) $$1 = z \otimes b \simeq z\ \partial_z = \text{и} \qquad i \simeq i\ z \otimes b \simeq i\ z\ \partial_z \simeq i\ \text{и}$$

$$h = \bar{z} \otimes z;$$

iv) φ is a tube-like Hermitian complex fibred isomorphism.

The normal bases and trivialisations will be refereed to as *quantum gauges*.

It can be proved that the bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ admits a bundle atlas constituted by normal charts. The associated cocycle takes its values in the group $U(1)$. Now on, we will always refer to such an atlas, without any explicit mention.

If (x^0, y^i, w^a) is a fibred chart $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, then the induced linear fibred chart of the vector bundle $J_1\boldsymbol{Q} \to \boldsymbol{E}$ will be denoted by

$$(x^0, y^i, w^a, w^a_\lambda).$$

The sections $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ are interpreted physically as the possible quantum histories.

For each $\Psi \in \mathscr{S}(\boldsymbol{Q} \to \boldsymbol{E})$, we shall write locally

$$\Psi = \psi\, b, \qquad\qquad \psi := z \circ \Psi \in \mathscr{M}(\boldsymbol{E}, \mathbb{C}).$$

II.1.2. Quantum densities

The sections of the quantum bundle are sufficient for our starting purposes. However, in several contexts it is useful or necessary to multiply them by the space-time or space-like half-volume forms. For this reason, we introduce the notion of quantum half-densities. We have a natural bijection between quantum sections and quantum half-densities.

Let us consider the bundles

$$\mathbb{T}^{1/2} \otimes \mathbb{A}^{3/4} \otimes \sqrt{\overset{4}{\wedge} T^* \boldsymbol{E}} \to \boldsymbol{E} \qquad \mathbb{A}^{3/4} \otimes \sqrt{\overset{3}{\wedge} V^* \boldsymbol{E}} \to \boldsymbol{E},$$

where $\sqrt{\ }$ denotes the square root of the 1-dimensional positive semi-vector bundle induced by the positive orientation (see § III.1.3).

Thus, we have the sections (see § I.1.2)

$$\sqrt{\upsilon} : \boldsymbol{E} \to \mathbb{T}^{1/2} \otimes \mathbb{A}^{3/4} \otimes \sqrt{\overset{4}{\wedge} T^* \boldsymbol{E}} \qquad\qquad \sqrt{\eta} : \boldsymbol{E} \to \mathbb{A}^{3/4} \otimes \sqrt{\overset{3}{\wedge} V^* \boldsymbol{E}}$$

and, by definition, we obtain

$$\sqrt{\upsilon} \otimes \sqrt{\upsilon} = \upsilon \qquad\qquad \sqrt{\eta} \otimes \sqrt{\eta} = \eta.$$

The *space-time half-densities quantum bundle* and the *space-like half-densities quantum bundle* are defined to be, respectively, the bundles

$$\pi^\upsilon : \boldsymbol{Q}^\upsilon := \mathbb{T}^{1/2} \otimes \mathbb{A}^{3/4} \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \sqrt{\overset{4}{\wedge} T^* \boldsymbol{E}}) \to \boldsymbol{E}$$

$$\pi^{\eta} : \boldsymbol{Q}^{\eta} := \mathbb{A}^{3/4} \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \sqrt{\overset{3}{\wedge} V^{*}\boldsymbol{E}}) \to \boldsymbol{E},$$

The above bundles turn out to be 2-dimensional real vector bundles and inherit the complex and Hermitian structures from the Hermitian complex bundle $\boldsymbol{Q} \to \boldsymbol{E}$.

If $\Psi \in \mathscr{S}(\boldsymbol{Q})$, then we obtain the local sections

$$\Psi^{\upsilon} := \Psi \otimes \sqrt{\upsilon} : \boldsymbol{E} \to \boldsymbol{Q}^{\upsilon} \qquad \Psi^{\eta} := \Psi \otimes \sqrt{\eta} : \boldsymbol{E} \to \boldsymbol{Q}^{\eta}$$

with coordinate expressions

$$\Psi^{\upsilon} = \psi^{\eta}\, b \otimes \sqrt{(u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3)} \qquad \Psi^{\eta} = \psi^{\eta}\, b \otimes \sqrt{(\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3)},$$

where we have set

$$\psi^{\eta} \equiv \sqrt[4]{|g|}\ \psi.$$

Of course, we have the natural linear sheaf isomorphisms

$$\mathscr{S}(\boldsymbol{Q}) \to \mathscr{S}(\boldsymbol{Q}^{\upsilon}) : \Psi \mapsto \Psi^{\upsilon} \qquad \mathscr{S}(\boldsymbol{Q}) \to \mathscr{S}(\boldsymbol{Q}^{\eta}) : \Psi \mapsto \Psi^{\eta}.$$

II.1.3. Systems of connections

In view of the introduction of the quantum connection, we need a few recalls on systems of connections (see [MM], [Mo1]).

Let $p:\boldsymbol{F} \to \boldsymbol{B}$ be a fibred manifold.

A *system of connections* is defined to be a fibred morphism over $\boldsymbol{F}$

$$\xi : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to J_1\boldsymbol{F} \subset T^{*}\boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} T\boldsymbol{F},$$

where $q:\boldsymbol{C} \to \boldsymbol{B}$ is a fibred manifold.

The system ξ maps in a natural way sections of $q:\boldsymbol{C} \to \boldsymbol{B}$ into connections of $p:\boldsymbol{F} \to \boldsymbol{B}$

$$\boldsymbol{\xi} : c \mapsto \xi_c := \xi \circ c^{\uparrow},$$

where $c^{\uparrow}$ denotes the pullback of c (see § III.1).

In other words, a system of connections is a smooth family of connections parametrised by the sections of the bundle $\boldsymbol{C}$. The connections of the fibred manifold $p:\boldsymbol{F}\to\boldsymbol{B}$, which are of the type ξ_c, are the distinguished connections of the system.

Let (x^λ,z^i) and (x^λ,a^u) be fibred charts of $\boldsymbol{F}$ and $\boldsymbol{C}$, whose domains project over the same open subset of $\boldsymbol{B}$. Then, the coordinate expression of ξ is of the type

$$\xi = d^\lambda\otimes\partial_\lambda + \xi_\lambda{}^i\, d^\lambda\otimes\partial_i \qquad \xi_\lambda{}^i\in\mathcal{F}(\boldsymbol{C}\underset{\boldsymbol{B}}{\times}\boldsymbol{F}).$$

Given a system of connections ξ, we obtain the “universal connection” in the following way.

Let us consider the pullback bundle over $\boldsymbol{C}$

$$p^\uparrow : \boldsymbol{F}^\uparrow := \boldsymbol{C}\underset{\boldsymbol{B}}{\times}\boldsymbol{F} \to \boldsymbol{C}.$$

REMARK II.1.3.1. We can easily exhibit a natural inclusion

$$\iota : (J_1\boldsymbol{F})^\uparrow := \boldsymbol{C}\underset{\boldsymbol{B}}{\times}J_1\boldsymbol{F} \hookrightarrow J_1(\boldsymbol{F}^\uparrow).$$

with coordinate expression

$$(x^\lambda,a^u,z^i;z_\lambda{}^i,z_u{}^i)\circ\iota = (x^\lambda,a^u,z^i;z_\lambda{}^i,0).$$

Next, we can easily see that the map

$$\Lambda_\xi := \iota\circ\xi^\uparrow : \boldsymbol{F}^\uparrow \to J_1(\boldsymbol{F}^\uparrow)$$

is a section. Thus, Λ_ξ is a connection of the fibred manifold $p^\uparrow:\boldsymbol{F}^\uparrow\to\boldsymbol{C}$, with coordinate expression

$$\Lambda_\xi = d^\lambda\otimes\partial_\lambda + d^u\otimes\partial_u + \xi_\lambda{}^i\, d^\lambda\otimes\partial_i.$$

The connection Λ_ξ is said to be the *universal connection* of the system ξ because every connection ξ_c of the system can be obtained from Λ_ξ by pull-back:

$$\xi_c = c^*\Lambda_\xi. \qquad ❑$$

We can characterise the universal connection in the following way.

REMARK II.1.3.2. i) Let ξ be a system of connections. Then the coordinate expression of the universal connection Λ_ξ of the system ξ shows that the linear fibred morphism over $\boldsymbol{F}^\uparrow \to \boldsymbol{F}$

$$V\boldsymbol{C} \underset{\boldsymbol{C}}{\times} \boldsymbol{F}^\uparrow \hookrightarrow T\boldsymbol{C} \underset{\boldsymbol{C}}{\times} \boldsymbol{F}^\uparrow \xrightarrow{\Lambda_\xi} T(\boldsymbol{F}^\uparrow) \longrightarrow T\boldsymbol{F}$$

vanishes, i. e., for each vertical vector field $X:\boldsymbol{B}\to V\boldsymbol{C}$,

$$X \lrcorner \Lambda_\xi = X.$$

In coordinates, this condition reads

$$\Lambda_u{}^i = 0.$$

ii) Conversely, let $q:\boldsymbol{C}\to\boldsymbol{B}$ be a fibred manifold and

$$\Lambda:\boldsymbol{F}^\uparrow \to T^*\boldsymbol{C} \underset{\boldsymbol{F}^\uparrow}{\otimes} T(\boldsymbol{F}^\uparrow)$$

a connection, which fulfills the above condition. Then, we can prove that there is a unique system of connections

$$\xi_\Lambda : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to J_1\boldsymbol{F},$$

whose universal connection is Λ. ❑

Next, let us go back to the system ξ. The curvature of the connection Λ_ξ is the vertical valued 2-form (see § III.5 and [MM2], [Mo2])

$$\Omega_\xi := \tfrac{1}{2}[\Lambda_\xi, \Lambda_\xi] : \boldsymbol{F}^\uparrow \to \overset{2}{\wedge} T^*\boldsymbol{C} \underset{\boldsymbol{F}}{\otimes} V\boldsymbol{F},$$

with coordinate expression

$$\Omega_\xi = \big((\partial_\lambda \xi_\mu{}^i + \xi_\lambda{}^j \, \partial_j \xi_\mu{}^i) \, d^\lambda \wedge d^\mu + \partial_u \xi_\mu{}^i \, d^u \wedge d^\mu\big) \otimes \partial_i.$$

The curvature Ω_ξ is said to be the *universal curvature* of the system ξ, because the curvature

$$R_{\xi_c} := \tfrac{1}{2}[\xi_c, \xi_c] : \boldsymbol{F} \to \overset{2}{\wedge} T^*\boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} V\boldsymbol{F}$$

of every connection ξ_c of the system can be obtained from Ω_ξ by pullback:

$$R_{\xi_c} = c^*\Omega_\xi .$$

Next, let $p:\boldsymbol{F}\to\boldsymbol{B}$ be a Hermitian line bundle. Then, a connection[16]

$$\text{ч}:\boldsymbol{F}\to T^*\boldsymbol{B}\underset{\boldsymbol{B}}{\otimes}T\boldsymbol{F}$$

is said to be *Hermitian* if it is (real) linear and preserves the Hermitian product h. A Hermitian connection turns out to be also complex linear. The coordinate expression of a Hermitian connection ч is of the type

$$\text{ч} = d^\lambda\otimes(\partial_\lambda + i\,\text{ч}_\lambda\,\text{и}) \qquad \text{ч}_\lambda\in\mathcal{F}(\boldsymbol{B}).$$

The curvature of a Hermitian connection ч can be regarded as an imaginary 2-form

$$R_\text{ч}:\ \boldsymbol{B}\to\overset{2}{\wedge}T^*\boldsymbol{B}\otimes\mathbb{C}$$

with coordinate expression

$$R_\text{ч} = i\,\partial_\lambda\text{ч}_\mu\,d^\lambda\wedge d^\mu\otimes\text{и}.$$

REMARK II.1.3.3. Let $p:\boldsymbol{F}\to\boldsymbol{B}$ be a Hermitian complex vector bundle and ξ a system of connections. Then also $p^\uparrow:\boldsymbol{F}^\uparrow\to\boldsymbol{C}$ turns out to be a Hermitian complex vector bundle.

Moreover, ξ is a system of Hermitian connections if an only if Λ_ξ is a Hermitian connection. ❑

II.1.4. The quantum connection

Now, we are in the position to introduce the quantum connection, which constitutes our basically unique assumption of the quantum theory.

We observe that the quantum bundle lives on the space-time $\boldsymbol{E}$. However, $\boldsymbol{E}$ does not carry sufficient information of the classical structure; for instance, Γ, γ and Ω live on $J_1\boldsymbol{E}$. Therefore, we are led to consider the pullback bundle

---

[16] ч is the Cyrillic character corresponding to “ch”.

$$\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$$

of the quantum bundle $\boldsymbol{Q} \to \boldsymbol{E}$, with respect to $J_1\boldsymbol{E} \to \boldsymbol{E}$. In the present context, this bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ has to be interpreted as the target space of classical observers (see § I.1.1).

Then, we make the following main assumptions.

ASSUMPTION Q2. We assume the *Planck constant* to be an element

$$\boldsymbol{\hbar} \in \mathbb{T}^{+*} \otimes \mathbb{A} \otimes \mathbb{M}. \qquad \diamond$$

Moreover, given $u_0 \in \mathbb{T}^{+}$, we set

$$\hbar := \boldsymbol{\hbar}(u_0) \in \mathbb{A} \otimes \mathbb{M}.$$

ASSUMPTION Q3. We assume the *quantum connection* to be a connection on the bundle $\boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$

$$\text{ч} : \boldsymbol{Q}^{\uparrow} \to T^* J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} T\boldsymbol{Q}^{\uparrow},$$

with the following properties:

i) ч is Hermitian

ii) ч is universal,

iii) the curvature

$$R_{\text{ч}} : J_1\boldsymbol{E} \to \overset{2}{\wedge} T^* J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} (\boldsymbol{Q}^* \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q})$$

of ч is given by

$$(R_{\text{ч}}) \qquad R_{\text{ч}} = i\,\frac{m}{\boldsymbol{\hbar}}\,\Omega \otimes \text{и}. \qquad \diamond$$

We stress that our assumption on the closure of the classical contact 2-form Ω turns out to be an essential integrability condition for the existence of the quantum connection. In fact, the equality

$$d\Omega = 0$$

can be regarded as the Bianchi identity for the connection ч.

PROPOSITION II.1.4.1. Our assumption Q3) on the quantum connection can be re-formulated by saying that we assume a system of Hermitian connections, parametrised by the observers $o : \boldsymbol{E} \to J_1\boldsymbol{E}$,

$$\xi : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{Q},$$

whose curvature is given, for each observer o, by

$$R_{\xi_o} = \frac{1}{2}\, i\, \frac{m}{\hbar}\, \Phi \otimes \text{и}.$$

PROOF. It follows immediately from the properties of universal connection and curvature of a system of connections (see § II.1.3). ❑

COROLLARY II.1.4.1. For each observer o, we obtain the connection

$$\xi_o : \boldsymbol{Q} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{Q}$$

whose coordinate expression, in adapted coordinates, is

$$\xi_o = d^\lambda \otimes \partial_\lambda + i\, a_\lambda\, d^\lambda \otimes \text{и},$$

where a is a distinguished (see, later, Rem II.1.4.1) choice of the potential of Φ.

PROOF. It follows immediately from the coordinate expression of the curvature of a Hermitian connection and the definition of a. ❑

LEMMA II.1.4.1. Let b be a quantum gauge. Then, in the tube of the trivialisation, there is a unique flat Hermitian and universal connection

$$\text{ч}'' : \boldsymbol{Q}^\uparrow \to T^*J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} T\boldsymbol{Q}^\uparrow,$$

such that for each section $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$, which is constant with respect to φ, we have $\nabla''\Psi = 0$. ❑

We say that $\text{ч}''$ is the *background connection* associated with b. We stress that $\text{ч}''$ is not a quantum connection because its curvature vanishes (and it is local).

THEOREM II.1.4.1. Let b be a quantum gauge and o an observer. Then, in the tube of the trivialisation, the quantum connection can be uniquely written as

$$(QC) \qquad \text{ч} = \text{ч}'' + i\,\frac{1}{\hbar}\,(\boldsymbol{G} + \boldsymbol{P} + m\,a)\,\text{и},$$

where $\boldsymbol{G}$ and $\boldsymbol{P}$ are the classical kinetic energy and momentum forms associated with the observer o (see Rem. I.5.2.1) and a is a potential of the 2-form $\Phi := 2\,o^*\Omega$ (see Prop. I.2.5.1 and Theor. I.4.3.2), which depends on b and, obviously, on o.

Thus, with reference to the normal chart associated with b and to any space-time chart adapted to o, we obtain the following coordinate expression

$$(QC)' \quad \text{ч} = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + i\,\frac{m}{\hbar}\,\left(-\frac{1}{2}\,g_{ij}\,y^i_0\,y^j_0\,d^0 + g_{ij}\,y^i_0\,d^j + a_\lambda\,d^\lambda\right) \otimes \text{и},$$

hence (see Rem. I.5.2.2)

$$\text{ч}_0 = -\,H/\hbar \qquad \text{ч}_j = p_j/\hbar \qquad \text{ч}^0_j = 0.$$

Proof. Let us refer to the normal chart associated with b and to any space-time chart adapted to o.

Condition ii) reads in coordinates as

$$\text{ч}^0_j = 0.$$

Moreover, in virtue of condition i), the coordinate expression of the connection ч can be written (without loosing in generality) as

$$\text{ч} = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + i\,\frac{m}{\hbar}\,\left(-\frac{1}{2}\,g_{ij}\,y^i_0\,y^j_0\,d^0 + g_{ij}\,y^i_0\,d^j + a_\lambda\,d^\lambda\right) \otimes \text{и},$$

where

$$a := a_\lambda\,u^0 \otimes d^\lambda : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T^*\boldsymbol{E}$$

is a suitable fibred morphism over $\boldsymbol{E}$, which depends on φ and the space-time chart.

Eventually, a computation in coordinates shows that condition iii) implies that a depends on $J_1\boldsymbol{E}$ only through $\boldsymbol{E}$ and that it is a potential of Φ.

Moreover, we can easily see that a change of space-time chart adapted to the same observer o leaves the term

$$d^{\lambda}\otimes\partial_{\lambda} + d^{i}_{0}\otimes\partial^{0}_{i} + i\,\frac{m}{\hbar}\,(-\tfrac{1}{2}\,g_{ij}\,y^{i}_{0}\,y^{j}_{0}\,d^{0} + g_{ij}\,y^{i}_{0}\,d^{j})\otimes\text{и}$$

unchanged. Hence, a depends on the space-time chart only through the observer o. ❑

COROLLARY II.1.4.1. The composition of the connections γ and ч

$$\gamma\lrcorner\text{ч}:\boldsymbol{Q}^{\uparrow}\to\mathbb{T}^{*}\otimes T\boldsymbol{Q}^{\uparrow}$$

is a connection on the fibred manifold $\boldsymbol{Q}^{\uparrow}\to\boldsymbol{T}$.

Let b be a quantum gauge and o an observer. Then, in the tube of the trivialisation, we can write

$$\gamma\lrcorner\text{ч} = \gamma\lrcorner\text{ч}'' + i\,\frac{1}{\hbar}\,\boldsymbol{L}\,\text{и},$$

where $\boldsymbol{L}$ is the classical Lagrangian form associated with the observer o (see Rem. I.5.2.1) and with the potential a of Φ fixed by the above theorem.

Thus, with reference to the normal chart associated with b and to any space-time chart adapted to o, we obtain the following coordinate expression

$$\gamma\lrcorner\text{ч} = u^{0}\otimes(\partial_{0} + y^{i}_{0}\,\partial_{i} + \gamma^{i}\,\partial^{0}_{i} + i\,L/\hbar\;\text{и}),$$

hence

$$(\gamma\lrcorner\text{ч})_{0} = L/\hbar.$$ ❑

PROPOSITION II.1.4.2. Let o be an observer. Let b and b' be two quantum gauges and set

$$e^{i\,\vartheta} := b'/b \in \mathcal{M}(\boldsymbol{E},\mathbb{C}).$$

Let a and a' be the potentials of the form $\Phi := 2o^{*}\Omega$ which appear in the expressions of the quantum connection ч related to the quantum gauges b and b', respectively, according to formula (QC).

Then, we obtain

$$a' = a - \frac{\hbar}{m}\,d\vartheta.$$

PROOF. Let (x^{0},y^{i},z) and $(x^{0'},y^{i'},z')$ be charts of $\boldsymbol{Q}$ associated with (o,b) and (o,b'), respectively. Then, we can write

$$d^{\lambda'} \otimes \partial_{\lambda'} + d^{i'}_{0'} \otimes \partial^{0'}_{i'} = d^{\lambda} \otimes \partial_{\lambda} + d^{i}_{0} \otimes \partial^{0}_{i} - i\, d\vartheta \otimes \mathbb{I}.$$

Hence, the comparison of formula (QC) in the two charts yields the result.❑

PROPOSITION II.1.4.3. Let b be a quantum gauge. Let o and o' be two observers and set

$$v := o' - o : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}.$$

Let a and a' be the potentials of the forms $\Phi := 2o^*\Omega$ and $\Phi' := 2o'^*\Omega$ which appear in the expressions of the quantum connection ч related to the quantum gauge b and the two observers o and o', respectively, according to formula (QC).

Then, we obtain

$$(*) \qquad a' = a - \tfrac{1}{2}\, g\circ(v,v) + \nu_o{}^* \lrcorner\, v^\flat,$$

where $v^\flat := g^\flat \circ v$, $\nu_o{}^*: V^*\boldsymbol{E} \to T^*\boldsymbol{E}$ is the transpose of ν_o (see Rem. I.1.1.3) and $g\circ(v,v)$ is regarded as a form

$$g\circ(v,v) : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes T^*\boldsymbol{E}.$$

In particular, by vertical restriction, we obtain

$$\check{a}' = \check{a} + v^\flat.$$

Hence, if (x^0, y^i) and (x^0, y'^i) are space-time charts[17] adapted to o and o', respectively, then we can write

$$a'_{0'} = a_0 + (-\tfrac{1}{2} v_i + a_i)\, v^i$$

$$a'_{i'} = \partial_{i'} y'^j\, (a_j + v_j),$$

where we have set

$$a' = a'_{\lambda'}\, u^0 \otimes d^{\lambda'} \qquad a' = a'_{\lambda}\, u^0 \otimes d^{\lambda} \qquad a = a_{\lambda}\, u^0 \otimes d^{\lambda} \qquad v = v^i\, u^0 \otimes \partial_i,$$

with

[17] For the sake of simplicity, we take $x'^0 = x^0$.

$$a_\lambda\,,\ a'_{\lambda'}\,,\ a'_\lambda \in \mathcal{M}(\boldsymbol{E},\mathbb{A}) \qquad v^i \in \mathcal{F}(\boldsymbol{E}).$$

PROOF. Formula (∗) follows from formula (QC) and Rem. I.5.2.3.

Then, we can write

$$a'_0 = a_0 - \tfrac{1}{2} g_{ij}\, v^i\, v^j$$

$$a'_i = a_i + g_{ij}\, v^j$$

in the space-time chart (x^0, y^i). On the other hand, we have

$$a'_{0'} = a'_0 + \partial_{0'} y^i\, a'_i \qquad a'_{i'} = \partial_{i'} y^j\, a_j.$$

Moreover, by taking the composition of the transition map

$$y^i_0 = \partial_{j'} y^i\, y^{j'}_{0'} + \partial_{0'} y^i$$

with $o' = o + v$, we obtain

$$v^i := y^i_0 \circ v \ = \partial_{0'} y^i. \qquad \square$$

The above results have delicate aspects which deserve an explanation.

REMARK II.1.4.1. In the above expressions of the quantum connection we deal with two different kinds of observers. First, in order to write an explicit expression of ч, we have chosen an observer o. Then, in order to parametrise the connections of the system ξ, we deal with all observers (including o itself). These observers are spanned by the coordinates y^i_0; in particular, the observer o itself is characterised by $y^i_0 \circ o = 0$.

We stress that a does not depend on the family of observers of the system, but it depends only on the chosen observer o. ❑

REMARK II.1.4.2. In the classical context, we can refer to any local potential a of the observer dependent form Φ. Conversely, in the quantum context, ч is a global and intrinsic object. Hence, formula (QC) determines, for each quantum gauge, the choice of the local potential a related to the observer o. Now on, we shall refer to the above choice of a and we say that it is *quantistically gauged* . ❑

REMARK II.1.4.3. We stress that locally a quantum connection always exists; on the other hand, our assumption of a global existence of a quantum connection may imply global conditions on the quantum bundle.

We do not discuss here these conditions and just assume, as a postulate, that the compatibility of our assumptions is fulfilled. The special relativistic space-time (see § I.6.3) and the two-body space-time provide important examples. ❑

The requirement of universality of the quantum connection is very important, because it allows us to skip the well known problem of the choice of polarisations.

II.1.5. Quantum covariant differentials

Next, we study the covariant differentials of the quantum sections, with respect to the quantum connection.

Let us consider a section $\Psi \in \mathcal{S}(\boldsymbol{Q} \to \boldsymbol{E})$ and its pullback $\Psi^{\uparrow} \in \mathcal{S}(\boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E})$.

PROPOSITION II.1.5.1. The quantum covariant differential of $\Psi^{\uparrow}$ is a 1-form

$$\nabla_{\text{ч}} \Psi^{\uparrow} : J_1\boldsymbol{E} \to T^* J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} \boldsymbol{Q}^{\uparrow} .$$

However, as the section $\Psi^{\uparrow} : J_1\boldsymbol{E} \to \boldsymbol{Q}^{\uparrow}$ is the pull-back of a section $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ and the connection ч is universal, we can write

$$\nabla_{\text{ч}} \Psi^{\uparrow} : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q} .$$

Moreover, for each observer $o : \boldsymbol{E} \to J_1\boldsymbol{E}$, we have (see § II.1.3)

$$\nabla_{\text{ч}} \Psi^{\uparrow} \circ o = \nabla_{\xi(o)} \Psi : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q} .$$ ❑

Then, we introduce the following notions.

DEFINITION II.1.5.1. We define the *covariant differential* of Ψ as the fibred morphism over $\boldsymbol{E}$

$$\nabla\Psi := \nabla_{\text{ч}}\Psi^{\uparrow} : J_1\boldsymbol{E} \to T^*\boldsymbol{E}\underset{\boldsymbol{E}}{\otimes}\boldsymbol{Q},$$

and the *time-like* and *space-like covariant differentials* of Ψ as the fibred morphisms over $\boldsymbol{E}$

$$\mathring{\nabla}\Psi := \text{д}\lrcorner\nabla\Psi : J_1\boldsymbol{E} \to \mathbb{T}^*\otimes\boldsymbol{Q} \qquad \check{\nabla}\Psi : J_1\boldsymbol{E} \to V^*\boldsymbol{E}\underset{\boldsymbol{E}}{\otimes}\boldsymbol{Q}. \quad \clubsuit$$

REMARK II.1.5.1. We have the coordinate expressions

$$\nabla\Psi = \nabla_\lambda\psi\, d^\lambda\otimes b = \big((\partial_0\psi + i\,H/\hbar\,\psi)\,d^0 + (\partial_j\psi - i\,p_j/\hbar\,\psi)\,d^j\big)\otimes b$$

$$\mathring{\nabla}\Psi = (\text{д}_0.\psi - i\,(L/\hbar)\,\psi)\,u^0\otimes b \qquad \check{\nabla}\Psi = \big(\partial_i\psi - i\,(p_i/\hbar)\,\psi\big)\,\check{d}^i\otimes b. \quad \square$$

DEFINITION II.1.5.2. If o is an observer, then we define the *observed covariant differential* of Ψ as the section

$$\nabla^o\Psi := \nabla\Psi\circ o : \boldsymbol{E} \to T^*\boldsymbol{E}\otimes\boldsymbol{Q},$$

the *observed time-like* and *observed space-like differentials* as the sections

$$\mathring{\nabla}^o\Psi := \mathring{\nabla}\Psi\circ o = o\lrcorner\nabla^o\Psi : \boldsymbol{E} \to \mathbb{T}^*\otimes\boldsymbol{Q} \qquad \check{\nabla}^o\Psi := \check{\nabla}\Psi\circ o : \boldsymbol{E} \to V^*\boldsymbol{E}\underset{\boldsymbol{E}}{\otimes}\boldsymbol{Q}. \quad \clubsuit$$

REMARK II.1.5.2. In any space-time chart adapted to the observer o, we have the coordinate expressions

$$\nabla^o\Psi = \nabla^o{}_\lambda\psi\, d^\lambda\otimes b = (\partial_\lambda\psi - i\,\tfrac{m}{\hbar}\,a_\lambda\,\psi)\,d^\lambda\otimes b$$

$$\nabla^o\Psi = \nabla^o{}_0\psi\, u^0\otimes b \qquad \check{\nabla}\Psi = \nabla^o{}_i\psi\,\check{d}^i\otimes b. \quad \square$$

With reference to a quantum chart (z), we shall use the standard notation for the local complex conjugation of the covariant differential and write

$$\overline{\nabla^o{}_i\psi} = \bar{\nabla}^o{}_i\bar{\psi} \qquad \bar{\nabla}^o{}_i := \partial_i + i\,\frac{m}{\hbar}\,a_i.$$

PROPOSITION II.1.5.2. The classical connection $\check{K}$ (see Rem. I.2.1.3) and the quantum connection ч yield the covariant differentials of $\check{\nabla}\Psi$ and $\check{\nabla}^o\Psi$

$$\check{\nabla}\check{\nabla}\Psi : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q} \qquad \check{\nabla}^o\check{\nabla}^o\Psi : \boldsymbol{E} \to V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}\,.$$

Hence, we obtain the *Laplacian* and the *observed Laplacian*

$$\check{\Delta}\Psi := \langle \bar{g}, \check{\nabla}\check{\nabla}\Psi \rangle : J_1\boldsymbol{E} \to \mathbb{A}^*\otimes\boldsymbol{Q} \qquad \check{\Delta}^o\Psi := \langle \bar{g}, \check{\nabla}^o\check{\nabla}^o\Psi \rangle : \boldsymbol{E} \to \mathbb{A}^*\otimes\boldsymbol{Q}\,,$$

with coordinate expressions

$$\check{\Delta}\Psi = g^{ij}\left(\nabla_i\nabla_j\psi + K_i{}^h{}_j\,\nabla_h\psi\right) b \qquad \check{\Delta}^o\Psi = g^{ij}\left(\nabla^o{}_i\nabla^o{}_j\psi + K_i{}^h{}_j\,\nabla^o{}_h\psi\right) b\,,$$

where

$$g^{ij}\,\nabla^o{}_i\nabla^o{}_j\psi = g^{ij}\,(\partial_{ij}\psi - 2\,i\,\frac{m}{\hbar}\,a_i\,\partial_j\psi - (i\,\frac{m}{\hbar}\,\partial_i a_j + \frac{m^2}{\hbar^2}\,a_i\,a_j)\,\psi). \qquad \square$$

Analogously, we can define the covariant differential of the space-time half-density Ψ^{υ} and the vertical covariant differential of the space-like half-density Ψ^{η}. Moreover, we obtain the vertical covariant Laplacians.

In particular, we shall be involved (see Cor. II.2.3.1) with the equality

$$(\check{\Delta}^o\Psi)^{\eta} = \check{\Delta}^o(\Psi^{\eta}).$$

Therefore, we shall write locally, without ambiguity,

$$(\check{\Delta}^o\psi)^{\eta} = \check{\Delta}^o\psi^{\eta} = \check{\Delta}^o(\psi^{\eta}).$$

II.1.6. The principle of projectability

We conclude this chapter with a criterion, which will be our heuristic guideline for the following developments.

Our only essential assumption for quantum mechanics is the quantum connection. This will be the source of all other quantum structures, including the quantum Lagrangian, total equation and operators.

But, the quantum connection lives on the bundle $\boldsymbol{Q}^{\uparrow}$ over $J_1\boldsymbol{E}$, while we require that the physically significant objects live on the bundle $\boldsymbol{Q}$ over $\boldsymbol{E}$. In fact, in a sense, $\boldsymbol{Q}^{\uparrow}$ involves all observers, while our quantum theory must be explicitly independent of any observer.

We shall solve this problem by means of a *projectability criterion*. Namely, each time we are looking for a physical object on $\boldsymbol{Q}$, we shall meet two canonical analogous objects on $\boldsymbol{Q}^{\uparrow}$ and we shall prove that there is a unique (up to a scalar factor) combination of them, which projects on $\boldsymbol{Q}$. Then, we shall assume such a combination as the searched physical object.

This procedure works pretty well in all cases and yields an effective heuristic method. Thus, it can be regarded as a new way for implementing the principle of general relativity.

II.2 - The generalised Schrödinger equation

This chapter is devoted to the study of the quantum total equation and related objects.

The main observer independent objects will be achieved by means of the projectability criterion. We shall follow two independent ways: the Lagrangian approach and a geometrical approach based on the quantum covariant differential.

II.2.1. The quantum Lagrangian

First, we look for the quantum Lagrangian by means of the projectability criterion.

The time-like and space-like differentials of a quantum section yield naturally two real valued functions.

Lemma II.2.1.1. If $\Psi \in \mathscr{S}(\boldsymbol{Q})$, then we obtain the following natural fibred morphisms over $\boldsymbol{E}$

$$\overset{0}{\mathscr{L}}_{\Psi} := \frac{1}{2}\big(h(\Psi,\, i\,\overset{\circ}{\nabla}\Psi) + h(i\,\overset{\circ}{\nabla}\Psi,\, \Psi)\big)\,\upsilon : J_1\boldsymbol{E} \to \mathbb{A}^{3/2}\otimes \overset{4}{\wedge} T^*\boldsymbol{E}$$

$$\check{\mathscr{L}}_{\Psi} := \frac{1}{2}\frac{\hbar}{m}\big((\bar{g}\otimes h)(\check{\nabla}\Psi,\, \check{\nabla}\Psi)\big)\,\upsilon : J_1\boldsymbol{E} \to \mathbb{A}^{3/2}\otimes \overset{4}{\wedge} T^*\boldsymbol{E},$$

with coordinate expressions

$$\overset{0}{\mathscr{L}}_{\Psi} = \frac{1}{2}\big(i\,(\bar{\psi}\;\partial_0.\psi - \partial_0.\bar{\psi}\;\psi) + 2\,(L/\hbar)\;\bar{\psi}\;\psi\big)\,u^0\otimes\upsilon$$

$$\check{\mathscr{L}}_{\Psi} = \frac{\hbar}{2m}\,g^{ij}\,\big(\partial_i\bar{\psi}\;\partial_j\psi - i\,(p_i/\hbar)\,(\partial_j\bar{\psi}\;\psi - \bar{\psi}\;\partial_j\psi) + (p_i/\hbar)\,(p_j/\hbar)\;\bar{\psi}\;\psi\big)\,u^0\otimes\upsilon. \qquad \square$$

Theorem II.2.1.1. If $\Psi \in \mathscr{S}(\boldsymbol{Q})$, then, there is a unique combination $\mathscr{L}_{\Psi}$ (up to a scalar factor) of $\overset{0}{\mathscr{L}}_{\Psi}$ and $\check{\mathscr{L}}_{\Psi}$, which projects on $\boldsymbol{E}$; namely

$$\mathcal{L}_{\Psi} = \overset{0}{\mathcal{L}}_{\Psi} - \check{\mathcal{L}}_{\Psi} : \boldsymbol{E} \to \mathbb{A}^{3/2} \otimes \overset{4}{\wedge} T^{*}\boldsymbol{E},$$

with coordinate expression

$$\mathcal{L}_{\Psi} = \frac{1}{2}\Big(-\frac{h}{m}\, g^{ij}\, \partial_i\bar{\psi}\, \partial_j\psi - i\,(\partial_0\bar{\psi}\,\psi - \bar{\psi}\,\partial_0\psi) + i\, a^i\,(\partial_i\bar{\psi}\,\psi - \bar{\psi}\,\partial_i\psi) +$$

$$+ \frac{m}{\hbar}\,(2\,a_0 - a_i\,a^i)\,\bar{\psi}\,\psi\Big)\, u^0 \otimes \upsilon.$$

PROOF. This is the unique combination which makes the coordinates y_0^i disappear in its coordinate expression. ❑

Thus, we denote by[18]

$$\mathcal{L} : J_1\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{4}{\wedge} T^{*}\boldsymbol{E},$$

the fibred morphism over $\boldsymbol{E}$, which is characterised by

$$\mathcal{L}_{\Psi} = \mathcal{L} \circ j_1\Psi,$$

for each section $\Psi \in \mathcal{S}(\boldsymbol{Q})$. Its coordinate expression is

$$\mathcal{L} = \frac{1}{2}\Big(-\frac{\hbar}{m}\, g^{ij}\,\bar{z}_i\, z_j - i\,(\bar{z}_0\, z - \bar{z}\, z_0) +$$

$$+ i\, a^i\,(\bar{z}_i\, z - \bar{z}\, z_i) + \frac{m}{\hbar}\,(2\,a_0 - a_i\,a^i)\,\bar{z}\, z\Big)\,\sqrt{|g|}\; d^0 \wedge d^1 \wedge d^2 \wedge d^3,$$

i.e.

$$\mathcal{L} = \Big(-\frac{\hbar}{2m}\, g^{ij}\,(w_i{}^1\, w_j{}^1 + w_i{}^2\, w_j{}^2) + (w_0{}^1\, w^2 - w^1\, w_0{}^2) +$$

$$- a^i\,(w_i{}^1\, w^2 - w^1\, w_i{}^2) + \frac{m}{\hbar}\,(a_0 - \frac{1}{2}\,a_i\,a^i)\,(w^1\, w^1 + w^2\, w^2)\Big)\,\sqrt{|g|}\; d^0 \wedge d^1 \wedge d^2 \wedge d^3.$$

Moreover, we set

$$\mathcal{L} = \ell\; d^0 \wedge d^1 \wedge d^2 \wedge d^3 \qquad\qquad \ell \in \mathcal{M}(J_1\boldsymbol{Q}, \mathbb{A}^{3/2}).$$

---

[18] Here and later, we denote the jet space of order r of the fibred manifold $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ by $J_r\boldsymbol{Q}$.

We stress that $\mathscr{L}$ is a real map.

ASSUMPTION Q4. We assume $\mathscr{L}$ to be the *quantum Lagrangian* responsible of the quantum total equation. ◇

Before analysing the consequences of this assumptions, we conclude this section with a digression.

We prefer to develop the Lagrangian formalism directly on the quantum bundle. However, the reader may wish to develop an equivalent theory on the quantum principal bundle. Here, we just give a hint to find a distinguished Lagrangian on the quantum principal bundle.

Let $\boldsymbol{P} \to \boldsymbol{E}$ be the *quantum principal bundle*, i.e. the principal bundle with structure group $U(1)$, whose associated bundle is the quantum bundle $\boldsymbol{Q} \to \boldsymbol{E}$. Let $\vartheta : \boldsymbol{P} \to \mathbb{R}$ be the normal chart associated with the quantum gauge b.

LEMMA II.2.1.2. The form associated with the quantum connection on the quantum principal bundle is the fibred morphism

$$\nu_{\text{ч}} : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{P} \to T^*\boldsymbol{P}$$

with coordinate expression

$$\nu_{\text{ч}} = d\vartheta - i\,\frac{m}{h}\,\left(-\tfrac{1}{2}\,g_{ij}\,y^i_0\,y^j_0\,d^0 + g_{ij}\,y^i_0\,d^j + a_\lambda\,d^\lambda\right).$$

Then, we obtain the fibred morphism over $\boldsymbol{E}$

$$G_{\text{ч}} : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{P} \to \mathbb{T} \otimes (T^*\boldsymbol{P} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{P})$$

defined as

$$G_{\text{ч}} := \nu_{\text{ч}} \otimes dt + dt \otimes \nu_{\text{ч}},$$

with coordinate expression

$$G_{\text{ч}} = u_0 \otimes \Big(d\vartheta \otimes d^0 + d^0 \otimes d\vartheta +$$

$$- i\,\frac{m}{h}\,\left(-g_{ij}\,y^i_0\,y^j_0\,d^0 \otimes d^0 + g_{ij}\,y^i_0\,(d^j \otimes d^0 + d^0 \otimes d^j) + a_\lambda\,(d^\lambda \otimes d^0 + d^0 \otimes d^\lambda)\right)\Big). \qquad \square$$

LEMMA II.2.1.3. The pullback of the vertical metric is the fibred morphisms over $\boldsymbol{E}$

$$\vartheta^* g : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E})$$

with coordinate expression

$$\vartheta^* g = m\ g_{ij}\ (d^i - y^i_0\ d^0) \otimes (d^j - y^j_0\ d^0).$$

Then, we obtain the fibred morphism over $\boldsymbol{E}$

$$G^{\uparrow} : J_1\boldsymbol{E} \to \mathbb{T} \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E})$$

defined as

$$G^{\uparrow} := \frac{m}{\hbar}\ \vartheta^* g,$$

with coordinate expression

$$G^{\uparrow} = \frac{m}{\hbar}\ g_{ij}\ u_0 \otimes (d^i - y^i_0\ d^0) \otimes (d^j - y^j_0\ d^0). \qquad \square$$

The two above objects live on $J_1\mathbf{E}$, i.e. they are observer dependent. Next, we show that there is a unique minimal coupling of them, which is projectable on $\boldsymbol{P}$, i.e. observer independent.

THEOREM II.2.1.2. There is a unique combination of $G_{\text{ч}}$ and $G^{\dagger}$, which projects on $\boldsymbol{P}$; namely

$$G := G_{\text{ч}} - i\ G^{\uparrow}\ : \boldsymbol{P} \to \mathbb{T} \otimes (T^*\boldsymbol{P} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{P}),$$

with coordinate expression

$$G = u_0 \otimes \Big(d\vartheta \otimes d^0 + d^0 \otimes d\vartheta - i\ \frac{m}{h}\ \big(g_{ij}\ d^i \otimes d^j + 2\ a_0\ d^0 \otimes d^0 + a_i\ (d^i \otimes d^0 + d^0 \otimes d^i)\big)\Big).$$

Thus, G is a non-degenerate $\mathbb{T}$-valued metric on the quantum principal bundle. $\square$

The above metric G yields naturally a Lagrangian on the quantum principal bundle, which is equivalent to the above Lagrangian $\mathscr{L}$.

II.2.2. The quantum momentum

Next, we study the (observer independent) four dimensional quantum momentum. We follow two independent ways: we deduce it from the Lagrangian formalism and from another geometrical construction based on the differential of the quantum section.

DEFINITION II.2.2.1. The *quantum momentum* is defined to be the vertical derivative of $\mathscr{L}$

$$V_{\boldsymbol{Q}}\mathscr{L} : J_1\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \overset{4}{\wedge} T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}^*. \qquad ♣$$

PROPOSITION II.2.2.1. The quantum momentum can be naturally regarded as a fibred morphism over $\boldsymbol{E}$

$$\mathfrak{p} : J_1\boldsymbol{Q} \to \mathbb{T}^* \otimes T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}.$$

Then, for each $\Psi \in \mathscr{S}(\boldsymbol{Q})$, we obtain the observer independent section

$$\mathfrak{p}_\Psi := \mathfrak{p} \circ j_1\Psi : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q},$$

with coordinate expression

$$\mathfrak{p}_\Psi = u^0 \otimes \big(\psi\, \partial_0 - i\,\tfrac{\hbar}{m}\, g^{ij}\, (\partial_j\psi - i\,\tfrac{m}{\hbar}\, a_j\, \psi)\, \partial_i\big) \otimes b.$$

PROOF. It follows by using the natural fibred isomorphisms[19]

$$i\ \mathrm{re}(h) : \boldsymbol{Q}^* \to \boldsymbol{Q} \qquad\qquad \langle\ ,\bar{\upsilon}\rangle : \overset{4}{\wedge} T^*\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A}^{*3/2} \otimes T\boldsymbol{E}. \qquad \square$$

We can recover geometrically the quantum momentum in the following way.

LEMMA II.2.2.1. If $\Psi \in \mathscr{S}(\boldsymbol{Q})$, then we have the following natural fibred morphisms over $\boldsymbol{E}$

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[19] The conventional multiplicative factor i in the isomorphism $\boldsymbol{Q}^* \to \boldsymbol{Q}$ has been chosen in such a way to obtain an expression of momentum in agreement with Prop. II.4.2.1.

$$\overset{\circ}{\mathfrak{p}}_\Psi := \text{д}\otimes\Psi : J_1\boldsymbol{E} \to \mathbb{T}^*\otimes T\boldsymbol{E}\underset{\boldsymbol{E}}{\otimes}\boldsymbol{Q} \qquad \check{\mathfrak{p}}_\Psi := \frac{\hbar}{m}\,\check{\nabla}^{\neq}\Psi : J_1\boldsymbol{E} \to \mathbb{T}^*\otimes T\boldsymbol{E}\underset{\boldsymbol{E}}{\otimes}\boldsymbol{Q},$$

with coordinate expressions

$$\overset{\circ}{\mathfrak{p}}_\Psi := \psi\, u^0\otimes(\partial_0 + y_0^i\,\partial_i)\otimes b \qquad \check{\mathfrak{p}}_\Psi := \frac{\hbar}{m}\, g^{ij}\left(\partial_j\psi - i\,(p_j/\hbar)\,\psi\right)\partial_i\otimes b.$$

THEOREM II.2.2.1. If $\Psi\in\mathscr{S}(\boldsymbol{Q})$, then $\mathfrak{p}_\Psi$ is the unique combination (up to a scalar factor) of $\overset{\circ}{\mathfrak{p}}_\Psi$ and $\check{\mathfrak{p}}_\Psi$, which projects over $\boldsymbol{E}$; namely,

$$\mathfrak{p}_\Psi = \overset{\circ}{\mathfrak{p}}_\Psi - i\,\check{\mathfrak{p}}_\Psi : \boldsymbol{E} \to \mathbb{T}^*\otimes T\boldsymbol{E}\underset{\boldsymbol{E}}{\otimes}\boldsymbol{Q}.$$

PROOF. This is the unique combination which makes the coordinates y_0^i disappear in its coordinate expression. ❑

We stress that the quantum momentum is an observer independent four dimensional object.

The time component of the quantum momentum is defined without reference to any observer; actually, it coincides with the quantum section itself. On the other hand, the choice of an observer allows us to define the spacelike component of the quantum momentum.

COROLLARY II.2.2.1. For each $\Psi\in\mathscr{S}(\boldsymbol{Q})$, we obtain the observer independent section

$$\mathfrak{v}_\Psi := \operatorname{re} h(\Psi,\mathfrak{p}_\Psi) : \boldsymbol{E} \to \mathbb{T}^*\otimes T\boldsymbol{E}.$$

Moreover, in the domain where Ψ does not vanish, the section

$$o_\Psi := \mathfrak{v}_\Psi / h(\Psi,\Psi) : \boldsymbol{E} \to \mathbb{T}^*\otimes T\boldsymbol{E}$$

projects over $1_{\boldsymbol{T}}$, hence it is an observer, with coordinate expression

$$o_\Psi = u^0\otimes\Big(\partial_0 - \operatorname{re} i\,\frac{\hbar}{m}\, g^{ij}\,\frac{\bar\psi\,(\partial_j\psi - i\,\frac{m}{\hbar}\,a_j\,\psi)}{h(\Psi,\Psi)}\,\partial_i\Big).$$ ❑

Thus, we have obtained a distinguished observer associated with Ψ. It is possible to interpret the physical meaning of this observer in agreement with the uncertainty principle; in fact, we can assume that o_Ψ is determined experimentally only by means of a statistical procedure which involves many particles (see [Me]).

II.2.3. The generalised Schrödinger equation

Next, we study the (observer independent) quantum total equation. We follow two independent ways: we deduce it from the Lagrangian formalism and from another geometrical construction based on the differentials of the quantum section and momentum.

For this purpose, we need an intrinsic construction of the Euler-Lagrange operator. We shall follow the procedure of [Co].

REMARK II.2.3.1. The quantum momentum can be regarded as a fibred morphism over $\boldsymbol{E}$

$$\mathcal{P} : J_1\boldsymbol{Q} \to \overset{4}{\wedge} T^*\boldsymbol{Q}.$$

PROOF. It follows by applying to $V_{\boldsymbol{Q}}\mathcal{L}$ the natural linear fibred morphisms

$$\langle\, ,\, \rangle : T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \overset{4}{\wedge} T^*\boldsymbol{E} \to \wedge^3 T^*\boldsymbol{E} \qquad \vartheta_{\boldsymbol{Q}}{}^* : J_1\boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}^* \to T^*\boldsymbol{Q}$$

over $\boldsymbol{E}$ and $J_1\boldsymbol{E} \to \boldsymbol{Q}$, respectively. ❑

REMARK II.2.3.2. We obtain the fibred morphism over $\boldsymbol{E}$

$$\mathcal{E} := d\mathcal{L} + d_h\mathcal{P} : J_2\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{5}{\wedge} T^*\boldsymbol{Q},$$

where d is the exterior differential and d_h the contact horizontal exterior differential (see [Co]). ❑

REMARK II.2.3.3. By considering the linear epimorphism $T^*\boldsymbol{Q} \to V^*\boldsymbol{Q}$ over $\boldsymbol{Q}$, $\mathcal{E}$ can be characterised by a fibred morphism over $\boldsymbol{E}$

$$\check{\mathcal{E}} : J_2\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{4}{\wedge} T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}^*.$$

Moreover, by taking into account the real component of the Hermitian metric and the space-time volume form υ, $\check{\mathcal{E}}$ is characterised by the fibred morphism over $\boldsymbol{E}$

$$*\check{\mathcal{E}}^{\neq} : J_2\boldsymbol{Q} \to \mathbb{T}^*\otimes\boldsymbol{Q},$$

with coordinate expression

$$(EL) \qquad *\check{\mathcal{E}}^{\neq} = \frac{1}{\sqrt{|g|}}\left(\partial_a\ell - (\partial_\lambda + w_\lambda{}^b\,\partial_b + w_{\lambda\mu}{}^b\,\partial_b^\mu).\partial_a^\lambda\ell\right) u^0\otimes b_a. \qquad \square$$

Eventually, we can prove (see [Co]) that $*\check{\mathcal{E}}^{\neq}$ is nothing but the intrinsic expression of the standard Euler-Lagrange operator associated with the Lagrangian $\mathscr{L}$.

So, we obtain the following important result.

THEOREM II.2.3.1. *(Generalised Schrödinger equation)* The coordinate expression of the Euler-Lagrange equation, in the unknown $\Psi\in\mathscr{S}(\boldsymbol{Q})$, is

$$0 = *\check{\mathcal{E}}^{\neq}\circ j_2\Psi = 2\Big(i\,\partial_0\psi + \frac{m}{\hbar}\,a_0\,\psi + \frac{1}{2}\,i\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi +$$

$$+ \frac{\hbar}{2m}\,(g^{ij}\,(\partial_{ij}\psi - 2\,i\,\frac{m}{\hbar}\,a_i\,\partial_j\psi - (i\,\frac{m}{\hbar}\,\partial_i a_j + \frac{m^2}{\hbar^2}\,a_i\,a_j)\,\psi) +$$

$$+ \frac{\partial_i(g^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_j\psi - i\,\frac{m}{\hbar}\,a_j\,\psi))\Big)\,u^0\otimes b.$$

PROOF. Formula (EL) yields

$$*\check{\mathcal{E}}^{\neq} = 2\Big(-\,w_0{}^2 - \frac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,w^2 + \frac{\hbar}{2m}\,g^{ij}\,w_{ij}{}^1 + a^i\,w_i{}^2 + \frac{\hbar}{2m}\,\frac{\partial_i(g^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,w_j{}^1 +$$

$$+ \frac{m}{\hbar}\,(a_0 - \frac{1}{2}\,a_i\,a^i)\,w^1 + \frac{1}{2}\,\frac{\partial_i(a^i\sqrt{|g|})}{\sqrt{|g|}}\,w^2\Big)\,u^0\otimes b_1 +$$

$$+ 2\Big(w_0{}^1 + \frac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,w^1 + \frac{\hbar}{2m}\,g^{ij}\,w_{ij}{}^2 - a^i\,w_i{}^1 + \frac{\hbar}{2m}\,\frac{\partial_i(g^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,w_j{}^2 +$$

$$+ \frac{m}{\hbar} (a_0 - \tfrac{1}{2} a_i \, a^i) \; w^2 - \tfrac{1}{2} \frac{\partial_i (a^i \sqrt{|g|})}{\sqrt{|g|}} \; w^1 \Big) \, u^0 \otimes b_2 . \qquad \square$$

Thus, the above formula expresses the total equation for a spinless quantum particle with a given mass and charge, in a curved space-time, with absolute time, under the action of a given gravitational and electromagnetic field.

In order to interpret the above equation correctly, we must take into account that here a includes both the gravitational and electromagnetic potentials (see Theor. I.4.3.2).

In the special relativistic Galilei case, the above equation reduces exactly to the standard Schrödinger equation referred to a given quantum gauge.

We can write the Schrödinger equation in a more compact way.

For this purpose, we introduce the following natural derivation.

REMARK II.2.3.4. Let $o: \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ be an observer.

Then, by functorial prolongation (see § III.3 and [MM2]), we obtain the section

$$o' := r_1 \circ J_1 o : J_1 \boldsymbol{E} \to \mathbb{T}^* \otimes T J_1 \boldsymbol{E},$$

with coordinate expression in adapted coordinates

$$o' = u^0 \otimes \partial_0 .$$

Next, the quantum connection yields the vector field

$$\xi(o') : \boldsymbol{Q} \to \mathbb{T}^* \otimes T\boldsymbol{Q},$$

with coordinate expression in adapted coordinates

$$\xi(o') = u^0 \otimes (\partial_0 + i \, \frac{m}{\hbar} \, a_0 \, \text{и}).$$

Hence, for each $\Psi \in \mathscr{S}(\boldsymbol{Q})$, we set (see Rem. I.1.2.1)

$$D^o \Psi := \langle \sqrt{\bar{\upsilon}}, \, L_{\xi(o')} (\Psi \otimes \sqrt{\upsilon}) \rangle : \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q}$$

and we obtain the coordinate expression

$$D^o\Psi = (\partial_0\psi - \frac{m}{\hbar} i\, a_0\, \psi + \frac{1}{2}\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi)\, u^0 \otimes b. \qquad \square$$

We could achieve $D^o\Psi$, in a simpler but less elegant way, by means of the observed covariant differential of Ψ and the Lie derivative of $\sqrt{\upsilon}$.

COROLLARY II.2.3.1. If o is an observer, then the Euler-Lagrange equation, in the unknown $\Psi \in \mathcal{S}(\boldsymbol{Q})$, can be written as (see Prop. II.1.5.2)

$$2\,(i\, D^o\Psi + \frac{\hbar}{2m}\check{\Delta}^o\Psi) = 0.$$

PROOF. We have

$$\check{\Delta}^o\Psi = \Big(g^{ij}\,(\partial_{ij}\psi - 2\, i\, \frac{m}{\hbar} a_i\, \partial_j\psi - (i\,\frac{m}{\hbar}\partial_i a_j + \frac{m^2}{\hbar^2} a_i\, a_j)\,\psi) + K_i{}^h{}_j\,(\partial_h\psi - i\,\frac{m}{\hbar} a_h\psi)\Big)\, b$$

and (see formula $***$) in Cor. I.2.3.2)

$$g^{ij}\, K_i{}^h{}_j = \frac{\partial_j(g^{hj}\sqrt{|g|})}{\sqrt{|g|}}. \qquad \square$$

We stress that both terms of the left hand side of the above equation depend essentially on the observer o; however, their sum turns out to be observer independent.

Warning: we might be tempted to write the coordinate expression of the Schrödinger equation as

$$2\,(i\,\nabla^o_0\psi^\eta + \frac{\hbar}{2m} g^{ij}\,\nabla_i\nabla_j\psi^\eta) = 0;$$

but, unfortunately, we cannot find a serious and consistent interpretation of the symbols yielding the above formula.

We can recover geometrically the generalised Schrödinger equation in the following way.

LEMMA II.2.3.1. If $\Psi \in \mathcal{S}(\boldsymbol{Q})$, then we have the following natural fibred morphisms over $\boldsymbol{E}$

$$\overset{o}{\nabla}\Psi := д \lrcorner \nabla\Psi : J_1\boldsymbol{E} \to \mathbb{T}^*\otimes\boldsymbol{Q} \qquad \delta\,\mathfrak{p}_\Psi := \langle\bar{\upsilon}, d\langle\upsilon, \mathfrak{p}_\Psi\rangle\rangle : J_1\boldsymbol{E} \to \mathbb{T}^*\otimes\boldsymbol{Q},$$

where d and δ are the covariant differential and codifferentials induced by ч.❑

THEOREM II.2.3.2. If $\Psi \in \mathscr{S}(\boldsymbol{Q})$, then $*\check{\mathscr{E}}^{\neq}{}_\Psi$ is the unique combination (up to a scalar factor) of $\overset{o}{\nabla}\Psi$ and $\delta\mathfrak{p}_\Psi$, which projects over $\boldsymbol{E}$; namely,

$$*\check{\mathscr{E}}^{\neq}{}_\Psi = \overset{o}{\nabla}\Psi + \delta\,\mathfrak{p}_\Psi : \boldsymbol{E} \to \mathbb{T}^*\otimes\boldsymbol{Q}.$$

PROOF. This is the unique combination which makes the coordinates y^i_0 disappear in its coordinate expression. ❑

In view of later developments, it is convenient to rescale the Euler-Lagrange fibred morphism in such a way that it can be interpreted as a connection in the infinite dimensional setting of the quantum theory (see, later, § II.6.2).

DEFINITION II.2.3.1. The *Schrödinger operator* is defined to be the sheaf morphism

$$\mathfrak{S}^\eta : \mathscr{S}(\boldsymbol{Q}^\eta \to \boldsymbol{E}) \to \mathscr{S}(\mathbb{T}^*\otimes\boldsymbol{Q}^\eta \to \boldsymbol{E}) : \Psi^\eta \mapsto -\,i\,\tfrac{1}{2}\,(*\check{\mathscr{E}}^{\neq}{}_\Psi)\otimes\sqrt{\eta}. \qquad ♣$$

We obtain the coordinate expression

$$\mathfrak{S}^\eta(\Psi^\eta) = (\partial_0\psi^\eta - i\,\frac{\hbar}{2m}\,\check{\Delta}^o\psi^\eta - i\,\frac{m}{\hbar}\,a_0\,\psi^\eta)\,u^0\otimes b\otimes\sqrt{\check{d}^1\wedge\check{d}^2\wedge\check{d}^3},$$

where (see § II.1.5)

$$\partial_0\psi^\eta := \partial_0(\psi^\eta) \qquad (\check{\Delta}^o\psi)^\eta = \check{\Delta}^o\psi^\eta = \check{\Delta}^o(\psi^\eta).$$

II.2.4. The quantum probability current

We conclude this chapter by studying the quantum probability current. As usual, its conservation is an essential requirement of the probabilistic interpretation of quantum mechanics.

First we introduce the Poincaré-Cartan form associated with the quantum

Lagrangian (see [GS], [Ga], [Co]).

REMARK II.2.4.1. According to the general Lagrangian theory, the *Poincaré-Cartan* form associated with $\mathscr{L}$ is the form

$$\Theta : J_1\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{4}{\wedge} T^*\boldsymbol{Q}$$

defined as

$$\Theta := \mathscr{L} + \vartheta \barwedge V_{\boldsymbol{Q}}\mathscr{L},$$

with coordinate expression

$$\Theta = \mathscr{L} + \partial^{\lambda}_{a}\ell \; \vartheta_{\boldsymbol{Q}}{}^{a} \wedge \omega_{\lambda},$$

where

$$\vartheta_{\boldsymbol{Q}}{}^{a} := (dw^a - w^a_{\lambda}\, d^{\lambda}) \qquad \omega_{\lambda} := \partial_{\lambda} \lrcorner (d^0 \wedge d^1 \wedge d^2 \wedge d^3).$$

So, if $\Psi \in \mathscr{S}(\boldsymbol{Q})$, then we obtain the section

$$\Theta_{\Psi} : \boldsymbol{E} \to \mathbb{A}^{3/2} \otimes \overset{4}{\wedge} T^*\boldsymbol{Q}$$

defined as

$$\Theta_{\Psi} := \Theta \circ j_1\Psi.$$

with coordinate expression

$$\Theta_{\Psi} = \sqrt{|g|}\; \tfrac{1}{2}\Big(\big(\tfrac{\hbar}{m}\, g^{ij}\, \partial_i\bar{\psi}\; \partial_j\psi + \tfrac{m}{\hbar}\,(a_0 - \tfrac{1}{2}\, a^2)\; \bar{\psi}\; \psi\big)\; d^0 \wedge d^1 \wedge d^2 \wedge d^3 +$$

$$+ \, i\; (\bar{\psi}\; dz - \psi\; d\bar{z}) \wedge d^1 \wedge d^2 \wedge d^3 +$$

$$- (-1)^i\, \tfrac{\hbar}{m}\, g^{ij}\, \big((\partial_j\bar{\psi} + i\, \tfrac{m}{\hbar}\, a_j\; \bar{\psi})\; dz + (\partial_j\psi - i\, \tfrac{m}{\hbar}\, a_j\; \psi)\; d\bar{z}\big)\Big) \wedge d^0 \wedge d^1 \ldots \wedge \hat{d}^i \ldots \wedge d^3\Big). \qquad \square$$

Then, by considering the invariance of the Lagrangian under the action of the group $U(1)$, the Nöther theorem yields a conserved current (see [GS], [Ga], [Co]), which will be interpreted as *probability current*.

REMARK II.2.4.2. The quantum Lagrangian $\mathscr{L}$ is invariant under the action of the group $U(1)$, hence it is invariant with respect to its infinitesimal generator, namely with respect to the vertical vector field[20]

$$- J_1(i\,\text{и}) : J_1\boldsymbol{Q} \to V J_1 \boldsymbol{Q},$$

with coordinate expression

$$- J_1(i\,\text{и}) = w^2\, \partial w_1 - w^1\, \partial w_2 + w^2_\lambda\, \partial w^\lambda_1 - w^1_\lambda\, \partial w^\lambda_2 .$$

Proof. The complex coordinate expression of $\mathscr{L}$ is invariant with respect to the jet prolongation of a constant change of phase of $\boldsymbol{Q}$. The infinitesimal generator of constant changes of phase is $i\,\text{и}$. So, $\mathscr{L}$ is invariant with respect to the jet prolongation of $i\,\text{и}$. On the other hand, this last fact can be checked directly by means of the real coordinate expression of $\mathscr{L}$. ❑

PROPOSITION II.2.4.1. The Nöther theorem yields the following conserved current

$$j : J_1\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{3}{\wedge} T^*\boldsymbol{E}$$

defined as

$$j := - i\,\text{и} \lrcorner \Theta .$$

If $\Psi \in \mathscr{S}(\boldsymbol{Q})$ is a solution of the Schrödinger equation, then the form

$$j_\Psi := \tfrac{1}{2} \langle \upsilon,\, h(\Psi, \mathfrak{p}_\Psi) - h(\mathfrak{p}_\Psi, \Psi) \rangle : \boldsymbol{E} \to \mathbb{A}^{3/2} \otimes \overset{3}{\wedge} T^*\boldsymbol{E},$$

with coordinate expression

$$j_\Psi = \sqrt{|g|}\, \Big(\bar{\psi}\, \psi\, d^1 \wedge d^2 \wedge d^3 +$$

$$+ (-1)^h \big(- i\, \frac{\hbar}{2m}\, g^{hk}\, (\bar{\psi}\, \partial_k \psi - \partial_k \bar{\psi}\, \psi) - a^h\, \bar{\psi}\, \psi\big)\, d^0 \wedge d^1 \ldots \wedge \hat{d}^h \ldots \wedge d^3 \Big),$$

is closed

$$d j_\Psi = 0 .$$ ❑

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[20] We recall the canonical fibred isomorphism $V J_1 \boldsymbol{Q} \to J_1 V \boldsymbol{Q}$ over $J_1 \boldsymbol{Q} \underset{\boldsymbol{Q}}{\times} V \boldsymbol{Q}$, where V is taken with respect to the base space $\boldsymbol{E}$.

II.3 - Quantum vector fields

Now, we introduce the quantum vector fields, in view of quantum operators (see, later, § II.4.2 and § II.6.3).

II.3.1. The Lie algebra of quantisable functions

Quantum vector fields require some preliminary facts. Accordingly, we present a few further important results of classical mechanics.

Let us consider the sheaves of local functions, of local vector fields and of local forms of $J_1\boldsymbol{E}$, respectively,

$$\mathcal{F}(J_1\boldsymbol{E}) := \{f : J_1\boldsymbol{E} \to \mathbb{R}\}$$

$$\mathcal{T}(J_1\boldsymbol{E}) := \{X : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}\} \qquad \mathcal{T}^*(J_1\boldsymbol{E}) := \{\varphi : J_1\boldsymbol{E} \to T^*J_1\boldsymbol{E}\}.$$

Moreover, let

$$\mathcal{T}_{\boldsymbol{\tau}}(J_1\boldsymbol{E}) \subset \mathcal{T}(J_1\boldsymbol{E}) \qquad \mathcal{T}^*_{\gamma}(J_1\boldsymbol{E}) \subset \mathcal{T}^*(J_1\boldsymbol{E})$$

be the subsheaves of local vector fields whose time component is a given map $\boldsymbol{\tau} : J_1\boldsymbol{E} \to \mathbb{T}$ (which will be referred to as a *time scale*) and of local forms which vanish on γ, respectively.

The coordinate expressions of $X \in \mathcal{T}_{\boldsymbol{\tau}}(J_1\boldsymbol{E})$ and $\varphi \in \mathcal{T}^*_{\gamma}(J_1\boldsymbol{E})$ are of the type

$$X = X^0\,(\partial_0 + y^i_0\,\partial_i + \gamma^i\,\partial^0_i) + X^i\,\partial_i + X^i_0\,\partial^0_i \qquad \varphi = \varphi_i\,\vartheta^i + \varphi^0_i\,(d^i_0 - \gamma^i\,d^0),$$

with

$$X^0 = \tau := \langle u^0, \boldsymbol{\tau}\rangle \in \mathcal{F}(J_1\boldsymbol{E}).$$

LEMMA II.3.1.1. The contact 2-form

$$\frac{m}{\hbar}\,\Omega : J_1\boldsymbol{E} \to \overset{2}{\wedge} T^*J_1\boldsymbol{E}$$

maps each vector field of $J_1\boldsymbol{E}$ into a 1-form orthogonal to γ.

PROOF. It follows immediately from $\gamma \lrcorner \Omega = 0$ (see Cor. I.2.5.1). ❑

PROPOSITION II.3.1.1. For each time scale $\boldsymbol{\tau} : J_1\boldsymbol{E} \to \mathbb{T}$, we obtain the sheaf isomorphism

$$\Omega^\flat_{\boldsymbol{\tau}} : \mathcal{T}_{\boldsymbol{\tau}}(J_1\boldsymbol{E}) \to \mathcal{T}^*_\gamma(J_1\boldsymbol{E}) : X \mapsto \frac{m}{\hbar}\, i_X \Omega$$

with coordinate expression

$$\Omega^\flat_{\boldsymbol{\tau}}(X) = \frac{m}{\hbar}\big((g_{ij}\, X^j_0 + (\Gamma_{ij} - \Gamma_{ji})\, X^j)\, \vartheta^i - g_{ij}\, X^j\, (d^i_0 - \gamma^i\, d^0)\big).$$

The inverse isomorphism $\Omega^\#_{\boldsymbol{\tau}}$ has coordinate expression

$$\Omega^\#_{\boldsymbol{\tau}}(\varphi) = \tau\, (\partial_0 + y^i_0\, \partial_i + \gamma^i\, \partial^0_i) + \frac{\hbar}{m}\, g^{ij}\, \big(-\varphi^0_j\, \partial_i + (\varphi_j + g^{hk}\, (\Gamma_{jh} - \Gamma_{hj})\, \varphi^0_k)\, \partial^0_i\big).$$ ❑

We recall that (see Theor. I.4.3.2)

$$\Gamma_{jh} - \Gamma_{hj} = -\,(\partial_j g_{hl} - \partial_h g_{jl})\, y^l_0 - \Phi_{jh}.$$

Then, we obtain the following *Hamiltonian lift* of functions.

LEMMA II.3.1.2. For each $f \in \mathcal{F}(J_1\boldsymbol{E})$, we obtain naturally the form

$$d_\gamma f := df - \gamma \lrcorner df \in \mathcal{T}^*_\gamma(J_1\boldsymbol{E})$$

with coordinate expression

$$d_\gamma f = \partial_i f\, \vartheta^i + \partial^0_i f\, (d^i_0 - \gamma^i\, d^0).$$ ❑

COROLLARY II.3.1.1. For each time scale $\boldsymbol{\tau} : J_1\boldsymbol{E} \to \mathbb{T}$, we obtain the following sheaf morphism

$$\mathcal{F}(J_1\boldsymbol{E}) \to \mathcal{T}_{\boldsymbol{\tau}}(J_1\boldsymbol{E}) : f \mapsto f^\#_{\boldsymbol{\tau}} := \Omega^\#_{\boldsymbol{\tau}}(d_\gamma f).$$

with coordinate expression

$$f^\#_{\boldsymbol{\tau}} = \tau\, (\partial_0 + y^i_0\, \partial_i + \gamma^i\, \partial^0_i) + \frac{\hbar}{m}\, g^{ij}\, \big(-\partial^0_j f\, \partial_i + (\partial_j f + (\Gamma_j{}^k - \Gamma^k{}_j)\, \partial^0_k f)\, \partial^0_i\big).$$

We have the following properties

$$f^{\#}_{\tau} = \gamma(\tau) + f^{\#}_{0}$$

$$(f' + f'')^{\#}_{0} = f'^{\#}_{0} + f''^{\#}_{0} \qquad (f'\, f'')^{\#}_{0} = f'\, f''^{\#}_{0} + f'^{\#}_{0}\, f''. \qquad \square$$

The Hamiltonian lift yields the generalised *Poisson Lie bracket*.

PROPOSITION II.3.1.1. The sheaf morphism

$$\mathcal{F}(J_1\boldsymbol{E})\times\mathcal{F}(J_1\boldsymbol{E}) \to \mathcal{F}(J_1\boldsymbol{E}) : (f', f'') \mapsto \{f', f''\} := \frac{m}{\hbar}\, i_{f'^{\#}_{0}} i_{f''^{\#}_{0}} \Omega$$

is a Lie bracket.

Its coordinate expression is

$$\{f', f''\} = \frac{\hbar}{m}\, g^{ij}\left(\partial_i f'\, \partial^0_j f'' - \partial^0_i f'\, \partial_j f'' + g^{hk}\,(\Gamma_{jh} - \Gamma_{hj})\, \partial^0_k f'\, \partial^0_i f''\right).$$

PROOF. The proof can be achieved analogously to the standard case, but with more difficulties, because we have to replace the standard d with d_γ. $\square$

This bracket has some interesting properties. In particular, we have

$$\{f', f''\}^{\#}_{0} = [f'^{\#}_{0}, f''^{\#}_{0}].$$

However, this bracket has no relativistically covariant role in classical mechanics.

On the other hand, we can prove the following important result based on a criterion of projectability for classical Hamiltonian lift, which later will play an essential role in quantum mechanics.

Let us consider the subsheaf of tube-like functions with respect to the fibring $J_1E \to \boldsymbol{E}$

$$\mathcal{F}_t(J_1\boldsymbol{E}) \subset \mathcal{F}(J_1\boldsymbol{E}).$$

PROPOSITION II.3.1.2. If $f \in \mathcal{F}_t(J_1E)$, then the following conditions are equivalent:

i) the vector field $f^{\#}_{\boldsymbol{\tau}}$ is projectable over $\boldsymbol{E}$;

ii) the function f is, with respect to the fibres of $J_1\boldsymbol{E}\to\boldsymbol{E}$, a polynomial of degree 2, whose second derivative is of the form

$$\boldsymbol{f}'' \frac{m}{\hbar} g : \boldsymbol{E} \to (\mathbb{T}\otimes\mathbb{T})\otimes V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E},$$

where

$$\boldsymbol{f}'' = \boldsymbol{\tau};$$

iii) the coordinate expression of f is of the following type

$$f = f'' \frac{m}{2\hbar} g_{ij}\, y^i_0\, y^j_0 + f_i\, y^i_0 + f_\circ \qquad\qquad f'', f_\circ, f_i \in \mathcal{F}(\boldsymbol{E}),$$

where

$$f'' = \tau.$$

PROOF. The vector field $f^{\#}_{\boldsymbol{\tau}}$ is projectable if and only if

$$\partial^0_h \tau = 0 \qquad\qquad \tau\, \partial^0_h y^i_0 - \frac{\hbar}{m} g^{ij}\, \partial^{00}_{hj} f = 0$$

i.e., if and only if

$$\partial^0_h \tau = 0 \qquad\qquad \partial^{00}_{ij} f = \tau\, \frac{m}{\hbar} g_{ij}$$

i.e. if and only if

$$\tau : \boldsymbol{E} \to \mathbb{R} \qquad\qquad f = \tau\, \frac{m}{2\hbar} g_{ij}\, y^i_0\, y^j_0 + f_i\, y^i_0 + f_\circ . \qquad \square$$

Then, we introduce the following definition, for a reason which will be clear soon.

DEFINITION II.3.1.1. We define the *quantisable functions* to be the functions of the above type. ♣

If f is a quantisable function, then the associated time scale

$$\boldsymbol{\tau} := \boldsymbol{f}'' : \boldsymbol{E} \to \mathbb{T}$$

is said to be its *time component*.

If f is a quantisable function, then the corresponding projectable Hamiltonian lift and the associated projection are denoted by

$$f^{\#} := f^{\#}_{\tau} \in \mathcal{T}_{\tau}(J_1\boldsymbol{E}) \qquad f^{H} \in \mathcal{T}_{\tau}(\boldsymbol{E}).$$

Thus, given $u^0 \in \mathbb{T}^{+*}$, we have implicitly set

$$f'' := \langle u^0, \boldsymbol{f}'' \rangle.$$

We stress that we have assumed no relation between $f'' := \tau$ and $f_{\circ}$.

Remark II.3.1.2. If f is a quantisable function, then we obtain the coordinate expression

$$f^{\#} = f'' \, \partial_0 - \frac{\hbar}{m} f^i \, \partial_i +$$

$$+ g^{ij} \big(\tfrac{1}{2} \, \partial_j f'' \, g_{hk} \, y^h_0 \, y^k_0 + \big(\tfrac{\hbar}{m} \, \partial_j f_h + \tfrac{\hbar}{m} \, (\partial_k g_{jh} - \partial_j g_{hk}) \, f^k - f'' \, \partial_0 g_{jh} \big) \, y^h_0 +$$

$$+ \frac{\hbar}{m} \, \partial_j f_{\circ} + \frac{\hbar}{m} \, \Phi_{hj} \, f^h + f'' \, \Phi_{j0} \big) \, \partial^0_i,$$

hence

$$f^{H} = f'' \, \partial_0 - \frac{\hbar}{m} f^i \, \partial_i. \qquad \square$$

The quantisable functions constitute a sheaf, which is denoted by

$$\mathcal{Q}(J_1\boldsymbol{E}) \subset \mathcal{F}_t(J_1\boldsymbol{E}).$$

Moreover, we shall be concerned with the subsheaves of quantisable functions whose time component is constant and of quantisable functions whose time component vanishes

$$\mathcal{Q}_c(J_1\boldsymbol{E}) \subset \mathcal{Q}(J_1\boldsymbol{E}) \qquad \mathcal{Q}_0(J_1\boldsymbol{E}) \subset \mathcal{Q}_c(J_1\boldsymbol{E}).$$

Remark II.3.1.1. All functions $f \in \mathcal{F}(\boldsymbol{E})$, which depend only on space-time (for instance, the space-time coordinates), are quantisable functions. Moreover, all functions $f \in \mathcal{A}(J_1\boldsymbol{E})$, which are affine with respect to the fibring $J_1\boldsymbol{E} \to \boldsymbol{E}$ (for instance, the components of the classical momentum), are quantisable functions. The remaining quantisable functions (for instance, the classical kinetic energy, Hamiltonian and Lagrangian) have a quadratic term with respect to the fibring $J_1\boldsymbol{E} \to \boldsymbol{E}$, which is proportional to the metric, through an arbitrary coefficient dependent only on space-time.

Thus, we have the sheaf monomorphisms

$$\mathcal{F}(\boldsymbol{E}) \subset \mathcal{A}(J_1\boldsymbol{E}) \subset \mathcal{Q}(J_1\boldsymbol{E}). \qquad \square$$

THEOREM II.3.1.2. The sheaf $\mathcal{Q}(J_1\boldsymbol{E})$ is a sheaf of Lie algebras with respect to the bracket

$$[f', f''] := \{f', f''\} + \gamma(\boldsymbol{f}^{\prime\,\|}).f'' - \gamma(\boldsymbol{f}^{\prime\prime\,\|}).f'.$$

Moreover, the sheaves $\mathcal{Q}_c(J_1\boldsymbol{E})$ and $\mathcal{Q}_0(J_1\boldsymbol{E})$ are subsheaves of Lie algebras.

The coordinate expression of the above Lie bracket is given by

$$\begin{aligned}
[f', f''] = {} & \Big(f^{\prime\,\|}\, \partial_0 f^{\prime\prime\,\|} - f^{\prime\prime\,\|}\, \partial_0 f^{\prime\,\|} - \tfrac{\hbar}{m}\,(f^{\prime i}\, \partial_i f^{\prime\prime\,\|} - f^{\prime\prime i}\, \partial_i f^{\prime\,\|})\Big)\, \tfrac{m}{2\hbar}\, g_{hk}\, y^h_0\, y^k_0 + \\
& + g_{hk}\, \Big(f^{\prime\,\|}\, \partial_0 f^{\prime\prime k} - f^{\prime\prime\,\|}\, \partial_0 f^{\prime k} - \tfrac{\hbar}{m}\,(f^{\prime i}\, \partial_i f^{\prime\prime k} - f^{\prime\prime i}\, \partial_i f^{\prime k})\Big)\, y^h_0 + \\
& + f^{\prime\,\|}\, \partial_0 f''_{\circ} - f^{\prime\prime\,\|}\, \partial_0 f'_{\circ} - \tfrac{\hbar}{m}\,(f^{\prime h}\, \partial_h f''_{\circ} - f^{\prime\prime h}\, \partial_h f'_{\circ}) + \\
& + (f^{\prime\,\|}\, f^{\prime\prime h} - f^{\prime\prime\,\|}\, f^{\prime h})\, \Phi_{h0} + \tfrac{\hbar}{m}\, f^{\prime h}\, f^{\prime\prime k}\, \Phi_{hk}.
\end{aligned}$$

PROOF. The explicit expression of the bracket follows from a long computation in coordinates. Then, this expression shows that $\mathcal{Q}(J_1\boldsymbol{E})$ is closed under the above bracket.

The Jacobi property could be checked by a very long direct computation in coordinates. However, in the next section we can obtain an intrinsic proof as a corollary of an important result of the quantum theory (Cor. II.3.2.2). $\square$

COROLLARY II.3.1.2. The subsheaf $\mathcal{A}(J_1\boldsymbol{E}) \subset \mathcal{Q}(J_1\boldsymbol{E})$ is a subsheaf of Lie algebras. $\square$

COROLLARY II.3.1.3. The sheaf morphism

$$H : \mathcal{Q}(J_1\boldsymbol{E}) \to \mathcal{C}(\boldsymbol{E}) : f \mapsto f^H$$

is a morphsim of Lie algebras, i.e.

$$[f', f'']^H = [f'^H, f''^H]. \qquad \square$$

EXAMPLE II.3.1.1. We have the following Hamiltonian lifts

$$(x^0)^H = 0 \qquad (y^i)^H = 0$$

$$(p_i/\hbar)^H = -\partial_i \qquad (H/\hbar)^H = \partial_0 \qquad (L/\hbar)^H = \partial_0 - a^i\,\partial_i. \qquad \square$$

EXAMPLE II.3.1.2. We have the following brackets

$$[x^0, f] = -f'' \qquad [y^i, f] = \frac{\hbar}{m} f^i$$

$$[p_i/\hbar, p_j/\hbar] = 0 \qquad [H/\hbar, p_i/\hbar] = \frac{m}{\hbar}\Phi_{0i}. \qquad \square$$

II.3.2. The Lie algebra of quantum vector fields

In this section, we show how the geometrical structure of the quantum bundle yields naturally a distinguished Lie algebra of vector fields, which will be called quantum vector fields. Moreover, we exhibit a natural Lie algebra isomorphism between quantisable functions and quantum vector fields.

Later, the quantum vector fields will be the source of quantum operators (see, later, § II.4.2 and § II.6.3).

We start by considering the vector fields of $\boldsymbol{Q}^{\uparrow}$, which preserve the basic quantum structures. We need to start from this bundle, because the quantum connection lives on it.

We need the covariant differential of a vector field, which is a particular case of the differential of tangent valued forms defined in [Mo2], [Mo3] (see also § III.5). We stress that this differential cannot be understood neither in the sense of linear connections on a manifold, nor in the sense of derivation laws.

REMARK II.3.2.1. Let $p:\boldsymbol{F}\to\boldsymbol{B}$ be a fibred manifold and

$$c:\boldsymbol{F}\to T^*\boldsymbol{B}\underset{\boldsymbol{F}}{\otimes}T\boldsymbol{F}$$

a connection. If $X:\boldsymbol{F}\to T\boldsymbol{F}$ is a vector field projectable on $\underline{X}:\boldsymbol{B}\to T\boldsymbol{B}$, then its covariant differential is defined to be the vertical valued 1-form

$$d_c X := [c,X] : \boldsymbol{F} \to T^*\boldsymbol{B}\underset{\boldsymbol{F}}{\otimes}V\boldsymbol{F},$$

where [,] is the Frölicher-Nijenhuis bracket, given by

$$(d_c X)(u) = [c(u),\, X] - c([u,\, \underline{X}]), \qquad \forall u\in\mathcal{T}(\boldsymbol{B}),$$

and with coordinate expression

$$d_c X = (-\,\partial_\lambda X^\mu\, c_\mu{}^i - \partial_\mu c_\lambda{}^i\, X^\mu + \partial_\lambda X^i + c_\lambda{}^j\, \partial_j X^i - \partial_j c_\lambda{}^i\, X^j)\, d^\lambda\otimes\partial_i.$$

Moreover, we recall the formula

$$(R) \qquad d_c(c(\underline{X})) = -\,2\,\underline{X}\lrcorner R_c,$$

where $R_c:\boldsymbol{F}\to\overset{2}{\wedge}T^*\boldsymbol{B}\underset{\boldsymbol{F}}{\otimes}V\boldsymbol{F}$ is the curvature of c. ❑

Next, let $p:\boldsymbol{F}\to\boldsymbol{B}$ be a line bundle. Then, a projectable vector field X is said to be *Hermitian* if it is (real) linear over its projection $\underline{X}$ and preserves the Hermitian product. A Hermitian vector field turns out to be also complex linear. The coordinate expression of a Hermitian vector field X is of the type

$$X = X^\lambda\, \partial_\lambda + i\, f\, \text{и} \qquad f,\, X^\lambda\in\mathcal{F}(\boldsymbol{B}).$$

Then, we introduce the following concept.

DEFINITION II.3.2.1. An *upper quantum vector field* is defined to be a (tube-like) vector field[21]

$$X^\uparrow:\boldsymbol{Q}^\uparrow\to T\boldsymbol{Q}^\uparrow$$

which

[21] Here, the arrow "↑" in $X^\uparrow$ does not refer to any pullback of a possible section X, but just reminds the pullback bundle $\boldsymbol{Q}^\uparrow$.

i) projects onto a (local) vector field

$$\underline{X}^{\uparrow} : J_1\boldsymbol{E} \to T J_1\boldsymbol{E},$$

ii) is Hermitian complex linear over $\underline{X}^{\uparrow}$,
iii) has a $\boldsymbol{T}$-horizontal covariant differential, i.e.

$$d_{\text{ч}}X^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{T}^{*} \otimes \boldsymbol{Q}) \subset T^{*}J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} \boldsymbol{Q}^{\uparrow}. \qquad ♣$$

$$d_{\text{ч}}X^{\uparrow} = \gamma \lrcorner d_{\text{ч}}X^{\uparrow}. \qquad ♣$$

Condition iii) can be reformulated in a useful way through γ.

REMARK II.3.2.2. Condition iii) is clearly equivalent to

$$pr \circ d_{\text{ч}}X^{\uparrow} = 0,$$

where $pr : T^{*}J_1\boldsymbol{E} \to V^{*}J_1\boldsymbol{E}$ is the canonical linear epimorphism over $J_1\boldsymbol{E}$.
Thus, condition iii) reads in coordinates as

$$d_{\text{ч}i}X^{\uparrow} = 0 \qquad d_{\text{ч}i}^{\;0}X^{\uparrow} = 0.$$

Moreover, condition iii) is equivalent to

iii)′
$$d_{\text{ч}}X^{\uparrow} = \gamma \lrcorner d_{\text{ч}}X^{\uparrow}. \qquad ❑$$

The upper quantum vector fields constitute a sheaf and the upper quantum vector fields, with a given time component $\tau : J_1\mathbf{E} \to \mathbb{T}$, constitute a subsheaf, which are denoted by

$$\mathcal{Q}(\boldsymbol{Q}^{\uparrow}) \subset \mathcal{T}(\boldsymbol{Q}^{\uparrow}) \qquad\qquad \mathcal{Q}_{\tau}(\boldsymbol{Q}^{\uparrow}) \subset \mathcal{Q}(\boldsymbol{Q}^{\uparrow}).$$

The above sheaves are not closed with respect to the Lie bracket.
On the other hand, the upper quantum vector fields can be classified, up to the choice of an arbitrary time scale, by the functions of the classical jet space, through the following formula.

THEOREM II.3.2.1. For each time scale $\tau : J_1\mathbf{E} \to \mathbb{T}$, we have an $\mathbb{R}$-linear sheaf isomorphism

$$q^{\uparrow}{}_{\tau} : \mathcal{F}(J_1\boldsymbol{E}) \to \mathcal{Q}_{\tau}(\boldsymbol{Q}^{\uparrow}) : f \mapsto X^{\uparrow}{}_{f,\tau}$$

given by

$$(U) \qquad X^{\uparrow}{}_{f,\tau} := \text{ч}(f^{\#}_{\tau}) + i\, f\, \text{и}.$$

Proof. 1) Let $X^{\uparrow}$ be an upper quantum vector field whose time-component is τ and let us prove that it is of the type

$$X^{\uparrow} := \text{ч}(f^{\#}_{\tau}) + i\, f\, \text{и},$$

for some function $f \in \mathcal{F}(J_1\boldsymbol{E})$.

Let us decompose $X^{\uparrow}$ into its horizontal and vertical components

$$(\text{ч}) \qquad X^{\uparrow} = \text{ч}(\underline{X}^{\uparrow}) + \nu_{\text{ч}}(X^{\uparrow}).$$

Conditions i) and ii) imply that $\nu_{\text{ч}}(X^{\uparrow})$ is of the type

$$(\upsilon) \qquad \nu_{\text{ч}}(X^{\uparrow}) = i\, f\, \text{и}, \qquad f \in \mathcal{F}(J_1\boldsymbol{E}).$$

Therefore, condition iii)′ can be written as

$$\text{iii)''} \qquad i\,(df - \gamma.f) \otimes \text{и} = \gamma \lrcorner d_{\text{ч}}(\text{ч}(\underline{X}^{\uparrow})) - d_{\text{ч}}(\text{ч}(\underline{X}^{\uparrow})).$$

Moreover, formulas (R) and ($R_{\text{ч}}$) (see Rem. II.3.2.1) yield

$$d_{\text{ч}}(\text{ч}(\underline{X}^{\uparrow})) = -\, 2\, i\, \frac{m}{\hbar}\, \underline{X}^{\uparrow} \lrcorner \Omega \otimes \text{и}.$$

Hence, condition iii)" can be written as

$$(df - \gamma.f) = \Omega^{\flat}_{\tau}(\underline{X}^{\uparrow})$$

i.e.

$$\text{h)} \qquad \underline{X}^{\uparrow} = f^{\#}_{\tau}.$$

Thus, formulas (ч), (υ) and (h) yield

$$X^{\uparrow} = \text{ч}(f^{\#}_{\tau}) + i\, f\, \text{и}$$

2) Conversely, if $f \in \mathcal{F}(J_1\boldsymbol{E})$, then analogous computations prove that formula (U) yields an upper quantum vector field, whose time-component is τ. ❑

PROPOSITION II.3.2.1. The coordinate expression of $X^{\uparrow}{}_{f,\tau}$ is

$$X^{\uparrow}{}_{f,\tau} = \tau\,(\partial_0 + y^i_0\,\partial_i + \gamma^i\,\partial^0_i) +$$

$$+ \frac{\hbar}{m}\,g^{ij}\left(-\,\partial^0_j f\,\partial_i + (\partial_j f + g^{hk}\,(\Gamma_{jh} - \Gamma_{hj})\,\partial^0_k f)\,\partial^0_i\right) +$$

$$+\,i\left(f + \tau\,L/\hbar - \frac{\hbar}{m}\,g^{ij}\,(p_j/\hbar)\,\partial^0_i f\right) \text{и}. \qquad \square$$

Formula (U) clearly recalls a well known formula of geometric quantisation (see, for instance, [St], [Wo]). However, there are important differences between our approach and geometric quantisation; they are basically related to the general covariance and the role of time. In particular, we shall see that an important difference will arise later in the construction of the quantum operator associated with energy. Indeed, we stress that our upper quantum vector fields need not to be time-vertical.

The above formula looks nice, but we have got two problems. In fact, we have been forced to search for distinguished vector fields on $\boldsymbol{Q}^{\uparrow}$ and not on $\boldsymbol{Q}$, just because the quantum connection lives on $\boldsymbol{Q}^{\uparrow}$. But, these vector fields map sections of $\boldsymbol{Q}$ into sections of $\boldsymbol{Q}^{\uparrow}$. This problem can be solved if, additionally, the above vector fields are projectable over $\boldsymbol{E}$. Moreover, the above isomorphism depends on the choice of a time scale and we need a reasonable criterion to make this choice. Luckily, the two problems can be solved together. In fact, we can prove the following result.

If $X^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T\boldsymbol{Q}^{\uparrow}$ is any vector field, then the canonical fibred epimorphism $pr : T\boldsymbol{Q}^{\uparrow} \to T\boldsymbol{Q}$ yields the fibred morphism over $\boldsymbol{Q}$

$$X := pr \circ X^{\uparrow} : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T\boldsymbol{Q}.$$

The vector field $X^{\uparrow}$ is said to be projectable over $\boldsymbol{E}$ if X can be written as

$$X : \boldsymbol{Q} \to T\boldsymbol{Q}.$$

THEOREM II.3.2.2. If $X^{\uparrow}{}_{f,\tau}$ is an upper quantum vector field, then the following conditions are equivalent:

i) $X^{\uparrow}{}_{f,\tau}$ is projectable over $\boldsymbol{E}$;

ii) f is a quantisable function and its time component is τ.

Proof. i) $\Rightarrow$ ii). We can prove the assertion directly.

If $X^{\uparrow}{}_{f,\tau}$ is projectable over $\boldsymbol{E}$, then

$$\partial^0_h \tau = 0 \qquad \tau\, \partial^0_h y^i_0 - \frac{\hbar}{m}\, g^{ij}\, \partial^{00}_{hj} f = 0$$

i.e.

$$\partial^0_h \tau = 0 \qquad \partial^{00}_{ij} f = \tau\, \frac{m}{\hbar}\, g_{ij}$$

i.e.

$$\tau : \boldsymbol{E} \to \mathbb{R} \qquad f = \tau\, \frac{m}{2\hbar}\, g_{ij}\, y^i_0\, y^j_0 + f_i\, y^i_0 + f_\circ .$$

On the other hand we obtain the same result as a consequence of the theorem concerning the projectability of the Hamiltonian lift of functions (see Prop. II.3.1.2).

ii) $\Rightarrow$ i). We can easily see that all components of $X_{f,\tau}$, including the imaginary component, do not depend on the coordinates y^i_0. ❑

Definition II.3.2.2. A *quantum vector field* is defined to be the vector field

$$X : \boldsymbol{Q} \to T\boldsymbol{Q}$$

associated with a projectable upper quantum vector field $X^{\uparrow}$. ♣

The quantum vector fields constitute a sheaf, which is denoted by

$$\mathcal{Q}(\boldsymbol{Q}) \subset \mathcal{T}(\boldsymbol{Q}).$$

Moreover, we shall be concerned with the subsheaves of quantum vector fields with constant and vanishing time component

$$\mathcal{Q}_c(\boldsymbol{Q}) \subset \mathcal{Q}(\boldsymbol{Q}) \qquad \mathcal{Q}_0(\boldsymbol{Q}) \subset \mathcal{Q}_c(\boldsymbol{Q}).$$

So, we are in the position to achieve the following important formula.

Corollary II.3.2.1. The coordinate expression of the quantum vector field associated with the quantisable function

$$f = f'' \frac{m}{2\hbar} g_{ij}\, y^i_0\, y^j_0 + f_i\, y^i_0 + f_\circ$$

is

$$X_f = f''\, \partial_0 - \frac{\hbar}{m} f^i\, \partial_i + i\, (\frac{m}{\hbar} f''\, a_0 - f^i\, a_i + f_\circ)\, \text{и}.$$

PROOF. It follows from the coordinate expression of $X^\uparrow$ and f. ❑

COROLLARY II.3.2.2. The map

$$q : \mathcal{Q}(J_1\boldsymbol{E}) \to \mathcal{Q}(\boldsymbol{Q}) : f \mapsto X_f$$

is a sheaf linear isomorphism. In particular, $\mathcal{Q}_c(J_1\boldsymbol{E})$ and $\mathcal{Q}_0(J_1\boldsymbol{E})$ are isomorphic, respectively, to $\mathcal{Q}_c(\boldsymbol{Q})$ and $\mathcal{Q}_0(\boldsymbol{Q})$. ❑

Eventually, we can prove the following important result.

LEMMA II.3.2.1. The sheaf $\mathcal{Q}(\boldsymbol{Q})$ of quantum vector fields and the subsheaves $\mathcal{Q}_c(\boldsymbol{Q})$ and $\mathcal{Q}_0(\boldsymbol{Q})$ of quantum vector fields with constant and vanishing time component are closed under the Lie bracket.

PROOF. It follows from a long computation in coordinates. ❑

THEOREM II.3.2.3. The map (see Theor. II.3.1.2)

$$\mathcal{Q}(J_1\boldsymbol{E}) \to \mathcal{Q}(\boldsymbol{Q}) : f \mapsto X_f$$

is an isomorphism of sheaves of Lie algebras.

Namely, for each $k \in \mathbb{R}$, $f, f', f'' \in \mathcal{Q}(J_1\boldsymbol{E})$, we have

$$X_{kf} = k\, X_f \qquad X_{f'+f''} = X_{f'} + X_{f''}$$

$$[X_{f'},\, X_{f''}] = X_{[f',\, f'']}.$$ ❑

II.4 - Quantum Lie operators

In this chapter, we show how the quantum vector fields act naturally on the quantum sections and yield a Lie algebra of operators, which will be called quantum Lie operators. Moreover, we exhibit a natural Lie algebra isomorphism between quantisable functions and quantum Lie operators.

This is our first approach to the subject of quantum operators and to the principle of correspondence; in this step, the classical Hamiltonian corresponds to the time derivative. A further development of the theory will be achieved later in the framework of the quantum Hilbert bundle (see § II.6.3); in this context, the classical Hamiltonian will correspond to the standard operator, naturally generalised to our curved space-time.

The quantum vector fields act naturally on the quantum sections as Lie derivatives; so, we might introduce the quantum operators directly in this way. The result would be quite interesting; but, in order to obtain symmetric (possibly self-adjoint) operators (see § II.6.3), we need a little more complicated approach. Namely, we have to consider quantum half-densities. More precisely, in the space-like integration procedure (see § II.6.1), we shall be involved with space-like half-densities. On the other hand, in the construction of operators, we need to consider space-time half-densities.

II.4.1. Lie operators

This section is devoted to a preliminary discussion about the action of linear projectable vector fields on quantum densities.

First, we introduce the notion of Lie derivative of sections of a vector bundle.

Let $p:\boldsymbol{F}\to\boldsymbol{B}$ be a vector bundle.

Remark II.4.1.1. A vertical vector field $Y:\boldsymbol{F}\to V\boldsymbol{F}:=\boldsymbol{F}\underset{\boldsymbol{B}}{\times}\boldsymbol{F}$ is said to be *basic* if it projects over a section $s:\boldsymbol{B}\to\boldsymbol{F}$ through the following commutative diagram

$$
\begin{tikzcd}
\boldsymbol{F} \arrow[r, "Y"] \arrow[d, "p"'] & V\boldsymbol{F} := \boldsymbol{F} \times_{\boldsymbol{B}} \boldsymbol{F} \arrow[d, "pr_2"] \\
\boldsymbol{B} \arrow[r, "s"] & \boldsymbol{F}
\end{tikzcd}
$$

Conversely each section $s:\boldsymbol{B}\to\boldsymbol{F}$ can be regarded as a basic vertical vector field $Y:\boldsymbol{F}\to V\boldsymbol{F}$, which is projectable over the section itself. This correspondence between sections and basic vector fields is a bijection. ❑

Lemma II.4.1.1. Let $X:\boldsymbol{F}\to T\boldsymbol{F}$ be a linear projectable vector field and let us denote its projection by $\underline{X}:\boldsymbol{B}\to T\boldsymbol{B}$.

For each section $s:\boldsymbol{B}\to\boldsymbol{F}$, the Lie bracket

$$[X,s]:\boldsymbol{F}\to V\boldsymbol{F}$$

is a basic vector field, hence determines the section

$$X.s := [X,s] : \boldsymbol{B} \to \boldsymbol{F},$$

with coordinate expression

$$X.s = (X^\lambda \partial_\lambda s^i - X^i_j s^j)\, b_i. \qquad ❑$$

So, if $X:\boldsymbol{F}\to T\boldsymbol{F}$ is a linear projectable vector field, then we define the associated *Lie operator* as the sheaf morphism

$$X. : \mathscr{S}(\boldsymbol{F}) \to \mathscr{S}(\boldsymbol{F}) : s \mapsto X.s\,.$$

Lemma II.4.1.2. The map

$$X \mapsto X.$$

is injective. Moreover, the map

$$(X,s) \mapsto X.s$$

has the following properties

$$X.(s+s') = X.s + X.s' \qquad X.(fs) = f\,(X.s) + \underline{X}.f\; s$$

$$(X + X').s = X.s + X'.s \qquad (fX).s = f\,(X.s)$$

$$[X., X'.](s) := X.X'.s - X'.X.s = [X,X'].s$$

for each linear projectable vector field $X, X' : \boldsymbol{F} \to T\boldsymbol{F}$, section $s, s' : \boldsymbol{B} \to \boldsymbol{F}$ and function $f : \boldsymbol{B} \to \mathbb{R}$. ❑

We can re-interpret the above results in terms of connections.

If we assume a linear connection

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \underset{\boldsymbol{B}}{\otimes} T\boldsymbol{F}$$

on the vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$, then we have the following result.

LEMMA II.4.1.3. Let $\underline{X} : \boldsymbol{B} \to T\boldsymbol{B}$ be a vector field and consider its horizontal prolongation

$$X := \underline{X} \lrcorner c : \boldsymbol{F} \to T\boldsymbol{F},$$

which is linear over its projection $\underline{X}$.

Then, for each section $s : \boldsymbol{B} \to \boldsymbol{F}$, we obtain the formula

$$X.s = \nabla_{\underline{X}} s,$$

with coordinate expression

$$X.s = (X^\lambda\, \partial_\lambda s^i - X^i{}_j\, s^j)\, b_i = X^\lambda\, (\partial_\lambda s^i - c_\lambda{}^i{}_j\, s^j)\, b_i = \nabla_{\underline{X}} s.$$ ❑

Now, let us apply the above results to our quantum framework.

PROPOSITION II.4.1.1. Let $X : \boldsymbol{Q} \to T\boldsymbol{Q}$ be a linear projectable vector field and let us denote its projection by $\underline{X} : \boldsymbol{E} \to T\boldsymbol{E}$. Then, we obtain, via the above Lie derivative, the sheaf morphism

$$X. : \mathscr{S}(\boldsymbol{Q}) \to \mathscr{S}(\boldsymbol{Q}) : \Psi \mapsto [X, \Psi].$$

Moreover, we obtain, via a standard Lie derivative, the sheaf morphism

$$X. : \mathscr{S}(\sqrt{\overset{4}{\wedge}} T^*\boldsymbol{E}) \to \mathscr{S}(\sqrt{\overset{4}{\wedge}} T^*\boldsymbol{E}) : \sqrt{\upsilon} \mapsto L(X)\sqrt{\upsilon},$$

which depends only on $\underline{X}$.

Hence, we can extend, via the Leibnitz rule, the Lie derivative to the space-time densities and obtain the sheaf morphism

$$X. : \mathscr{S}(\boldsymbol{Q}^\upsilon) \to \mathscr{S}(\boldsymbol{Q}^\upsilon) : \Psi \mapsto X.\Psi^\upsilon := [X, \Psi] \otimes \sqrt{\upsilon} + \Psi \otimes L(X)\sqrt{\upsilon}.$$

Furthermore, let $X : \boldsymbol{Q} \to V\boldsymbol{Q}$ be a linear vertical vector field. Then, in an analogous way, we obtain, via a vertical Lie derivative, the sheaf morphism

$$X. : \mathcal{S}(\boldsymbol{Q}^{\eta}) \to \mathcal{S}(\boldsymbol{Q}^{\eta}) : \Psi \mapsto X.\Psi^{\eta} := [X, \Psi] \otimes \sqrt{\eta} + \Psi \otimes L(X) \sqrt{\eta}. \qquad \square$$

PROPOSITION II.4.1.2. If $X, X' : \boldsymbol{Q} \to T\boldsymbol{Q}$ are linear projectable vector fields, and we consider their action both on $\mathcal{S}(\boldsymbol{Q}^{\upsilon})$ and $\mathcal{S}(\boldsymbol{Q}^{\eta})$, then the standard identity for the commutator of Lie derivatives yields

$$[X., X'.] = [X, X']. \, . \qquad \square$$

II.4.2. The general Lie algebras isomorphism

So, we are in the position to introduce the quantum Lie operators and the Lie algebra isomorphism between quantisable functions and quantum Lie operators.

DEFINITION II.4.2.1. Let $f \in \mathcal{Q}(J_1\boldsymbol{E})$ be a quantisable function and $X_f \in \mathcal{Q}(\boldsymbol{Q})$ the corresponding quantum vector field. Then, the *quantum Lie operator* associated with f is defined to be the sheaf morphism

$$Y_f := i\, X_f. : \mathcal{S}(\boldsymbol{Q}^{\eta}) \to \mathcal{S}(\boldsymbol{Q}^{\eta}) : \Psi^{\eta} \mapsto i\, (X_f.\Psi^{\upsilon}) \otimes \frac{1}{\sqrt{\upsilon}} \otimes \sqrt{\eta}. \qquad \clubsuit$$

We have already explained why we apply X_f to $\mathcal{S}(\boldsymbol{Q}^{\eta})$ and not directly to $\mathcal{S}(\boldsymbol{Q})$: we consider Ψ^{η} in view of the integration on the fibres of space-time and of the symmetry of the quantum Lie operator (see § II.6.3). Moreover, we are forced to pass through Ψ^{υ} because, in the general case, the action of X_f on $\sqrt{\eta}$ is not defined. Furthermore, the reason of the multiplication by the imaginary unit will appear later (see § II.6.3), when we prove the symmetry of the quantum operators.

REMARK II.4.2.1. In the particular case when the quantisable function f is affine with respect to the fibres of $J_1\mathbf{E} \to \boldsymbol{E}$, hence the quantum vector field X_f is vertical, we obtain, more directly,

$$Y_f = i\, X_f. : \mathcal{S}(\boldsymbol{Q}^{\eta}) \to \mathcal{S}(\boldsymbol{Q}^{\eta}) : \Psi^{\eta} \mapsto i\, X_f.\Psi^{\eta}. \qquad \square$$

The quantum Lie operators constitute a sheaf, which is denoted by

$$\mathcal{L}(\boldsymbol{Q}^\eta).$$

Moreover, we shall be concerned with the subsheaves of quantum Lie operators corresponding to quantisable functions with constant and vanishing time component

$$\mathcal{L}_c(\boldsymbol{Q}^\eta) \subset \mathcal{L}(\boldsymbol{Q}^\eta) \qquad \mathcal{L}_0(\boldsymbol{Q}^\eta) \subset \mathcal{L}_c(\boldsymbol{Q}^\eta).$$

LEMMA II.4.2.1. The map

$$\mathcal{Q}(J_1\boldsymbol{E}) \to \mathcal{L}(\boldsymbol{Q}^\eta) : f \mapsto Y_f$$

is a sheaf isomorphism. ❑

LEMMA II.4.2.2. The sheaf $\mathcal{L}(\boldsymbol{Q}^\eta)$ is a sheaf of Lie algebras with respect to the bracket

$$\mathcal{L}(\boldsymbol{Q}^\eta)\times\mathcal{L}(\boldsymbol{Q}^\eta) \to \mathcal{L}(\boldsymbol{Q}^\eta) : (Y_{f'}, Y_{f''}) \mapsto [Y_{f'}, Y_{f''}] := -i\,(Y_{f'}\circ Y_{f''} - Y_{f''}\circ Y_{f'}).$$

Moreover, $\mathcal{L}_c(\boldsymbol{Q}^\eta)$ and $\mathcal{L}_0(\boldsymbol{Q}^\eta)$ are subsheaves of Lie algebras. ❑

So, we are in the position to state the following important result.

THEOREM II.4.2.1. The map

$$\mathcal{Q}(J_1\boldsymbol{E}) \to \mathcal{L}(\boldsymbol{Q}^\eta) : f \mapsto Y_f$$

is an isomorphism of sheaves of Lie algebras.

Namely, for each $k\in\mathbb{R}$, $f, f', f''\in\mathcal{Q}(J_1\boldsymbol{E})$, we have

$$Y_{kf} = k\,Y_f \qquad Y_{f'+f''} = Y_{f'} + Y_{f''}$$

$$[Y_{f'}, Y_{f''}] = Y_{[f', f'']}.$$ ❑

PROPOSITION II.4.2.1. For each observer o, the action of the quantum Lie operator Y_f on Ψ^η is given by

$$Y_f(\Psi^\eta) = i\,(f^H \lrcorner \nabla^o\Psi)^\eta + \big(f\circ o + i\,\tfrac{1}{2}(\operatorname{div} f^H)\big)\,\Psi^\eta,$$

where $f^H : \boldsymbol{E} \to T\boldsymbol{E}$ is the projection of the Hamiltonian lift of f (see § II.3.1).

In other words, we have the following coordinate expression

$$Y_f(\Psi^\eta) =$$

$$= i\left(f'' \nabla^o{}_0\psi^\eta - \frac{\hbar}{m} f^i \nabla^o{}_i\psi^\eta - i\, f_\circ\, \psi^\eta + \frac{1}{2}\,(\partial_0 f'' - \frac{\hbar}{m}\,\partial_i f^i)\;\psi^\eta\right) b \otimes \sqrt{\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3},$$

where

$$\nabla^o{}_\lambda\psi^\eta := \partial_\lambda\psi^\eta - i\,\frac{m}{\hbar}\,a_\lambda\,\psi^\eta. \qquad \square$$

II.4.3. Main examples

We conclude this chapter by computing the main examples of quantum Lie operators.

Let us choose a frame of reference (u^0, o) and a related fibred space-time chart (x^0, y^i).

As usual, we denote by ∇^o the quantum covariant differential associated with o (see Def. II.1.5.2).

LEMMA II.4.3.1. The quantum Lie operator associated with every quantisable function of the type

$$f \in \mathcal{F}(\boldsymbol{E}) \subset \mathcal{Q}(J_1\boldsymbol{E})$$

is

$$Y_f(\Psi^\eta) = f\;\Psi^\eta. \qquad \square$$

EXAMPLE II.4.3.1. We have the following quantum Lie operators

$$Y_{x^0}(\Psi^\eta) = x^0\;\Psi^\eta \qquad\qquad Y_{y^i}(\Psi^\eta) = y^i\;\Psi^\eta$$

$$Y_{p_i/\hbar}(\Psi^\eta) = -\,i\;\partial_i\psi^\eta\; b \otimes \sqrt{\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3}$$

$$Y_{H/\hbar}(\Psi^\eta) = i\;\partial_0\psi^\eta\; b \otimes \sqrt{\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3}. \qquad \square$$

EXAMPLE II.4.3.2. For each quantisable function f, we have

$$[Y_{x^0}, Y_f] = -f'' \qquad [Y_{y^i}, Y_f] = \frac{\hbar}{m} f^i.$$

Moreover, we have

$$[Y_{p_i/\hbar}, Y_{x^0}] = 0 \qquad [Y_{p_i/\hbar}, Y_{y^j}] = -\delta_i^j \qquad [Y_{p_i/\hbar}, Y_{H/\hbar}] = 0.$$ ❑

II.5 - Systems of double fibred manifolds

In view of further developments of the quantum theory, we need some preliminary notions and results concerning the system associated with a double fibred manifold. The theory of finite dimensional systems has been extensively studied in [MM], [Mo2] and has been already used in this paper (see § II.1.3). In this chapter, we introduce the infinite dimensional system of all smooth sections by means of a functorial construction.

II.5.1. The system

We start by introducing the notion of system of sections associated with a double fibred manifold and studying its basic properties.

First we need a few preliminaries about the concept of smoothness due to Frölicher and the notion of fibred set.

We shall be concerned with some sets constructed geometrically from some functional spaces. We could define a topology in order to achieve a structure of infinite dimensional manifold on such sets; but great difficulties would arise. On the other hand the concept of *smoothness* due to Frölicher ([Fr]) is very suitable for our purposes, as it allows us to achieve the geometrical constructions that we need, avoiding all troubles related to infinite dimensionality.

DEFINITION II.5.1.1. A *smooth space* in the sense of Frölicher (see [Fr], [Sl]), can be defined as a pair

$$(\boldsymbol{S}, C),$$

where $\boldsymbol{S}$ is a set and

$$C := \{c : \mathbb{R} \to \boldsymbol{S}\}$$

is a set of curves which will be called *smooth*.

If $(\boldsymbol{S}, C)$ and $(\boldsymbol{S}', C')$ are smooth spaces, then a map $f : \boldsymbol{S} \to \boldsymbol{S}'$ is said to be *smooth* if, for each smooth curve $c : \mathbb{R} \to \boldsymbol{S}$, the curve

$$c' := f \circ c : \mathbb{R} \to \boldsymbol{S}'$$

is smooth. ♣

In particular, each classical manifold $\boldsymbol{M}$ becomes a smooth space by assuming as smooth curves, in the sense of Frölicher, just the smooth curves in the classical sense. Then, it can be proved (see [CCKM], [Bo]) that a map between classical manifolds is smooth in the classical sense if and only if it is smooth in the sense of Frölicher.

If $(\boldsymbol{S}, C)$ is a smooth space, then by abuse of language we shall also say that $\boldsymbol{S}$ is a smooth space.

If $\boldsymbol{S}$ and $\boldsymbol{S}'$ are smooth spaces, then $\boldsymbol{S} \times \boldsymbol{S}'$ becomes a smooth space in a natural way.

A *fibred set* is defined to be a set $\boldsymbol{S}$ together with a surjective map

$$\sigma : \boldsymbol{S} \to \boldsymbol{B}$$

of $\boldsymbol{S}$ onto a set $\boldsymbol{B}$. When $\boldsymbol{B}$ is a manifold, we denote by

$$\mathcal{S}^{\square}(\boldsymbol{S} \to \boldsymbol{B})$$

the sheaf of local sections of the fibred set $\sigma : \boldsymbol{S} \to \boldsymbol{B}$. Moreover, if $\boldsymbol{S}$ is a smooth space, then we denote by

$$\mathcal{S}(\boldsymbol{S} \to \boldsymbol{B}) \subset \mathcal{S}^{\square}(\boldsymbol{S} \to \boldsymbol{B})$$

the subsheaf of smooth local sections of the fibred set $\sigma : \boldsymbol{S} \to \boldsymbol{B}$.

Now, let us consider a (smooth) double fibred manifold

$$\boldsymbol{F} \xrightarrow{q} \boldsymbol{E} \xrightarrow{p} \boldsymbol{B}$$

and denote the typical double fibred charts of $\boldsymbol{F}$ by

$$(x^\lambda, y^i, z^a) \qquad 1 \le \lambda \le m, \ 1 \le i \le l, \ 1 \le a \le n.$$

We denote by

$$\mathcal{S}_t^{\square}(\boldsymbol{F} \to \boldsymbol{E} \to \boldsymbol{B}) \quad \text{and} \quad \mathcal{S}_t(\boldsymbol{F} \to \boldsymbol{E} \to \boldsymbol{B})$$

the sheaf of tube-sections which are smooth along the fibres but possibly non-smooth with respect to the base space $\boldsymbol{B}$ and the subsheaf of smooth tube-sections.

DEFINITION II.5.1.2. The *system* associated with the double fibred manifold $\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B}$ is defined to be the pair

$$(\sigma : S\boldsymbol{F}\to\boldsymbol{B},\ \varepsilon),$$

where

i) $S\boldsymbol{F}$ is the set

$$S\boldsymbol{F} := \coprod_{x\in\boldsymbol{B}} S_x\boldsymbol{F},$$

where

$$S_x\boldsymbol{F} := \{\Psi_x : \boldsymbol{E}_x \to \boldsymbol{F}_x\}$$

denotes the set of smooth fibre-sections related to $x\in\boldsymbol{B}$;

ii) σ is the natural surjective map

$$\sigma : S\boldsymbol{F} \to \boldsymbol{B} : \Psi_x \mapsto x;$$

iii) ε is the evaluation fibred morphism over $\boldsymbol{E}$

$$\varepsilon : S\boldsymbol{F}\underset{\boldsymbol{B}}{\times}\boldsymbol{E} \to \boldsymbol{F} : (\Psi_x,\ y) \mapsto \Psi_x(y). \qquad ♣$$

REMARK II.5.1.1. The evaluation fibred morphism ε yields the natural sheaf isomorphism

$$\boldsymbol{\varepsilon} : \mathscr{S}^{\square}(S\boldsymbol{F}\to\boldsymbol{B}) \to \mathscr{S}_t{}^{\square}(\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B}) : \hat{\Psi} \mapsto \Psi := \boldsymbol{\varepsilon}(\hat{\Psi}) := \varepsilon\circ\hat{\Psi}^{\uparrow},$$

where "$\uparrow$" denotes the pullback.

The inverse natural sheaf isomorphism is

$$\boldsymbol{\varepsilon}^{-1} : \mathscr{S}_{\mathrm{t}}{}^{\square}(\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B}) \to \mathscr{S}^{\square}(S\boldsymbol{F}\to\boldsymbol{B}) : \Psi \mapsto \hat{\Psi},$$

where

$$\hat{\Psi} : x \mapsto \Psi_x := \Psi_{|\boldsymbol{E}_x}. \qquad \square$$

Next, we introduce a smooth structure in the sense of Frölicher on the set $S\boldsymbol{F}$ by the following definition (see also [Ko]).

DEFINITION II.5.1.3. A curve $\hat{c} : \mathbb{R}\to S\boldsymbol{F}$ is said to be *smooth*, in the sense of Frölicher, if the induced maps

$$\sigma\circ\hat{c}:\mathbb{R}\to\boldsymbol{B} \qquad c:(\sigma\circ\hat{c})^*\boldsymbol{E}\to\boldsymbol{F}:y_\lambda\mapsto\hat{c}(\lambda)(y_\lambda)$$

are smooth in the classical sense. ♣

REMARK II.5.1.2. If $\boldsymbol{M}$ is a manifold, then we can easily prove that a map $\hat{c}:\boldsymbol{M}\to S\boldsymbol{F}$ is smooth in the sense of Frölicher if and only if the induced maps

$$\sigma\circ\hat{c}:\boldsymbol{M}\to\boldsymbol{B} \qquad c:(\sigma\circ\hat{c})^*\boldsymbol{E}\to\boldsymbol{F}$$

are smooth in the classical sense.

Moreover, if $\boldsymbol{S}'$ is a further smooth space, then we can easily prove that the map $f:S\boldsymbol{F}\to\boldsymbol{S}'$ is smooth in the sense of Frölicher if and only if, for each smooth map $\hat{c}:\boldsymbol{M}\to S\boldsymbol{F}$, the map

$$f\circ\hat{c}:\boldsymbol{M}\to\boldsymbol{S}'$$

is smooth in the sense of Frölicher. ❑

PROPOSITION II.5.1.1. The maps σ and ε turn out to be smooth.

Moreover, a local section

$$\hat{\Psi}:\boldsymbol{B}\to S\boldsymbol{F}$$

of the fibred set $S\boldsymbol{F}\to\boldsymbol{B}$ turns out to be smooth if and only if the associated tube-section

$$\Psi := \boldsymbol{\varepsilon}(\hat{\Psi}) : \boldsymbol{E}\to\boldsymbol{F}$$

is smooth.

Thus, the sheaf isomorphism $\boldsymbol{\varepsilon}$ restricts to a sheaf isomorphism

$$\boldsymbol{\varepsilon}:\mathscr{S}(S\boldsymbol{F}\to\boldsymbol{B})\to\mathscr{S}_t(\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B}).$$ ❑

The following remark has an important role in the following (see § II.6.1).

REMARK II.5.1.3. Let $(\sigma':S'\boldsymbol{F}\to\boldsymbol{B},\ \varepsilon')$ be the subsystem of $(\sigma:S\boldsymbol{F}\to\boldsymbol{B},\ \varepsilon)$ constituted by any subset of smooth-fibre sections of the double fibred manifold $\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B}$. Then, all above constructions can be easily repeated for this subsystem. Moreover, the inclusion $S'\boldsymbol{F}\hookrightarrow S\boldsymbol{F}$ turns out to be a smooth map in the sense of Frölicher. ❑

II.5.2. The tangent prolongation of the system

We pursue our presentation of systems by studying the tangent space of the system and the tangent prolongation of sections. We refer to the double fibred manifold and the associated system as in the previous section.

In order to define the tangent space of the smooth space $\boldsymbol{SF}$ let us study the tangent map of smooth curves $\mathbb{R}\to\boldsymbol{SF}$.

LEMMA II.5.2.1. i) Let

$$\hat{c}:\mathbb{R}\to\boldsymbol{SF}$$

be a smooth curve defined in a neighbourhood of $0\in\mathbb{R}$. Let us consider the induced curves

$$\sigma\circ\hat{c}:\mathbb{R}\to\boldsymbol{B} \qquad c:(\sigma\circ\hat{c})^*\boldsymbol{E}\to\boldsymbol{F}$$

and set (see § III.2)

$$x := (\sigma\circ\hat{c})(0) \in \boldsymbol{B} \qquad u := \partial(\sigma\circ\hat{c}) := T(\sigma\circ\hat{c})(0,1) \in T_x\boldsymbol{B} \qquad \hat{\Psi}_x := \hat{c}(0).$$

Then, the restriction of the tangent map $Tc:T(\sigma\circ\hat{c})^*T\boldsymbol{E}\to T\boldsymbol{F}$ to the fibre over $(0,1)\in\mathbb{R}\times\mathbb{R}$ turns out to be an affine fibred morphism over $\hat{\Psi}_x$

$$\Phi_u := \partial c := Tc_{|(0,1)} : (T\boldsymbol{E})_u \to (T\boldsymbol{F})_u,$$

whose fibre derivative is the linear fibred morphism over $\hat{\Psi}_x$

$$D\Phi_u = T(\hat{\Psi}_x) : (V\boldsymbol{E})_x \to (V\boldsymbol{F})_x.$$

In other words, if $v\in(T\boldsymbol{E})_u$, then we can write

$$\Phi_u : (T\boldsymbol{E})_u \to (T\boldsymbol{F})_u : (v + w) \mapsto \Phi_u(v) + T(\hat{\Psi}_x)(w) \qquad \forall w\in(V\boldsymbol{E})_x.$$

We have the coordinate expression

$$(y^i, z^a; \dot{x}^\lambda, \dot{y}^i, \dot{z}^a)\circ\Phi_u = (y^i, \Psi_x{}^a; u^\lambda, \dot{y}^i, \Phi_u{}^a + \partial_i\Psi_x{}^a\,\dot{y}^i),$$

with

$$u^\lambda = \partial c^\lambda \in \mathbb{R} \qquad \Phi_u{}^a = \partial c^a : \boldsymbol{E}_x \to \mathbb{R}.$$

ii) Conversely, let

$$x \in \boldsymbol{B} \qquad u \in T_x \boldsymbol{B} \qquad \hat{\Psi}_x \in S_x \boldsymbol{F}$$

and

$$\Phi_u : (T\boldsymbol{E})_u \to (T\boldsymbol{F})_u$$

be a map of the above type. Then, we can prove (see [CK]) that there is a smooth curve $\hat{c} : \mathbb{R} \to \boldsymbol{SF}$, such that

$$\partial c = \Phi_u. \qquad \square$$

Then, we are led to the following definition.

DEFINITION II.5.2.1. The *tangent space* of the smooth space $\boldsymbol{SF}$ is defined to be the set

$$T\boldsymbol{SF} := \coprod_{\hat{\Psi}_x \in \boldsymbol{SF}} T_{\hat{\Psi}_x} \boldsymbol{SF},$$

where

$$T_{\hat{\Psi}_x} \boldsymbol{SF} := \{\Phi_u\}$$

is the set of smooth sections of the type

$$\Phi_u : (T\boldsymbol{E})_u \to (T\boldsymbol{F})_u, \qquad u \in T_x \boldsymbol{B},\ x := \sigma(\hat{\Psi}_x) \in \boldsymbol{B},$$

such that Φ_u is an affine fibred morphism over $\hat{\Psi}_x$ and its fibre derivative is the linear fibred morphism over $\hat{\Psi}_x$

$$D\Phi_u = T(\hat{\Psi}_x) : (V\boldsymbol{E})_x \to (V\boldsymbol{F})_x. \qquad \clubsuit$$

The basic geometrical constructions which hold for the tangent space of the standard finite dimensional manifolds can be repeated in our "infinite dimensional" case.

PROPOSITION II.5.2.1. We have the natural surjective maps

$$\pi_{\boldsymbol{SF}} : T\boldsymbol{SF} \to \boldsymbol{SF} : \Phi_u \mapsto \hat{\Psi}_x, \qquad x := \pi_{\boldsymbol{B}}(u),$$

and

$$T\sigma : T\boldsymbol{SF} \to T\boldsymbol{B} : \Phi_u \mapsto u.$$

For each $\hat{\Psi}_x \in \boldsymbol{SF}$, the tangent space $T_{\hat{\Psi}_x}\boldsymbol{SF}$ of $\boldsymbol{SF}$ at $\hat{\Psi}_x$ turns out to be a vector space in a natural way.

Namely, if

$$\Phi_u,\ \Phi'_{u'},\ \Phi''_{u''} \in T_{\hat{\Psi}_x}\boldsymbol{SF} \qquad \lambda \in \mathbb{R},$$

then

$$\lambda\, \Phi'_{u'} \in (T_{\hat{\Psi}_x}\boldsymbol{SF})_{\lambda u'} \qquad \Phi'_{u'} + \Phi''_{u''} \in (T_{\hat{\Psi}_x}\boldsymbol{SF})_{u'+u''}$$

are well defined by

$$(\lambda\, \Phi_u) : (T\boldsymbol{E})_{\lambda u} \to (T\boldsymbol{F})_{\lambda u} : (\lambda\, v + \lambda\, w) \mapsto \lambda\, \Phi_u(v) + T(\hat{\Psi}_x)(\lambda\, w)$$

$$(\Phi'_{u'} + \Phi''_{u''}) : (T\boldsymbol{E})_{u'+u''} \to (T\boldsymbol{F})_{u'+u''} :$$

$$: (v' + v'' + w) \mapsto \Phi'_{u'}(v') + \Phi''_{u''}(v'') + T(\hat{\Psi}_x)(w) \qquad \forall w \in (V\boldsymbol{E})_x$$

where $v \in (T\boldsymbol{E})_u$, $v' \in (T\boldsymbol{E})_{u'}$, $v'' \in (T\boldsymbol{E})_{u''}$.

Hence, the map $T\sigma$ turns out to be a linear fibred morphism over σ. ❑

Remark II.5.2.1. We have the natural evaluation fibred morphism over $T\boldsymbol{E}$

$$T\varepsilon : T\boldsymbol{SF} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{E} \to T\boldsymbol{F} : (\Phi_u, v) \mapsto \Phi_u(v).$$

Thus, $(T\sigma : T\boldsymbol{SF} \to T\boldsymbol{B},\ T\varepsilon)$ turns out to be a subsystem of the system associated with the double fibred manifold $T\boldsymbol{F} \to T\boldsymbol{E} \to T\boldsymbol{B}$.

Therefore, all results of the above section apply to this subsystem. ❑

Remark II.5.2.2. The vector structure of the fibred set $T\boldsymbol{SF} \to \boldsymbol{SF}$ turns out to be smooth in the sense that the fibred morphisms over $\boldsymbol{SF}$

$$\cdot : \mathbb{R} \times T\boldsymbol{SF} \to T\boldsymbol{SF} \qquad + : T\boldsymbol{SF} \times T\boldsymbol{SF} \to T\boldsymbol{SF}$$

are smooth. ❑

We can define the tangent map of sections of the space of fibred sections in the following way.

REMARK II.5.2.3. If $\hat{\Psi} \in \mathscr{S}(\boldsymbol{SF} \to \boldsymbol{B})$, then

$$T\Psi : T\boldsymbol{E} \to T\boldsymbol{F}$$

yields the smooth tube-section

$$\widehat{T\Psi} := (T\boldsymbol{\varepsilon})^{-1}(T\Psi) : T\boldsymbol{B} \to T\boldsymbol{SF}. \qquad \square$$

DEFINITION II.5.2.2. The *tangent prolongation* of the section $\hat{\Psi} \in \mathscr{S}(\boldsymbol{SF} \to \boldsymbol{B})$ is defined to be the section $T\hat{\Psi} \in \mathscr{S}(T\boldsymbol{SF} \to T\boldsymbol{B})$ given by

$$T\hat{\Psi} := \widehat{T\Psi}. \qquad ♣$$

II.5.3. Connections on the system

We conclude this chapter by introducing the notion of a connection on our system. We still refer to the double fibred manifold and the associated system as in the first section.

Analogously to the finite dimensional case (see § III.2), we have the exact sequence over $\boldsymbol{SF}$

$$0 \longrightarrow V\boldsymbol{SF} \longrightarrow T\boldsymbol{SF} \longrightarrow \boldsymbol{SF} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \twoheadrightarrow 0.$$

DEFINITION II.5.3.1. A *connection* of the fibred set $\boldsymbol{SF} \to \boldsymbol{B}$ is defined to be a smooth splitting[22] $\hat{k}$ of the above sequence, i.e. a smooth section

$$\hat{k} : \boldsymbol{SF} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{SF},$$

which is linear over $\boldsymbol{SF}$ and is projectable over $1_{\boldsymbol{B}} : T\boldsymbol{B} \to T\boldsymbol{B}$. ♣

We can interpret a connection $\hat{k}$ as an operator which acts on the tube-sections of the double fibred manifold $\boldsymbol{F} \to \boldsymbol{E} \to \boldsymbol{B}$, in the following way.

PROPOSITION II.5.3.1. Let $\hat{k}$ be a connection.

If $\hat{\Psi} \in \mathscr{S}(\boldsymbol{SF} \to \boldsymbol{B})$, then, we obtain the section

[22] k is a Chinese character, whose romanization in Pinyin is "kě".

$$k(\Psi): TE \to TF,$$

given by the composition

$$TE \xrightarrow{\hat{\Psi}^{\uparrow}} SF \underset{B}{\times} TE \xrightarrow{\hat{k}^{\uparrow}} TSF \underset{TB}{\times} TE \xrightarrow{T\varepsilon} TF,$$

with coordinate expression

$$(x^{\lambda}, y^{i}, z^{a}; \dot{x}^{\lambda}, \dot{y}^{i}, \dot{z}^{a}) \circ k(\Psi) = (x^{\lambda}, y^{i}, \Psi^{a}; \dot{x}^{\lambda}, \dot{y}^{i}, k_{\mu}{}^{a}(\Psi)\, \dot{x}^{\mu} + \partial_{j}\Psi^{a}\, \dot{y}^{j}),$$

where

$$k_{\mu}{}^{a}(\Psi) \in \mathcal{F}(E).$$

Moreover, we obtain the smooth section

$$\widehat{k(\Psi)} = \hat{k}(\hat{\Psi}) : TB \to TSF.$$

Therefore, the sheaf morphism

$$k : \mathcal{S}_{t}(F \to E \to B) \to \mathcal{S}_{t}(TF \to TE \to TB) : \Psi \mapsto k(\Psi)$$

characterises $\hat{k}$ itself. ❑

A connection $\hat{k}$ induces the covariant differential in the standard way.

LEMMA II.5.3.1. Let $\hat{k}$ be a connection and $\hat{\Psi} \in \mathcal{S}(SF \to B)$. Then, the map

$$T\hat{\Psi} - \hat{k} \circ \hat{\Psi}^{\uparrow} : TB \to TSF$$

takes its values in VSF, is projectable over $\hat{\Psi}$ and is a linear fibred morphism over $\hat{\Psi}$. Hence, it can be can regard as a smooth section

$$\nabla_{\hat{k}} \hat{\Psi} := T\hat{\Psi} - \hat{k} \circ \hat{\Psi}^{\uparrow} : B \to T^{*}B \underset{SF}{\otimes} VSF,$$

which is projectable over $\hat{\Psi}$. ❑

DEFINITION II.5.3.2. Let $\hat{k}$ be a connection. Then, the *covariant differential* of $\hat{\Psi} \in \mathcal{S}(SF \to B)$ is defined to be the smooth local section

$$\nabla_{\hat{k}} \hat{\Psi} := T\hat{\Psi} - \hat{k} \circ \hat{\Psi}^{\uparrow} : B \to T^{*}B \underset{SF}{\otimes} VSF. \quad ♣$$

We can interpret the covariant differential $\nabla_{\hat{k}}\hat{\Psi}$ in terms of the tube-section Ψ, in the following way.

PROPOSITION II.5.3.2. Let k be a connection and $\hat{\Psi}\in\mathcal{S}(\boldsymbol{SF}\to\boldsymbol{B})$. Then, the map

$$T\Psi - k(\Psi) : T\boldsymbol{E} \to T\boldsymbol{F}$$

is a smooth linear tube-fibred morphism over Ψ, which factorizes through a smooth linear tube-fibred morphism over Ψ

$$\nabla_k\Psi : \boldsymbol{E}\underset{\boldsymbol{B}}{\times}T\boldsymbol{B}\to V\boldsymbol{F};$$

conversely, the map $\nabla_k\Psi$ characterises $T\Psi - k(\Psi)$.

Moreover, we have

$$\nabla_{\hat{k}}\hat{\Psi} = \widehat{T\Psi - k(\Psi)} : \boldsymbol{B} \to T^*\boldsymbol{B}\underset{\boldsymbol{SF}}{\otimes}V\boldsymbol{SF}.$$

The coordinate expression of $\nabla_k\Psi$ is

$$\nabla_k\Psi = \left(\partial_\lambda\Psi^a - k_\mu{}^a(\Psi)\right) d^\mu\otimes(\partial_a\circ\Psi).$$ ❑

A connection $\hat{k}$ is said to be *of order k* if, for each smooth tube-section $\Psi:\boldsymbol{E}\to\boldsymbol{F}$, the map $k(\Psi)$ depends on Ψ through its vertical jet up to order k.

Now, let us consider the case when $\boldsymbol{F}\to\boldsymbol{E}$ is a vector bundle, hence $\boldsymbol{SF}\to\boldsymbol{B}$ and $T\boldsymbol{SF}\to T\boldsymbol{B}$ turn out to be smoothly equipped with a vector structure on their fibres.

The connection k is said to be *linear* if it is linear as a fibred morphism over $\boldsymbol{B}$. In other words, the connection k is linear if the operator

$$k : \mathcal{S}_t(\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B}) \to \mathcal{S}_t(T\boldsymbol{F}\to T\boldsymbol{E}\to T\boldsymbol{B}) : \Psi \mapsto k(\Psi)$$

is linear.

PROPOSITION II.5.3.3. Let $\hat{k}$ be a connection of order k. Then, the coordinate expression of the linear operator $k:\Psi\mapsto k(\Psi)$ is of the type

$$k_\mu{}^a(\Psi) = k_\mu{}^a{}_b\,\Psi^b + k_\mu{}^{aj}{}_b\,\partial_j\Psi^b + \dots + k_\mu{}^{aj_1\dots j_k}{}_b\,\partial_{j_1\dots j_k}\Psi^b,$$

where

$$(c) \qquad k_{\mu}{}^{a}{}_{b},\ k_{\mu}{}^{aj}{}_{b},\ \dots,\ k_{\mu}{}^{aj_1\dots j_k}{}_{b} \in \mathcal{F}(\boldsymbol{E}).$$

Thus, in the linear case, the coordinate expression of $\nabla_k\Psi$ is of the type

$$\nabla_k\Psi = (\partial_\mu\Psi^a - k_{\mu}{}^{a}{}_{b}\,\Psi^b - k_{\mu}{}^{aj}{}_{b}\,\partial_j\Psi^b - \dots - k_{\mu}{}^{aj_1\dots j_k}{}_{b}\,\partial_{j_1\dots j_k}\Psi^b)\, d^\mu\otimes\partial_a\,.$$

Thus, the equation

$$\nabla_k\Psi = 0,$$

in the unknown section $\Psi\in\mathcal{S}_t(\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B})$, turns out to be a linear differential equation of order k in the fibre derivatives and of order 1 in the base derivatives of Ψ. ❑

We observe that for any arbitrary choice of the above coefficients (c), we obtain a local connection. Hence global connections can be obtained by means of the partition of unity.

We finish this chapter by observing that all above geometrical constructions are compatible, in a natural way, with restrictions to the subsystem associated with a subsheaf of the sheaf of smooth tube-sections (see Rem. II.5.1.3). In particular, we shall be concerned with the subsystem of compact support tube-sections, on a vector double fibred manifold.

II.6 - The infinite dimensional quantum system

So far, the quantum theory has been developed on the quantum bundle $\boldsymbol{Q}$, which is based over the space-time $\boldsymbol{E}$ (except for a temporary extension of the base space to $J_1\boldsymbol{E}$) and has one dimensional complex fibres.

On the other hand, the theory of systems suggests to translate the main concepts and results of the above quantum theory in terms of a new bundle $S\boldsymbol{Q}^{\eta}$ which is based over the time $\boldsymbol{T}$ and has infinite dimensional complex fibres.

Such a geometrical development of our quantum theory yields interesting physical results and interpretations. In particular, it yields the Hilbert bundle and the generalisation to a curved space-time of the standard Hamiltonian quantum operator and commutators.

II.6.1. The quantum system

We start by introducing the system associated with the double fibred manifold of quantum space-like densities (see § II.1.2), according to the ideas of the previous chapter.

So, we consider the system

$$(\sigma : S\boldsymbol{Q}^{\eta} \to \boldsymbol{T}\,,\ \varepsilon)$$

associated with the double fibring

$$\eta \quad \boldsymbol{Q} \xrightarrow{\pi^{\eta}} \boldsymbol{E} \xrightarrow{t} \boldsymbol{T}\,.$$

In order to be able to perform integrations over the fibres of $S\boldsymbol{Q}^{\eta} \to \boldsymbol{T}$, we need to consider the subsystem of sections with compact support. So doing, we miss sections of physical interest, but they can be recovered later by means of a completion procedure. So, we introduce the following definition.

Definition II.6.1.1. The *pre-quantum system* is defined to be the subsystem

$$(\sigma^c : S^c \boldsymbol{Q}^\eta \to \boldsymbol{T}, \ \varepsilon^c)$$

where

$$S^c \boldsymbol{Q}^\eta := \bigsqcup_{\tau \in \boldsymbol{T}} S^c{}_\tau \boldsymbol{Q}^\eta \subset S \boldsymbol{Q}^\eta,$$

and

$$S^c{}_\tau \boldsymbol{Q}^\eta \ \subset \ S_\tau \boldsymbol{Q}^\eta := \{\Psi^\eta{}_\tau : \boldsymbol{E}_\tau \to \boldsymbol{Q}^\eta{}_\tau\}$$

is the subset of sections with compact support. ♣

Let $f : \boldsymbol{E} \to \mathbb{C}$ is a tube-function such that, for each $\tau \in \boldsymbol{T}$, its restriction $f_\tau : \boldsymbol{E}_\tau \to \mathbb{C}$, has compact support; then we define the "space-like partial integral" of f as

$$\int f\,\eta : \boldsymbol{T} \to \mathbb{C} : \tau \mapsto (\int f\,\eta)(\tau) := \int_{\boldsymbol{E}_\tau} f_\tau\,\eta_\tau .$$

PROPOSITION II.6.1.1. On the fibres of $\sigma^c : S^c \boldsymbol{Q}^\eta \to \boldsymbol{T}$, we obtain the $\mathbb{A}^{3/2}$-valued Hermitian structure, given by

$$\hat{h} : S^c \boldsymbol{Q}^\eta \underset{\boldsymbol{B}}{\times} S^c \boldsymbol{Q}^\eta \to \mathbb{C} \otimes \mathbb{A}^{3/2} : (\Phi^\eta{}_\tau, \Psi^\eta{}_\tau) \mapsto \langle \Phi^\eta{}_\tau \mid \Psi^\eta{}_\tau \rangle := \int_{\boldsymbol{E}_\tau} h\,(\Phi_\tau, \Psi_\tau)\,\eta_\tau . \qquad \square$$

Hence, $\sigma^c : S^c \boldsymbol{Q}^\eta \to \boldsymbol{T}$ turns out to be a pre-Hilbert fibred set. Moreover, by completion (fibre by fibre), we obtain a Hilbert fibred set

$$H \boldsymbol{Q}^\eta \to \boldsymbol{T} .$$

By abuse of language, we say that $S^c \boldsymbol{Q}^\eta \to \boldsymbol{T}$ is the *pre-Hilbert quantum bundle* and $H \boldsymbol{Q}^\eta \to \boldsymbol{T}$ the *Hilbert quantum bundle*.

II.6.2. The Schrödinger connection on the quantum system

Next, we re-interpret the Schrödinger operator in the present framework (see Def. II.2.3.1) and study its basic properties.

THEOREM II.6.2.1. There is a unique linear connection

$$\hat{k}: \boldsymbol{SQ}^{\eta} \to \mathbb{T}^{*} \otimes T\boldsymbol{SQ}^{\eta}$$

on the vector fibred set $\sigma: \boldsymbol{SQ}^{\eta} \to \boldsymbol{T}$, such that the Schrödinger operator $\mathfrak{S}^{\eta}$ can be written as

$$\nabla_{k} = \mathfrak{S}^{\eta}.$$

The coordinate expression of k is given by

$$k_{0}(\Psi^{\eta}) = i\,\big(\tfrac{\hbar}{2m}\,\check{\Delta}^{o}\psi^{\eta} + \tfrac{m}{\hbar}\,a_{0}\,\psi^{\eta}\big).$$

PROOF. It follows immediately from a comparison of coordinate expressions of ∇_{k} and $\mathfrak{S}^{\eta}$. ❑

We can easily see that the connection k restricts to the subsystem

$$\sigma^{c}: S^{c}\boldsymbol{Q}^{\eta} \to \boldsymbol{T}.$$

REMARK II.6.2.1. Let o be an observer. Then, the observed vertical Laplacian can be regarded as a fibred morphism over $\boldsymbol{T}$

$$\check{\Delta}^{o}: \boldsymbol{SQ}^{\eta} \to \boldsymbol{SQ}^{\eta}.$$

Moreover, we can easily see that $\check{\Delta}^{o}$ restricts to the subsystem

$$\sigma^{c}: S^{c}\boldsymbol{Q}^{\eta} \to \boldsymbol{T}.$$ ❑

LEMMA II.6.2.1. Let o be an observer. Then, the observed vertical Laplacian is Hermitian. Namely, for each $\hat{\Phi}^{\eta}, \hat{\Psi}^{\eta} \in \mathscr{S}(S^{c}\boldsymbol{Q}^{\eta} \to \boldsymbol{T})$, we have

$$\langle \check{\Delta}^{o}\hat{\Phi}^{\eta} \mid \hat{\Psi}^{\eta} \rangle = \langle \hat{\Phi}^{\eta} \mid \check{\Delta}^{o}\hat{\Psi}^{\eta} \rangle.$$

PROOF. We have

$$g^{ij}\,(\bar{\nabla}^{o}{}_{i}\bar{\nabla}^{o}{}_{j}\bar{\varphi}\;\psi + K_{i}{}^{h}{}_{j}\;\bar{\nabla}^{o}{}_{h}\bar{\varphi}\;\psi - \bar{\varphi}\;\nabla^{o}{}_{i}\nabla^{o}{}_{j}\psi - \bar{\varphi}\;K_{i}{}^{h}{}_{j}\;\nabla^{o}{}_{h}\psi) =$$

$$= g^{ij}\,\big(\bar{\nabla}^{o}{}_{i}(\bar{\nabla}^{o}{}_{j}\bar{\varphi}\;\psi) + K_{i}{}^{h}{}_{j}\;\bar{\nabla}^{o}{}_{h}\bar{\varphi}\;\psi - \bar{\nabla}^{o}{}_{j}\bar{\varphi}\;\partial_{i}\psi\;+$$

$$-\;\nabla^{o}{}_{i}(\bar{\varphi}\;\nabla^{o}{}_{j}\psi) - \bar{\varphi}\;K_{i}{}^{h}{}_{j}\;\nabla^{o}{}_{h}\psi + \partial_{i}\bar{\varphi}\;\nabla^{o}{}_{j}\psi\big) =$$

$$= g^{ij} \big(\partial_i(\bar\nabla^o{}_j\bar\varphi\;\psi) + K_i{}^h{}_j\;\bar\nabla^o{}_h\bar\varphi\;\psi - \bar\nabla^o{}_j\bar\varphi\;\partial_i\psi + i\,\frac{m}{h}\,a_i\,(\bar\nabla^o{}_j\bar\varphi\;\psi)$$

$$- \partial_i(\bar\varphi\;\nabla^o{}_j\psi) - \bar\varphi\;K_i{}^h{}_j\;\nabla^o{}_h\psi + \partial_i\bar\varphi\;\nabla^o{}_j\psi + i\,\frac{m}{h}\,a_i\,(\bar\varphi\;\nabla^o{}_j\psi)\big) =$$

$$= g^{ij}\,\big(\partial_i(\bar\nabla^o{}_j\bar\varphi\;\psi) + K_i{}^h{}_j\;\bar\nabla^o{}_h\bar\varphi\;\psi - \partial_i(\bar\varphi\;\nabla^o{}_j\psi) - \bar\varphi\;K_i{}^h{}_j\;\nabla^o{}_h\psi\big) =$$

$$= g^{ij}\,\frac{\partial_i(\bar\nabla^o{}_j\bar\varphi\;\psi\;\sqrt{|g|}\,)}{\sqrt{|g|}} - g^{ij}\,\frac{\partial_i(\bar\varphi\;\nabla^o{}_j\psi\;\sqrt{|g|}\,)}{\sqrt{|g|}}.$$

Moreover, the integral on $\boldsymbol{E}_\tau$, through a partition of the unity, of each of the above terms vanishes, in virtue of the Gauss theorem. ❑

THEOREM II.6.2.2. The connection k is Hermitian. Namely, for each $\hat\Phi^\eta$, $\hat\Psi^\eta \in \mathcal{S}(S^c\boldsymbol{Q}^\eta \to \boldsymbol{T})$, we have

$$d\langle\hat\Phi^\eta \mid \hat\Psi^\eta\rangle = \langle\nabla_{\hat k}\hat\Phi^\eta \mid \hat\Psi^\eta\rangle + \langle\hat\Phi^\eta \mid \nabla_{\hat k}\hat\Psi^\eta\rangle.$$

PROOF. We have

$$\partial_0(\bar\varphi^\eta\;\psi^\eta) = \partial_0\bar\varphi^\eta\;\psi^\eta + \bar\varphi^\eta\;\partial_0\psi^\eta =$$

$$= \bar\nabla_{k0}\bar\varphi^\eta\;\psi^\eta + \bar\varphi^\eta\;\nabla_{k0}\psi^\eta - i\,\frac{\hbar}{2m}\,(\bar{\check\Delta}{}^o\bar\varphi^\eta\;\psi^\eta - \bar\varphi^\eta\;\check\Delta^o\psi^\eta).$$

Hence, the Lemma II.6.2.1 yields the result. ❑

II.6.3. Quantum operators on the quantum system

Now we are in a position to translate our quantum Lie operators (see § II.4.2) into quantum operators, in the infinite dimensional context of systems.

REMARK II.6.3.1. A fibred morphism over $\boldsymbol{T}$

$$\hat\Xi : S\boldsymbol{Q}^\eta \to S\boldsymbol{Q}^\eta$$

can be regarded as a sheaf morphism

$$\hat{\Xi}. : \mathcal{S}(\boldsymbol{SQ}^{\eta}\to\boldsymbol{T}) \to \mathcal{S}(\boldsymbol{SQ}^{\eta}\to\boldsymbol{T}) : \hat{\Psi}^{\eta} \mapsto \hat{\Xi}.\hat{\Psi}^{\eta} := \hat{\Xi}\circ\hat{\Psi}^{\eta}.$$

Moreover, $\hat{\Xi}$ yields the sheaf morphism

$$\Xi : \mathcal{S}(\boldsymbol{Q}^{\eta}\to\boldsymbol{E}) \to \mathcal{S}(\boldsymbol{Q}^{\eta}\to\boldsymbol{E}),$$

characterised by

$$\widehat{\Xi(\Psi^{\eta})} = \hat{\Xi}.\hat{\Psi}^{\eta}.$$

The map

$$\hat{\Xi} \mapsto \Xi$$

is a bijection. ❑

Henceforth, in order to deal with objects globally defined on the fibres of $t:\boldsymbol{E}\to\boldsymbol{T}$, we need to restrict our attention to quantisable functions, which are tube-like with respect to the fibring $J_1\boldsymbol{E}\to\boldsymbol{T}$.

We shall denote the corresponding sheaves by

$$\mathcal{Q}_t(J_1\boldsymbol{E}) \subset \mathcal{Q}(J_1\boldsymbol{E}) \qquad \mathcal{Q}_{tc}(J_1\boldsymbol{E}) \subset \mathcal{Q}_c(J_1\boldsymbol{E}) \qquad \mathcal{Q}_{t0}(J_1\boldsymbol{E}) \subset \mathcal{Q}_0(J_1\boldsymbol{E})$$

$$\mathcal{L}_t(\boldsymbol{Q}^{\eta}) \subset \mathcal{L}(\boldsymbol{Q}^{\eta}) \qquad \mathcal{L}_{tc}(\boldsymbol{Q}^{\eta}) \subset \mathcal{L}_c(\boldsymbol{Q}^{\eta}) \qquad \mathcal{L}_{t0}(\boldsymbol{Q}^{\eta}) \subset \mathcal{L}_0(\boldsymbol{Q}^{\eta}).$$

PROPOSITION II.6.3.1. Let $f\in\mathcal{Q}_t(J_1\boldsymbol{E})$ and consider $Y_f\in\mathcal{L}_t(\boldsymbol{Q}^{\eta})$.
Then, we obtain the sheaf morphism

$$\hat{Y}_f : \mathcal{S}(\boldsymbol{SQ}^{\eta}\to\boldsymbol{T}) \to \mathcal{S}(\boldsymbol{SQ}^{\eta}\to\boldsymbol{T}) : \hat{\Psi}^{\eta} \mapsto \widehat{Y_f\Psi^{\eta}}.$$ ❑

COROLLARY II.6.3.1. In the particular case when $f\in\mathcal{Q}_{t0}(J_1\boldsymbol{E})$, $\hat{Y}_f$ acts pointwisely on the sections of $\boldsymbol{SQ}^{\eta}\to\boldsymbol{T}$, hence it can be regarded as a local fibred morphism over $\boldsymbol{T}$

$$\hat{Y}_f : \boldsymbol{SQ}^{\eta}\to\boldsymbol{SQ}^{\eta}.$$ ❑

If $f'' \neq 0$, then $\hat{Y}_f$ is a differential operator of first order on the sections of $\boldsymbol{SQ}^{\eta}\to\boldsymbol{T}$.

However, we are looking for a pointwise operator for all quantisable functions. We shall solve the problem again by means of a criterion of projectability, according to the following important result.

THEOREM II.6.3.1. For any $f \in \mathcal{Q}_t(J_1\boldsymbol{E})$, the sheaf morphism

$$\hat{\Xi}_f := \hat{Y}_f - i\, \boldsymbol{f}'' \lrcorner \nabla_{\hat{k}} : \mathcal{S}(\boldsymbol{SQ}^\eta \to \boldsymbol{T}) \to \mathcal{S}(\boldsymbol{SQ}^\eta \to \boldsymbol{T})$$

acts pointwisely on the sections of $\boldsymbol{SQ}^\eta \to \boldsymbol{T}$, hence it can be regarded as a local fibred morphism over $\boldsymbol{T}$

$$\hat{\Xi}_f := \hat{Y}_f - i\, \boldsymbol{f}'' \lrcorner \nabla_{\hat{k}} : \boldsymbol{SQ}^\eta \to \boldsymbol{SQ}^\eta.$$

Its coordinate expression is

$$\Xi_f(\Psi^\eta) =$$

$$= \Big(f_o\, \psi^\eta + i\, \tfrac{1}{2}\, (\partial_0 f'' - \tfrac{\hbar}{m}\, \partial_i f^i)\, \psi^\eta - i\, \tfrac{\hbar}{m}\, f^i\, \nabla^o{}_i \psi^\eta - f''\, \tfrac{\hbar}{2m}\, \check{\Delta}^o \psi^\eta\Big)\, b \otimes \sqrt{\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3}.$$

In the particular case when $f \in \mathcal{Q}_{t0}(J_1\boldsymbol{E})$, we recover

$$\hat{\Xi}_f := \hat{Y}_f. \qquad \square$$

Of course, the above operator Ξ_f is intrinsic, by construction. However, we stress that its coordinate expression cannot be easily guessed by means of standard arguments of differential geometry. Actually, this result is a "miracle" produced by the specific structure of the quantum bundle and its quantum connection.

Then, we are led to introduce the following definition.

DEFINITION II.6.3.1. A *quantum operator* is defined to be the local fibred morphism over $\boldsymbol{T}$

$$\hat{\Xi}_f := \hat{Y}_f - i\, \boldsymbol{f}'' \lrcorner \hat{\nabla}_k : \boldsymbol{SQ}^\eta \to \boldsymbol{SQ}^\eta,$$

associated with a quantisable function $f \in \mathcal{Q}_t(J_1\boldsymbol{E})$. ♣

The quantum operators constitute a sheaf (with respect to the tube-topology of the fibred set $\boldsymbol{SQ}^\eta \to \boldsymbol{T}$), which is denoted by

$$\mathcal{Q}(\boldsymbol{SQ}^{\eta}).$$

Moreover, we shall be concerned with the subsheaves of quantum operators corresponding to quantisable functions with constant and vanishing time component

$$\mathcal{Q}_c(\boldsymbol{SQ}^{\eta}) \subset \mathcal{Q}(\boldsymbol{SQ}^{\eta}) \qquad \mathcal{Q}_0(\boldsymbol{SQ}^{\eta}) \subset \mathcal{Q}_c(\boldsymbol{SQ}^{\eta}).$$

THEOREM II.6.3.2. The map

$$\mathcal{Q}(J_1\boldsymbol{E}) \to \mathcal{Q}(\boldsymbol{SQ}^{\eta}) : f \mapsto \hat{\Xi}_f$$

is a sheaf isomorphism.

PROOF. It follows from the coordinate expression of $\Xi_f(\Psi^{\eta})$. ❑

Thus, this is our implementation of the correspondence principle.

Now, let us compute the main examples of quantum Lie operators.

Let us choose a frame of reference (u^0, o) and a related fibred space-time chart (x^0, y^i). We observe that, the definition of the quantum operators associated with the functions y^i and $p_i/\hbar$ requires that these functions be tube-like of the fibred manifold $\boldsymbol{E} \to \boldsymbol{T}$; hence the space-time chart must be defined on a space-time tube.

EXAMPLE II.6.3.1. We have

$$\hat{\Xi}_{x^0}(\hat{\Psi}^{\eta}) = x^0\, \hat{\Psi}^{\eta} \qquad \hat{\Xi}_{y^i}(\hat{\Psi}^{\eta}) = y^i\, \hat{\Psi}^{\eta}$$

$$\Xi_{p_i/\hbar}(\Psi^{\eta}) := -\, i\, \partial_i \psi^{\eta}\, b \otimes \sqrt{\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3}$$

$$\Xi_{H/\hbar}(\Psi^{\eta}) := -\left(\frac{\hbar}{2m}\, \check{\Delta}^o \psi^{\eta} + \frac{m}{\hbar}\, a_0\, \psi^{\eta} \right) b \otimes \sqrt{\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3}. \quad ❑$$

In the special relativistic Galilei case, the above operators coincide with the corresponding operators of standard quantum mechanics.

II.6.4. Commutators of quantum operators

Next, we study the commutators of quantum operators.

If $f, f' \in \mathcal{Q}(J_1\boldsymbol{E})$ are quantisable functions, then we define the bracket of the associated quantum operators to be the sheaf morphism

$$[\hat{\Xi}_f \,,\, \hat{\Xi}_{f'}] := -\, i\, (\hat{\Xi}_f \circ \hat{\Xi}_{f'} - \hat{\Xi}_{f'} \circ \hat{\Xi}_f) : \boldsymbol{SQ}^{\eta} \to \boldsymbol{SQ}^{\eta}.$$

However, we stress that, in general, the bracket of two quantum operators needs not to be a quantum operator.

Lemma II.6.4.1. Let $f, f' \in \mathcal{Q}_t(J_1\boldsymbol{E})$ be quantisable functions. Then, we obtain

$$[\hat{\Xi}_f \,,\, \hat{\Xi}_{f'}] =$$

$$= \hat{Y}_{[f,f']} - \big(\hat{Y}_f \circ (\boldsymbol{f}'^{\shortparallel} \lrcorner \hat{\nabla}_k) - \hat{Y}_{f'} \circ (\boldsymbol{f}^{\shortparallel} \lrcorner \hat{\nabla}_k)\big) - \big((\boldsymbol{f}^{\shortparallel} \lrcorner \hat{\nabla}_k) \circ \hat{Y}_{f'} - (\boldsymbol{f}'^{\shortparallel} \lrcorner \hat{\nabla}_k) \circ \hat{Y}_f\big) +$$

$$+\, i\, \big((\boldsymbol{f}^{\shortparallel} \lrcorner \hat{\nabla}_k) \circ (\boldsymbol{f}'^{\shortparallel} \lrcorner \hat{\nabla}_k) - (\boldsymbol{f}'^{\shortparallel} \lrcorner \hat{\nabla}_k) \circ (\boldsymbol{f}^{\shortparallel} \lrcorner \hat{\nabla}_k)\big). \qquad \square$$

Henceforth, we shall restrict our attention to quantisable functions with constant time component. In fact, this hypothesis will yield important results. On the other hand, this is a reasonable restriction from the viewpoint of physics, because all functions which are relevant for physics are of the above type.

Theorem II.6.4.1. Let $f, f' \in \mathcal{Q}_{tc}(J_1\boldsymbol{E})$ be quantisable functions with constant time component. Then, the above commutator reduces to

$$[\hat{\Xi}_f \,,\, \hat{\Xi}_{f'}] =$$

$$= \hat{Y}_{[f,f']} + \big(\hat{Y}_{f'} \circ (\boldsymbol{f}^{\shortparallel} \lrcorner \hat{\nabla}_k) - \hat{Y}_f \circ (\boldsymbol{f}'^{\shortparallel} \lrcorner \hat{\nabla}_k)\big) - \big((\boldsymbol{f}^{\shortparallel} \lrcorner \hat{\nabla}_k) \circ \hat{Y}_{f'} - (\boldsymbol{f}'^{\shortparallel} \lrcorner \hat{\nabla}_k) \circ \hat{Y}_f\big). \qquad \square$$

Corollary II.6.4.1. Let $f, f' \in \mathcal{Q}_{t0}(J_1\boldsymbol{E})$ be quantisable functions with vanishing time component. Then, we obtain

$$[\hat{\Xi}_f\,,\,\hat{\Xi}_{f'}] := [\hat{Y}_f\,,\,\hat{Y}_{f'}] = \hat{Y}_{[f,f']}. \qquad \square$$

COROLLARY II.6.4.2. Let $f \in \mathcal{Q}_{tc}(J_1\boldsymbol{E})$ be a quantisable function with constant time component and $f' \in \mathcal{Q}_{t0}(J_1\boldsymbol{E})$ a quantisable functions with vanishing time component. Then, we obtain

$$[\hat{\Xi}_f\,,\,\hat{\Xi}_{f'}] = \hat{Y}_{[f,f']} + \big(\hat{Y}_{f'}\circ(\boldsymbol{f}''\lrcorner\hat{\nabla}_k) - (\boldsymbol{f}''\lrcorner\hat{\nabla}_k)\circ\hat{Y}_{f'}\big). \qquad \square$$

EXAMPLE II.6.4.1. For each quantisable function $f \in \mathcal{Q}_t(\mathrm{J}_1\mathbf{E})$, we have

$$[\hat{\Xi}_{x^0},\,\hat{\Xi}_f] = 0.$$

For each quantisable function $f \in \mathcal{Q}_{tc}(\mathrm{J}_1\mathbf{E})$, we have

$$[\hat{\Xi}_{y^i},\,\hat{\Xi}_f](\Psi^\eta) = \frac{\hbar}{m}\Big(f^i\,\psi^\eta + f''\,\big(g^{ij}\,\nabla^o{}_j\psi^\eta + \tfrac{1}{2}\,\partial_j g^{ij}\,\psi^\eta\big)\Big)\,b\otimes\sqrt{\check{d}^1\wedge\check{d}^2\wedge\check{d}^3}.$$

In particular, we obtain

$$[\hat{\Xi}_{y^i},\,\hat{\Xi}_{p_j/\hbar}] = \delta^i_j. \qquad \square$$

Eventually, we can state the following important result.

LEMMA II.6.4.2. For each quantisable function $f \in \mathcal{Q}_t(J_1\boldsymbol{E})$, the corresponding quantum operator restricts to the system of tube-sections with constant support

$$\hat{\Xi}_f : S^c\boldsymbol{Q}^\eta \to S^c\boldsymbol{Q}^\eta. \qquad \square$$

THEOREM II.6.4.2. Let $f \in \mathcal{Q}_{tc}(J_1\boldsymbol{E})$ be a quantisable function with constant time component. Then, the corresponding quantum operator $\hat{\Xi}_f$ is symmetric, i.e., for each $\hat{\Phi}^\eta,\,\hat{\Psi}'^\eta \in \mathcal{S}(S^c\boldsymbol{Q}^\eta \to \boldsymbol{T})$,

$$\langle\hat{\Xi}_f\hat{\Phi}^\eta \mid \hat{\Psi}^\eta\rangle = \langle\hat{\Phi}^\eta \mid \hat{\Xi}_f\hat{\Psi}^\eta\rangle.$$

PROOF. We have

$$\langle \hat{\Xi}_f \hat{\Phi}^\eta \mid \hat{\Psi}^\eta \rangle =$$

$$= \int \Big(i \, \frac{\hbar}{2m} \, \frac{\partial_i (f^i \sqrt{|g|})}{\sqrt{|g|}} \, \bar{\varphi} + i \, \frac{\hbar}{m} \, f^i \, \partial_i \bar{\varphi} + (f_\circ - f^i \, a_i) \, \bar{\varphi} - f'' \, \frac{\hbar}{2m} \, \check{\Delta}^o \bar{\varphi} \Big) \, \psi \, \eta =$$

$$= \int \Big(i \, \frac{\hbar}{m} \, \frac{\partial_i (f^i \, \bar{\varphi} \, \psi \, \sqrt{|g|})}{\sqrt{|g|}} - i \, \frac{\hbar}{2m} \, \frac{\partial_i (f^i \sqrt{|g|})}{\sqrt{|g|}} \, \bar{\varphi} \, \psi - i \, \frac{\hbar}{m} \, \bar{\varphi} \, f^i \, \partial_i \psi +$$

$$+ (f_\circ - f^i \, a_i) \, \bar{\varphi} \, \psi - f'' \, \frac{\hbar}{2m} \, \bar{\check{\Delta}}^o \bar{\varphi} \, \psi \Big) \, \eta .$$

In an analogous way, we obtain

$$\langle \hat{\Phi}^\eta \mid \hat{\Xi}_f \hat{\Psi}^\eta \rangle =$$

$$= \int \Big(- i \, \frac{\hbar}{2m} \, \frac{\partial_i (f^i \sqrt{|g|})}{\sqrt{|g|}} \, \bar{\varphi} \, \psi - i \, \frac{\hbar}{m} \, \bar{\varphi} \, f^i \, \partial_i \psi + (f_\circ - f^i \, a_i) \, \bar{\varphi} \, \psi - f'' \, \frac{\hbar}{2m} \, \bar{\varphi} \, \check{\Delta}^o \bar{\psi} \Big) \, \eta .$$

Hence, the proof follows from the Gauss theorem, Lemma II.6.4.2 and Lemma II.6.2.1. ❑

Hence, under reasonable hypothesis on the quantisable function f, the associated quantum operator $\hat{\Xi}_f$ can be extended to a self-adjoint operator on the quantum Hilbert $H\boldsymbol{Q} \to \boldsymbol{T}$ bundle obtained by completion of $S^c\boldsymbol{Q} \to \boldsymbol{T}$.

In this way, we can apply the standard probabilistic interpretation of quantum mechanics to our approach. The only caution to be taken concerns the fact that we do not deal with a unique Hilbert space, but with a fibred set of Hilbert spaces equipped with a connection.

II.6.5. The Feynmann path integral

Eventually, we show how the Feynmann path integral principle can be formulated in the present framework. Of course, we are aware of the serious problems dealing with the existence of an appropriate measure and we do not offer any help to overcome these difficulties. However, we can exhibit a nice geometrical interpretation of the Feynmann amplitudes.

Let us consider the parallel transport on the fibred manifold

$$\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$$

associated with the quantum connection ч (see § II.1.4).

Namely, let $s : \boldsymbol{T} \to \boldsymbol{E}$ be a (smooth) motion. Then, consider the equation

$$(*) \qquad (T j_1 s) \lrcorner \nabla_{ч} \Psi^{\uparrow} = 0$$

in the unknown section $\Psi \circ s : \mathbf{T} \to \boldsymbol{Q}$, projectable on s; its coordinate expression is

$$\partial_0(\psi \circ s) - i\,(L/\hbar) \circ j_1 s\;(\psi \circ s) = 0.$$

Equation $(*)$ can be integrated in any finite interval[23] $\boldsymbol{I} \subset \boldsymbol{T}$. Its solution is of the type

$$(**) \qquad q(\tau) := (\psi \circ s)(\tau) = q_0\, exp\Big(\frac{i}{\hbar} \int_{[\tau_0, \tau]} (\boldsymbol{L} \circ j_1 s)\Big), \qquad q_0 \in \mathbb{C},$$

for each $\tau_0, \tau \in \boldsymbol{T}$ such that $s([\tau_0, \tau]) \subset \boldsymbol{E}$ belongs to the domain of a quantum base $b : \boldsymbol{E} \to \boldsymbol{Q}$. For greater times, the solution will be obtained by adding the gauged analogous contributions of the different quantum charts.

Hence, the section $s : \boldsymbol{T} \to \boldsymbol{E}$ yields the complex linear isometry

$$\Pi_{(s,\tau_0,\tau)} : \boldsymbol{Q}_{s(\tau_0)} \to \boldsymbol{Q}_{s(\tau)} : q_0\, b(\tau_0) \mapsto q(\tau)\, b(\tau),$$

for any $\tau_0, \tau \in \boldsymbol{T}$.

The above map can be easily extended to the case when $s : \boldsymbol{T} \to \boldsymbol{E}$ is "broken", i.e. continuous and almost everywhere smooth.

Each map $\Pi_{(s,\tau_0,\tau)}$ as above is said to be a *Feynmann amplitude*.

The Feynmann integral can be expressed heuristically as follows.

For $e_0, e \in \boldsymbol{E}$, one might wish to define the complex linear isometry

$$\Pi_{(e_0,e)} := \sum \Pi_{(s,\tau_0,\tau)} : \boldsymbol{Q}_{e_0} \to \boldsymbol{Q}_e,$$

---

[23] According to our assumptions (smooth fields and motions), the integral $\int(\boldsymbol{L} \circ j_1 s)$ exists on any finite time interval.

with

$$\tau_0 := t(e_0) \in \boldsymbol{T} \qquad t(e) := \tau \in \boldsymbol{T},$$

where the "sum" (taken with respect to a suitable measure[24]) is extended to all broken motions such that

$$s(\tau_0) = e_0 \qquad s(\tau) = e.$$

We set

$$\Pi : \boldsymbol{E}\times\boldsymbol{E} \to L(\boldsymbol{Q},\boldsymbol{Q}) : (e_0,e) \mapsto \Pi_{(e_0,e)}.$$

Then, for $\tau_0,\tau\in\boldsymbol{T}$, one might wish to define the complex linear map

$$K_{(\tau_0,\tau)} : H\boldsymbol{Q}^{\eta}{}_{\tau_0} \to H\boldsymbol{Q}^{\eta}{}_{\tau} : \hat{\Psi}^{\eta}{}_{\tau_0} \mapsto \hat{\Psi}^{\eta}{}_{\tau},$$

where

$$\hat{\Psi}_{\tau} : \boldsymbol{E}_{\tau} \to \boldsymbol{Q}_{\tau} : e \mapsto \int_{\boldsymbol{E}_{\tau_0}} \Pi_{(\,\cdot\,,e)}\circ\hat{\Psi}_{\tau_0}\,\eta.$$

We remark that, if Π exists and is not too singular, then, for each $e\in\boldsymbol{E}_{\tau}$, the above integral makes sense because the integrand is a map on $\boldsymbol{E}_{\tau_0}$ with values into the fixed vector space $\boldsymbol{Q}_{\tau}$.

Moreover, suppose that the Schrödinger equation on the Hilbert bundle can be integrated locally. According to the interpretation of this equation as a connection, it yields, for $\tau\in\boldsymbol{T}$ sufficiently close to $\tau_0\in\boldsymbol{T}$, a parallel transport

$$S_{(\tau_0,\tau)} : H\boldsymbol{Q}^{\eta}{}_{\tau_0} \to H\boldsymbol{Q}^{\eta}{}_{\tau}.$$

Then, the Feynmann guess can be expressed by saying that

$$S_{(\tau_0,\tau)} = K_{(\tau_0,\tau)}.$$

[24] Unfortunately, such a measure is known not to exist at all!

II.6 - Quantum two-body mechanics

So far, we have been dealing with the quantum mechanics of a charged particle; accordingly, the quantum bundle has been based on the classical space-time associated with a classical particle.

Now, we modify slightly our model in order to describe a closed system constituted by two quantum particles interacting through the classical gravitational and electromagnetic fields. Thus, we no longer consider an external source of the classical fields, but the source is constituted by the particles themselves. For this purpose, it suffices to substitute the pattern space-time with the multi-space-time as base of the quantum bundle.

This scheme can be developed for any $n \geq 1$. However, we do it explicitly only for the case $n = 2$. The reader can generalise it without any difficulty.

We do not find the most general solution of field equations, but we just exhibit the simplest solution whose symmetries and boundary conditions are physically sensible. Then, the quantum dynamics follows easily. This solution is nothing but a Galilei general relativistic formulation of the well known standard quantum two body problem.

II.6.1. The quantum bundle and connection over the multi-space-time

We start by introducing the quantum bundle and connection over the multi-space-time associated with two classical particles.

ASSUMPTION QTB1. We assume the *quantum bundle* to be a Hermitian line bundle over reduced multi-space-time (see § I.7.1 and § II.1.1)

$$\pi : \boldsymbol{Q} \to \mathbb{E}' . \qquad \diamond$$

ASSUMPTION QTB2. We assume the *quantum connection* to be a connection on the bundle $\boldsymbol{Q}^\uparrow \to J_1\mathbb{E}'$ (see § II.1.4)

$$\Psi : \boldsymbol{Q}^\uparrow \to T^* J_1\mathbb{E}' \underset{J_1\mathbb{E}'}{\otimes} T\boldsymbol{Q}^\uparrow$$

with the following properties:

i) ч is Hermitian

ii) ч is universal,

iii) the curvature

$$R_{\text{ч}} : J_1\mathbb{E}' \to \overset{2}{\wedge} T^* J_1\mathbb{E}'' \underset{J_1\mathbb{E}'}{\otimes} (\boldsymbol{Q}^* \underset{\mathbb{E}'}{\otimes} \boldsymbol{Q})$$

of ч is given by

$$(R_{\text{ч}}) \qquad R_{\text{ч}} = i\,\frac{m}{\hbar}\,\Omega \otimes \text{и}. \qquad \diamond$$

II.6.2. The two-body solution for the quantum bundle and connection

Next, we exhibit a distinguished realisation of quantum bundle and quantum connection whose symmetries and boundary values are physically appropriate.

We consider a quantum bundle $\pi : \boldsymbol{Q} \to \mathbb{E}$, which is trivial (but without any distinguished trivialisation).

PROPOSITION II.6.2.1. Let b be a global quantum gauge and o a pattern Newtonian observer (see § I.7.1, I.7.2). Then, the connection

$$\text{ч} = \text{ч}'' + i\,\frac{1}{\hbar}\,(\mathbb{G} + \mathbb{P} + m\,@)\,\text{и},$$

where ч″ is the flat connection associated with the quantum gauge, $\mathbb{G}$ and $\mathbb{P}$ are the classical kinetic energy and momentum forms associated with the observer o (see § I.7.2) and @ is the potential described in Rem. I.7.2.2., is a quantum connection. ❑

COROLLARY II.6.2.1. With reference to the normal chart associated with b and to a Cartesian space-time chart adapted to o, we obtain the following coordinate expression (see Rem. I.7.2.3)

$$\text{ч} = d^0 \otimes \partial_0 + d_1{}^i \otimes \partial_{1i} + d_2{}^i \otimes \partial_{2i} + d_{10}{}^i \otimes \partial_{1i}{}^0 + d_{20}{}^i \otimes \partial_{2i}{}^0$$

$$+ i\,\frac{m}{\hbar}\,\big(-\tfrac{1}{2}\,g_{ij}\,(\mu_1\, y_{10}^i\, y_{10}^j + \mu_2\, y_{20}^i\, y_{20}^j)\,d^0 + g_{ij}\,(\mu_1\, y_{10}^j\, d_1{}^i + \mu_2\, y_{20}^j\, d_2{}^i)$$

$$- \tfrac{1}{m} \; \frac{\kappa m_1 m_2 - q_1 q_2}{r} \; d^0 \big) \otimes \text{и} =$$

$$= d^0 \otimes \partial_0 + d_c{}^i \otimes \partial_{ci} + d_r{}^i \otimes \partial_{ri} + d_{c0}{}^i \otimes \partial_{ci}^0 + d_{r0}{}^i \otimes \partial_{ri}^0$$

$$+ i \, \tfrac{m}{\hbar} \; g_{ij} \big(- \tfrac{1}{2} (y_{c0}{}^i \, y_{c0}{}^j + \mu_1 \, \mu_2 \; y_{r0}{}^i \, y_{r0}{}^j) \, d^0 + (y_{c0}{}^j \, d_c{}^i + \mu_1 \, \mu_2 \, y_{r0}{}^j \, d_r{}^i)$$

$$- \tfrac{1}{m} \; \frac{\kappa m_1 m_2 - q_1 q_2}{r} \; d^0 \big) \otimes \text{и}. \qquad \square$$

Then, the two body quantum theory can be developed in full analogy to the one body case and our results fit completely the standard ones. In particular, just as an example, we get the quantum Hamiltonian operator.

REMARK II.6.2.1. Let o be a pattern Newtonian observer and $s : \boldsymbol{T} \to \boldsymbol{E}$ any motion such that $j_1 s = o \circ s$. Moreover, let us consider the affine fibred morphisms over $\boldsymbol{T}$

$$\iota_1 : \mathbb{E} \to \mathbb{E} : e \mapsto \big(e_1, s(t(e)) \big) \qquad \iota_2 : \mathbb{E} \to \mathbb{E} : e \mapsto \big(s(t(e)), e_2 \big)$$

$$\iota_c : \mathbb{E} \to \mathbb{E} : e \mapsto (e_c, e_c) \qquad \iota_r : \mathbb{E} \to \mathbb{E} : e \mapsto \big(s(t(e)) + \mu_2 \, \boldsymbol{r}_1, \; s(t(e)) + \mu_1 \, \boldsymbol{r}_2 \big).$$

Then, we obtain the following "Laplacian" operators

$$\check{\Delta}_1 := \check{\Delta} \circ \iota_1 : \mathcal{M}(\mathbb{E}, \mathbb{C}) \to \mathcal{M}(\mathbb{E}, \mathbb{C}) \qquad \check{\Delta}_1 := \check{\Delta} \circ \iota_1 : \mathcal{M}(\mathbb{E}, \mathbb{C}) \to \mathcal{M}(\mathbb{E}, \mathbb{C})$$

$$\check{\Delta}_c := \check{\Delta} \circ \iota_c : \mathcal{M}(\mathbb{E}, \mathbb{C}) \to \mathcal{M}(\mathbb{E}, \mathbb{C}) \qquad \check{\Delta}_r := \check{\Delta} \circ \iota_r : \mathcal{M}(\mathbb{E}, \mathbb{C}) \to \mathcal{M}(\mathbb{E}, \mathbb{C})$$

which turn out to be independent of the choice of s.

With reference to any Cartesian space-time chart (x^0, y^i) adapted to o, we have the following coordinate expressions

$$\check{\Delta}_1 \psi = g^{i_1 j_1} \, \partial_{i_1 j_1} \psi \qquad \check{\Delta}_2 \psi = g^{i_2 j_2} \, \partial_{i_2 j_2} \psi$$

$$\check{\Delta}_c \psi = g^{i_c j_c} \, \partial_{i_c j_c} \psi \qquad \check{\Delta}_r \psi = g^{i_r j_r} \, \partial_{i_r j_r} \psi,$$

where, by abuse of language, the indices i_1, i_2, i_c, i_r refer to the induced coordinates y_1^i, y_2^i, y_c^i, y_r^i (see § I.7.2). ❑

COROLLARY II.6.2.2. With reference to the normal chart associated with b and to a Cartesian space-time chart adapted to o, we obtain the standard operator (see Rem. I.7.2.3)

$$\Xi_{\mathbb{H}/\hbar}(\Psi^{\eta}) = -\Big(\frac{\hbar}{2m} \big(\frac{1}{\mu_1} \check{\Delta}_1\psi + \frac{1}{\mu_2} \check{\Delta}_2\psi\big) + \frac{1}{\hbar} \frac{\kappa m_1 m_2 - q_1 q_2}{r} \psi\Big) b\otimes\sqrt{\eta}$$

$$= -\Big(\frac{\hbar}{2m} \big(\check{\Delta}_c\psi + \frac{1}{\mu_1\mu_2} \check{\Delta}_r\psi\big) + \frac{1}{\hbar} \frac{\kappa m_1 m_2 - q_1 q_2}{r} \psi\Big) b\otimes\sqrt{\eta}. \qquad ❑$$

III - APPENDIX

This appendix is aimed at recalling a few fundamental notions on manifolds, fibred manifolds, jets, tangent valued forms and general connections, in order to fix our basic terminology and notation.

Some of these ideas and results can be found in any standard book of differential geometry. However, we are concerned with some further geometrical techniques, which can be traced only in a more specialised literature. Then, we think that the reader will appreciate a brief sketch; further details can be found, for instance, in [CCKM], [MM1], [MM2], [Mo2], [Mo3].

The non standard techniques are required by some specific subjects of our theory, hence cannot be avoided. On the other hand, these techniques are part of a general approach to differential geometry, which is able to recover in a very compact scheme many standard notions and results. For this reason the quick summary below has a certain systematic character.

III.1. Fibred manifolds and bundles

In this section we recall a few basic notions on fibred manifolds. For further details, the reader could refer to [CCKM], [MM1], [MM2], [Mo2].

Throughout the paper, we deal with smooth manifolds and maps, unless a different statement is explicitly mentioned.

III.1.1. Fibred manifolds

Let us start with fibred manifolds and bundles without any additional structure.

Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be manifolds.
A local map defined on the open subset $\boldsymbol{U} \subset \boldsymbol{M}$, is often denoted (by abuse of

language) by

$$f : \boldsymbol{M} \to \boldsymbol{N}.$$

The local maps $f : \boldsymbol{M} \to \boldsymbol{N}$ constitute a sheaf

$$\mathcal{M}(\boldsymbol{M}, \boldsymbol{N}) := \{f : \boldsymbol{M} \to \boldsymbol{N}\}.$$

In particular, the sheaf of local real valued functions on $\boldsymbol{M}$ is denoted by

$$\mathcal{F}(\boldsymbol{M}) := \mathcal{M}(\boldsymbol{M}, \mathbb{R}) := \{f : \boldsymbol{M} \to \mathbb{R}\}.$$

We denote the typical manifold chart of $\boldsymbol{M}$ and $\boldsymbol{N}$ by (x^λ) and (y^i).

A *fibred manifold* is defined to be a manifold $\boldsymbol{F}$ together with a surjective map of maximum rank

$$p : \boldsymbol{F} \to \boldsymbol{B}.$$

Let $p : \boldsymbol{F} \to \boldsymbol{B}$ be a fibred manifold, with

$$n := \dim \boldsymbol{B} \qquad l := \dim \boldsymbol{F} - \dim \boldsymbol{B}.$$

The *fibre* over x is denoted by

$$\boldsymbol{F}_x := p^{-1}(x) \subset \boldsymbol{F}.$$

By the rank theorem, $\boldsymbol{F}$ admits a *local fibred splitting* in a neighbourhood of any $y \in \boldsymbol{F}$. Namely, there is an open neighbourhood $\boldsymbol{V} \subset \boldsymbol{F}$ of $y \in \boldsymbol{F}$, a manifold $\boldsymbol{F}_{\boldsymbol{V}}$ and a diffeomorphism

$$\Phi : \boldsymbol{V} \to p(\boldsymbol{V}) \times \boldsymbol{F}_{\boldsymbol{V}},$$

such that

$$pr_1 \circ \Phi = p.$$

We identify the real functions of $\boldsymbol{F}$, which are constant along the fibres with the corresponding functions of $\boldsymbol{B}$; hence, we have the natural inclusion

$$\mathcal{F}(\boldsymbol{B}) \subset \mathcal{F}(\boldsymbol{F}).$$

A *fibred chart* is defined to be a chart

$$(x^\lambda, y^i) \qquad 1 \le \lambda \le n, \ 1 \le i \le l$$

of $\boldsymbol{F}$, adapted to a local fibred splitting.

A *section* of $p:\boldsymbol{F}\to\boldsymbol{B}$ is defined to be a local map

$$s:\boldsymbol{B}\to\boldsymbol{F},$$

such that

$$p\circ s = \mathrm{id}_{\mathrm{B}}.$$

We denote the sheaf of sections of $p:\boldsymbol{F}\to\boldsymbol{B}$ by

$$\mathscr{S}(\boldsymbol{F}) := \mathscr{S}(\boldsymbol{F}\to\boldsymbol{B}) := \{s:\boldsymbol{B}\to\boldsymbol{F}\}.$$

We stress that a section s is assumed to be local, i.e. it is defined on an open subset of the base space; in the particular case when s is defined on the whole base space, we say that it is *global*.

Let $\boldsymbol{F}\xrightarrow{q}\boldsymbol{E}\xrightarrow{p}\boldsymbol{B}$ be a double fibred manifold.

A *tube-section* is defined to be a section of the type

$$s : p^{-1}(\boldsymbol{U}) \subset \boldsymbol{E} \to \boldsymbol{F}$$

where $\boldsymbol{U} \subset \boldsymbol{B}$ is an open subset.

The tube sections constitute a sheaf (with respect to the tube-topology)

$$\mathscr{S}_t(\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B}) \subset \mathscr{S}(\boldsymbol{F}\to\boldsymbol{E}).$$

Let $p:\boldsymbol{F}\to\boldsymbol{B}$ and $q:\boldsymbol{G}\to\boldsymbol{B}$ be fibred manifolds over the same base space.

Their *fibred product* over $\boldsymbol{B}$ is defined to be the fibred manifold

$$\boldsymbol{F}\underset{\boldsymbol{B}}{\times}\boldsymbol{G} := \coprod_{x\in\boldsymbol{B}} \boldsymbol{F}_x\times\boldsymbol{G}_x \to \boldsymbol{B}.$$

If $s\in\mathscr{S}(\boldsymbol{G}\to\boldsymbol{B})$, then we denote the pullback of s by

$$s^{\uparrow} : \boldsymbol{F} \to \boldsymbol{F}\underset{\boldsymbol{B}}{\times}\boldsymbol{G} : f \mapsto \big(f,\, s(p(f))\big).$$

Let $p:\boldsymbol{F}\to\boldsymbol{B}$ and $q:\boldsymbol{G}\to\boldsymbol{C}$ be fibred manifolds.

A *fibred morphism* is defined to be a pair of local maps

$$\Phi:\boldsymbol{F}\to\boldsymbol{G} \qquad \underline{\Phi}:\boldsymbol{B}\to\boldsymbol{C},$$

such that

$$q\circ\Phi = \underline{\Phi}\circ p.$$

Briefly, we say that Φ is a fibred morphism over $\underline{\Phi}$. In particular, if $\boldsymbol{C} = \boldsymbol{B}$ and $\underline{\Phi} = \mathrm{id}_{\boldsymbol{B}}$, then we say that Φ is a fibred morphism over $\boldsymbol{B}$.

Let $p{:}\boldsymbol{F}\to\boldsymbol{B}$ be a fibred manifold.

We say that a local fibred splitting Φ is a *local bundle splitting* if it is of the type

$$\Phi : \boldsymbol{V} = p^{-1}(\boldsymbol{U}) \to \boldsymbol{U}\times\boldsymbol{F}_{\boldsymbol{U}},$$

where $\boldsymbol{U} = p(\boldsymbol{V}) \subset \boldsymbol{B}$ is an open subset.

Then, we say that the fibred manifold $p{:}\boldsymbol{F}\to\boldsymbol{B}$ is a *bundle* if there exists a *trivialising bundle atlas*, i.e. a family of local bundle splittings

$$\{\Phi_\alpha{:}p^{-1}(\boldsymbol{U}_\alpha)\to\boldsymbol{U}_\alpha\times\boldsymbol{F}_0\}_{\alpha\in\mathcal{A}},$$

where $\{\boldsymbol{U}_\alpha\}_{\alpha\in\mathcal{A}}$ is an open covering of $\boldsymbol{B}$.

We are concerned with several bundles, whose fibres are smoothly endowed with algebraic structures; moreover, we need to prolong this bundles and their algebraic structures via the tangent and jet functors. This subject can be treated in several ways; the most standard approach is based on the technique of principal and associated bundles. However, in this report, we prefer to follow a more direct and intrinsic way, which is very suitable for our goals and fits the intrinsic spirit of our viewpoint. Here and in the following sections, we just sketch the main ideas and results; for a more general and detailed treatment the reader can refer to [CCKM].

III.1.2. Structured bundles

Next, we consider bundles whose fibres are equipped with an algebraic structure.

A *vector bundle* is defined to be a bundle $p{:}\boldsymbol{F}\to\boldsymbol{B}$ smoothly equipped with a vector structure on its fibres. Thus, by definition, each fibre $\boldsymbol{F}_x$ is a vector space; moreover, there exists a trivialising bundle atlas, whose type-fibre $\boldsymbol{F}_0$ is a vector space, such that the maps $\Phi_{\alpha x}{:}\boldsymbol{F}_x\to\boldsymbol{F}_0$, with $x\in\boldsymbol{U}_\alpha$, $\alpha\in A$, are linear.

A fibred morphism between vector bundles is said to be *linear* if it yields linear maps between the fibres.

If $p{:}\boldsymbol{F}\to\boldsymbol{B}$ is a vector bundle, then we obtain the global section and the fibred morphisms over $\boldsymbol{B}$

$$0:\boldsymbol{B}\to\boldsymbol{F} \qquad +:\boldsymbol{F}\underset{\boldsymbol{B}}{\times}\boldsymbol{F}\to\boldsymbol{F} \qquad \cdot:\mathbb{R}\times\boldsymbol{F}\to\boldsymbol{F},$$

which fulfill the standard algebraic properties and characterise the vector structure of the bundle.

An *affine bundle* is defined to be a bundle $p:\boldsymbol{F}\to\boldsymbol{B}$ smoothly equipped with an affine structure on its fibres. Thus, by definition, each fibre $\boldsymbol{F}_x$ is an affine space associated with a vector space $\bar{\boldsymbol{F}}_x$; moreover, there exists a trivialising bundle atlas, whose type-fibre $\boldsymbol{F}_0$ is an affine space associated with a vector space $\bar{\boldsymbol{F}}_0$, such that the maps $\Phi_{\alpha x}:\boldsymbol{F}_x\to\boldsymbol{F}_0$, with $x\in\boldsymbol{U}_\alpha$, $\alpha\in A$, are affine.

It can be proved that

$$\bar{\boldsymbol{F}} := \coprod_{x\in\boldsymbol{B}} \bar{\boldsymbol{F}}_x \to \boldsymbol{B}$$

has a natural structure of vector bundle; this is said to be the *vector bundle* associated with the affine bundle.

A fibred morphism between affine bundles is said to be *affine* if it yields affine maps between the fibres.

If $p:\boldsymbol{F}\to\boldsymbol{B}$ is an affine bundle, then we obtain the fibred morphism over $\boldsymbol{B}$

$$+:\boldsymbol{F}\underset{\boldsymbol{B}}{\times}\bar{\boldsymbol{F}}\to\boldsymbol{F},$$

which fulfills the standard algebraic properties and characterises the affine structure of the bundle.

Other algebraic structures on bundles can be defined in this way. For instance, the complex structure can be defined by considering a linear endomorphism ι, such that $\iota^2 = -1$.

III.1.3. *Positive semi-vector bundles*

Furthermore, in order to describe rigorously the units of measurement and to introduce the bundles of densities, we need the notions of a positive semi-vector space and a positive semi-vector bundle.

Let us consider $\mathbb{R}^+ := \{x\in\mathbb{R} \mid x>0\}$ as a *semi-field*, i.e. as an abelian semigroup with respect to the addition and a group with respect to the multiplication.

Then, we define a *semi-vector space* to be an abelian semi-group $\boldsymbol{U}$ with a scalar multiplication by $\mathbb{R}^+$, which fulfills the standard properties

$$(r+s)\,u = r\,u + s\,u \qquad (r\,s)\,u = r\,(s\,u)$$

$$r\,(u+v) = r\,u + r\,v \qquad 1\,u = u \qquad \forall r,s\in\mathbb{R}^+,\ u,v\in\boldsymbol{U}.$$

A vector space $\boldsymbol{V}$ and a basis B yield a semi-vector space in a natural way. In fact, the subset finitely generated by B over $\mathbb{R}^+$ turns out to be a semi-vector space. In particular, $\mathbb{R}^{+n}$, with $n\geq 1$, is a semi-vector space.

Moreover, any vector space $\boldsymbol{V}$ can be regarded as a semi-vector space in a natural way. On the other hand, a semi-vector space $\boldsymbol{U}$ is said to be a *positive semi-vector space* if the scalar multiplication cannot be extended neither to $\mathbb{R}^+\cup\{0\}$, nor to $\mathbb{R}$. Hence, a positive semi-vector space contains neither the zero element, nor the negative of any element. Thus, a semi-vector space is a vector space, or a positive semi-vector space, or a positive semi-vector space extended by the zero element.

Several concepts and results of standard linear and multi-linear algebra related to vector spaces (including linear and multi-linear maps, bases, dimension, tensor products and duality) can be easily repeated for semi- vector spaces and positive semi-vector spaces. The main caution to be taken is to avoid the formulations which involve the zero element. We shall use for semi-vector spaces the standard terminology used for vector spaces.

Let $\boldsymbol{U}$ be a semi-vector space and $\boldsymbol{V}$ a vector space. By regarding $\boldsymbol{V}$ as a semi-vector space, we can consider the tensor product over $\mathbb{R}^+$

$$\boldsymbol{U}\otimes\boldsymbol{V}.$$

We observe that the semi-vector space $\boldsymbol{U}\otimes\boldsymbol{V}$ turns out to be also a vector space, according to the formula

$$\mathbb{R}\times(\boldsymbol{U}\otimes\boldsymbol{V}) \to \boldsymbol{U}\otimes\boldsymbol{V} : (r,\, u\otimes v) \mapsto u\otimes(rv).$$

So, any positive semi-vector space $\boldsymbol{U}$ can be extended to the vector space $\boldsymbol{U}\otimes\mathbb{R}$, through the natural inclusion $\boldsymbol{U}\subset\boldsymbol{U}\otimes\mathbb{R}$.

In particular, we are concerned with 1-dimensional positive semi-vector spaces.

If $\boldsymbol{V}$ is an oriented 1-dimensional vector space, then the positively oriented subset $\boldsymbol{V}^{+} \subset \boldsymbol{V}$ is a positive space.

Moreover, if $\boldsymbol{W}$ is a further vector space, then we obtain the following useful canonical isomorphisms[25]

$$\boldsymbol{V}^{+} \otimes \boldsymbol{W} \simeq \boldsymbol{V} \otimes \boldsymbol{W}$$

and, in particular,

$$\boldsymbol{V}^{+} \otimes \mathbb{R} \simeq \boldsymbol{V} \otimes \mathbb{R} \simeq \boldsymbol{V}.$$

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be 1-dimensional positive semi-vector spaces.

In order to write formulas in a way apparently equal to the standard one used by physicists, it is convenient to treat the elements of 1-dimensional positive spaces as they were numbers and introduce some conventions.

So, we can treat the dual element $\upsilon \in \boldsymbol{U}^{*}$ of $u \in \boldsymbol{U}$ as its "inverse" and write

$$\boldsymbol{U}^{-1} := \boldsymbol{U}^{*} \qquad\qquad \upsilon = \frac{1}{u}.$$

Moreover, if $u \in \boldsymbol{U}$ and $v \in \boldsymbol{V}$, then we often omit the tensor product $\otimes$ and just write

$$u\, v := u \otimes v.$$

Furthermore, we make the canonical identification

$$\boldsymbol{U} \otimes \boldsymbol{U}^{*} \simeq \mathbb{R}^{+}.$$

Let $\mathbf{U}$ be a 1-dimensional positive semi-vector space.

A *square root* of $\mathbf{U}$ is defined to be a 1-dimensional positive semi-vector space $\sqrt{\boldsymbol{U}}$ together with a quadratic map $q : \sqrt{\boldsymbol{U}} \to \boldsymbol{U}$, such that, for each 1-dimensional positive semi-vector space $\sqrt{\boldsymbol{U}}\,'$ and each quadratic map $q' : \sqrt{\boldsymbol{U}}\,' \to \boldsymbol{U}$, there is a unique linear map $l : \sqrt{\boldsymbol{U}} \to \sqrt{\boldsymbol{U}}\,'$, which yields $q' \circ l = q$.

By a standard argument related to universal properties, the square root is unique up to a canonical isomorphism. Moreover, we can easily prove the ex-

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[25] The first tensor product is taken over $\mathbb{R}^{+}$ and the second one over $\mathbb{R}$.

istence of a square root. So, in the following, we shall refer to "the" square root $\sqrt{\boldsymbol{U}}$ of $\boldsymbol{U}$.

The universal property implies that the quadratic map q is a bijection. So, we obtain the inverse bijection $\sqrt{}: \boldsymbol{U} \to \sqrt{\boldsymbol{U}} : u \mapsto \sqrt{u}$.

Moreover, the universal property of the tensor product implies that there is a unique linear isomorphism $i: \sqrt{\boldsymbol{U}} \otimes \sqrt{\boldsymbol{U}} \to \boldsymbol{U}$, which yields $i \circ \delta = q$, where $\delta: \sqrt{\boldsymbol{U}} \to \sqrt{\boldsymbol{U}} \otimes \sqrt{\boldsymbol{U}}$ is the canonical map.

In an analogous way, for every natural number q, we can define the q-root $\boldsymbol{U}^{1/q}$ of $\boldsymbol{U}$.

Therefore, for every pair of natural numbers p and q, we write

$$\boldsymbol{U}^{p/q} := \underbrace{\boldsymbol{U}^{1/q} \otimes \dots \otimes \boldsymbol{U}^{1/q}}_{\text{p times}} \qquad \boldsymbol{U}^{0} := \mathbb{R} \qquad \boldsymbol{U}^{-p/q} := \underbrace{\boldsymbol{U}^{*1/q} \otimes \dots \otimes \boldsymbol{U}^{*1/q}}_{\text{p times}}.$$

If $\boldsymbol{V}$ is an oriented 1-dimensional vector space, then we set $\sqrt{\boldsymbol{V}} := \sqrt{\boldsymbol{V}^{+}}$.

In particular, if $\boldsymbol{W}$ is an oriented n-dimensional vector space, then we obtain the square root positive semi-vector space $\sqrt{\Lambda^{n}\boldsymbol{W}}$.

The above algebraic constructions on positive semi-vector spaces can be easily extended to bundles.

So, a *semi-vector bundle* and a *positive semi-vector bundle* can be defined analogously to a vector bundle. Moreover, if $p: \boldsymbol{F} \to \boldsymbol{B}$ is a 1-dimensional oriented vector bundle, then we obtain in a natural way the positive semi-vector bundle $p: \sqrt{\boldsymbol{F}} \to \boldsymbol{B}$ and the canonical fibred isomorphism $\sqrt{\boldsymbol{F}} \underset{\boldsymbol{B}}{\otimes} \sqrt{\boldsymbol{F}} \simeq \boldsymbol{F}^{+}$ over $\boldsymbol{B}$.

We observe that these constructions lead naturally to a generalisation of the half-densities due to de Rham (see [dR]).

III.2. Tangent prolongation of fibred manifolds

In this section we recall a few basic notions on the tangent prolongation of fibred manifolds. For further details, the reader could refer to [CCKM], [MM1], [MM2], [Mo2].

III.2.1 Tangent prolongation of manifolds

We start by considering just simple manifolds.

We denote the *tangent functor* from the category of manifolds to the category of vector bundles by T.

So, if $\boldsymbol{M}$ is a manifold, then we have the vector bundle

$$\pi_{\boldsymbol{M}} : T\boldsymbol{M} \to \boldsymbol{M}$$

and if $f \in \mathcal{M}(\boldsymbol{M}, \boldsymbol{N})$, then we have the linear fibred morphism over f

$$Tf : T\boldsymbol{M} \to T\boldsymbol{N},$$

or, equivalently, the section

$$df : \boldsymbol{M} \to T^*\boldsymbol{M} \underset{\boldsymbol{M}\times\boldsymbol{N}}{\otimes} T\boldsymbol{N} : x \mapsto T_x f.$$

We set

$$\mathcal{T}(\boldsymbol{M}) := \mathcal{S}(T\boldsymbol{M} \to \boldsymbol{M}) \qquad \mathcal{T}^*(\boldsymbol{M}) := \mathcal{S}(T^*\boldsymbol{M} \to \boldsymbol{M}).$$

We denote the induced linear fibred charts of $T\boldsymbol{M}$ and $T^*\boldsymbol{M}$ by

$$(x^\lambda, \dot{x}^\lambda) \qquad (x^\lambda, \dot{x}_\lambda)$$

and the induced bases of $\mathcal{T}(\boldsymbol{M})$ and $\mathcal{T}^*(\boldsymbol{M})$ by

$$(\partial_\lambda) := (\partial x_\lambda) \qquad (d^\lambda) := (dx^\lambda).$$

Then, we obtain the coordinate expressions

$$(y^i, \dot{y}^i) \circ Tf = (f^i, \partial_\lambda f^i \, \dot{x}^\lambda) \qquad df = \partial_\lambda f^i \, d^\lambda \otimes (\partial_i \circ f).$$

If $c : \mathbb{R} \to \boldsymbol{M}$ is a local map, then we set

$$dc : \mathbb{R} \to T\boldsymbol{M} : \lambda \mapsto Tc(\lambda, 1).$$

If $c : \mathbb{R}\times\boldsymbol{M} \to \boldsymbol{N}$ is a map defined in a neighbourhood of $\{0\}\times\boldsymbol{M} \subset \mathbb{R}\times\boldsymbol{M}$, then we set

$$\partial c : \boldsymbol{M} \to T\boldsymbol{N} : x \mapsto d(c_x)(0).$$

III.2.1 Tangent prolongation of fibred manifolds

Next, we consider fibred manifolds.

Let $p : \boldsymbol{F} \to \boldsymbol{B}$ be a fibred manifold.

We have the vector bundle

$$\pi_{\mathbf{F}} : T\mathbf{F} \to \mathbf{F}$$

and the fibred manifold

$$Tp : T\mathbf{F} \to T\mathbf{B}.$$

If $p:\mathbf{F}\to\mathbf{B}$ is a bundle, then also $Tp:T\mathbf{F}\to T\mathbf{B}$ is a bundle.

The induced fibred chart of $T\mathbf{F}$ is

$$(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i).$$

A vector field $X\in\mathcal{T}(\mathbf{F})$ is said to be *projectable* if it projects over a vector field $\underline{X}\in\mathcal{T}(\mathbf{B})$. The coordinate expression of a projectable vector field is of the type

$$X = X^\mu\, \partial_\mu + X^i\, \partial_i \qquad X^\mu\in\mathcal{F}(\mathbf{B}),\ X^i\in\mathcal{F}(\mathbf{F}).$$

The projectable vector fields constitute the subsheaf

$$\mathcal{P}(\mathbf{F}) \subset \mathcal{T}(\mathbf{F}).$$

We denote the *vertical functor* from the category of fibred manifolds to the category of vector bundles by V.

So, if $p:\mathbf{F}\to\mathbf{B}$ is a fibred manifold, then we have the vertical subbundle over $\mathbf{F}$

$$V\mathbf{F} := ker_{\mathbf{F}} Tp \subset T\mathbf{F},$$

and the exact sequence of vector bundles over $\mathbf{F}$

$$0 \longrightarrow V\mathbf{F} \longrightarrow T\mathbf{F} \longrightarrow \mathbf{F}\times_{\mathbf{B}} T\mathbf{B} \longrightarrow 0.$$

The induced fibred chart of $V\mathbf{F}$ is

$$(x^\lambda, y^i, \dot{y}^i).$$

A vector field $X\in\mathcal{T}(\mathbf{F})$ is said to be *vertical* if it is vertical valued, i.e. if it projects over a zero vector field $\underline{X}\in\mathcal{T}(\mathbf{B})$. The coordinate expression of a vertical vector field is of the type

$$X = X^i\, \partial_i \qquad X^i\in\mathcal{F}(\mathbf{F}).$$

The vertical vector fields constitute the subsheaf

$$\mathcal{V}(\boldsymbol{F}) \subset \mathcal{P}(\boldsymbol{F}) \subset \mathcal{C}(\boldsymbol{F}).$$

If $p: \boldsymbol{F} \to \boldsymbol{B}$ is a vector bundle, then the tangent prolongation of the distinguished section and fibred morphisms

$$T0: T\boldsymbol{B} \to T\boldsymbol{F} \qquad T+: T\boldsymbol{F} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{F} \qquad T\cdot: \mathrm{I\!R} \times T\boldsymbol{F} \to T\boldsymbol{F}$$

makes

$$Tp: T\boldsymbol{F} \to T\boldsymbol{B}$$

a vector bundle.

If $p: \boldsymbol{F} \to \boldsymbol{B}$ is an affine bundle associated with the vector bundle, $\bar{p}: \bar{\boldsymbol{F}} \to \boldsymbol{B}$, then the tangent prolongation of the distinguished fibred morphism

$$T+: T\boldsymbol{F} \underset{T\boldsymbol{B}}{\times} T\bar{\boldsymbol{F}} \to T\boldsymbol{F}$$

makes

$$Tp: T\boldsymbol{F} \to T\boldsymbol{B}$$

an affine bundle associated with the vector bundle

$$T\bar{p}: T\bar{\boldsymbol{F}} \to T\boldsymbol{B}.$$

The tangent prolongation of other algebraic structures on bundles can be obtained in this way. For instance, the complex structure can be prolonged by considering the linear endomorphism $T\iota$, which fulfills $(T\iota)^2 = -1$.

In fact, the algebraic constructions of previous sections concerning positive semi-vector spaces and positive spaces can be easily extended in a smooth way to any vector bundle (see [CCKM]).

The tangent prolongation and the related differential operators extend to semi-vector and positive bundles in a natural way. Here, we add a few observations, which are useful for a clear understanding of our procedures.

The tangent space of a positive space $\boldsymbol{U}$ turns out to be $T\boldsymbol{U} = \boldsymbol{U} \times (\mathrm{I\!R} \otimes \boldsymbol{U})$.

Hence, the vertical bundle of a positive bundle $\boldsymbol{U} \to \boldsymbol{B}$ turns out to be the vector bundle $V\boldsymbol{U} = \boldsymbol{U} \underset{\boldsymbol{B}}{\times} (\mathrm{I\!R} \otimes \boldsymbol{U})$.

The differential operations such as exterior differential, covariant differential, Lie derivative and so on, commute with the tensor product with a vector space or a positive semi-vector space. So, in our differential calculus, we extend the standard notation, such as d, ∇, L_X and so on, to the corresponding operators acting on the appropriate objects tensorialised with a scale factor.

III.3. Jet prolongation of fibred manifolds

In this section we recall a few basic notions on the tangent prolongation of fibred manifolds. For further details, the reader could refer to [CCKM], [MM1], [MM2], [Mo2].

Let $p : \boldsymbol{F} \to \boldsymbol{B}$ be a fibred manifold.
The *k-jet space* is defined to be the fibred manifold

$$p_k : J_k\boldsymbol{F} := \coprod_{x \in \boldsymbol{B}} J_{k\,x}\boldsymbol{F} \to \boldsymbol{B},$$

where

$$J_{k\,x}\boldsymbol{F} := \{[s]_{k\,x}\}_{s \in \mathscr{S}(\boldsymbol{F})}$$

is the set of equivalence classes of sections whose partial derivatives at x coincide up to order k, in any chart, according to the formula

$$s \underset{kx}{\backsim} s' \qquad \Leftrightarrow \qquad \partial_{\underline{\alpha}} s^i(x) = \partial_{\underline{\alpha}} s'^i(x),$$

for all multi-indices of length between 0 and k

$$\underline{\alpha} := (\alpha_1, \dots, \alpha_m) \qquad\qquad 0 \le |\underline{\alpha}| := \alpha_1 + \dots + \alpha_m \le k.$$

If $s : \mathscr{S}(\boldsymbol{F})$, then we obtain the section

$$j_k s : \boldsymbol{B} \to J_k\boldsymbol{F} : x \mapsto [s]_{k\,x}.$$

We have the following natural fibred epimorphisms

$$J_k\boldsymbol{F} \xrightarrow{p^k_h} \longrightarrow J_1\boldsymbol{F} \xrightarrow{p^1_0} J_0\boldsymbol{F} := \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}.$$

Accordingly, we obtain the following sequence of natural linear fibred monomorphisms over $J_k\boldsymbol{F}$

$$V_{k-1} J_k\boldsymbol{F} \hookrightarrow V_{k-2} J_k\boldsymbol{F} \hookrightarrow \dots V_0 J_k\boldsymbol{F} \hookrightarrow V$$

where $V_h J_k \boldsymbol{F}$, $0 \le h \le k-1$, and $V J_k \boldsymbol{F}$ denote the vertical bundles of the fibrings $J_k \boldsymbol{F} \to J_h \boldsymbol{F}$ and $J_k \boldsymbol{F} \to \boldsymbol{B}$, respectively.

Moreover, the fibring

$$J_k \boldsymbol{F} \to J_{k-1} \boldsymbol{F}$$

turns out to be naturally an affine bundle associated with the pullback over $J_{k-1} \boldsymbol{F}$ of the vector bundle

$$S^k T^* \boldsymbol{B} \underset{\boldsymbol{E}}{\otimes} V \boldsymbol{E},$$

where S^k denotes the symmetrised tensor product.

Let $p : \boldsymbol{F} \to \boldsymbol{B}$ and $q : \boldsymbol{G} \to \boldsymbol{B}$ be fibred manifolds over the same base space and $f : \boldsymbol{F} \to \boldsymbol{G}$ a fibred morphism over $\boldsymbol{B}$.

There exists a unique fibred morphism over $f : \boldsymbol{F} \to \boldsymbol{G}$

$$J_k f : J_k \boldsymbol{F} \to J_k \boldsymbol{G},$$

which makes the following diagram commutative, for each section $s \in \mathcal{S}(\boldsymbol{F})$,

$$\begin{tikzcd}
J_k \boldsymbol{F} \arrow[rr, "J_k f"] & & J_k \boldsymbol{G} \\
& \boldsymbol{B} \arrow[ul, "j_k s"] \arrow[ur, "j_k (f \circ s)"'] &
\end{tikzcd}$$

Thus, J_k is a contravariant functor from the category of fibred manifolds over a given base space into itself.

There is a unique fibred monomorphism over $\boldsymbol{F}$

$$\text{д}_k : J_k \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T \boldsymbol{B} \to T J_{k-1} \boldsymbol{F},$$

which makes commutative the following diagram for each section $s \in \mathcal{S}(\boldsymbol{F})$,

$$\begin{tikzcd}
J_k \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T \boldsymbol{B} \arrow[rr, "\text{д}_k"] & & T J_{k-1} \boldsymbol{F} \\
& T \boldsymbol{B} \arrow[ul, "(j_k s)^{\uparrow}"] \arrow[ur, "T j_{k-1} s"'] &
\end{tikzcd}$$

Moreover, д_k is a linear fibred monomorphism over $J_k \boldsymbol{F} \to J_{k-1} \boldsymbol{F}$, projectable over $1_{\boldsymbol{B}}:T\boldsymbol{B}\to T\boldsymbol{B}$ and a fibred monomorphism over $J_{k-1}\boldsymbol{F}\underset{\boldsymbol{B}}{\times}T\boldsymbol{B}$.

Hence, in virtue of the rank theorem, $J_k\boldsymbol{F}\to J_{k-1}\boldsymbol{F}$ turns out to be an affine subbundle over $J_{k-1}\boldsymbol{F}$ of the affine bundle

$$J_1 J_{k-1}\boldsymbol{F}\to J_{k-1}\boldsymbol{F}$$

associated with the vector subbundle

$$S_k T^*\boldsymbol{B}\underset{J_{k-1}\boldsymbol{F}}{\otimes} V\boldsymbol{F} \to T^*\boldsymbol{B}\underset{J_{k-1}\boldsymbol{F}}{\otimes} V J_{k-1}\boldsymbol{F}.$$

Furthermore, we obtain the complementary surjective linear fibred morphism over $J_k\boldsymbol{F}\to J_{k-1}\boldsymbol{F}$

$$\vartheta_k : J_k\boldsymbol{F}\underset{J_{k-1}\boldsymbol{F}}{\times} T J_{k-1}\boldsymbol{F}\to V J_{k-1}\boldsymbol{F}.$$

Thus, д_k and ϑ_k yield a splitting over $J_k\boldsymbol{F}$ of the exact sequence

$$0 \longrightarrow V J_{k-1}\boldsymbol{F} \longrightarrow T J_{k-1}\boldsymbol{F} \longrightarrow J_{k-1}\boldsymbol{F}\underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \longrightarrow 0.$$

Additionally, there is a natural fibred morphism over $J_k\boldsymbol{F}\underset{\boldsymbol{B}}{\times}J_kT\boldsymbol{B}\to J_k\boldsymbol{F}\underset{\boldsymbol{B}}{\times}T\boldsymbol{B}$

$$r_k : J_k T\boldsymbol{F}\to T J_k\boldsymbol{F},$$

which yields the prolongation of a vector field $X:\boldsymbol{F}\to T\boldsymbol{F}$ into the vector field

$$r_k\circ J_k X : J_k\boldsymbol{F}\to T J_k\boldsymbol{F}.$$

If $f\in\mathcal{F}(J_{k-1}\boldsymbol{F})$, then we can write, for each $s\in\mathcal{S}(\boldsymbol{F}\to\boldsymbol{B})$,

$$d(f\circ j_{k-1}s) = (\text{д}_k . f)\circ j_{k-1}f \,:\, \boldsymbol{B}\to T^*\boldsymbol{B}.$$

We denote the induced fibred chart of $J_k\boldsymbol{F}$ by

$$(x^\lambda, y_{\underline{\alpha}}{}^i)_{0\le|\underline{\alpha}|\le k} = (x^\lambda, y^i, y_\lambda{}^i, y_{\lambda\mu}, \dots)_{1\le\lambda\le\mu\le\dots\le m,\ \dots}.$$

Then, we obtain the coordinate expressions

$$\text{д}_k := d^\lambda\otimes\text{д}_{k\lambda} = d^\lambda\otimes(\partial_\lambda + y_{\underline{\alpha}+\lambda}{}^i\,\partial_i^{\underline{\alpha}})_{0\le|\underline{\alpha}|\le k-1} = d^\lambda\otimes(\partial_\lambda + y_\lambda{}^i\,\partial_i + y_{\lambda\mu}{}^i\,\partial_i^\mu + \dots)_{1\le\lambda\le\mu\le m,\ \dots}$$

$$\vartheta_k := \vartheta_{k\,\underline{\alpha}} \otimes \partial_i^{\underline{\alpha}} = (d^i - y_\lambda{}^i\, d^\lambda) \otimes \partial_i + (d_\mu{}^i - y_{\lambda\mu}{}^i\, d^\lambda) \otimes \partial_i^\mu + \ldots .$$

$$(x^\lambda, y_{\underline{\alpha}}{}^i) \circ j_k s = (x^\lambda, y^i, y_\lambda{}^i, y_{\lambda\mu}{}^i, \ldots) \circ j_k s = (x^\lambda, \partial_{\underline{\alpha}} s^i) = (x^\lambda, s^i, \partial_\lambda s^i, \partial_{\lambda\mu} s^i, \ldots)$$

$$(x^\lambda, z^i, z_\lambda{}^i, \ldots) \circ J_k f = (x^\lambda, f^i, \partial_\lambda f^i + y_\lambda{}^h\, \partial_h f^i, \ldots) = (x^\lambda, f^i, d_\lambda . f^i, \ldots).$$

If $p{:}\boldsymbol{F} \to \boldsymbol{B}$ is a vector bundle, then the k-jet prolongation of the distinguished section and fibred morphisms

$$J_k 0{:}\boldsymbol{B} \to J_k\boldsymbol{F} \qquad J_k{+}{:}J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} J_k\boldsymbol{F} \to J_k\boldsymbol{F} \qquad J_k{\cdot}{:}\mathbb{R} \times J_k\boldsymbol{F} \to J_k\boldsymbol{F}$$

makes

$$p_k{:}J_k\boldsymbol{F} \to \boldsymbol{B}$$

a vector bundle.

If $p{:}\boldsymbol{F} \to \boldsymbol{B}$ is an affine bundle associated with the vector bundle, $\bar{p}{:}\bar{\boldsymbol{F}} \to \boldsymbol{B}$, then the k-jet prolongation of the distinguished fibred morphism

$$J_k{+}{:}J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} J_k\bar{\boldsymbol{F}} \to J_k\boldsymbol{F}$$

makes

$$p_k{:}J_k\boldsymbol{F} \to \boldsymbol{B}$$

an affine bundle associated with the vector bundle

$$p_k{:}J_k\bar{\boldsymbol{F}} \to \boldsymbol{B}.$$

The k-jet prolongation of other algebraic structures on bundles can be obtained in this way. For instance, the complex structure can be prolonged by considering the linear endomorphism $J_k\iota$, which fulfills $(J_k\iota)^2 = -1$.

III.4. Tangent valued forms

In this section we recall a few basic notions on the graded Lie algebra of tangent valued forms. For further details, the reader could refer to [MM].

First, let us consider just a manifold, $\boldsymbol{M}$.

A *tangent valued form* of $\boldsymbol{M}$ is defined to be a section

$$\varphi : \boldsymbol{M} \to \wedge T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} T\boldsymbol{M}.$$

We have the coordinate expression

$$\varphi = \varphi_{\lambda_1 \dots \lambda_r}{}^{\mu}\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r} \otimes \partial_{\mu} \qquad \varphi_{\lambda_1 \dots \lambda_r}{}^{\mu} \in \mathcal{F}(\boldsymbol{M}).$$

In particular, the vector fields are tangent valued forms of degree 0.

The Lie bracket of vector fields can be naturally extended to a graded Lie bracket of tangent valued forms, which is called the *Frölicher-Nijenhuis bracket*.

Namely, there is a unique sheaf morphism

$$[,] : \overset{r}{\wedge} \mathcal{T}^*(\boldsymbol{M}) \underset{\mathcal{F}(\boldsymbol{M})}{\otimes} \mathcal{T}(\boldsymbol{M}) \times \overset{s}{\wedge} \mathcal{T}^*(\boldsymbol{M}) \underset{\mathcal{F}(\boldsymbol{M})}{\otimes} \mathcal{T}(\boldsymbol{M}) \to \overset{r+s}{\wedge} \mathcal{T}^*(\boldsymbol{M}) \underset{\mathcal{F}(\boldsymbol{M})}{\otimes} \mathcal{T}(\boldsymbol{M})$$

which on decomposable forms is given by

$$[\alpha \otimes u, \beta \otimes v] = \alpha \wedge \beta \otimes [u, v] +$$

$$+ \alpha \wedge L_u \beta \otimes v - (-1)^{|\alpha||\beta|} \beta \wedge L_v \alpha \otimes u + (-1)^r i_v \alpha \wedge d\beta \otimes u - (-1)^{|\alpha||\beta| + |\beta|}\, i_u \beta \wedge d\alpha \otimes v.$$

We have the coordinate expression

$$[\varphi, \psi] = (\varphi^{\rho}{}_{\lambda_1 \dots \lambda_r}\, \partial_{\rho} \psi^{\mu}{}_{\lambda_{r+1} \dots \lambda_{r+s}} - (-1)^{rs}\, \psi^{\rho}{}_{\lambda_1 \dots \lambda_s}\, \partial_{\rho} \varphi^{\mu}{}_{\lambda_{s+1} \dots \lambda_{r+s}} +$$

$$- r\, \varphi^{\mu}{}_{\lambda_1 \dots \lambda_{r-1} \rho}\, \partial_{\lambda_r} \psi^{\rho}{}_{\lambda_{r+1} \dots \lambda_{r+s}} + (-1)^{rs}\, s\, \psi^{\mu}{}_{\lambda_1 \dots \lambda_{s-1} \rho}\, \partial_{\lambda_s} \varphi^{\rho}{}_{\lambda_{s+1} \dots \lambda_{r+s}}) \cdot d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+s}} \otimes \partial_{\mu}.$$

Thus, $\wedge \mathcal{T}^*(\boldsymbol{M}) \underset{\mathcal{F}(\boldsymbol{M})}{\otimes} \mathcal{T}(\boldsymbol{M})$, together with the F-N bracket, is a sheaf of graded Lie algebras, namely we have

$$[\varphi + \varphi', \psi] = [\varphi, \psi] + [\varphi', \psi] \qquad [\varphi, \psi + \psi'] = [\varphi, \psi] + [\varphi, \psi']$$

$$[k\, \varphi, \psi] = k\, [\varphi, \psi] = [\varphi, k\, \psi]$$

$$[\varphi, \psi] = -(-1)^{|\varphi||\psi|}\, [\psi, \varphi]$$

$$[\vartheta, [\varphi, \psi]] = [[\vartheta, \varphi], \psi] + (-1)^{|\vartheta||\varphi|} [\varphi, [\vartheta, \psi]]$$

$$\varphi, \psi, \vartheta \in \wedge \mathcal{T}^*(\boldsymbol{M}) \underset{\mathcal{F}(\boldsymbol{M})}{\otimes} \mathcal{T}(\boldsymbol{M});\ k \in \mathbb{R}.$$

Now, let us replace the manifold $\boldsymbol{M}$ with the fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and consider the sheaf of tangent valued forms of $\boldsymbol{F}$

$$\wedge \mathcal{T}^*(\boldsymbol{F}) \underset{\mathcal{F}(\boldsymbol{F})}{\otimes} \mathcal{T}(\boldsymbol{F}).$$

A tangent valued form

$$\varphi : \boldsymbol{F} \to \wedge T^* \boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} T \boldsymbol{F},$$

is said to be *projectable* if it projects over a tangent valued form of $\boldsymbol{B}$

$$\underline{\varphi} : \boldsymbol{B} \to \wedge T^* \boldsymbol{B} \underset{\boldsymbol{B}}{\otimes} T \boldsymbol{B}.$$

The projectable tangent valued forms constitute the subsheaf

$$\wedge \mathcal{T}^*(\boldsymbol{B}) \underset{\mathcal{F}(\boldsymbol{B})}{\otimes} \mathcal{P}(\boldsymbol{F}) \subset \wedge \mathcal{T}^*(\boldsymbol{B}) \underset{\mathcal{F}(\boldsymbol{B})}{\otimes} \mathcal{T}(\boldsymbol{F}).$$

Their coordinate expression is of the type

$$\varphi = (\varphi_{\lambda_1 \dots \lambda_r}{}^{\mu}\, \partial_\mu + \varphi_{\lambda_1 \dots \lambda_r}{}^{i}\, \partial_i) \otimes d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r}$$

with

$$\varphi_{\lambda_1 \dots \lambda_r}{}^{\mu} \in \mathcal{F}(\boldsymbol{B}) \qquad \varphi_{\lambda_1 \dots \lambda_r}{}^{i} \in \mathcal{F}(\boldsymbol{F}).$$

As a special case, we have the *vertical* valued forms

$$\varphi : \boldsymbol{F} \to \wedge T^* \boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} V \boldsymbol{F},$$

which project over the zero tangent valued form of B

$$\underline{\varphi} = 0 : \boldsymbol{B} \to \wedge T^* \boldsymbol{B} \underset{\boldsymbol{B}}{\otimes} T \boldsymbol{B}.$$

The vertical valued forms constitute the subsheaf

$$\wedge \mathcal{T}^*(\boldsymbol{B}) \underset{\mathcal{F}(\boldsymbol{B})}{\otimes} \mathcal{V}(\boldsymbol{F}) \subset \wedge \mathcal{T}^*(\boldsymbol{B}) \underset{\mathcal{F}(\boldsymbol{B})}{\otimes} \mathcal{P}(\boldsymbol{F}).$$

Their coordinate expression is of the type

$$\varphi = \varphi_{\lambda_1 \dots \lambda_r}{}^{i} \, \partial_i \otimes d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r}.$$

The projectable tangent valued forms constitute a subalgebra of the algebra of tangent valued forms. Moreover, we have

$$\underline{[\varphi, \psi]} = [\underline{\varphi}, \underline{\psi}] \qquad \varphi, \psi \in \wedge \mathcal{T}^*(\boldsymbol{B}) \underset{\mathcal{F}(\boldsymbol{B})}{\otimes} \mathcal{P}(\boldsymbol{F}).$$

Let $p : \boldsymbol{F} \to \boldsymbol{B}$ be a vector bundle. A projectable tangent valued form

$$\varphi : \boldsymbol{F} \to \wedge T^* \boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} T \boldsymbol{F}$$

is said to be *linear* if it is a linear fibred morphism over $\underline{\varphi}$.

The coordinate expression of a linear projectable tangent valued form is of the type

$$\varphi = (\varphi_{\lambda_1 \dots \lambda_r}{}^{\mu} \, \partial_\mu + \varphi_{\lambda_1 \dots \lambda_r}{}^{i}{}_{j} \, y^j \, \partial_i) \otimes d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r}$$

with

$$\varphi_{\lambda_1 \dots \lambda_r}{}^{\mu}, \; \varphi_{\lambda_1 \dots \lambda_r}{}^{i}{}_{j} \in \mathcal{F}(\boldsymbol{B}).$$

The linear projectable tangent valued forms constitute a subalgebra of the algebra of projectable tangent valued forms.

Let $p : \boldsymbol{F} \to \boldsymbol{B}$ be an affine bundle. A projectable tangent valued form

$$\varphi : \boldsymbol{F} \to \wedge T^* \boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} T \boldsymbol{F}$$

is said to be *affine* if it is an affine fibred morphism over $\underline{\varphi}$.

The coordinate expression of an affine projectable tangent valued form is of the type

$$\varphi = \big(\varphi_{\lambda_1 \dots \lambda_r}{}^{\mu} \, \partial_\mu + (\varphi_{\lambda_1 \dots \lambda_r}{}^{i}{}_{j} \, y^j + \varphi_{\lambda_1 \dots \lambda_r}{}^{i}{}_{\circ}) \, \partial_i\big) \otimes d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r}$$

with

$$\varphi_{\lambda_1 \dots \lambda_r}{}^{\mu}, \; \varphi_{\lambda_1 \dots \lambda_r}{}^{i}{}_{j}, \; \varphi_{\lambda_1 \dots \lambda_r}{}^{i}{}_{\circ} \in \mathcal{F}(\boldsymbol{B}).$$

The affine projectable tangent valued forms constitute a subalgebra of the algebra of projectable tangent valued forms.

III.5. General connections

In this section we recall a few basic notions on general connections. For further details, the reader can refer to [MM].

Let $p : \boldsymbol{F} \to \boldsymbol{B}$ be a fibred manifold.

A *connection* is defined to be a section

$$c : \boldsymbol{F} \to J_1 \boldsymbol{F},$$

i.e. a tangent valued 1-form

$$c : \boldsymbol{F} \to T^* \boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} T \boldsymbol{F},$$

which projects onto

$$\underline{c} := 1_{\boldsymbol{B}} : \boldsymbol{B} \to T^* \boldsymbol{B} \underset{\boldsymbol{B}}{\otimes} T \boldsymbol{B},$$

i.e. a linear fibred morphism over $\boldsymbol{F}$

$$c : \boldsymbol{F} \underset{\boldsymbol{F}}{\times} T \boldsymbol{B} \to T \boldsymbol{F},$$

which projects onto

$$\underline{c} := 1_{\boldsymbol{B}} : T \boldsymbol{B} \to T \boldsymbol{B}.$$

Its coordinate expression is of the type

$$(x^\lambda, \, y^i, \, y_\lambda{}^i) \circ c = (x^\lambda, \, y^i, \, c_\lambda{}^i),$$

i.e.

$$c = d^\lambda \otimes (\partial_\lambda + c_\lambda{}^i \otimes \partial_i) \qquad\qquad c_\lambda{}^i \in \mathcal{F}(\boldsymbol{F}).$$

A connection c yields the linear fibred epimorphism over $\boldsymbol{F}$

$$\nu_c : T \boldsymbol{F} \to V \boldsymbol{F},$$

with coordinate expressions

$$\nu_c = (d^i - c_\lambda{}^i \, d^\lambda) \otimes \partial_i,$$

and the translation fibred isomorphism over $\boldsymbol{F}$

$$\nabla_c : J_1\boldsymbol{F} \to T^*\boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} V\boldsymbol{F},$$

with coordinate expression

$$\nabla_c = (y_\lambda{}^i - c_\lambda{}^i)\, d^\lambda \otimes \partial_i .$$

The maps ν_c and ∇_c characterise the connection c itself.

The *covariant differential* of a section $s \in \mathcal{S}(\boldsymbol{F})$ is defined to be the section

$$\nabla_c s := \nabla_c \circ j_1 s : \boldsymbol{B} \to T^*\boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} V\boldsymbol{F},$$

with coordinate expression

$$\nabla_c s = (\partial_\lambda s^i - c_\lambda{}^i \circ s)\, d^\lambda \otimes (\partial_i \circ s).$$

The *covariant differential* of a tangent valued form $\varphi \in \overset{r}{\wedge} \mathcal{T}^*(\boldsymbol{F}) \underset{\mathcal{F}(\boldsymbol{F})}{\otimes} \mathcal{T}(\boldsymbol{F})$ is defined to be the tangent valued form

$$d_c\varphi := [c, \varphi] : \boldsymbol{F} \to \overset{r+1}{\wedge} T^*\boldsymbol{F} \underset{\boldsymbol{F}}{\otimes} T\boldsymbol{F}.$$

In particular, if φ is projectable, then its covariant differential turns out to be vertical valued

$$d_c\varphi := [c, \varphi] : \boldsymbol{F} \to \overset{r+1}{\wedge} T^*\boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} V\boldsymbol{F}$$

and its coordinate expression is

$$d_c\varphi = (- \partial_{\lambda_1} \varphi_{\lambda_2 \dots \lambda_{r+1}}{}^\mu c_\mu{}^i - \partial_\mu c_{\lambda_1}{}^i \varphi_{\lambda_2 \dots \lambda_{r+1}}{}^\mu +$$

$$+ \partial_{\lambda_1} \varphi_{\lambda_2 \dots \lambda_{r+1}}{}^i + c_{\lambda_1}{}^j \partial_j \varphi_{\lambda_2 \dots \lambda_{r+1}}{}^i - \partial_j c_{\lambda_1}{}^i \varphi_{\lambda_2 \dots \lambda_{r+1}}{}^j)\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+1}} \otimes \partial_i .$$

For each $\varphi, \psi \in \overset{r}{\wedge} \mathcal{T}^*(\boldsymbol{F}) \underset{\mathcal{F}(\boldsymbol{F})}{\otimes} \mathcal{T}(\boldsymbol{F})$, we have the following properties

$$d_c(\varphi + \psi) = d_c\varphi + d_c\psi$$

$$d_c[\varphi, \psi] = [d_c\varphi, \psi] + (-1)^{|\varphi|} [\varphi, d_c\psi]$$

$$d_c(\omega \wedge \varphi) = d\omega \wedge \nu_c(\varphi) + (-1)^{|\omega|}\, \omega \wedge d_c \varphi \qquad \omega \in \mathcal{T}^*(\boldsymbol{B})$$

and, if φ is a projectable vector field

$$\big(d_c(\varphi)\big)(u) = [c(u),\, \varphi] - c([u, \underline{\varphi}]), \qquad \forall u \in \mathcal{T}(\boldsymbol{B}).$$

The *curvature* of c is defined to be the vertical valued 2-form

$$R_c := \tfrac{1}{2}\, d_c c \,:\, \boldsymbol{F} \to \overset{2}{\wedge} T^* \boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} V \boldsymbol{F},$$

with coordinate expression

$$R_c = (\partial_\lambda c_\mu{}^i + c_\lambda{}^j\, \partial_j c_\mu{}^i)\, d^\lambda \wedge d^\mu \otimes \partial_i .$$

The curvature is characterised by the following property

$$2\, R_c(X,Y) = [c(X), c(Y)] - c([X,Y]) \qquad X, Y \in \mathcal{T}(\boldsymbol{B}).$$

We have the *Bianchi identity*

$$d_c{}^2 \varphi = [R_c, \varphi] \qquad \varphi \in \wedge \mathcal{T}^*(\boldsymbol{F}) \underset{\mathcal{F}(\boldsymbol{F})}{\otimes} \mathcal{T}(\boldsymbol{F}).$$

In particular, we obtain

$$d_c R_c = 0,$$

in virtue of

$$[c, R_c] = d_c R_c := \tfrac{1}{2}\, d_c{}^2 c = \tfrac{1}{2}\, [R_c, c].$$

Moreover, if

$$\varphi := c(\underline{\varphi}) \,:\, \boldsymbol{B} \to \wedge T^* \boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} T \boldsymbol{F}$$

is a horizontal tangent valued form, then we obtain the formula[26]

$$d_c \varphi = -\, R_c \barwedge \varphi,$$

with coordinate expression

[26] Here $\barwedge$ denotes the exterior product combined with an interior product.

$$d_c\varphi = -2\, R_{c\ \mu\lambda_1}{}^{i}\, \varphi_{\lambda_2\ldots\lambda_{r+1}}{}^{\mu}\, \partial_i \otimes d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+1}}.$$

Given a section $\sigma : \boldsymbol{F} \to T^*\mathrm{B} \underset{\boldsymbol{F}}{\otimes} V\boldsymbol{F}$, the *torsion* of c is defined to be the 2-form

$$\tau_c := d_c\sigma : \boldsymbol{F} \to \overset{2}{\wedge} T^*\boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} V\boldsymbol{F}.$$

We have the *Bianchi identity*

$$d_c\tau_c = [R_c,\ \sigma].$$

Let $\boldsymbol{F} \to \boldsymbol{B}$ be a vector bundle. A connection c is said to be *linear* if, as a fibred morphism $c : \boldsymbol{F} \to J_1\boldsymbol{F}$ over $\boldsymbol{B}$, it is linear. A connection is linear if and only if, in a linear fibred chart, its coordinate expression is of the following type

$$c_\lambda{}^i = c_\lambda{}^i{}_j\, y^j \qquad c_\lambda{}^i{}_j \in \mathcal{F}(\boldsymbol{B}).$$

The covariant differential of a linear projectable tangent valued form with respect to a linear connection turns out to be a linear vertical valued form.

Let $\boldsymbol{F} \to \boldsymbol{B}$ be an affine bundle. A connection c is said to be *affine* if, as a fibred morphism $c : \boldsymbol{F} \to J_1\boldsymbol{F}$ over $\boldsymbol{B}$, it is affine. A connection is affine if and only if, in an affine fibred chart, its coordinate expression is of the following type

$$c_\lambda{}^i = c_\lambda{}^i{}_j\, y^j + c_\lambda{}^i{}_\circ \qquad c_\lambda{}^i{}_j,\ c_\lambda{}^i{}_\circ \in \mathcal{F}(\boldsymbol{B}).$$

The covariant differential of an affine projectable tangent valued form with respect to an affine connection turns out to be an affine vertical valued form.

Connections adapted to other algebraic structures on bundles can be obtained in a similar way. For instance, a complex connection can be defined as a real linear fibred morphism $c : \boldsymbol{F} \to J_1\boldsymbol{F}$ over $\boldsymbol{B}$, which makes the following diagram commute

$$\begin{array}{ccc} \boldsymbol{F} & \overset{c}{} & J_1\boldsymbol{F} \longrightarrow \\ \iota \downarrow & & \downarrow J_1\iota \\ \boldsymbol{F} & \underset{c}{} & J_1\boldsymbol{F} \longrightarrow \end{array}$$

IV - INDEXES

IV.1 - Main symbols

Tangent and jet functors

$\pi_{\boldsymbol{M}}: T\boldsymbol{M} \to \boldsymbol{M}$	tangent bundle of the manifold $\boldsymbol{M}$
$Tf: T\boldsymbol{M} \to T\boldsymbol{N}$	tangent prolongation of the map $f: \boldsymbol{M} \to \boldsymbol{N}$
$df: \boldsymbol{M} \to T^*\boldsymbol{M}$	differential of the function $f: \boldsymbol{M} \to \mathbb{R}$
$df: \boldsymbol{M} \to T\boldsymbol{M}$	differential of the curve $f: \mathbb{R} \to \boldsymbol{M}$
$X.f := \langle df, X \rangle \equiv X \lrcorner df$	action of the vector field X on the function f
$\pi_{\boldsymbol{E}}: V\boldsymbol{E} \to \boldsymbol{E}$	vertical bundle of the fibred manifold $p: \boldsymbol{E} \to \boldsymbol{B}$
$Vf: V\boldsymbol{E} \to V\boldsymbol{F}$	vertical prolongation of the fibred morphism $f: \boldsymbol{E} \to \boldsymbol{F}$ over $\boldsymbol{B}$
$p_k: J_k\boldsymbol{E} \to \boldsymbol{B}$	k-jet fibred manifold of the sections of the fibred manifold $p: \boldsymbol{E} \to \boldsymbol{B}$
${p^k}_h: J_k\boldsymbol{E} \to J_h\boldsymbol{E}$	bundle projection between jet spaces, with $0 \leq h < k$
$j_k s: \boldsymbol{B} \to J_1\boldsymbol{E}$	k-jet prolongation of the section $s: \boldsymbol{B} \to \boldsymbol{E}$
$J_k f: J_k\boldsymbol{E} \to J_k\boldsymbol{F}$	k-jet prolongation of the fibred morphism $f: \boldsymbol{E} \to \boldsymbol{F}$ over $\boldsymbol{B}$
$д_k: J_k\boldsymbol{E} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to TJ_{k-1}\boldsymbol{E}$	contact fibred morphism of order $1 \leq k$
$\vartheta_k: J_k\boldsymbol{E} \underset{\boldsymbol{B}}{\times} TJ_{k-1}\boldsymbol{E} \to VJ_{k-1}\boldsymbol{E}$	contact fibred morphism of order $1 \leq k$
$r_k: J_kT\boldsymbol{E} \to TJ_k\boldsymbol{E}$	exchange fibred morphism between jet and tangent functors

Sheaves

$\mathcal{M}(\boldsymbol{M},\boldsymbol{N}) := \{f:\boldsymbol{M}\to\boldsymbol{N}\}$	sheaf of local maps between the manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$
$\mathcal{F}(\boldsymbol{M}) := \{f:\boldsymbol{M}\to\mathbb{R}\}$	sheaf of local functions of the manifold $\boldsymbol{M}$
$\mathcal{S}(\boldsymbol{F}) \equiv \mathcal{S}(\boldsymbol{F}\to\boldsymbol{B}) := \{s:\boldsymbol{B}\to\boldsymbol{F}\}$	sheaf of local sections of the fibred manifold $\boldsymbol{F}\to\boldsymbol{B}$
$\mathcal{T}(\boldsymbol{M}) := \{X:\boldsymbol{M}\to T\boldsymbol{M}\}$	sheaf of local vector fields of the manifold $\boldsymbol{M}$
$\mathcal{T}^*(\boldsymbol{M}) := \{\varphi:\boldsymbol{M}\to T^*\boldsymbol{M}\}$	sheaf of local 1-forms of the manifold $\boldsymbol{M}$
$\mathcal{F}_{\boldsymbol{B}}(\boldsymbol{F},\boldsymbol{G}) := \{f:\boldsymbol{F}\to\boldsymbol{G}\}$	sheaf of local fibred morphisms *over* $\boldsymbol{B}$
$\mathcal{F}(J_1\boldsymbol{E}) := \{f:J_1\boldsymbol{E}\to\mathbb{R}\}$	sheaf of local functions of $J_1\boldsymbol{E}$
$\mathcal{T}(J_1\boldsymbol{E}) := \{X:J_1\boldsymbol{E}\to TJ_1\boldsymbol{E}\}$	sheaf of local vector fields of $J_1\boldsymbol{E}$
$\mathcal{T}^*(J_1\boldsymbol{E}) := \{\varphi:J_1\boldsymbol{E}\to T^*J_1\boldsymbol{E}\}$	sheaf of local 1-forms of $J_1\boldsymbol{E}$
$\mathcal{T}_{\boldsymbol{\tau}}(J_1\boldsymbol{E}) \subset \mathcal{T}(J_1\boldsymbol{E})$	subsheaf of local vector fields with time component $\boldsymbol{\tau}$
$\mathcal{T}^*_{\gamma}(J_1\boldsymbol{E}) \subset \mathcal{T}^*(J_1\boldsymbol{E})$	subsheaf of local forms which vanish on γ
$\mathcal{F}_t(J_1\boldsymbol{E}) \subset \mathcal{F}(J_1\boldsymbol{E})$	subsheaf of tube-like functions with respect to $J_1\boldsymbol{E}\to\boldsymbol{E}$
$\mathcal{Q}(J_1\boldsymbol{E}) \subset \mathcal{F}_t(J_1\boldsymbol{E})$	subsheaf of quantisable functions
$\mathcal{Q}_c(J_1\boldsymbol{E}) \subset \mathcal{Q}(J_1\boldsymbol{E})$	subsheaf of q. f. with constant time-component
$\mathcal{Q}_0(J_1\boldsymbol{E}) \subset \mathcal{Q}_c(J_1\boldsymbol{E})$	subsheaf of q. f. with vanishing time-component
$\mathcal{Q}_t(J_1\boldsymbol{E}) \subset \mathcal{Q}(J_1\boldsymbol{E})$	subsheaf of q. f., which are tube-like with respect to $J_1\boldsymbol{E}\to\boldsymbol{T}$
$\mathcal{Q}_{tc}(J_1\boldsymbol{E}) \subset \mathcal{Q}_c(J_1\boldsymbol{E})$	subsheaf of q. f. with constant time-component
$\mathcal{Q}_{t0}(J_1\boldsymbol{E}) \subset \mathcal{Q}_0(J_1\boldsymbol{E})$	subsheaf of q. f. with vanishing time-component
$\mathcal{Q}(\boldsymbol{Q}^{\uparrow}) \subset \mathcal{T}(\boldsymbol{Q}^{\uparrow})$	subsheaf of upper quantum vector fields
$\mathcal{Q}_{\boldsymbol{\tau}}(\boldsymbol{Q}^{\uparrow}) \subset \mathcal{Q}(\boldsymbol{Q}^{\uparrow})$	subsheaf of upper q. v. f. with time-component $\boldsymbol{\tau}$
$\mathcal{Q}(\boldsymbol{Q}) \subset \mathcal{T}(\boldsymbol{Q})$	subsheaf of quantum vector fields
$\mathcal{Q}_c(\boldsymbol{Q}) \subset \mathcal{Q}(\boldsymbol{Q})$	subsheaf of q. v. f. with constant time-component
$\mathcal{Q}_0(\boldsymbol{Q}) \subset \mathcal{Q}_c(\boldsymbol{Q})$	subsheaf of q. v. f. with vanishing time-component
$\mathcal{L}(\boldsymbol{Q}^{\eta})$	sheaf of quantum Lie operators
$\mathcal{L}(\boldsymbol{SQ}^{\eta})$	sheaf of quantum operators

$\mathcal{Q}_c(SQ^{\uparrow}) \subset \mathcal{Q}(SQ^{\uparrow})$	subsheaf of quantum operators corresponding to $\mathcal{Q}_c(J_1\boldsymbol{E})$
$\mathcal{Q}_0(SQ^{\uparrow}) \subset \mathcal{Q}_c(SQ^{\uparrow})$	subsheaf of quantum operators corresponding to $\mathcal{Q}_0(J_1\boldsymbol{E})$
$\mathcal{S}_t{}^{\square}(\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B})$	sheaf of tube-sections $\boldsymbol{E}\to\boldsymbol{F}$ (non-smooth with respect to $\boldsymbol{B}$)
$\mathcal{S}_t(\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{B})$	sheaf of smooth tube-sections $\boldsymbol{E}\to\boldsymbol{F}$
$\mathcal{S}^{\square}(\boldsymbol{A}\to\boldsymbol{B})$	sheaf of (non-smooth) local sections $\boldsymbol{B}\to\boldsymbol{A}$

Units of measurement

$\mathbb{T}$	vector space associated with the affine space $\boldsymbol{T}$
$\mathbb{A}$	1-dimensional positive semi-space of area units
$\mathbb{M}$	1-dimensional positive semi-vector space of masses
$\mathbb{Q} := \mathbb{T}^*\otimes\mathbb{A}^{3/4}\otimes\mathbb{M}^{1/2}$	1-dimensional vector space of charges
$u_0\in\mathbb{T}^+,\ u^0\in\mathbb{T}^{+*}$	time unit of measurement
$m \in \mathbb{M}$	mass
$\boldsymbol{q} \in \mathbb{Q}$	charge
$q := \boldsymbol{q}(u_0) \in \mathbb{A}^{3/4}\otimes\mathbb{M}^{1/2}$	charge (related to u_0)
$\boldsymbol{\hbar} \in \mathbb{T}^{+*}\otimes\mathbb{A}\otimes\mathbb{M}$	Plank constant
$\hbar := \boldsymbol{\hbar}(u_0) \in \mathbb{A}\otimes\mathbb{M}$	Plank constant (related to u_0)
$\boldsymbol{\kappa} \in \mathbb{T}^{*2}\otimes\mathbb{A}^{3/2}\otimes\mathbb{M}^*$	gravitational coupling constant

Space-time

$t:\boldsymbol{E}\to\boldsymbol{T}$	classical space-time fibred over time
$J_1\boldsymbol{E}\to\boldsymbol{E}\to\boldsymbol{T}$	first jet space of the space-time fibred manifold
$\pi_{\boldsymbol{E}}:T\boldsymbol{E}\to\boldsymbol{E}$	tangent bundle of the space-time manifold
$\pi_{\boldsymbol{E}}:V\boldsymbol{E}\to\boldsymbol{E}$	vertical bundle of the space-time fibred manifold
$dt:\boldsymbol{E}\to\mathbb{T}\otimes T^*\boldsymbol{E}$	space-time 1-form
$\text{д}:J_1\boldsymbol{E}\to\mathbb{T}^*\otimes T\boldsymbol{E}$	contact tangent valued form
$\vartheta:J_1\boldsymbol{E}\to T^*\boldsymbol{E}\otimes V\boldsymbol{E}$	contact vertical valued form

$s : \boldsymbol{T} \to \boldsymbol{E}$ classical motion

$j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E}$ velocity of the classical motion

$o : \boldsymbol{E} \to J_1\boldsymbol{E}$ observer

(u_0, o) frame of reference

$g : \boldsymbol{E} \to \mathbb{A} \otimes V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}$ vertical metric

$\bar{g} : \boldsymbol{E} \to \mathbb{A}^* \otimes V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}$ contravariant scaled vertical metric

$\upsilon : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{A}^{3/2}) \otimes \overset{4}{\wedge} T^*\boldsymbol{E}$ space-time volume form

$\eta : \boldsymbol{E} \to \mathbb{A}^{3/2} \otimes \overset{4}{\wedge} V^*\boldsymbol{E}$ space-like volume form

$\sqrt{\upsilon} : \boldsymbol{E} \to \mathbb{T}^{1/2} \otimes \mathbb{A}^{3/4} \otimes \sqrt{\overset{4}{\wedge}} T^*\boldsymbol{E}$ space-time half-density

$\sqrt{\eta} : \boldsymbol{E} \to \mathbb{A}^{3/4} \otimes \sqrt{\overset{3}{\wedge}} V^*\boldsymbol{E}$ space-like half-density

Space-time connection

$K : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E}$ space-time connection on the bundle $T\boldsymbol{E} \to \boldsymbol{E}$

$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E}$ space-time connection on the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$

$\nu_\Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T^*J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} V\boldsymbol{E}$ vertical valued form associated with Γ

$\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$ connection on the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$

$\Omega : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^*J_1\boldsymbol{E}$ contact 2-form

$\Phi : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^*\boldsymbol{E}$ 2-form associated with an observer

$\Gamma^\natural : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E}$ gravitational connection

Total objects

$F : \boldsymbol{E} \to \mathbb{B} \otimes \overset{2}{\wedge} T^*\boldsymbol{E}$ electromagnetic field

$\Omega := \Omega^\natural + \Omega^e := \Omega^\natural + \frac{1}{2}\frac{\boldsymbol{q}}{m} F$ total contact 2-form

$\Gamma = \Gamma^\natural + \Gamma^e$ total space-time connection

$\gamma = \gamma^{\natural} + \gamma^{e}$	total second order space-time connection
$\gamma^{e} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes (\mathbb{T}^* \otimes V\boldsymbol{E})$	Lorentz force
$\Gamma^{e} : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} (\mathbb{T}^* \otimes V\boldsymbol{E})$	electromagnetic soldering form
$a : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes T^*\boldsymbol{E}$	potential of Φ
$\tau := \tau^{\natural} + \tau^{e}$	total energy tensor
$r = r^{\natural} + r^{\natural e} + r^{e}$	total Ricci tensor
$\nabla_{\gamma} j_1 s : \boldsymbol{T} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes V\boldsymbol{E}$	covariant differential of the velocity

Classical kinematical functions

$G : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes \mathbb{R}$	classical kinetic energy (related to u_0)
$H : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes \mathbb{R}$	classical Hamiltonian (related to u_0)
$L : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes \mathbb{R}$	classical Lagrangian (related to u_0)
$p : J_1\boldsymbol{E} \to \mathbb{A} \otimes \mathbb{M} \otimes V^*\boldsymbol{E}$	classical momentum (related to u_0)

Quantum bundle

$\pi : \boldsymbol{Q} \to \boldsymbol{E}$	quantum bundle
$h : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{C}$	Hermitian product
$\Psi : \boldsymbol{E} \to \boldsymbol{Q}$	quantum section
$\Psi = \psi\, b$	local expression of a quantum section
$\boldsymbol{Q}^{\upsilon} := \mathbb{T}^{1/2} \otimes \mathbb{A}^{3/4} \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \sqrt{\overset{4}{\wedge} T^*\boldsymbol{E}})$	space of space-time quantum half-densities
$\boldsymbol{Q}^{\eta} := \mathbb{A}^{3/4} \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \sqrt{\overset{3}{\wedge} V^*\boldsymbol{E}})$	space of space-like quantum densities
$\Psi^{\upsilon} := \Psi \otimes \sqrt{\upsilon}$	space-time quantum half-density
$\Psi^{\eta} := \Psi \otimes \sqrt{\eta}$	space-like quantum half-density
$\text{и} : \boldsymbol{Q} \to V\boldsymbol{Q} = \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}$	Liouville vertical vector field
$\boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$	pullback of the quantum bundle
$\text{ч} : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} T\boldsymbol{Q}^{\uparrow}$	quantum connection

$R_{ч} = i\,\frac{m}{\hbar}\,\Omega \otimes и$	curvature of the quantum connection
$\nabla \Psi$	covariant differential of a quantum section
$\check{\nabla}^{\neq} \Psi : J_1\boldsymbol{E} \to \mathbb{A}^* \otimes V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}$	space-like covariant differential of a quantum section
$\mathring{\nabla} \Psi : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q}$	time-like covariant differential of a quantum section

Quantum dynamics

$\mathscr{L}_{\Psi} : \boldsymbol{E} \to \mathbb{A}^{3/2} \otimes \overset{4}{\wedge} T^*\boldsymbol{E}$	quantum Lagrangian along a quantum section
$\mathscr{L} : J_1\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{4}{\wedge} T^*\boldsymbol{E}$	quantum Lagrangian
$V_{\boldsymbol{Q}}\mathscr{L} : J_1\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{3}{\wedge} T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}^*$	quantum momentum
$\mathfrak{p} : J_1\boldsymbol{Q} \to \mathbb{T}^* \otimes T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}$	quantum momentum
$\Pi : J_1\boldsymbol{Q} \to \overset{4}{\wedge} T^*\boldsymbol{Q}$	quantum momentum
$\mathscr{E} : J_2\boldsymbol{Q} \to \mathbb{A}^{3/2} \otimes \overset{5}{\wedge} T^*\boldsymbol{Q}$	Euler-Lagrange form
$*\check{\mathscr{E}}^{\neq} : J_2\boldsymbol{Q} \to \mathbb{T}^* \otimes \boldsymbol{Q}$	Euler-Lagrange fibred morphism
$*\check{\mathscr{E}}^{\neq} \circ j_2\Psi = 0$	Schrödinger equation
$j_{\Psi} : \boldsymbol{E} \to \mathbb{A}^{3/2} \otimes \overset{3}{\wedge} T^*\boldsymbol{E}$	probability current

Quantum operators

$\Omega^{\flat}_{\boldsymbol{\tau}} : \mathcal{C}_{\boldsymbol{\tau}}(J_1\boldsymbol{E}) \to \mathcal{C}^*_{\gamma}(J_1\boldsymbol{E})$	Hamiltonian isomorphism
$\mathscr{F}(J_1\boldsymbol{E}) \to \mathcal{C}_{\boldsymbol{\tau}}(J_1\boldsymbol{E}) : f \mapsto f^{\neq}_{\boldsymbol{\tau}}$	Hamiltonian lift
$\{f', f''\}$	Poisson bracket
$[f', f'']$	Lie bracket of quantisable functions
$f^{\neq}_{\boldsymbol{\tau}} : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$	Hamiltonian lift of the function $f \in \mathscr{F}(J_1\boldsymbol{E})$
$f^H : \boldsymbol{E} \to T\boldsymbol{E}$	Hamiltonian lift of the quantisable function $f \in \mathscr{Q}(J_1\boldsymbol{E})$
$X^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T\boldsymbol{Q}^{\uparrow}$	upper quantum vector field
$X^{\uparrow}_{f,\boldsymbol{\tau}} : \boldsymbol{Q}^{\uparrow} \to T\boldsymbol{Q}^{\uparrow}$	upper q. v. f. associated with f and $\boldsymbol{\tau}$

$X_f : \boldsymbol{Q} \to T\boldsymbol{Q}$	q. v. f. associated with the quantisable function $f \in \mathcal{Q}(J_1\boldsymbol{E})$
$[X_{f'}, X_{f''}]$	Lie bracket of quantum vector fields
$X.s : \boldsymbol{B} \to \boldsymbol{F}$	Lie derivative of the section s with respect to the vector field X
$Y_f := i\, X_f .$	quantum Lie operator associated with the quantisable function f
$[Y_{f'}, Y_{f''}]$	bracket of quantum Lie operators

Quantum system

$\boldsymbol{F} \xrightarrow{q} \boldsymbol{E} \xrightarrow{p} \boldsymbol{B}$	double fibred manifold
$(\sigma : S\boldsymbol{F} \to \boldsymbol{B},\ \varepsilon)$	system of the double fibred manifold $\boldsymbol{F} \to \boldsymbol{E} \to \boldsymbol{B}$
$\varepsilon : S\boldsymbol{F} \underset{B}{\times} \boldsymbol{E} \to \boldsymbol{F}$	evaluation fibred morphism
$\hat{\Psi} : \boldsymbol{B} \to S\boldsymbol{F}$	section associated with $\Psi : \boldsymbol{E} \to \boldsymbol{F}$
$TS\boldsymbol{F}$	tangent space of the set $S\boldsymbol{F}$
$\widehat{T\Psi} : T\boldsymbol{B} \to TS\boldsymbol{F}$	section associated with $T\Psi : T\boldsymbol{E} \to T\boldsymbol{F}$
$\hat{k} : S\boldsymbol{F} \to T^*\boldsymbol{B} \otimes TS\boldsymbol{F}$	connection on the fibred set $S\boldsymbol{F} \to \boldsymbol{B}$
$k(\Psi) : T\boldsymbol{E} \to T\boldsymbol{F}$	fibred morphism associated with k and Ψ
$k_\mu{}^a(\Psi) : \boldsymbol{E} \to \mathbb{R}$	components of the connection k
$\nabla_{\hat{k}} \hat{\Psi} : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes VS\boldsymbol{F}$	covariant differential of $\hat{\Psi}$ with respect to k
$(\sigma : S\boldsymbol{Q}^\eta \to \boldsymbol{T},\ \varepsilon)$	system of space-like quantum half-densities
$\hat{h} : S^c\boldsymbol{Q}^\eta \underset{\boldsymbol{B}}{\times} S^c\boldsymbol{Q}^\eta \to \mathbb{A}^{3/2} \otimes \mathbb{C}$	Hermitian product
$H\boldsymbol{Q}^\eta \to \boldsymbol{T}$	Hilbert quantum bundle
$\mathfrak{S} : \mathcal{S}(\boldsymbol{Q}) \to \mathcal{S}(\mathbb{T}^* \otimes \boldsymbol{Q})$	Schrödinger operator
$\mathfrak{S}^\eta : \mathcal{S}(\boldsymbol{Q}^\eta) \to \mathcal{S}(\mathbb{T}^* \otimes \boldsymbol{Q}^\eta)$	Schrödinger operator
$\hat{k} : S\boldsymbol{Q}^\eta \to \mathbb{T}^* \otimes TS\boldsymbol{Q}^\eta$	Schrödinger connection
$\hat{Y}_f : S\boldsymbol{Q}^\eta \to S\boldsymbol{Q}^\eta S\boldsymbol{Q}^\eta \to S\boldsymbol{Q}^\eta$	operator associated with the q. f. $f \in \mathcal{Q}(J_1\boldsymbol{E})$
$\hat{\Xi}_f : S\boldsymbol{Q}^\eta \to S\boldsymbol{Q}^\eta$	quantum operator associated with the q. f. $f \in \mathcal{Q}(J_1\boldsymbol{E})$

Coordinates

(x^0, y^i)	fibred chart of $\boldsymbol{E}$
$(x^0, y^i, \dot{x}^0, \dot{y}^i)$	fibred chart of $T\boldsymbol{E}$
(x^0, y^i, y^i_0)	fibred chart of $J_1\boldsymbol{E}$
$(x^0, y^i, y^i_0, \dot{x}^0, \dot{y}^i, \dot{y}^i_0)$	fibred chart of $TJ_1\boldsymbol{E}$
(∂_0, ∂_i)	local base of $T\boldsymbol{E}$
$(\partial_0, \partial_i, \dot{\partial}_0, \dot{\partial}_i)$	local base of $TT\boldsymbol{E}$
$(\partial_0, \partial_i, \partial^0_i)$	local base of $J_1\boldsymbol{E}$
(d^0, d^i)	local base of $T^*\boldsymbol{E}$
$(d^0, d^i, d^0_{\cdot}, d^i_{\cdot})$	local base of $T^*T\boldsymbol{E}$
(d^0, d^i, d^i_0)	local base of $T^*J_1\boldsymbol{E}$
$д_0 := \partial_0 + y^i_0\, \partial_i$	component of the contact form
$\vartheta^i := d^i - y^i_0\, d^0$	component of the contact form
$\Gamma_\Phi{}^i := \Gamma_\Phi{}^i{}_h\, y^h_0 + \Gamma_\Phi{}^i{}_\circ$	components of the space-time connection Γ
$K_\Phi{}^i := K_\Phi{}^i{}_h\, \dot{y}^h + K_\Phi{}^i{}_0\, \dot{x}^0$	components of the space-time connection K
$\gamma^i := \Gamma_h{}^i{}_k\, y^h_0\, y^k_0 + 2\, \Gamma_h{}^i{}_\circ\, y^h_0 + \Gamma_0{}^i{}_\circ$	components of the connection γ
$ч_0 = -H/\hbar$	components of the quantum connection
$ч_j = p_j/\hbar$	components of the quantum connection
$ч^0_j = 0$	components of the quantum connection

IV.2 - Analytic index

IV.3 - References

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