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# Some Variations on the Notion of Connection (*).

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**Abstract.** – *Distributions on manifolds are studied in terms of jets of submanifolds and are interpreted as «pre-connections» or «almost-fibrings»; the associated differential calculus is developed in detail. A comparison with connections on fibred manifolds is analysed. Moreover, «higher order pre-connections», defined as pre-connections dependent on jets of arbitrary order, are introduced and studied. It is shown that infinite jets play an essential role in the associated differential calculus.*

## Introduction.

Being motivated by some geometrical and physical reasons, we investigate in this paper a neighbourhood of the notion of connection.

One of these reasons arises from the problem of unification of internal and external variables in the basic model of field theory. It is well known that the division of variables into internal and external ones is achieved by means of a fibred structure on the manifold of all variables. So, as far as connections are concerned, the unification problem leads us to the question: what should be the corresponding notion in absence of a fibred structure? An answer is proposed in Part I of this paper. Here, *connections without fibrings*, called *pre-connections*, are treated as $m$-dimensional distributions viewed as a part of the jet theory of $m$-dimensional submanifolds of the manifold of all variables, $m$ being the number of independent variables.

In Part I we focus our attention mainly on two questions: namely, whether the no-

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tion of covariant differential and that of curvature can be introduced also for pre-connections. First, we have shown that no fibred structure is necessary to define the covariant differential. So, this can be associated with any pre-connection as well. This results from the interpretation of the standard covariant differential in terms of generating functions of (higher order) contact transformations. On the other hand, only a weak analogue of the curvature, called *deviation*, can be defined for pre-connections. This «deviation» is, in fact, an interpretation of a known construction in the distribution theory (see, for instance [11]). However, we go further on and present the graded (or, «super», speaking physically) extension of this notion.

Unfortunately, the Fröhlicher-Nijenhuis bracket machinery, which is suitable for treating standard connections, cannot be applied directly to pre-connections. This, however, can be carried out naturally in presence of an almost-fibring transversal to a given pre-connection. In Part II we show, applying the Fröhlicher-Nijenhuis bracket that all basic formulas concerning the standard connections are also valid for pre-connections with respect to a chosen almost-fibring.

In Parts I and II we re-expose also some standard facts of the standard connection theory in a way suitable to prepare the passage to infinite jets, which is made in Part III. The last one is the most important point of the whole paper. In doing it we were motivated by a «general» principle of geometry of partial differential equations claiming that things become much more simple and transparent after being appropriately lifted to infinite jets ([17], [19]). In particular, we show that a lifting of the module of infinite order contact transformations can be associated with a given connection. This enables us to discover higher order analogues of standard connections by going the back direction, i.e. from splittings to connections. The concept of $k$-th order connection we are led this way is quite different from the commonly adopted one (see, for instance, ([7], [8])) under the same name. Namely, the latter is a standard connection defined on the fibring $J^k\pi \to J^{k-1}\pi$ of jets associated with a given fibring. On the contrary, our $k$-th order connections are defined on $\pi$ itself. In particular, the corresponding covariant differential acts on sections of $\pi$ and is a $k$-th order differential operator.

We note that every fibring $\pi\colon E \to B$ possesses a canonical $k$-th order connection for $k \geqslant 1$. The infinite lift of this connection coincides with the canonical flat connection in the fibring $J^\infty\pi \to B$ whose horizontal distribution is the standard infinite order contact structure on $J^\infty\pi$. This example indicates possible applications of higher order connections in geometry of partial differential equations. Very interesting applications of this kind one can find in the forthcoming paper by I. KRASIL'SHCHIK in which the Frölicher-Niejenhuis machinery applied to the mentioned canonical connection produces new important cohomological invariant of partial differential equations (super-symmetries, deformations, etc.).

Moreover, in Part III we deduce all basic formulas concerning the covariant differential and the curvature (deviation) of higher order (pre-)connection with respect to a given higher order almost-fibring. All these results are new to our knowledge.

In this paper we restrict ourselves only to a motivated presentation of the above mentioned new conceptions, leaving aforementioned applications to field theory and geometry of differential equations to the future.

Below, everything is supposed to be smooth.

## Part I. – Pre-connections.

### 1. – Jets of submanifolds.

Here necessary notion and formulas from the jet theory are collected; for details see [5].

Throughout the paper we consider a manifold $E$ and submanifolds $N \subset E$ of a fixed dimension, say $m$, with

$$\dim E = m + l, \qquad \dim N = m, \qquad m \geqslant 1, \; l \geqslant 1.$$

By definition, for $0 \leqslant k$, a *k-jet* of $m$-dimensional submanifolds of $E$ at $e \in E$ is an equivalence class $[N]_e^k$ of $m$-dimensional submanifolds $N \subset E$, passing through $e$ and touching each other at $e$ with contact of order $k$.

The set of all $k$-jets of $m$-dimensional submanifolds of $E$ can be supplied, in a natural way, with a smooth structure. The corresponding manifold is denoted by $J^k(E, m)$, or simply by $J^k$.

For $p \geqslant q \geqslant 0$, the natural projection

$$\pi_{p,q}\colon J^p(E, m) \to J^q(E, m)\colon [N]_e^p \mapsto [N]_e^q$$

makes $J^p$ a bundle over $J^q$. Obviously,

$$\pi_{q,r} \circ \pi_{p,q} = \pi_{p,r}, \qquad r \leqslant q \leqslant p.$$

Moreover, for a given $m$-dimensional submanifold $N \subset E$ and an integer $k \geqslant 0$, we have the map

$$j^k N\colon N \to J^k, \qquad e \mapsto [N]_e^k,$$

which is called the *k-th prolongation* of $N$. The map $j^k N$ is an embedding. So,

$$N^{(k)} := (j^k N)(N)$$

is an $m$-dimensional submanifold of $J^k$. Obviously,

$$\pi_{p,q} \circ j^p N = j^q N, \qquad p \geqslant q \geqslant 0. \tag{1}$$

We identify $J^0(E, m)$ with $E$. Next, we can identify $[N]_e^1$ with $T_e N$; in fact, $m$-dimensional submanifolds of $E$ touch each other at $e$ with first order contact if and only if they have the same tangent space at $e$. By this reason, the manifold $J^1(E, m)$ can be identified with the manifold Grass $(E, m)$ consisting of all $m$-dimensional subspaces of the tangent bundle $\tau_E$: $TE \to E$. The $m$-dimensional subspace corresponding to $\vartheta \in J^1$, under this identification, will be denoted by $L_\vartheta$. In other words,

$$L_\vartheta = T_e N \subset T_e E\,, \qquad \text{if } \vartheta = [N]^1_e\,,\ \ e = \pi_{1,0}(\vartheta)\,.$$

For $\vartheta \in J^k$, we adopt the notation

$$\underline{\vartheta} \equiv \pi_{k,0}(\vartheta) \in E\,, \qquad \tilde{\vartheta} \equiv \pi_{k,1}(\vartheta) \in E\,.$$

*Coordinates.*

A local chart on $E$ is said to be *divided* if the set of its coordinate functions is divided into two parts, consisting of $m$ and $l$ elements, respectively. The coordinate functions belonging to the first of them are interpreted as «independent variables» and the others as «dependent variables». Our typical notation for a divided chart will be

(2) $$(x^\lambda, y^i)\,, \qquad 1 \leqslant \lambda \leqslant m,\ \ 1 \leqslant i \leqslant l\,.$$

A divided chart (2) and an $m$-dimensional submanifold $N \subset E$ are said to be *concordant* if $x^\lambda|_N$, $1 \leqslant \lambda \leqslant m$, are (local) coordinates on $N$. If so, the submanifold $N \subset E$ can be expressed (locally) by formulas of the type

(3) $$y^i = f^i(x^1, \dots, x^m)\,, \qquad 1 \leqslant i \leqslant l\,.$$

Every divided chart, say (2), on $E$ determines canonically a local chart

(4) $$(x^\lambda, y^i_{\boldsymbol\sigma})\,, \qquad 1 \leqslant \lambda \leqslant m,\ \ 1 \leqslant i \leqslant l,\ \ 0 \leqslant |\boldsymbol\sigma| \leqslant k\,,$$

on $J^k$, which will be said to be *special*. Here, $\boldsymbol\sigma = (\sigma_1, \dots, \sigma_m)$ denotes a multi-index(1) and $|\boldsymbol\sigma| \equiv \sigma_1 + \dots + \sigma_m$. The coordinate functions $y^i_{\boldsymbol\sigma}$ are completely characterized by the equalities

$$y^i_{\boldsymbol\sigma} \circ j^k N = \frac{\partial^{|\boldsymbol\sigma|} f^i}{\partial x^{\boldsymbol\sigma}}$$

for every submanifold $N$ concordant with the chart (2); here we have used the short notation

$$\frac{\partial^{|\boldsymbol\sigma|} f^i}{\partial x^{\boldsymbol\sigma}} \equiv \frac{\partial^{|\boldsymbol\sigma|} f^i}{\partial x^{\sigma_1} \dots \partial x^{\sigma_m}}\,.$$

For $1 \leqslant \lambda \leqslant m$, we shall identify the index $\lambda$ with the multi-index $(0, \dots, 1, \dots, 0)$, with the unit at the $\lambda$-th place. According to this notation, the special chart on $J^1$ cor-

(1) Multi-indices will be denoted by boldfaces greek letters.

responding to (2) looks as follows:

$$(5) \qquad (x^\lambda, y^i, y^i_\lambda), \qquad 1 \leq \lambda \leq m, \ 1 \leq i \leq l.$$

Moreover, we put

$$\partial_\lambda \equiv \frac{\partial}{\partial x^\lambda}, \qquad \partial_i \equiv \frac{\partial}{\partial y^i}.$$

If $\vartheta \in J^1$, then the subspace $L_\vartheta \subset T_{\underline{\vartheta}} E$ is the span of the vectors

$$\partial_\lambda + \sum_i y^i_\lambda(\vartheta)\,\partial_i, \qquad 1 \leq \lambda \leq m,$$

where we refer to a special chart.

*Infinite jets.*

All above considerations hold also for $k = \infty$, of course, under the necessary cautions. First, we note that the above definition of $k$-jets remains meaningful for $k = \infty$. So, the set $J^\infty(E, m)$ is well-defined. It can be easily identified with the inverse limit of the sequence

$$E = J^0(E, m) \xleftarrow{\pi_{1,0}} J^1(E, m) \xleftarrow{\pi_{2,1}} \dots \xleftarrow{\pi_{k,k-1}} J^k(E, m) \xleftarrow{\pi_{k+1,k}} \dots.$$

Next, we define the «algebra of smooth functions» on $J^\infty(E, m)$ to be the direct limit of the sequence

$$C^\infty(E) = C^\infty(J^0) \xrightarrow{\pi^*_{1,0}} C^\infty(J^1) \xrightarrow{\pi^*_{2,1}} \dots \xrightarrow{\pi^*_{k,k-1}} C^\infty(J^k) \xrightarrow{\pi^*_{k+1,k}} \dots$$

i.e. $C^\infty(J^\infty) = \lim\limits_{k \to \infty} \operatorname{dir} C^\infty(J^k)$. In other words, smooth functions on $J^\infty$ are of the form $\pi^*_{\infty,k}(\varphi) = \varphi \circ \pi_{\infty,k}$, where $0 \leq k < \infty$ and $\varphi \in C^\infty(J^k)$.

The special local chart on $J^\infty$ corresponding to (2) is given by

$$(6) \qquad (x^\lambda, y^i_\sigma), \qquad 1 \leq \lambda \leq m, \ 1 \leq i \leq l, \ 0 \leq |\sigma| < \infty.$$

Locally, smooth functions on $J^\infty$ look as smooth functions of a finite number of variables (6).

The algebra $C^\infty(J^\infty)$ is filtered by the images of algebras $C^\infty(J^k)$, $0 \leq k < \infty$, under the maps $\pi^*_{\infty,k}\colon C^\infty(J^k) \to C^\infty(J^\infty)$ which are, evidently, monomorphisms. Speaking below on vector fields, differential operators, forms, etc., we refer to the corresponding objects of the differential calculus over this filtered commutative algebra as it is understood, say, in [5].

## 2. - Infinite order contact transformations and their generating functions.

In this section we recall necessary facts on generating functions of infinitesimal contact transformations (i.e. contact vector fields) of $J^\infty$. Later, we shall use them twice: first, for defining the covariant differential of a connection and, secondly, for interpreting the notion of a connection from the viewpoint of infinite jet theory. For details and motivations see [5] and [18].

Let $\tau_E \colon TE \to E$ be the tangent vector bundle of $E$ and

$$\tau_E^k \colon T^{(k)} \equiv T^{(k)}(E, m) \to J^k, \qquad 0 \leqslant k \leqslant \infty,$$

be its pullback via the map $\pi_{k,0}$. Thus, $T^{(k)}$ is the submanifold of $J^k \times TE$

$$T^{(k)} := \{(\vartheta, u) \subset J^k \times TE \,|\, u \in T_{\underline{\vartheta}} E\} \subset J^k \times TE.$$

Moreover, we can define the vector subbundle of $\tau_E^k$

$$c^k \colon C^k \equiv C^k(E, m) \to J^k, \qquad 1 \leqslant k \leqslant \infty,$$

by putting

$$C^k := \{(\vartheta, u) \in T^{(k)} \,|\, u \in L_{\bar{\vartheta}}\} \subset T^{(k)}.$$

Furthermore, we consider the quotient bundle

$$w^k \colon W^k \equiv W^k(E, m) \to J^k, \qquad 1 \leqslant k \leqslant \infty,$$

of $\tau_E^k$ with respect to $c^k$. By definition, we have the following short exact sequence of vector bundles over $J^k$

$$0 \to C^k \xrightarrow{\text{д}^k} T^{(k)} \xrightarrow{\mathrm{r}^k} W^k \to 0 \tag{7}$$

where $\mathrm{r}^k$ denotes the quotient map.

Thus, the fibres of the bundles $\tau_E^k$, $c^k$ and $w^k$ over a point $\vartheta \in J^k$ are identified with $T_{\underline{\vartheta}} E$, $L_{\bar{\vartheta}}$ and $T_{\underline{\vartheta}} E / L_{\bar{\vartheta}}$, respectively, and the map $\mathrm{r}^k$ reads

$$\mathrm{r}^k(\vartheta, u) = u \bmod L_{\bar{\vartheta}}, \qquad (\vartheta, u) \in T^{(k)}.$$

If $k \geqslant s$, then the bundles $\tau_E^k$, $c^k$ and $w^k$ are the pullbacks via $\pi_{k,s}$ of the bundles $\tau_E^s$, $c^s$ and $w^s$, respectively. Therefore, $\pi_{k,s}$ induces the conclusions

$$\pi_{k,s}^* \colon \Gamma(\alpha^s) \hookrightarrow \Gamma(\alpha^k),$$

where $\alpha^i$ is one of the bundles $\tau_E^k$, $c^k$ or $w^k$ and $\Gamma(\alpha)$ stands for the set of all sections of the bundle $\alpha$. In particular, we have the sequence of embeddings

$$\ldots \hookrightarrow \Gamma(\alpha^k) \hookrightarrow \Gamma(\alpha^{k+1}) \hookrightarrow \ldots \hookrightarrow \Gamma(\alpha^\infty).$$

In other words, the $C^\infty(J^\infty)$-module $\Gamma(\alpha^\infty)$ is filtered by the $C^\infty(J^k)$-submodules $\Gamma(\alpha^k)$ and

$$\Gamma(\alpha^\infty) \equiv \lim_{k \to \infty} \operatorname{dir} \Gamma(\alpha^k). \tag{8}$$

■

The bundle $w^\infty \colon W^\infty \to J^\infty$ is called the *generating functions bundle*. Its sections are called *generating functions* of infinitesimal contact transformations (i.e. of contact vector fields) of $J^\infty$ (see below and also [5]). We put

$$\kappa^k \equiv \Gamma(w^k), \qquad 1 \leqslant k \leqslant \infty .$$

Then, according to (8),

$$\kappa^\infty = \lim_{k \to \infty} \operatorname{dir} \kappa^k .$$

Denote by $\mathcal{D}(M)$ the $C^\infty(M)$-module of (local) vector fields on a manifold $M$.
A canonical homomorphism of $C^\infty(J^k)$-modules

$$\mathfrak{g}^k\colon \mathcal{D}(J^k) \to \kappa^k, \qquad 1 \leqslant k \leqslant \infty ,$$

is generated by the following morphism of linear bundles over $J^k$:

$$g^k\colon TJ^k \to W^k, \qquad 1 \leqslant k \leqslant \infty ,$$

where, for $\xi \in T_\vartheta J^k$,

$$g^k(\xi) := \mathrm{r}^k(\vartheta, u), \qquad u \equiv d_\vartheta \Pi_{k,0}(\xi).$$

In virtue of (8), we have

$$\mathfrak{g}^\infty = \lim_{k \to \infty} \operatorname{dir} \mathfrak{g}^k .$$

Moreover, by considering the natural lift $\mathcal{D}(E) \hookrightarrow \mathcal{D}(J^1)$, we obtain the canonical homomorphism of $C^\infty(J^k)$-modules

$$\mathfrak{g}^0\colon \mathcal{D}(E) \to \kappa^1 ,$$

which is the composition $\mathcal{D}(E) \hookrightarrow \mathcal{D}(J^1) \to \kappa^1$; namely, we can write

$$\mathfrak{g}^0(X)(\vartheta) := \mathrm{r}^1(\vartheta, u), \qquad \vartheta \in J^1, \; u \equiv X_{\underline{\vartheta}} .$$

REMARK. - Sometimes it is useful to interpret $\mathrm{r}^k$ as a section of the bundle $(\tau_E^k)^* \otimes w^k = \pi_{k,0}^*(\tau_E^*) \otimes w^k$. In other words, $\mathrm{r}^k$ can be regarded as a $\kappa^k$-valued first order differential form on $E$.

Now we are ready to explain how all these constructions are connected with the theory of contact transformations of $J^\infty$ [5].

Recall that the manifold $J^\infty$ is canonically equipped with an $m$-dimensional integrable distribution, which is called the *Cartan distribution*, or the *infinite contact structure.*

Namely, this is the distribution $\vartheta \hookrightarrow C_\vartheta \subset T_\vartheta(J^\infty)$, where

$$C_\vartheta = T_\vartheta(N^{(\infty)}), \qquad \text{for } \vartheta = [N]_e^\infty .$$

The $C^\infty(J^\infty)$-module of all vector fields belonging to the Cartan distribution is denoted by $\mathcal{CD}(J^\infty)$, or, simply, by $\mathcal{CD}$. We can easily prove that $\mathcal{CD}$ is a Lie subalgebra of

$\mathcal{D}(J^\infty)$. So, the Cartan distribution is integrable in the sense it satisfies the conditions of the Frobenius theorem.

We say that a vector field $X \in \mathcal{D}(J^\infty)$ preserves (infinitesimally) the Cartan distribution iff $[X, Y] \in \mathcal{C}\mathcal{D}$, for any $Y \in \mathcal{C}\mathcal{D}$; in such a case, $X$ is said to be a *$\mathcal{C}$-field* or an *infinitesimal contact transformation* of infinite order. $\mathcal{C}$-fields form a Lie subalgebra of $\mathcal{D}(J^\infty)$, which is denoted by $\mathcal{D}_\mathcal{C}(J^\infty)$, or, simply, by $\mathcal{D}_\mathcal{C}$. Because of integrability of the Cartan distribution we have $\mathcal{C}\mathcal{D} \subset \mathcal{D}_\mathcal{C}$. In fact, $\mathcal{C}\mathcal{D}$ is an ideal of $\mathcal{D}_\mathcal{C}$ and we have the following result which is central for our purposes.

PROPOSITION. – The quotient Lie algebra $\mathcal{D}_\mathcal{C}/\mathcal{C}\mathcal{D}$ is canonically isomorphic to the $C^\infty(J^\infty)$-module $\kappa$.

The module $\kappa$ inherits a Lie algebra structure via this isomorphism. Its explicit construction is given by the map

$$X(\mathrm{mod}\ \mathcal{C}\mathcal{D}) \mapsto \mathrm{g}^0(X) = X \lrcorner\, \mathrm{r}^\infty \tag{9}$$

where $X \in \mathcal{D}_\mathcal{C}$. Here $\mathrm{r}^\infty$ is regarded as a $\kappa$-valued differential 1-form.

Every vector field $X \in \mathcal{D}(E)$ can be canonically prolonged to a $\mathcal{C}$-field $X^\infty \in \mathcal{D}_\mathcal{C}$. Then

$$\mathrm{g}^\infty(X^\infty) = \mathrm{g}^0(X).$$

*The fibred case.*

Now, let us consider a submanifold $N \subset E$. In virtue of (1), pullbacks of the fibre bundles $w^k$ via $j^kN$, $1 \leqslant k \leqslant \infty$, are canonically isomorphic each other. The corresponding bundle over $N$ is denoted by

$$w_N \colon W_N \to N.$$

This bundle can be easily identified with the co-normal bundle of the submanifold $N \subset E$, i.e. with the quotient bundle of $\tau_E/N$ by $\tau_N$.

The above considerations allow us to interpret pullbacks $(j^kN)^*\varphi$, $\varphi \in \kappa^k$, as elements of $\kappa_N := \Gamma(w_N)$.

Suppose now that $E$ is equipped with a fibred structure $\pi \colon E \to B$, where $\dim B = m$. The $k$-th jet $[s]^k_x$ of a local section $s\colon U \to E$, $U \subset B$, of $\pi$ at the point $x \in U$ can be defined as

$$[s]^k_x = [s(U)]^k_{s(x)} \in J^k(E, m), \qquad 0 \leqslant k \leqslant \infty. \tag{10}$$

The manifold of all $k$-th order jets of local sections of $\pi$ is denoted by $J^k\pi$. Definition (10) shows $J^k\pi$ to be an open and every where dense subset of $J^k(E, m)$, as it can be easily seen. By this reason, all above definitions and constructions are valid for the manifolds $J^k\pi$ as well. And, moreover, the fibred structure on $E$ produces some additional useful identifications.

In particular, let

$$\pi_k := \pi \circ \pi_{k,0} \colon J^k\pi \to B\,, \qquad 0 \leqslant k \leqslant \infty\,.$$

Then, the bundle $c^k\colon C^k \to J^k\pi$ is identified with the pullback $\pi_k^*(\tau_B)$, $1 \leqslant k \leqslant \infty$. This identification is given by means of the isomorphisms

$$d_{\underline{\vartheta}}\pi\colon L_{\tilde{\vartheta}} \to T_{\pi(\underline{\vartheta})}B\,, \qquad \vartheta \in J^k\pi\,,$$

taking into consideration that $\pi_k(\vartheta) = \pi(\underline{\vartheta})$ and that $L_{\underline{\vartheta}}$ and $T_{\pi(\underline{\vartheta})}B$ are fibres over $\vartheta$ of $c^k$ and $\pi_k^*(\tau_B)$, respectively.

Similarly, let

$$\upsilon_\pi\colon V_\pi \to E$$

be the *vertical subbundle* of $\tau_E$ consisting of all vectors tangent to the fibres of $\pi$. Then, $w^k$ is identified with the pullback $\pi_k^*(\upsilon^\pi)$. This results from the decomposition

$$T_{\underline{\vartheta}}E = \upsilon_\pi^{-1}(\underline{\vartheta}) \oplus L_{\tilde{\vartheta}}\,,$$

because of the identifications $\upsilon_\pi^{-1}(\tilde{\vartheta}) =$ (the fibre of $\pi_{\infty,0}^*(\upsilon_\pi)$ over $\vartheta \in J^k\pi$) and $T_{\underline{\vartheta}}E/L_{\tilde{\vartheta}} =$ (the fibre of $w^k$ over $\vartheta$).

Below, the pullback of $\upsilon_\pi$ via $\pi_{k,0}$ is denoted by

$$\upsilon_\pi^k\colon V_\pi^{(k)} \to J^k\pi\,.$$

*Coordinates.*

The local chart of $c^k$ naturally associated with the special chart (4) on $J^k$ is

$$(x^\lambda, y^i_\sigma, z^\lambda)\,, \qquad z^\lambda(\vartheta, u) \equiv u^\lambda\,, \qquad 1 \leqslant \lambda \leqslant m,\ 1 \leqslant i \leqslant l,\ 0 \leqslant |\sigma| \leqslant k\,.$$

Moreover, the vector fields

$$b_\lambda := \partial_\lambda + y^i_\lambda \partial_i\,, \qquad 1 \leqslant \lambda \leqslant m\,,$$

constitute the local basis of $c^1\colon C^1 \to J^1$, which is naturally associated with the local chart (5) of $J^1$.

Then, we obtain the following coordinate expression of $\text{Д}^k$

$$(x, y^i_\sigma, z^\lambda) \mapsto (x, y^i_\sigma;\ z^\lambda, z^\lambda y^i_\lambda)$$

i.e.

$$\text{Д}^k = dz^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i)\,.$$

The local chart of $w^k$ naturally associated with the special chart (4) on $J^k$ is

$$(x^\lambda, y^i_\sigma, z^\lambda)\,, \qquad z^i(\Gamma^k(\vartheta, u)) \equiv u^i - u^\lambda y^i_\lambda(\vartheta)\,.$$

The associated local basis of sections of $w^k$ consists of the generating functions

$$\zeta_i \equiv \mathrm{g}^0(\partial_i) \in \kappa^1 \subset \kappa^k, \qquad 1 \leq i \leq l,$$

of the vector fields $\partial_i \equiv \partial/\partial y^i$.

By interpreting $\mathrm{r}^k$ as a section of the bundle $\pi^*_{\infty,0}(\tau^*_E) \otimes w^k$, we obtain the following coordinate expression of $\mathrm{r}^k$:

$$\mathrm{r}^k = (dy^i - y^i_\lambda dx^\lambda) \otimes \zeta_i. \tag{11}$$

Now, by joining together (10) and (11), we get the coordinate expression of $\mathrm{g}^k$:

$$\mathrm{g}^k(X) = (X^i - X^\lambda y^i_\lambda)\,\zeta_i,$$

where $X = X^i \partial_i + X^\lambda \partial_\lambda + X^i_\sigma \partial/\partial y^i_\sigma$.

The pullback $\mathrm{r}_N$ of $\mathrm{r}^k$ via $j^k N$ (which does not depend on $k$) looks as

$$\mathrm{r}_N = (j^k N)^* \mathrm{r}^k = (dy^i - df^i) \otimes \zeta_{i,N}, \tag{12}$$

assuming that $N$ is given by (3) and $\zeta_{i,N} \equiv (j^k N)^* \zeta_i$. Here, $dy^i$ should be regarded as a section of $\tau^*_E|_N$.

The coordinate description of the Cartan distribution on $J^\infty$ can be given by means of the so-called full derivatives

$$D_\lambda = \partial_\lambda + \sum_{i,\sigma} y^i_{\sigma+\lambda} \frac{\partial}{\partial y^i_\sigma}, \qquad \lambda = 1, \ldots, m. \tag{13}$$

Namely, the subspace $C_\vartheta \subset T_\vartheta(J^\infty)$ for $\vartheta \in J^\infty$ is the span of vectors $D_{\lambda,\vartheta}$, $\lambda = 1, \ldots, m$, where

$$D_{\lambda,\vartheta} = \partial_\lambda + \sum_{i,\sigma} y^i_{\sigma+\lambda} \frac{\partial}{\partial y^i_\sigma}.$$

Now, we can see that (locally):

$$\mathcal{C}\mathcal{D}(J^\infty) \ni y \Leftrightarrow Y = \alpha^\lambda D_\lambda, \qquad \text{for some } \alpha^\lambda \in C^\infty(J^\infty).$$

The dual way to give the Cartan distribution is by means of the infinite Pfaff system

$$\omega^i_\sigma := dy^i_\sigma - y^i_{\sigma+\lambda} dx^\lambda = 0, \qquad \text{for all } i, \ \sigma.$$

Every $X \in \mathcal{D}_\mathcal{C}$ can be presented uniquely in term form

$$X = \mathfrak{Э}_\varphi + Y, \qquad Y \in \mathcal{C}\mathcal{D}, \tag{14}$$

where $\varphi \equiv (\varphi^1, \ldots, \varphi^l)$, $\varphi^i \in C^\infty(J^\infty)$, is the generating function of $X$ and

$$\mathfrak{Э}_\varphi = \sum_{\lambda,\sigma} D_\sigma(\varphi^i) \frac{\partial}{\partial y^i_\sigma}. \tag{15}$$

Here, $D_\sigma \equiv D_1^{i_1} \circ \ldots \circ D_m^{i_m}$, supposing that $\sigma \equiv (i_1, \ldots, i_m)$.

Finally, if $Z = \alpha^\lambda \partial_\lambda + \beta^i \partial_i$, $\alpha^\lambda, \beta^i \in C^\infty(E)$, is a vector field on $E$, then its infinite prolongation $Z^\infty \in \mathcal{D}(J^\infty)$ looks as

$$Z^\infty = \Im_\varphi + \alpha^\lambda D_\lambda ,$$

where $\varphi \equiv (\varphi^1, \ldots, \varphi^m)$ and $\varphi^i \equiv \beta^i - \alpha^\lambda y^i_\lambda$. Then, finite prolongations $Z^k \in \mathcal{D}(J^k)$ of $Z$ are projections of $Z^\infty$ onto $J^k$, $k = 1, 2, \ldots$, via $\pi_{\infty, k}$.

## 3. – Pre-connections.

By taking in mind the well-known definition of connection on fibred manifolds, we adopt the following

DEFINITION. – A *pre-connection* (on the space of all $m$-dimensional submanifolds of $E$) is a section

$$\gamma\colon E \to J^1(E, m)$$

of the bundle $\pi_{1,0}\colon J^1 \to E$.

The distribution

$$E \ni e \mapsto L_{\gamma(e)} \subset T_e E$$

on $E$ is naturally associated with $\gamma$. The corresponding $m$-dimensional vector subbundle of $TE$ will be denoted by

$$h_\gamma\colon H_\gamma \to E$$

and said to be *horizontal* with respect to $\gamma$. Obviously, $\gamma$ is uniquely characterized by this subbundle. Moreover, we have

$$h_\gamma = \gamma^*(c^1).$$

The subspace $h_\lambda^{-1}(e) = L_{\gamma(e)}$ of $T_e E$ is said to be *horizontal* with respect to $\gamma$ at $e \in E$.

REMARK. – An arbitrary $m$-distribution on $E$ can be, evidently, interpreted as a section of the bundle $\pi_{1,0}\colon J^1(E, m) \to E$, i.e. as a pre-connection. So, it seems to be no difference between distributions and pre-connections. In fact, the notion of distribution is used below in its «relative meaning», i.e. with relation to the role it plays in the study of the «manifold» of all $m$-dimensional submanifolds of a given manifold $E$. To underline this polarization of mind, we use the term «pre-connection» instead of «distribution». The subsequent exposition with clarify our terminology.

We define the *vertical* bundle

$$\upsilon_\gamma : V_\gamma \to E$$

of $\gamma$, to be quotient vector bundle of $\tau_E$ by $h_\gamma$. By definition, the fibre of $\upsilon_\gamma$ over $e \in E$ is $T_e E / L_{\gamma(e)}$. From the definitions it follows directly that

$$\upsilon_\gamma = \gamma^*(w^1).$$

Moreover, we have the following exact sequence of vector bundles over $E$

$$0 \to H_\gamma \xrightarrow{\iota_\gamma} TE \xrightarrow{\nu_\gamma} V_\gamma \to 0\,, \tag{16}$$

where $\iota_\gamma$ and $\nu_\gamma$ are the natural inclusion and quotient maps, respectively. Obviously, (16) is the pullback of (7), for $k = 1$, via $\gamma^*$.

*The fibred case.*

A fibred structure $\pi: E \to B$, $\dim B = m$, is said to be *transversal* to $\gamma$ if the horizontal subspaces of $\gamma$ are transversal to the corresponding fibres of $\pi$. In this case, $\gamma(E) \subset J^1\pi \subset J^1(E, m)$ and $\gamma$ can be considered as a section $\gamma: E \to J^1\pi$ of the bundle $\pi_{1,0}: J^1\pi \to E$, i.e. as a connection on $\pi$. In other words, a pre-connection $\gamma$ on $E$ becomes a connection with respect to any transversal fibred structure on $E$ (see [8]). Moreover, such a structure allows us to identify:

— the vertical bundle $\upsilon_\gamma$ of $\gamma$ with the vertical subbundle $\upsilon_\pi: V_\pi \to B$ of $\pi$;

— the horizontal bundle $h_\gamma$ of $\gamma$ with $\pi^*(\tau_B)$.

This results from the identification of $w^1$ with $\pi^*_{1,0}(\upsilon_\pi)$ and of $c^1$ with $\pi^*_1(\tau_B)$, which have been introduced in the previous section:

$$\upsilon_\gamma = \gamma^*(w^1) = \gamma^*(\pi^*_{1,0}(\upsilon_\pi)) = (\pi_{1,0} \circ \gamma)^*(\upsilon_\pi) = id^*_E(\upsilon_\pi) = \upsilon_\pi\,,$$

$$h_\gamma = \gamma^*(c^1) = \gamma^*(\pi^*_1(\tau_B)) = (\pi_1 \circ \gamma)^*(\tau_B) = \pi^*(\tau_B)\,.$$

Moreover, the bundle $w_N: W_N \to W$ is identified with the bundle $\upsilon_\pi|_N: V_\pi|_N \to N$ in a similar way.

The above identification map of vertical bundles can be regarded as a splitting of the short exact sequence (16). We shall see that the only point in which the theory of pre-connections differs from that of connections is the absence of a natural splitting of (16).

*Coordinates.*

A section $\gamma: E \to J^1$ of $\pi_{1,0}$ is described, in terms of an admissible divided chart (2) on $E$ and the corresponding coordinates (5) on $J^1$, by means of the local functions on $E$

$$\gamma^i_\lambda := y^i_\lambda \circ \gamma\,.$$

Here, «admissible» means that $H_\gamma$ is transversal to the coordinate submanifolds $x = \text{const}$ of the considered local chart.

The Pfaff equations determining the horizontal distribution $h_\gamma$ are

$$dy^i - \gamma^i_\lambda dx^\lambda = 0\,, \qquad 1 \leqslant \lambda \leqslant m\,.$$

The local vector fields

$$B_\lambda := \partial_\lambda + \gamma^i_\lambda \partial_i\,, \qquad 1 \leqslant \lambda \leqslant m\,, \tag{17}$$

constitute a local basis of $\Gamma(h_\gamma)$.

The local sections

$$Q_i := \gamma^*(\zeta_i) = \partial_i \operatorname{mod} H_\gamma\,, \qquad 1 \leqslant \lambda \leqslant m\,, \tag{18}$$

constitute a local basis of $\Gamma(\upsilon_\gamma)$.

By interpreting the $C^\infty(E)$-module homomorphism $\nu_\gamma\colon \mathcal{D}(E) \to \Gamma(\upsilon_\gamma)$ as an element of $\Gamma(\tau^*_E \otimes \upsilon_\gamma)$, we obtain the following coordinate expression

$$\nu_\gamma = (dy^i - \gamma^i_\lambda dx^\lambda) \otimes Q_i\,. \tag{19}$$

## 4. - Coavariant differential of a pre-connection.

Now we shall show that the basic notions of covariant differential and curvature can be defined just at the level of pre-connections.

Let

$$h^k_\gamma\colon H^{(k)}_\gamma \to J^k \qquad \text{and} \qquad \upsilon^k_\gamma\colon V^{(k)}_\gamma \to J^k$$

be the pullbacks of $h_\gamma$ and of $\upsilon_\gamma$ via $\pi_{k,0}$, respectively. Then, we have the canonical maps

$$\iota^k_\gamma := \pi^*_{k,0}(\iota_\gamma) \to T^{(k)}E \qquad \text{and} \qquad \nu^k_\gamma := \pi^*_{k,0}(\nu_\gamma)\colon T^{(k)}E \to V^{(k)}_\gamma$$

constituting the exact sequence

$$0 \to H^{(k)}_\gamma \xrightarrow{\iota^k_\gamma} T^{(k)}E \xrightarrow{\nu^k_\gamma} V^{(k)}_\gamma \to 0$$

which is the pullback of (16) via $\pi_{k,0}$.

The case $k = 1$ will be of particular interest for us.

DEFINITION. - The *covariant differential* of a pre-connection $\gamma$ is the composition

$$\nabla_\gamma := \Gamma^1 \circ \iota^1_\gamma\colon H^{(1)}_\gamma \to w^1\,,$$

or, passing to sections, the corresponding homomorphism of $C^\infty(J^\infty)$-modules

$$\nabla_\gamma : \Gamma(h^1_\gamma) \to \kappa^1 .$$

Sometimes, it is useful to interpret $\nabla_\gamma$ as a section of the bundle $(h^1_\gamma)^* \otimes w^1$.

From the above considerations it follows that the covariant differential can be regarded as a $\kappa^1$-valued differential form defined on the distribution $H_\gamma$. Therefore, the insertion operation $\mathcal{D}(E) \supset \Gamma(h_\gamma) \to \kappa^1$

$$X \mapsto \nabla_{\gamma, X} := X \lrcorner \nabla_\gamma$$

is well defined. This gives another interpretation of $\nabla_\gamma$ as the operation which assigns to every vector field on $E$, belonging to the distribution $H_\gamma$, its generating function.

Moreover, we can associate with $\nabla_{\gamma, X}$ the first order differential operator, denoted by the same symbol,

$$\nabla_{\gamma, X} : N \mapsto (j^1 N)^* \nabla_{\gamma, X} \in \Gamma(w_N), \tag{20}$$

which acts on $m$-dimensional submanifolds of $E$.

*The fibred case.*

As we have already seen in the previous section, a pre-connection $\gamma$ becomes a connection in presence of a fibring $\pi: E \to B$ transversal to $\gamma$.

PROPOSITION. – In this case, the covariant differential of $\gamma$, as defined above, can be identified with the standard covariant differential of $\gamma$ regarded as a connection.

PROOF. – To do it, one has:

i) to identify $w_N$ with $\upsilon_\pi|_N$ for $N = s(B)$, $s \in \Gamma(\pi)$;

ii) to pass from submanifolds of $E$ to sections of $\pi$ and from vector fields belonging to $H_\gamma$ to horizontal lifts of vector fields on $B$.

These changes transform definition (20) into

$$\nabla_Y : s \mapsto (j^1 s)^* (\nabla_{\gamma, \bar{Y}}) \in \Gamma(s^*(\upsilon_\pi)) \tag{21}$$

where $s \in \Gamma(\pi)$, $Y \in \mathcal{D}(B)$ and $\bar{Y}$ is horizontal lift of $Y$ with respect to $\gamma$. Now, it is easy to see that the operator $\nabla_\gamma$ defined by (21) coincides with the standard covariant derivative operator along the vector field $Y$ on $B$ assigned to the connection $\gamma$ (see [8]). It can be also seen coordinate-wisely from the local expression of $\gamma$ presented below. ■

Observe, also, the identification of $h_\gamma^k$ with $\pi_k^*(\tau_B)$:

$$h_\gamma^k = \pi_{k,0}^*(h_\gamma) = \pi_{k,0}^*(\pi^*(\tau_B)) = (\pi \circ \pi_{k,0})^*(\tau_B) = \pi_k^*(\tau_B).$$

In its turn, this allows us to identify $(h_\gamma^1)^* \otimes w^1$ and $\pi_1^*(\tau_B^*) \otimes \upsilon_\pi^1$. So, the covariant differential of the pre-connection $\gamma$ can be viewed as an element

$$\nabla_\gamma \in \Gamma(\pi_1^*(\tau_B)) \otimes \Gamma(\upsilon_\pi^1),$$

i.e. as a horizontal (with respect to $\pi_1$) $\upsilon_\pi^1$-valued differential form of $J^1\pi$. Then, the $C^\infty(B)$-linear map

$$\mathcal{D}(B) \ni Y \mapsto \bar{Y} \lrcorner \nabla_\gamma \in \Gamma(\upsilon_\pi^1)$$

defines a $\upsilon_\pi^1$-valued differential 1-form on $B$, which will be denoted by

$$\nabla_\gamma \in \Gamma(\tau_B^*) \underset{C^\infty(B)}{\otimes} \Gamma(\upsilon_\pi^1).$$

This gives an alternative description of the covariant differential of the connection $\gamma$.

Let $\Box_\gamma := \nu^1 \circ \text{Л}^1 \colon C^1 \to V_\gamma^{(1)}$. Collecting now together all basic maps introduced above, we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& C^1 \arrow[rr, "\Box_\gamma"] \arrow[d] \arrow[dr, "\text{Л}^1"] & & V_\gamma^{(1)} \arrow[ur] & \\
0 & \pi_1^*(TB) \arrow[l] & T^{(1)}E \arrow[l] \arrow[ur, "\nu_\gamma^1"] \arrow[dr, "r^1"'] & V_\pi^{(1)} \arrow[l] \arrow[u] \arrow[d] & 0 \arrow[l] \\
& H_\gamma^1 \arrow[u] \arrow[ur, "\iota_\gamma^1"] \arrow[rr, "\nabla_\gamma"'] & & W^1 \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd} \tag{22}$$

where the vertical arrows are, by definition, compositions of usual ones belonging to the same triangle. If $\pi\colon E \to B$ is transversal to $\gamma$, then they coincide, as it is easy to see, with the identifications made above:

$$C^1 \Leftrightarrow \pi_1^*(TB) \Leftrightarrow H_\gamma^{(1)} \quad \text{and} \quad V_\gamma^{(1)} \Leftrightarrow V_\pi \Leftrightarrow W^1.$$

Moreover, it is easy to check that, by performing these identifications, we have

$$\Box_\gamma = -\nabla_\gamma.$$

By this reason, $\square_\gamma$ is an alternative candidate for «pre-covariant differential». But the choice we made here seems to be more preferable, in particular, because of its direct relation to the theory of generating functions.

*Coordinates.*

Below we interpret $\nabla_\gamma$ as an element of $\Gamma(h_\gamma^{1*} \otimes w^1)$. Let $(B^\mu)$ be the basis of $h_\gamma^*$ dual to $(B_\lambda)$, (17). Then, using (11) and (17), we get the coordinate expression of $\nabla_\gamma$:

$$\nabla_\gamma = (\gamma_\lambda^i - y_\gamma^i) B^\lambda \otimes \zeta_i \,, \tag{23}$$

where $B^\lambda$ stands for $\pi_{1,0}^*(B^\lambda)$.

Let $(b^\lambda)$ be the local basis of $(c^1)^*$ dual to $(b_\lambda)$. Then, the following coordinate expression of $\square_\gamma$ results from (19)

$$\square_\gamma = (y_\lambda^i - \gamma_\lambda^1) b^\lambda \otimes Q_i \,, \tag{24}$$

where we interpret $\square_\gamma$ as a section of $(c^1)^* \otimes h_\gamma^1$.

In coordinates, the identifications $H_\gamma^{(1)} \leftrightarrow C^1$ and $W^1 \leftrightarrow H_\gamma^{(1)}$ described above look as

$$B_\lambda \leftrightarrow b_\lambda \quad \text{and} \quad \zeta_i \leftrightarrow Q_i$$

if the initial divided chart (2) on $E$ is coherent with $\pi$, i.e. if $x^\lambda$ are «base coordinates» and $y^i$ are «fibre coordinates».

## 5. - Deviation of a pre-connection.

The analogue of the notion of curvature for pre-connections is that of *deviation*, which will be introduced below.

For this purpose, we note that the map

$$\delta_\gamma : \Gamma(h_\gamma) \times \Gamma(h_\gamma) \to \Gamma(\upsilon_\gamma) : (u, \upsilon) \mapsto \upsilon_\gamma([u, \upsilon]) \tag{25}$$

is $C^\infty(E)$-linear and, obviously, skew-symmetric. In fact, for $f \in C^\infty(E)$,

$$\upsilon_\gamma([fu, \upsilon]) = \upsilon_\gamma(f[u, \upsilon]) - \upsilon_\gamma(\upsilon \cdot fu) = f\upsilon_\gamma([u, \upsilon]) \,,$$

because $\upsilon_\gamma$ is $C^\infty(E)$-linear and $\upsilon_\gamma(\Gamma(h_\gamma)) = 0$.

DEFINITION. - The skew-symmetric and $C^\infty(E)$-bilinear form

$$\delta_\gamma : \Gamma(h_\gamma) \times \Gamma(h_\gamma) \to \Gamma(\upsilon_\gamma)$$

is called the *deviation* of $\gamma$.

The integrability condition

$$[\Gamma(h_\gamma), \Gamma(h_\gamma)] \subset \Gamma(h_\gamma)$$

of the distribution $H_\gamma$ is, obviously, equivalent to the vanishing of $\delta_\gamma$. So, $\delta_\gamma$ measures the deviation of $H_\gamma$ from being a completely integrable distribution.

We shall regard $\delta_\gamma$ as an element of $\Gamma(\Lambda^2(h_\gamma) \otimes \upsilon_\gamma)$, where $\Lambda^k$ denotes the $k$-th exterior power of a vector bundle.

REMARK. – Let $\gamma$ be a connection of a bundle $\pi\colon E \to B$. Then, the deviation of the pre-connection $\gamma$ is related to the curvature of the connection $\gamma$ of $\pi$ in the following way.

Let $X, Y \in \mathcal{D}(B)$ and $u, \upsilon \in \mathcal{D}(E)$ be the horizontal lifts of $X$ and $Y$, respectively. Then, we have

$$R_\gamma(X, Y) = \delta_\gamma(u, \upsilon),$$

where $R_\gamma$ is the curvature of the connection $\gamma$ and $\nu_\gamma$ is identified with the vertical bundle of $\pi$.

*Coordinates.*

By choosing $(B_\lambda)$ as a local basis of $\Gamma(h_\gamma)$, we have

$$[\alpha^\lambda B_\lambda, \beta^\mu B_\lambda] = (\alpha^\lambda \beta^\mu - \alpha^\mu \beta^\lambda)(\partial_\lambda \gamma^i_\mu + \gamma^j_\lambda \partial_j \gamma^i_\mu)\,\partial_i \bmod H_\gamma .$$

Therefore,

$$\delta_\gamma = (\partial_\lambda \gamma^i_\mu + \gamma^j_\lambda \partial_j \gamma^i_\mu) B^\lambda \wedge B^\mu \otimes Q_i . \tag{25}$$

This formula becomes the usual coordinate expression of the curvature if we refer to fibred coordinates.

*The graded («super») extension of deviation.*

The above expression of deviation can be naturally expressed in terms of the following general machinery.

Namely, let $\Lambda^i \mathcal{H}^*_\gamma$ denote the $C^\infty(E)$-module of $C^\infty(E)$-valued and skew-symmetric forms on $H_\gamma$ and let $\mathcal{H}_\gamma \equiv \Gamma(h_\gamma)$, $\mathcal{V}_\gamma \equiv \Gamma(\upsilon_\gamma)$. Then, for a given $\gamma$, the natural differential-like maps

$$\boldsymbol{\delta}_\gamma\colon \Lambda^i \mathcal{H}^*_\gamma \otimes \mathcal{H}_\gamma \to \Lambda^{i+1} \mathcal{H}^*_\gamma \otimes \mathcal{V}_\gamma , \qquad i = 0, 1, \ldots, m ,$$

are defined by means of the formula

$$(\boldsymbol{\delta}_\gamma \varphi)(u_1, \ldots, u_{i+1}) = \frac{1}{i+1} \sum_{0 \leq k \leq i+1} (-1)^{k-1} \nu_\gamma [u_k, \varphi(u_1, \ldots, \widehat{u}_k, \ldots, u_{i+1})],$$

for $u_k \in \mathcal{H}_\gamma$, $\varphi \in \Lambda^i \mathcal{H}^*_\gamma \otimes \mathcal{H}_\gamma$.

If $id_\gamma \in \mathcal{H}^*_\gamma \otimes \mathcal{H}_\gamma = \mathrm{Hom}_{C^\infty(E)}(\mathcal{H}_\gamma, \mathcal{H}_\gamma)$ is the identity operator, then

$$\delta_\gamma = \boldsymbol{\delta}_\gamma(id_\gamma), \tag{26}$$

as it is easily seen from the definitions.

We note that the $C^\infty(E)$-module

$$\Lambda^* \mathcal{H}^*_\gamma \otimes \mathcal{H}_\gamma = \Big(\sum_{i \geq 0} \Lambda^i \mathcal{H}^*_\gamma\Big) \otimes \mathcal{H}_\gamma = \sum_{i \geq 0} (\Lambda^i \mathcal{H}^*_\gamma \otimes \mathcal{H}_\gamma)$$

can be also regarded as a graded $\Lambda^* \mathcal{H}_\gamma$-module, where $\Lambda^* \mathcal{H}^*_\gamma \equiv \sum_{i \geq 0} \Lambda^i \mathcal{H}^*_\gamma$ is viewed as a graded algebra, with respect to the standard exterior product:

$$(\omega, \rho \otimes u) \mapsto (\omega \wedge \rho) \otimes u\,, \qquad \omega, \rho \in \Lambda^* \mathcal{H}^*_\gamma\,, \ \ u \in \mathcal{H}_\gamma\,.$$

Similarly, the $C^\infty(E)$-module

$$\Lambda^* \mathcal{H}^*_\gamma \otimes \mathcal{V}_\gamma = \sum_{i \geq 0} (\Lambda^i \mathcal{H}^*_\gamma \otimes \mathcal{V}_\gamma)$$

can be viewed as a graded $\Lambda^* \mathcal{H}^*_\gamma$-module. Then, it is easy to see that

$$\boldsymbol{\delta}_\gamma(\omega \wedge \rho) = (-1)^i \omega \wedge \text{Д}_\gamma(\varphi)\,, \qquad \omega \in \Lambda^i \mathcal{H}^*_\gamma\,, \ \ \varphi \in \Lambda^* \mathcal{H}^*_\gamma \otimes \mathcal{H}_\gamma\,.$$

In other words, interpreting $\boldsymbol{\delta}_\gamma$ to be a graded map

$$\boldsymbol{\delta}_\gamma \colon \Lambda^* \mathcal{H}^*_\gamma \otimes \mathcal{H}_\gamma \to \Lambda^* \mathcal{H}^*_\gamma \otimes \mathcal{V}_\gamma$$

of degree 1, we see that it is a homomorphism of graded $\Lambda^* \mathcal{H}^*_\gamma$-modules.

Moreover, we define the $\lrcorner$-product

$$(\Lambda^i \mathcal{H}_\gamma \otimes \mathcal{H}_\gamma) \otimes (\Lambda^i \mathcal{H}^*_\gamma \otimes \mathcal{V}_\gamma) \to \Lambda^{i+j-1} \mathcal{H}^*_\gamma \otimes \mathcal{V}_\gamma$$

by means of the formula

$$(\omega \otimes u) \lrcorner (\rho \otimes \upsilon) = (\omega \wedge (u \lrcorner \rho)) \otimes \upsilon\,,$$

where $\omega \in \Lambda^i \mathcal{H}^*_\gamma$, $\rho \in \Lambda^j \mathcal{H}^*_\gamma$, $u, \upsilon \in \mathcal{H}_\gamma$ and $(u \lrcorner \rho)(u_1, \ldots, u_{j-1}) := \rho(u, u_1, \ldots, u_{j-1})$. Then, we see that, for $u \in \mathcal{H}_\gamma$,

$$\boldsymbol{\delta}_\gamma(u)(u_1) = \upsilon_\gamma([u_1, u]) = \delta_\gamma(u_1, u) = -(u \lrcorner \delta_\gamma)(u_1)\,,$$

i.e.

$$\boldsymbol{\delta}_\gamma(u) = -u \lrcorner \delta_\gamma\,.$$

More generally, we have the following

PROPOSITION. – If $\varphi = \omega \otimes u \in \Lambda^i \mathcal{H}^*_\gamma \otimes \mathcal{H}_\gamma$, then

$$\boldsymbol{\delta}_\gamma(\varphi) = \boldsymbol{\delta}_\gamma(\omega \otimes u) = (-1)^i \omega \wedge \boldsymbol{\delta}_\gamma(u) = (-1)^{i-1} \omega \wedge (u \lrcorner \delta_\gamma) = (-1)^{i-1} \varphi \lrcorner \delta_\gamma\,,$$

i.e.

$$\boldsymbol{\delta}_\gamma(\varphi) = (-1)^{i-1} \varphi \lrcorner \delta_\gamma\,. \tag{27}$$

Formula (27) shows that the deviation $\delta_\gamma$ determines $\boldsymbol{\delta}_\gamma$ completely.

From the point of view of the graded (or «super») calculus it is natural to define the degree of $\Lambda^i \mathcal{H}_\gamma^* \otimes \mathcal{H}_\gamma$ as well as $\Lambda^i \mathcal{H}_\gamma^* \otimes \mathcal{V}_\gamma$ to be equal to $i-1$ (see, for example, [20]). Then the degree of $\partial_\gamma$ is equal to 1 and we see from (26) that the deviation is the right operator corresponding to the left operator $\boldsymbol{\delta}_\gamma$. This allows us to interpret $\boldsymbol{\delta}_\gamma$ as the graded (or «super») extension of $\partial_\gamma$.

# Part II. - (Pre)-connections on almost-fibrings.

## 1. - Almost fibrings.

As we have already seen, the covariant differential and the deviation of a pre-connection $\gamma$ on $E$, in presence of a fibred structure $\pi$: $E \to B$, turn out to be the covariant differential and the curvature of the corresponding connection, respectively. More exactly, this is achieved by means of appropriate identifications, the main of which are those of $V_\gamma$ and $V_\pi$, which allow us to realize $V_\gamma$ as subbundle of $TE$. The last one, however, can be done with the help of an $l$-dimensional distribution on $E$ transversal to $H_\gamma$. So, having in mind the role that such a distribution could play in the theory of pre-connections, we adopt the following terminology.

DEFINITION. - An $l$-dimensional distribution $\text{п}$: $V_\text{п} \to E$, $V_\text{п} \subset TE$, on $E$ is said to be an *almost-fibring* of $E$ (with respect to the «space» of all $m$-dimensional submanifolds of $E$).

REMARK. - This notion is of the same «relative» nature as that of pre-connection and its introduced to keep the necessary polarization in mind.

It is natural to call *almost-sections* of an almost-fibring п on $E$ all $m$-dimensional submanifolds of $E$ which are transversal to п. Jets of a given order $k$ of almost-sections of п constitute an open and every where dense set of $J^k(E, m)$ which we denote by $J^k\text{п}$. In such a way, jet manifolds are associated with almost-fibrings. It is easy to see that the standard generalities of the jet theory of fibrings can be carried over onto jets of almost-fibrings without changing a word. For instance, we have the maps

$$\pi_{k,l}\colon J^k\text{п} \to J^l\text{п} \quad \text{and} \quad j_k N\colon N \to J^k\text{п}, \qquad 0 \leqslant l \leqslant k \leqslant \infty,$$

supposing that $N \subset E$ is an almost-section of п.

REMARK. - It seems promising to study the geometry of pairs $(J^k(E, m), J^k\text{п})$, $k > 0$, supplied with the Cartan distribution in view of further applications to the theory of distributions (i.e. of п).

For a given almost-fibring $\pi$, we denote by

$$h_\pi : H_\pi \to E$$

the quotient bundle of $\tau_E$ by $\pi$. Also

$$\pi^k : V_\pi^{(k)} \to J^k \quad \text{and} \quad h_\pi^k : H_\pi^{(k)} \to J^k$$

denote pullbacks of $\pi$ and $h_\pi$ via $\pi_{k,0}$, respectively.

A divided chart (2) is concordant with an almost fibring $\pi$ if the distribution $V_\pi$ is transversal to the coordinate submanifolds $y = \text{const}$. In this case, $V_\pi$ can be described with the aid of functions $\pi_i^\lambda$, which define vector fields of the form

$$B_i = \partial_i + \pi_i^\lambda \partial_\lambda, \qquad i = 1, \ldots, l.$$

## 2. - Pre-connections on almost-fibrings.

Now, we consider two transversal distributions on $E$, say $\gamma$ and $\pi$, of dimension $m$ and $l$, respectively, and regard the first of them as a pre-connection and the second one as an almost-fibring. The addition of an almost-fibring to a pre-connection enlarges algebraic and analytic tools to work with and, in particular, it allows us to use the machinery of the Frölicher-Nijenhuis bracket in the spirit of [10].

The compositions

$$H_\gamma \hookrightarrow TE \xrightarrow{\text{quotient along } V_\pi} H_\pi$$

$$V_\gamma \hookrightarrow TE \xrightarrow{\text{quotient along } H_\pi} V_\pi$$

are, evidently, isomorphisms of vector bundles if $\gamma$ and $\pi$ are transversal and, therefore, lead to the identifications

$$h_\gamma^k \sim h_\pi^k \quad \text{and} \quad v_\gamma^k \sim \pi^k, \qquad 0 \leqslant k \leqslant \infty.$$

Let now $P_\gamma : TE \to TE$ and $P_\pi : TE \to TE$ be the projections of $TE$ onto $H_\gamma \subset TE$ and $V_\pi \subset TE$, respectively, which correspond to the direct decomposition $TE = H_\gamma \oplus V_\pi$. Then, for all $\vartheta \in J^1\pi$, the linear maps

$$TE \supset L_\vartheta \xrightarrow{P_\gamma} h^{-1}(\underline{\vartheta}) \subset TE$$

are isomorphisms for all $\vartheta \in J^1\pi$ and generate an isomorphism of the linear bundles $\pi^1$ and $w^1$.

The isomorphism obtained in this way leads, via $p_{k,0}$, $1 \leqslant k \leqslant \infty$, to the isomorphisms

$$c^k \sim h_\gamma^k \quad \text{and} \quad \pi^k \sim w^k.$$

So, in presence of an almost-fibring $\pi$ transversal to $\gamma$, we have the following analogue of diagram (22):

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& C^1 \arrow[rr, "\square_\gamma"] \arrow[d] \arrow[dr, "J^1"] & & V_\gamma^{(1)} \arrow[ur] & \\
0 & H_\pi^{(1)} \arrow[l] & T^{(1)}E \arrow[l] \arrow[ur, "\nu_\gamma^1"'] \arrow[dr, "\Gamma^1"] & V_\pi^{(1)} \arrow[l] \arrow[u] \arrow[d] & 0 \arrow[l] \\
& H_\gamma^1 \arrow[u] \arrow[ur, "\iota_\gamma^1"'] \arrow[rr, "\nabla_\gamma"'] & & W^1 \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
\tag{28}
$$

Here, all bundles entering in it are supposed to be defined on $J^1\pi$ and all vertical arrows coincide with the corresponding isomorphisms defined above. Moreover, the identification $\upsilon_\gamma \sim \pi$ allows us to interpret $\delta_\gamma$ as an $V_\pi$-valued bilinear form on $H_\gamma$, i.e. as *the curvature tensor of $\gamma$ with respect to* $\pi$. We denote it by $R_\gamma^\pi$:

$$R_\gamma^\pi(u, \upsilon) \in \Gamma(\pi), \qquad u, \upsilon \in \Gamma(h_\gamma).$$

*Coordinates.*

We suppose the considered divided chart $(x^\lambda, y^i)$ on $E$ to be concordant both with $\gamma$ and $\pi$, i.e. that

$$B_\lambda = \partial_\lambda + \gamma_\lambda^i \partial_i, \qquad B_i = \partial_i + \pi_i^\lambda \partial_\lambda$$

are bases of $H_\gamma$ and $V_\pi$, respectively. Then,

$$\partial_\lambda = S_\lambda^\mu B_\mu + S_\lambda^j B_j, \qquad \partial_i = S_i^\mu B_\mu + S_i^j B_j, \tag{29}$$

where

$$
\begin{cases}
\|S_\lambda^\mu\| = \|\delta_\lambda^\mu - \gamma_\lambda^i \pi_i^\mu\|^{-1}, & \|S_i^j\| = \|\delta_i^j - \pi_i^\mu \gamma_\mu^j\|^{-1}, \\
S_\lambda^i = -\gamma_\lambda^j S_j^i = -S_\lambda^\mu \gamma_\mu^i, & S_j^\mu = -\pi_j^\lambda S_\lambda^\mu = -S_j^i \pi_i^\mu .
\end{cases}
\tag{30}
$$

Consider the basis $q_i := \partial_i \operatorname{mod} \Gamma(\pi)$, $i = 1, \dots, l$, of $H_\pi$. Then, the above identifications have the following coordinate description:

$$
(31)\qquad
\begin{cases}
\text{п} \sim \upsilon_\gamma\colon B_i \leftrightarrow (\delta_i^j - \text{п}_i^\mu \gamma_\mu^j)\, Q_j\,, & Q_j \leftrightarrow S_j^i B_i\,,\\
\text{п}^1 \sim w^1\colon B_i \leftrightarrow (\delta_i^j - \text{п}_i^\mu \gamma_\mu^j)\, \zeta_j\,, & \\
\upsilon_\gamma^1 \sim w^1\colon Q_j \leftrightarrow S_j^i (\delta_i^k - \text{п}_i^\lambda y_\gamma^k)\, \zeta_k\,, & \\
h_\gamma \sim h^{\text{п}}\colon B_\lambda \leftrightarrow (\delta_\lambda^\mu - y_\lambda^i \text{п}_i^\mu)\, q_\mu\,, & q_\mu \leftrightarrow S_\mu^\lambda B_\lambda\,,\\
c^1 \sim h_{\text{п}}^1\colon b_\lambda \leftrightarrow (\delta_\lambda^\mu - y_\lambda^i \text{п}_i^\mu)\, q_\mu\,, & \\
c^1 \sim h_\lambda^1\colon b_\lambda \leftrightarrow (\delta_\lambda^\mu - y_\lambda^i \text{п}_i^\mu)\, S_\mu^\nu B_\nu\,. &
\end{cases}
$$

Now we obtain from (25) and (31) the coordinate description of the curvature tensor $R_\gamma^{\text{п}}$ of $\gamma$ with respect to п:

$$R_\gamma^{\text{п}} = (\partial_\lambda \gamma_\mu^i + \gamma_\lambda^j \partial_j \gamma_\mu^i)\, S_i^k B^\lambda \wedge B^\mu \otimes B_k\,.$$

In other words, the components of this tensor with respect to the basis $(B_\lambda, B_i)$ of $TE$ are

$$(R_\gamma^{\text{п}})_{\lambda\mu}^k = (\partial_\lambda \gamma_\mu^i + \gamma_\lambda^j \partial_j \gamma_\mu^i - \partial_\mu \gamma_\lambda^i - \gamma_\mu^j \partial_j \gamma_\lambda^i)\, S_i^k\,.$$

We see that this expression coincides with the standard one when $S_j^k = \delta_j^k$, i.e. when $V_{\text{п}} = V_\pi$ for a suitable fibring $\pi\colon E \to B$ (see [10]).

## 3. – Thre Frölicher-Nijenhuis machinery.

The pair consisting of a pre-connection $\gamma$ and an almost-fibring п transversal to it can be given by means of each of the projections $P_\lambda, P_{\text{п}}\colon TE \to TE$ (see previous section), or by each of the corresponding $C^\infty(E)$-linear maps

$$\omega_\gamma, \omega_{\text{п}}\colon \mathcal{D}(E) \to \mathcal{D}(E)\,.$$

It is evident that

$$\omega_\gamma^2 \omega_\gamma\,, \qquad \omega_{\text{п}}^2 = \omega_{\text{п}}\,, \qquad \omega_\gamma + \omega_{\text{п}} = id_{\mathcal{D}(E)}\,,$$

$$\Gamma(h_\gamma) = \operatorname{im} \omega_\gamma = \ker \omega_{\text{п}}\,, \qquad \Gamma(\text{п}) = \ker \omega_\gamma = \operatorname{im} \omega_{\text{п}}\,.$$

Below we interpret these maps as vector-valued differential forms on $E$: $\omega_\gamma, \omega_{\text{п}} \in \Lambda^1(E) \otimes \mathcal{D}(E)$. This enables us to apply the machinery of Frölicher-Nijenhuis bracket (see [3]) which was found to be rather useful in the context of connections (see [1], [2], [4], [8], [9], [10], [11], [12], [13], [15]).

From (29), (30) we deduce the coordinate expressions of $\omega_\gamma$ and $\omega_{\text{п}}$:

$$\omega_\gamma = (S_\lambda^\mu\, dx^\lambda + S_i^\mu\, dy^i) \otimes B_\mu = (S_\lambda^\mu\, dx^\lambda + S_i^\mu\, dy^i) \otimes \partial_\mu + (S_i^\mu \gamma_\mu^k\, dx^i - S_\lambda^k\, dy^\lambda) \otimes \partial_k\,,$$

$$\omega_{\text{п}} = (S_\lambda^k\, dx^\lambda + S_i^k\, dy^i) \otimes B_k = (S_\lambda^k \text{п}_k^\mu\, dx^\lambda - S_i^\mu\, dy^i) \otimes \partial_\mu + (S_\lambda^k\, dx^\lambda + S_i^k\, dy^\lambda) \otimes \partial_k\,.$$

We recall that the Frölicher-Nijenhuis (F-N) bracket on $E$ is a set of pairings

$$[\cdot,\cdot]\colon \Lambda^r(E)\otimes \mathcal{D}(E)\times \Lambda^s(E)\otimes \mathcal{D}(E)\to \Lambda^{r-s}(E)\otimes \mathcal{D}(E)$$

which supplies the $\Lambda^*(E)$-module $\mathcal{N}(E):=\Lambda^*(E)\otimes \mathcal{D}(\mathcal{E})$ with a graded Lie algebra-structure. Here, $\Lambda^*(E):=\sum_{i\geq 0}\Lambda^i(E)$ stands for the exterior algebra of differential forms on $E$ and tensor products are taken over $C^\infty(E)$. On decomposable elements of $\mathcal{N}(E)$ the F-N bracket is given by the formula:

$$[\alpha\otimes u,\beta\otimes v]=\alpha\wedge\beta\otimes[u,v]+\alpha\wedge L_u\beta\otimes v-(-1)^{rs}\beta\wedge L_v\alpha\otimes u+$$
$$+(-1)^r i_v\alpha\wedge d\beta\otimes u-(-1)^{rs+s}i_u\beta\wedge d\alpha\otimes v,$$

where $\alpha\in\Lambda^r(E)$, $\beta\in\Lambda^s(E)$, $u, v\in\mathcal{D}(E)$ and $L_w$ and $i_w$ denote the Lie derivative and the insertion operator along $w\in\mathcal{D}(E)$, respectively. For an alternative approach, see [20].

The graded skew-commutativity and the Jacobi identity for F-N bracket looks as

$$[\alpha,\beta]=-(-1)^{|\alpha||\beta|}[\beta,\alpha],$$

$$(-1)^{|\alpha||\rho|}[\alpha,[\beta,\rho]]+(-1)^{|\beta||\alpha|}[\beta,[\rho,\alpha]]+(-1)^{|\rho||\beta|}[\rho,[\alpha,\beta]]=0,$$

where we put $|\omega|=s$ for $\omega\in\Lambda^s(E)\otimes\mathcal{D}(E)$.

With any $\omega\in\mathcal{N}(E)$ we associate a differential-like map

$$d_\omega\colon \mathcal{N}(E)\to\mathcal{N}(E),\qquad d_\omega(\rho):=[\omega,\rho].$$

Because of the Jacoby identity, $d_\omega$ turns out to be a derivation of degree $|\omega|$ of the Lie algebra $\mathcal{N}(E)$:

$$d\omega[\alpha,\beta]=[d_\omega\alpha,\beta]+(-1)^{|\omega||\alpha|}[\alpha,d_\omega\beta].$$

The Jacobi identity can be also presented in the form

$$d_\omega\circ d_\rho-(-1)^{|\omega||\rho|}d_\rho\circ d_\omega=d_{[\omega,\rho]}.$$

In particular, for $\omega=\rho$, we have

$$\frac{1}{2}(1-(-1)^{|\omega|})\,d_\omega^2=d_{r_\omega},$$

where

$$r_\omega:=\frac{1}{2}\,d_\omega\omega=\frac{1}{2}[\omega,\omega].$$

From the skew-symmetricity of F-N bracket it follows that

$$r_\omega=0\quad\text{if } |\omega| \text{ is even}.$$

On the contrary, in general $r_\omega \neq 0$ if $|\omega|$ is odd. Moreover, if $\omega \in \Lambda^1(E) \otimes \mathcal{D}(E)$ is a connection, then $r_\omega$ is its curvature (see [10]).

Another consequence of the skew-symmetricity of the F-N bracket is

$$[r_\omega, r_\omega] = 0\,.$$

which is valid for arbitrary $\omega \in \mathcal{N}(E)$, but is non-trivial for odd $|\omega|$ only. Furthermore, the following generalized Bianchi identity results from the Jacobi identity

$$d_\omega r_\omega = 0\,; \tag{32}$$

it is non-trivial for odd $|\omega|$.

## 4. - Curvature with respect to an almost-fibring.

To illustrate in which way the Frölicher-Nijenhuis techniques can be applied to (pre-connections), we present below the expressions of the curvature and the torsion in terms of the F-N bracket.

To start with, we recall the following general formula, which is valid for arbitrary $\alpha, \beta \in \mathrm{Hom}_{C^\infty(E)}(\mathcal{D}(E), \mathcal{D}(E)) \equiv \Lambda^1(E) \underset{C^\infty(E)}{\otimes} \mathcal{D}(E)$ and $u, v \in \mathcal{D}(E)$ (see [14], [10]):

$$\begin{aligned}[\alpha, \beta](u, v) = {} & [\alpha(u), \beta(v)] + [\beta(u), \alpha(v)] - \alpha([u, \beta(v)]) - \beta([u, \alpha(v)]) + \\ & + \alpha([v, \beta(u)]) + \beta([v, \alpha(u)]) + \alpha(\beta([u, v])) + \beta(\alpha([u, v]))\,.\end{aligned} \tag{33}$$

The immediate consequences of this formula, for a projection operator $\alpha$, i.e. such that $\alpha^2 = \alpha$, are:

$$\begin{cases} [\alpha, \alpha](u, v) = 2\alpha([u, v])\,, & \text{for } u, v \in \ker \alpha\,, \\ [\alpha, \alpha](u, v) = 2[u, v] - 2\alpha([u, v])\,, & \text{for } u, v \in \operatorname{im} \alpha\,. \end{cases} \tag{34}$$

Then, by putting

$$d_\gamma := d_{\omega_\gamma}\,, \qquad d_\Pi := d_{\omega_\Pi}\,,$$

and $\alpha = \omega_\Pi$ or $\omega_\gamma$, we see from (34) that

$$d_\gamma(\omega_\gamma)(u, v) = d_\Pi(\omega_\Pi)(u, v) = 2\omega_\Pi([u, v])\,, \qquad u, v \in \Gamma(h_\gamma)\,.$$

In other words, we have

$$R_\gamma^\Pi = \frac{1}{2}\, d_\gamma(\omega_\gamma)|_{H_\gamma} = \frac{1}{2}\, d_\Pi(\omega_\Pi)|_{H_\gamma}$$

and the Bianchi identity for $\gamma$ with respect to $\Pi$ follows from (32) by observing that

$$R_\gamma^\Pi = r_{\omega_\gamma} = r_{\omega_\Pi}\,.$$

Let now $\mu$ be another pre-connection on $E$, transversal to the same almost fibring $\Pi$. Then, the difference between $\gamma$ and $\mu$ can be given by means of the so-called *soldering form* $\sigma \in \mathrm{Hom}_{C^\infty(E)}(\Gamma(h_\gamma), \Gamma(\Pi)) \equiv \Gamma(h_\gamma)^* \underset{C^\infty(E)}{\otimes} \Gamma(\Pi)$. By definition $\sigma(u) \in \Gamma(\Pi)$, for $u \in \Gamma(h_\gamma)$, is the unique «vertical» vector field such that $u + \sigma(u) \in \Gamma(h_\mu)$.

One can regard the composition $\sigma \circ \omega_\gamma \colon \mathcal{D}(\mathcal{E}) \to \Gamma(\Pi)$ to be a map into $\mathcal{D}(\mathcal{E})$ because of the inclusion $\Gamma(\pi) \subset \mathcal{D}(\mathcal{E})$. Then, by applying (33) to $\alpha = \omega_\gamma$, $\beta = \sigma \circ \omega_\gamma$, we obtain:

$$[\omega_\gamma, \sigma \circ \omega_\gamma](u, v) = \omega_\Pi([u, \sigma(v)]) - \omega_\Pi([v, \sigma(u)]) - \sigma(\omega_\gamma([u, v])) \tag{35}$$

for $u, v \in \Gamma(h_\gamma)$. We denote the right hand side of (35) by $\tau(u, v) = \tau_{\Pi, \sigma}(u, v)$ and interpret it as the *torsion of* $\gamma$ *with respect to* $\sigma$ *and* ($\Pi$) (see [10], [12]). Now, we can rewrite (35) as

$$d_\gamma(\omega_\gamma \circ \sigma)\big|_{H_\gamma} = \tau_{\Pi, \sigma}.$$

This is the desired expression of the torsion in terms of the F-N bracket.

## Part III. - Higher order pre-connections.

### 1. - The infinite prolongation of pre-connections and almost-fibrings.

In this section the above theory on pre-connections and almost-fibrings is lifted to $J^\infty$. This is the necessary step in order to interpret connections and their generalizations in the framework of the category of differential equations (see [5], [19]). We recall that objects of this category are (locally) infinitely prolonged differential equations and, in particular, infinite jet spaces.

To start with, we need some elementary notions from the theory of smooth manifolds.

Let $M$, $N$ be manifolds and $F \colon M \to N$ be a smooth map. Then, the algebra $C^\infty(M)$ can be considered as a $C^\infty(N)$-module according to the multiplication:

$$(f, \varphi) \mapsto F^*(f) \cdot \varphi, \qquad f \in C^\infty(N), \quad \varphi \in C^\infty(M).$$

An *$M$-valued vector field on $N$ along $F$* is by definition a derivation of the algebra $C^\infty(N)$ with values in the algebra $C^\infty(M)$, regarded as a $C^\infty(N)$-module. The set of all such fields will be denoted by $\mathcal{D}(N; M, F)$. If

$$X \colon C^\infty(N) \to C^\infty(M)$$

is an $M$-valued field on $N$ and $f \in C^\infty(M)$, then the operator $fX \colon C^\infty(N) \to C^\infty(M)$, $\varphi \mapsto f \cdot X(\varphi)$, $\varphi \in C^\infty(N)$, is also an $M$-valued field. This fact shows that the multiplication $(f, X) \mapsto fX$ supplies $\mathcal{D}(N; M, F)$ with a structure of $C^\infty(M)$-module.

For $X \in \mathcal{D}(N; M, F)$ and $a \in M$, we can define the vector $X_a \in T_{F(a)}N$ by the rule

$$X_a(\varphi) \equiv X(\varphi)(a), \qquad \forall \varphi \in C^\infty(N).$$

From this formula it follows that the $M$-valued vector field $X$ can be interpreted as the section $a \mapsto X_a$ of the pullback $F^*(\tau_N)$ of the tangent bundle $\tau_N: TN \to N$ along $F$. This gives the identification

$$\mathcal{D}(N; M, F) = \Gamma(F^*(\tau_N)).$$

Let now $H$ be a distribution on $N$:

$$N \ni b \mapsto H_b \subset T_b N.$$

An $M$-valued vector filed is said to belong to $H$ if $X_a \in H_{F(a)}$, $\forall a \in M$. Obviously, all such fields constitute a submodule of $\mathcal{D}(N; M, F)$ denoted by $\mathcal{D}_H(N; M, F)$.

For each $X \in \mathcal{D}(N)$ (resp., $Y \in \mathcal{D}(M)$), we obtain the $M$-valued vector field $F^* \circ X$ (resp., $Y \circ F^*$) defined by $(F^* \circ X)(\varphi) = F^*(X(\varphi))$ (resp., $(Y \circ F^*)(\varphi) = X(F^*(\varphi))$, where $\varphi \in C^\infty(N)$.

If $x = (x^1, \dots, x^m)$ and $y = (y^1, \dots, y^n)$ are local coordinated on $M$ and $N$, respectively, then the corresponding local expression of $X \in \mathcal{D}(N; M, F)$ looks as

$$X = \sum_{1 \le i \le n} X^i(x)\left(F^* \circ \frac{\partial}{\partial y^i}\right), \tag{36}$$

where the $X^i(x)$'s are the components of the vector $X_a$ with respect to the basis $\partial/\partial y^1, \dots, \partial/\partial y^1$ of $T_{F(a)}N$ and the $x$'s are the coordinates of $a \in M$. This results directly from the above definition of $X_a$.

Below we will use the simplified notation

$$X = \sum_{1 \le i \le n} X^i(x) \frac{\partial}{\partial y^i},$$

instead of (36).

A map $G: M' \to M$ generates the map

$$G^0: \mathcal{D}(N; M, F) \to \mathcal{D}(N; M', F \circ G),$$

where

$$G^0(X)(\varphi) \equiv G^*(X(\varphi)), \qquad X \in \mathcal{D}(N; M, F), \quad \varphi \in C^\infty(N).$$

In particular, in such way, we get the map

$$F^0: \mathcal{D}(N) \to \mathcal{D}(N; M, F)$$

by identifying $\mathcal{D}(N)$ with $\mathcal{D}(N; N, id_N)$. Evidently, $F^0(X) = F^* \circ X$, for $X \in \mathcal{D}(N)$.

Note that $G^0$ is injective if $G$ is surjective.

Returning to jets, we define the $C^\infty(J^\infty)$-module

$$\mathcal{D}(E; J^\infty, \pi_{\infty,0})$$

to be the direct limit of the sequence

$$\mathcal{D}(E) \xrightarrow{\pi^0_{1,0}} \mathcal{D}(E; J^1, \pi_{1,0}) \xrightarrow{\pi^0_{2,1}} \dots \xrightarrow{\pi^0_{k,k-1}} \mathcal{D}(E; J^k, \pi_{k,0}) \xrightarrow{\pi^0_{k+1,k}} \dots .$$

Then, we have the following fundamental isomorphism

$$\mathcal{D}_{\mathcal{C}}(J^\infty(E, m)) = \mathcal{D}(E; J^\infty(E, m), \pi_{\infty,0}) \tag{37}$$

realized by means of the map:

$$\mathcal{D}_{\mathcal{C}}(J^\infty) \ni Y \mapsto Y \circ \pi^*_{\infty,0} \in \mathcal{D}(E; J^\infty, \pi_{\infty,0})$$

(see [5]). Below we identify $\mathcal{D}_{\mathcal{C}}(J^\infty)$ and $\mathcal{D}(E; J^\infty, \pi_{\infty,0})$.

Now we are able to associate a submodule of $\mathcal{D}_{\mathcal{C}}(J^\infty)$, denoted by $\mathcal{D}_\gamma$, with a given pre-connection $\gamma$ on $E$:

$$\mathcal{D}_\gamma := \mathcal{D}_{H_\gamma}(E; J^\infty, \pi_{\infty,0}) \subset \mathcal{D}_{\mathcal{C}}.$$

We need also the quotient $C^\infty(J^\infty)$-module

$$\kappa_\gamma = \mathcal{D}_{\mathcal{C}} / \mathcal{D}_\gamma.$$

The following identifications result directly from definitions:

$$\begin{aligned} &\mathcal{D}_{\mathcal{C}}(J^\infty) = \mathcal{D}(E; J^\infty, \pi_{\infty,0}) = \Gamma(\tau^\infty_E), \\ &\mathcal{D}_\gamma = \Gamma(h^\infty_\gamma), \qquad \kappa_\gamma = \Gamma(\upsilon^\infty_\gamma), \end{aligned} \tag{38}$$

(see Sec. 2 for the definition of $\Gamma(\alpha^\infty)$).

Because $h^\infty_\gamma$ and $\upsilon^\infty_\gamma$ are pullbacks via $\pi_{\infty,0}$ of $h_\gamma$ and $\upsilon_\gamma$, respetively, it is possible to lift the notion of deviation to $J^\infty$ by applying the pullback operation $\pi^*_{\infty,0}$ to the definition of $\delta_\gamma$ (see sec. 5). This leads us to the following defintion of the lifted deviation $\delta^\infty_\gamma$:

$$\delta^\infty_\gamma(X, Y) := \nu^\infty_\gamma([X, Y]), \qquad X, Y \in \mathcal{D}_\gamma, \tag{39}$$

where $\nu^\infty_\gamma$ can be interpreted either as the quotient map $\mathcal{D}_{\mathcal{C}} \to \kappa_\gamma$, or as the map $\Gamma(h^\infty_\gamma) \to \Gamma(\upsilon^\infty_\gamma)$ which corresponds to the $\pi_{\infty,0}$-pullback of $\nu_\gamma$ (see Sec. 3).

Similarly, the pullback operation $\pi^*_{\infty,0}$, applied to the definition of the covariant differential of a pre-connection, lifts this notion to $J^\infty$. Taking into account the above identifications, we can see that this lifted differential $\nabla^\infty_\gamma$ is the composition

$$\begin{tikzcd} \mathcal{D}_\gamma \arrow[r, hook, "i^\infty_\gamma"] \arrow[rr, bend right, "\nabla^\infty_\gamma"'] & \mathcal{D}_{\mathcal{C}} \arrow[r] & \kappa = \mathcal{D}_{\mathcal{C}} / \mathcal{C}\mathcal{D} \end{tikzcd}$$

In other words, the lifted covariant differential $\nabla^\infty_\gamma$ assigns to every $\mathcal{C}$-field belonging to $\mathcal{D}_\gamma$ its generating function.

Finally, by the same reasons, the $\pi_{\infty,0}$-lifting $\square_\gamma^\infty$ of the operator $\square_\gamma$ can be identified with the composition

$$\begin{tikzcd}
\mathcal{C}\mathcal{D} \arrow[r, hook] \arrow[rr, bend right, "\square_\gamma^\infty"'] & \mathcal{D}_{\mathcal{C}} \arrow[r, "\nu_\gamma^\infty"] & \kappa_\gamma
\end{tikzcd}$$

More generally, we can associate with a distribution $V \subset TE$ on $E$ the following two $C^\infty(J^\infty)$-modules:

$$\mathcal{D}_V := \mathcal{D}_V(E; J^\infty, \pi_{\infty,0}) \subset \mathcal{D}_{\mathcal{C}}$$

and

$$\kappa_V := \mathcal{D}_{\mathcal{C}} / \mathcal{D}_V .$$

As above, we have the natural identifications

$$\mathcal{D}_V = \Gamma(\upsilon^\infty), \qquad \kappa_V = \Gamma(\bar{\upsilon}^\infty), \tag{40}$$

where $\upsilon = \tau_E|_V \colon V \to E$ and $\bar{\upsilon}\colon TE/V \to E$ is the quotient bundle.

By putting together all maps and modules defined previously, we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dl] \\
& \mathcal{C}\mathcal{D} \arrow[rr, "\square_\gamma^\infty"] \arrow[d] \arrow[dr] & & \kappa_\gamma & \\
0 & \kappa_V \arrow[l] & \mathcal{D}_{\mathcal{C}} \arrow[l, "\nu_V^\infty"'] \arrow[ur, "\nu_\gamma^\infty"] \arrow[dr] & \mathcal{D}_V \arrow[l, "i_V^\infty"'] \arrow[u] \arrow[d] & 0 \arrow[l] \\
& \mathcal{D}_\gamma \arrow[u] \arrow[ur, "\iota_\gamma^\infty"] \arrow[rr, "\nabla_\gamma^\infty"'] & & \kappa \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd} \tag{41}$$

where $i_V^\infty : \mathcal{D}_V \to \mathcal{D}_{\mathcal{C}}$ and $\nu_V^\infty : \mathcal{D}_{\mathcal{C}} \to \kappa_V$ are the inclusion and the quotient map, respectively, and the vertical arrows close, by defintion, the corresponding triangles.

If $V$ is a distribution complementary to $H_\gamma$, then all these vertical arrows are isomorphisms, as it is easily seen from identifications (38), (40). So, in this case we have the identifications

$$\Gamma D_\gamma \leftrightarrow \kappa_V \leftrightarrow \mathcal{C}\mathcal{D}, \qquad \kappa \leftrightarrow \mathcal{D}_V \leftrightarrow \kappa_\gamma .$$

This fact shows (41) to be the lifted variant of (22).

## 2. – Higher order connections.

The «lifted» point of view on connections presented in the previous section reveals one of its advantages in that it leads straightforwardly to the higher order generalization of this concept. One of the possible generalizations of the notion of connection, to which «higher order» is applicable, is geometrically evident; this is a sections of a fibring $\pi_{k,k-1}: J^k\pi \to J^{k-1}\pi$, $k > 1$ (see Sec. 3 and, for example [7], [8]). But, below, these words are used for a different notion, which might be more worthy than the standard one.

The mentioned higher order generalization of the notion of connection results directly from the observation that the basic diagram (41) preserves its meaning when $\mathcal{D}_\gamma$ is substituted by an arbitrary $n$-dimensional projective submodule of $\mathcal{D}_\mathcal{C}$.

To perform this passage geometrically, we need the «relative» analogue of the notion of distribution when an underlying base manifold, say $M$ (absolute case) is replaced by a map, say $F: M \to N$ (relative case). As we have already seen in the previous section, the $C^\infty(M)$-module $\mathcal{D}(N; M, F)$ is the relative analogue of the $C^\infty(M)$-module $\mathcal{D}(M)$ of all vector fields on $M$. Moreover, we can treat an «absolute» distribution on $M$ as the projective submodule of $\mathcal{D}(M)$ consisting of all vecotr fields belonging to this distribution. Therefore, from this point of view, it is natural to interpret projective submodules of $\mathcal{D}(N; M, F)$ as relative distributions (with respect to $F$). To finish our motivations, it remains to observe that, in virtue of the isomorphism $\mathcal{D}(N; M, F) = \Gamma(F^*(\tau_N))$, every projective submodule of $\mathcal{D}(N; M, F)$ can be realized geometrically in the form $\Gamma(\mu)$, where $\mu: H_\mu \to M$ is a vector subbundle of the pullback $F^*(\tau_N)$.

Thus, we have motivated the following definition.

DEFINITION. – A vector subbundle of $F^*(\tau_N)$ is said to be a *relative distribution on $N$ (with respect to $F$)*.

Then, coming back to jets, we get the $k$-th order analogues of pre-connections on $E$, by passing from absolute $m$-dimensional distributions on $E$ to the relative ones with respet to $\pi_{k,0}$:

DEFINITION. – An $m$-dimensional vector subbundle of $\tau_E^k: T^{(k)}(E, m) \to J^k(E, m)$ is said to be a *k-th order pre-connection on E.*

The fibre $\mu^{-1}(\vartheta)$ of a $k$-th order pre-connection $\mu: H_\mu \to J^k(E, m)$ at the point $\vartheta \in J^k(E, m)$ can be naturally viewed as an $m$-dimensional subspace of $T_{\underline{\vartheta}}E$, where $\underline{\vartheta} \equiv \pi_{k,0}(\vartheta)$.

DEFINITION. – A $k$-th order pre-connection on $E$ is said to be a *k-th order connection* of a fibring $\pi: E \to B$, $\dim B = m$, if its fibres are transversal to the fibres of $\pi$.

It is easy to see that the «usual» (pre-)connections are exactly 0-order (pre-)connections in the sense of this definition.

Below, we adopt for $k$-th order (pre-)connections notations previously used for «usual» (pre-)connections. For instance, $\gamma\colon H_\gamma \to J^k$, where $H_\gamma$ is an $m$-dimensional vecotr subbundle of $T^{(k)}E$, will be the standard notations for $k$-th (pre-)connections.

EXAMPLE. – There is a unique canonical $k$-th order pre-connection for $k \geqslant 1$; namely, it is given by the subbundle $c^k\colon C^k(E, m) \to J^k(E, m)$ of $\tau_E^k$. As being restricted on $J^k\pi \subset J^k(E, m)$, where $\pi\colon E \to B$ is a fibring, it gives the canonical $k$-th order connection on $\pi$.

The *covariant differential* of a $k$-order pre-connection $\gamma\colon H_\gamma \to J^k$, $k \geqslant 1$, is defined to be the composition

$$\nabla_\gamma := \mathrm{r}^k \circ i_\gamma\colon H_\gamma \to W^k,$$

where $i_\gamma\colon H_\gamma \to T^{(k)}$ is the natural inclusion. By passing to sections, we can interpret $\nabla_\gamma$ as an element of the $C^\infty(J^k)$-module

$$\mathrm{Hom}_{C^\infty(J^k)}(\Gamma(\gamma), \kappa^k) = \Gamma^*(\gamma) \underset{C^\infty(J^k)}{\otimes} \kappa^k,$$

i.e. as a $\kappa^k$-valued $C^\infty(J^k)$-linear form on $\Gamma(\gamma)$. Below, we write $X \lrcorner \nabla_\gamma$ for $X \in \Gamma(\gamma)$ instead of $\nabla_\gamma(X)$. This is to stress the interpretation of $X$ as a relative vector field with respect to $\pi_{k,0}$, which results from the inclusion

$$\Gamma(\gamma) \subset \Gamma(\tau_E^k) = \mathcal{D}(E; J^k, \pi_{k,0}).$$

The evolution operator corresponding to the generating function $X \lrcorner \nabla_\gamma \in \kappa^k$, $X \in \Gamma(\gamma)$, is called the *pre-covariant derivative along* $X$ (with respect to $\gamma$).

PROPOSITION. – If $\gamma$ is a $k$-th order connection of the fibring $\pi\colon E \to B$, then every vector field $Y \in \mathcal{D}(B)$ can be lifted to the section $\bar{Y} \in \Gamma(\gamma)$. Namely, if $\vartheta \in J^k$, then the subspace $\gamma^{-1}(\vartheta) \subset T_\vartheta E$ is projected isomorphically onto $T_{\pi(\vartheta)}B$ by means of $d_\vartheta \pi$. We define the vector field $\bar{Y}$ to be the image of $Y_{\pi(\vartheta)} \in T_{\pi(\vartheta)}B$ along this isomorphism.

The evolution operator corresponding to the generating fucntion

$$\nabla_{\gamma, Y} := \bar{Y} \lrcorner \nabla_\gamma \in \kappa^k \tag{42}$$

is called the *covariant differential along* $Y$ (with respect to $\gamma$). From (42) it follows that

$$\nabla_{\gamma, fY} = f\nabla_{\gamma, Y}, \qquad \text{for } f \in C^\infty(B).$$

This formula reproduces, for $k$-th order connections, the well-known property of the «usual» covariant derivatives.

Next, the quantity

$$\nabla_{\gamma, Y}(s) := (j^k s)^* \nabla_{\gamma, Y},$$

for $s \in \Gamma(p)$, gives the *velocity of the parallel transport of $s$ along $Y$ in virtue of $\gamma$*. This is a direct consequence of the theory of higher order contact transformations (see [5], [18]). Contrary to the «usual» connections, the operator $s \mapsto \nabla_{\gamma, Y} s$ is, in general, a $k$-th order differential operator if $k > 0$. So, $k$-th order connections are characterized by the fact that the corresponding covariant derivatives are differential operators of $k$-th order.

EXAMPLE. – The covarinat differential of the canonical $k$-th order connection is equal to zero.

## 3. – Infinite lifting of higher order connections.

Thus, we see that there is no need to rise to $J^\infty$ in order to define covariant differentials and covariant derivatives of $k$-th order connections. On the contrary, this is absolutely necessary in what concerns the deviation or the curvature. This seems to be the reason why $k$-th order connections as they are defined in this work have not been introduced earlier. The lift of $k$-th order connections on $J^\infty$ is carried out as follows.

Let $\alpha: H_\alpha \to J^k(E, m)$ be a vector subbundle of $\tau_E^k$ (not necessarily $m$-dimensional). We consider its pullback

$$\alpha^{(l)}: H_\alpha^{(l)} \to J^l(E, m), \qquad l \geqslant k,$$

along $\pi_{l,k}$ to be a subbundle of $\tau_E^l$. Therefore,

$$\Gamma(\alpha^{(l)}) \subset \Gamma(\tau_E^l) = \mathcal{D}(E; J^l, \pi_{l,0})$$

and

$$\Gamma(\alpha^{(\infty)}) \subset \mathcal{D}(E; J^\infty, \pi_{\infty,0}),$$

where $\Gamma(\alpha^{(\infty)}) = \lim\limits_{l \to \infty} \operatorname{dir} \Gamma(\alpha^{(l)})$. Then, by applying the fundamental identification (37), we can identify $\Gamma(\alpha^{(\infty)})$ with a submodule of $\mathcal{D}_{\mathcal{C}}(J^\infty)$, denoted by $\mathcal{D}_\alpha$. It is easy to see that $\mathcal{D}_\alpha$ consists of all vecotr fields $X \in \mathcal{D}_{\mathcal{C}}(J^\infty)$ such that,

$$T_\vartheta \pi_{\infty,0}(X_\vartheta) \in \alpha^{-1}(\pi_{\infty,k}(\vartheta)), \qquad \text{for any } \vartheta \in \mathrm{J}^\infty,$$

where the fibre $\alpha^{-1}(\xi)$, $\xi \in J^k$, is viewed as a subspace of $T_\xi E$.

Introducing the $C^\infty(J^\infty)$-module $\kappa_\alpha = \mathcal{D}_{\mathcal{C}}/\mathcal{D}_\alpha$, we get the following short exact sequence

$$0 \longrightarrow \mathcal{D}_\gamma \xrightarrow{i_\alpha^\infty} \mathcal{D}_{\mathcal{C}} \xrightarrow{\nu_\alpha^\infty} \kappa_\alpha \longrightarrow 0,$$

where $i_\alpha^\infty$ and $\nu_\alpha^\infty$ are the inclusion and the quotient maps, respectively.

DEFINITION. – The $C^\infty(J^\infty)$-bilinear form on $\mathcal{D}_\alpha$

$$\delta_\alpha^\infty(X, Y) = \nu_\alpha^\infty([X, Y]), \qquad X, Y \in \mathcal{D}_\alpha$$

is called *the deviation of* $\alpha$. In particular, if $\gamma\colon H_\gamma \to J^k$ is a $k$-th order pre-connection, then the $\delta_\gamma^\infty$ is called its *deviation.*

EXAMPLE. – For the canonical $k$-th order (pre-)connection $c_k\colon H_{c_k} \to J^k$, we have $\mathcal{D}_{c_k} = \mathcal{C}\mathcal{D}$. Therefore, its deviation (curvature) vanishes identically.

The graded (or «super») extension $\boldsymbol{\delta}_\alpha^\infty$ of $\delta_\alpha^\infty$ can also be defined by following the lines of Sec. 5. Namely, let $\Lambda^i \mathcal{D}_\alpha^*$ denote the $C^\infty(J^\infty)$-module of $C^\infty(J^\infty)$-linear, skew-symmetric $i$-forms on $\mathcal{D}_\alpha$ and let $\Lambda^* \mathcal{D}_\alpha^* \equiv \sum_i \Lambda^i \mathcal{D}_\alpha^*$. Consider the map

$$\boldsymbol{\delta}_\alpha^\infty : \Lambda^i \mathcal{D}_\alpha^* \otimes \mathcal{D}_\alpha \to \Lambda^{i+1} \mathcal{D}_\alpha^* \otimes \kappa_\alpha, \qquad i = 0, 1, \dots, m,$$

(tensor products are taken over $C^\infty(J^\infty)$ defined by the formula

$$(\boldsymbol{\delta}_\alpha^\infty \varphi)(X_1, \dots, X_{i+1}) = \frac{1}{i+1} \sum_{0 \le k \le i+1} (-1)^k \nu_\alpha^\infty [X_k, \varphi(X_1, \dots, \widehat{X}_k, \dots, X_{i+1})]),$$

where $X_k \in \mathcal{D}_\alpha$ and $\varphi \in \Lambda^i \mathcal{D}_\alpha^* \otimes \mathcal{D}_\alpha$. Then, word by word repetition of the arguments of sec. 5 leads us to the formulas

$$\begin{aligned} &\delta_\alpha^\infty = \boldsymbol{\delta}_\alpha^\infty (id_{\alpha(\infty)}) \\ &\boldsymbol{\delta}_\alpha^\infty(\varphi) = (-1)^{i-1} \varphi \lrcorner \delta_\alpha^\infty, \qquad \text{for } \varphi \in \Lambda^i \mathcal{D}_\alpha^* \otimes \mathcal{D}_\alpha. \end{aligned} \tag{43}$$

Here, $\lrcorner$-product and other relevant constructions copy the corresponding ones of Sec. 5. From (43) it follows that $\boldsymbol{\delta}_\alpha^\infty$, regarded as the graded map

$$\boldsymbol{\delta}_\alpha^\infty : \Lambda^* \mathcal{D}_\alpha^* \otimes \mathcal{D}_\alpha \to \Lambda^* \mathcal{D}_\alpha^* \otimes \kappa_\alpha,$$

of $\Lambda^* \mathcal{D}_\alpha$-modules is graded (or «super»-)differential operator of bi-order (0, 1).

The idea of almost-fibring can be also realized in the high order variant. Namely, we say that an $l$-dimensional subbundle of $\tau_E^k$ is a *$k$-th order almost fibring of $E$.*

Let now $\alpha\colon H_\alpha \to J^k$, $\beta\colon H_\beta \to J^s$ be some vector subbundles of $\tau_E^k$ and $\tau_E^s$, respectively. They are said to be *transversal* each other if, for any $\vartheta \in J^\infty$, the fibres $\alpha^{-1}(\vartheta_k)$, $\vartheta_k = \pi_{\infty,k}(\vartheta)$ and $\beta^{-1}(\vartheta_s)$, $\vartheta_s \in \pi_{\infty,s}(\vartheta)$, are transversal as subspaces of $T_\vartheta E$. This is equivalent to the fact that $\mathcal{D}_\alpha + \mathcal{D}_\beta$ is a projective submodule of $\mathcal{D}_\mathcal{C}$ and $\mathcal{D}_\alpha \cap \mathcal{D}_\beta = 0$. In particular, we have

PROPOSITION. – A $k$-th order connection $\gamma\colon H_\gamma \to J^k$ and an $s$-th order almost-fibring $\alpha\colon H_\alpha \to J^s$ are transversal iff $\mathcal{D}_\mathcal{C} = \mathcal{D}_\alpha \oplus \mathcal{D}_\beta$.

Now, by putting together the above constructions, we obtain the following diagram which generalizes (41)

(44)
$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dl] \\
& \mathcal{C}\mathcal{D} \arrow[rr, "\square_\gamma^\infty"] \arrow[d] \arrow[dr] & & \kappa_\gamma & \\
0 & \kappa_\alpha \arrow[l] & \mathcal{D}_{\mathcal{C}} \arrow[l, "\nu_\alpha^\infty"'] \arrow[ur, "\nu_\gamma^\infty"] \arrow[dr] & \mathcal{D}_\alpha \arrow[l, "i_\alpha^\infty"'] \arrow[u] \arrow[d] & 0 \arrow[l] \\
& \mathcal{D}_\gamma \arrow[rr, "\nabla_\gamma^\infty"'] \arrow[u] \arrow[ur, "\iota_\gamma^\infty"] & & \kappa \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

Here, arrows directed forward or backward of $\mathcal{D}_{\mathcal{C}}$ are defined previously. The others, are, by definition, the composition of these ones. For instance, the lifted covariant differential $\nabla_\gamma^\infty$ is defined as the composition

$$\mathcal{D}_\gamma \xrightarrow{i^\infty} {}_\gamma\mathcal{D}_{\mathcal{C}} \longrightarrow \kappa .$$

As in Sec. 4, we observe that all double arrows in (44) are isomorphisms iff $\gamma$ and $\alpha$ are transversal. In this case we can identify $\mathcal{D}_\gamma$ with $\mathcal{C}\mathcal{D}$ and $\kappa_\gamma$ with $\kappa$.

PROPOSITION. – Under the above identifications the covariant differential is seen to be the map

$$\nabla_\gamma^\infty : \mathcal{C}\mathcal{D} \to \kappa$$

and the deviation $\delta_\gamma^\infty$ to be an element of $\Lambda^2 \mathcal{C}\mathcal{D}^* \otimes \kappa$.

Next, by applying the isomorphism $\Lambda^i \mathcal{C}\mathcal{D}^* = \bar{\Lambda}^i(J^\infty)$ (see [5]), where

$$\bar{\Lambda}(J^\infty) \equiv \Lambda^i(J^\infty)/\mathcal{C}\Lambda^i(J^\infty)$$

and $\mathcal{C}\Lambda^i(J^\infty)$ consists of differential $i$-forms vanishing on the Cartan distribution of $J^\infty$, we can consider $\delta_\gamma^\infty$ as an element of $\bar{\Lambda}^2(J^\infty) \otimes \kappa$. So interpreted, $\delta_\gamma^\infty$ is worthy to be named *the curvature of $\gamma$ with respect to the almost fibring $\alpha$*. Similarly, the graded version of the curvature is the map

$$\delta_\gamma^\infty \text{—} \bar{\Lambda}^*(J^\infty) \otimes \mathcal{C}\mathcal{D} \to \bar{\Lambda}^*(J^\infty) \otimes \kappa$$

of $\bar{\Lambda}^*(J^\infty)$-modules which is given by the formula

$$\varphi \mapsto (-1)^{i-1} \varphi \lrcorner \delta_\gamma^\infty ,$$

supposing that $\varphi \in \bar{\Lambda}^*(J^\infty)$, $\delta_\gamma^\infty \in \bar{\Lambda}^2(J^\infty) \otimes \kappa$.

It is natural to look for the higher order generalization of the approach to pre-connections on almost-fibrings based on the Frölicher-Nijenhuis calculus as it was presented in sec. 6-9. Our final observation is that this cannot be achieved without the proper high order generalization of this calculus itself. By this reason, we do not touch this problem here. We conclude with the remark that this generalization can be extracted more or less straightforwardly from the «unification» technique of the work [20] applied to the higher order de Rahm complexes [16].

## 4. - Higher order (pre-)connections coordinate-wisely.

Here we collect local coordinate expressions of basic objects of the theory of higher order (pre-)connections. Since the basic bundles to work with, say $c^k$, $h_\gamma^h$, etc., are pullbacks of bundles defined either on $E$ or on $J^1$, we can adopt as their local bases pullbacks of the corresponding bases of the original bundles. For simplicity, we denote pullback bases by the same symbols, writing, say, $B_\lambda$ instead of $\pi_k^*(B_\lambda)$, $\zeta_i$ instead of $\pi_k^*(\zeta_i)$, etc.

Let now $\gamma$ and $\Pi$ be some $k$-th order connection and $s$-th order connection and $s$-th order almost-fibring, respectively. A local chart (2) is said to be *concordant* with $\gamma$ (resp., with $\Pi$) in a point $\vartheta \in J^k$ if the subspace $\gamma^{-1}(\vartheta) \subset T_\vartheta E$ (resp., $\Pi^{-1}(\vartheta) \subset T_\vartheta E$) is transversal to the $y$-coordinates (resp., to the $x$-coordinates). If so, a local basis of $\Gamma(h_\gamma)$ (resp., of $\Gamma(\Pi)$) can be chosen in a neighbourhood $U$ of $\vartheta$ in the form

$$B_\lambda = \partial_\lambda + \gamma_{\lambda i}^i , \qquad \lambda = 1, \ldots, m$$

resp.,

$$B_i = \partial_i + \Pi_i + \Pi_i^\lambda \partial_\lambda , \qquad i = 1, \ldots, l ,$$

where $\gamma_\lambda^i \in C^\infty(U)$ (resp., $\Pi_i^\lambda \in C^\infty(U)$). In other words, for higher order (pre-)connections we have

$$\gamma_\lambda^i = \gamma_\lambda^i(x^\mu, y^j, \ldots, y_\sigma^j, \ldots) , \qquad |\sigma| \leqslant k ,$$

and. also,

$$\Pi_i^\lambda = \Pi_i^\lambda(x^\mu, y^j, \ldots, y_\sigma^j, \ldots) , \qquad |\sigma| \leqslant s .$$

So, from this point of view, the theory of higher order (pre-)connections differs from the «usual» one by the only fact that quantities $\gamma_\lambda^i$ (and, also, $\Pi_i^\lambda$) can depend on arbitrary high order derivatives.

Similarly, if the chart (2) is concordant both with $\gamma$ and $\Pi$, then formulae (29)-(31)

are still valid for higher order connections as well. Also, it is easy to see that the local expressions (19) of $\nu_\gamma$, (23) of $\nabla_\gamma$ and (24) of $\Box_\gamma$ remain to be local expressions or $\nu_\gamma^\infty$, $\nabla_\gamma^\infty$ and $\Box_\gamma^\infty$, respectivley, if $\gamma$ is a higher order (pre-)connection.

On the contrary, the form of local expression of the deviation and the curvature change when passing to higher order (pre-)connections and some preliminaries are to be listed before.

First, we recall the Lie algebra structure $\{\cdot,\cdot\}$ on the space of generating functions:

$$\{\varphi, \psi\}^i = \ni_\varphi(\psi^i) - \ni_\psi(\varphi^i), \tag{45}$$

where $\varphi = (\varphi^1, \ldots, \varphi^l)$, $\psi = (\psi^1, \ldots, \psi^l)$ and $\varphi^i, \psi^j \in C^\infty(J^\infty)$ (see Sec. 2) and, also, [5], [18]). Then, we can write

$$[\ni_\varphi, \ni_\psi] = \ni_{\{\varphi,\psi\}}. \tag{46}$$

Next, we observe that the equalities

$$[D_\lambda, \ni_\varphi]) = 0 \quad \text{and} \quad [\partial_\lambda, \ni_\varphi] = \ni_{\partial_\lambda \varphi}, \tag{47}$$

take place for every $\lambda$, $\varphi$ and, also,

$$D_\lambda = \ni_{y_\lambda} + \partial_\lambda$$

where $y_\lambda \equiv (y_\lambda^1, \ldots, y_\lambda^l)$ (see (14), (15)).

From (9) and (14) we see that the $\mathcal{C}$-field $\Psi_\lambda$ (see Sec. 2), corresponding to the derivation $B_\lambda \in \mathcal{D}(E; J^\infty(E, m), \pi_{\infty,0})$ via fundamental isomorphism (37), is given by

$$\Psi_\lambda = D_\lambda + \ni_{\gamma_\lambda - y_\lambda} = \ni_{\gamma_\lambda} + \partial_\lambda, \qquad \gamma_\lambda \equiv (\gamma_\lambda^1, \ldots, \gamma_\lambda^l),$$

since $B_\lambda \lrcorner (dy^i - y_\lambda^i dx^\lambda) = \gamma_\lambda^i - y_\lambda^i$. Then, accounting (46)-(47), we get

$$[\Psi_\lambda, \Psi_\mu] = [\ni_{\gamma_\lambda} + \partial_\lambda, \ni_{\gamma_\mu} + \partial_\mu] = \ni_{(\partial_\lambda \gamma_\mu - \partial_\mu \gamma_\lambda + \{\gamma_\lambda, \gamma_\mu\})}.$$

Now, the desired local expression of $\delta_\gamma^\infty$ is deduced straightforwardly from its definition given in the previous section:

$$\delta_\gamma^\infty = (\partial_\lambda \gamma_\mu^i - \partial_\mu \gamma_\lambda^i + \{\gamma_\lambda, \gamma_\mu\}^i) S_i^k B^\lambda \wedge B^\mu \otimes Q_i.$$

(compare it with (25)). Similarly, the local expression of the curvature of $\gamma$ with respect to the almost-fibring $\Pi$ is given by

$$R_\gamma^\Pi = (\partial_\lambda \gamma_\mu^i - \partial_\mu \gamma_\lambda^i + \{\gamma_\lambda, \gamma_\mu\}^i) S_i^k B^\lambda \wedge B^\mu \otimes B_k.$$

Finally, from (15) and (45) we see that

$$\{\gamma_\lambda, \gamma_\mu\}^i = D_\sigma(\gamma_\lambda^k) \frac{\partial \gamma_\mu^i}{\partial y_\sigma^k} - D_\sigma(\gamma_\mu^k) \frac{\partial \gamma_\lambda^i}{\partial y_\sigma^k}.$$

From the last formula we can conclude that the deviation of $k$-th order connection depends, in general, on all derivatives of order $\leqslant 2k$.

**List of main symbols.**

| | |
|---|---|
| $E$ | manifold of dim $m+1$ |
| $N \subset E$ | submanifold of dim $m$ |
| $J^k \equiv J^k(E, m)$ | space of $k$-jets of submanifolds of dim $m$ of $E$ |
| $\pi_{p,q}: J^p(E, m) \to J^q(E, m)$ | projection of jet spaces |
| $j^k N: N \to J^k,\ e \mapsto [N]^k_e$ | $k$-jet prolongation of the submanifold $N$ |
| $\tau_E: TE \to E$ | tangent bundle |
| $L_\vartheta \subset TE$ | $m$-dimensional subspace corresponding to $\vartheta \in J^1$ |
| $\underline{\vartheta} \equiv \pi_{k,0}(\vartheta)$ | projection of $\vartheta \in J^k$ on $J^0$ |
| $\tilde{\vartheta} \equiv \pi_{k,1}(\vartheta)$ | projection of $\vartheta \in J^k$ on $J^1$ |
| $(x^\lambda, y^i),\ 1 \leqslant \lambda \leqslant m,\ 1 \leqslant i \leqslant l$ | divided chart of $E$ |
| $(x^\lambda, y^i_\sigma)$ | induced chart on $J^k$ |
| $\partial_\lambda \equiv \dfrac{\partial}{\partial x^\lambda}\ \partial_i \equiv \dfrac{\partial}{\partial y^i}$ | basis of vector fields |
| $\tau^k_E: T^{(k)} \equiv T^{(k)}(E, m) \to J^k$ | pullback of the tangent bundle on $J^k$ |
| $c^k: C^k \equiv C^k(E, m) \to J^k$ | contact subbundle $C^k := \{(\vartheta, u) \in T^{(k)} \mid u \in L_{\tilde{\vartheta}}\}$ |
| $w^k: W^k \equiv W^k(E, m) \to J^k$ | quotient bundle $W^k := T^{(k)}/C^k$ |
| $\text{д}^k: C^k \hookrightarrow T^{(k)}$ | canonical inclusion |
| $\text{г}^k: T^{(k)} \to W^k$ | canonical projection |
| $\kappa^k := \Gamma(w^k)$ | space of sections of $w^k$ |
| $\Gamma(\alpha)$ | space of sections of the bundle $\alpha$ |
| $\mathcal{D}(M)$ | module of vectors fields of the manifold $M$ |
| $\mathfrak{g}^k: TJ^k \to W^k$ | canonical projection |
| $\mathfrak{g}^k: \mathcal{D}(J^k) \to \kappa^k$ | canonical projection |
| $\mathfrak{g}^0: \mathcal{D}(E) \to \kappa^1$ | canonical projection |
| $\mathcal{C}\mathcal{D} \subset \mathcal{D}(J^\infty)$ | Cartan distribution |
| $\mathcal{D}_{\mathcal{C}} \subset \mathcal{D}(J^\infty)$ | Lie algebra of infinites. contact transformations |
| $w_N: W_N \to N$ | pullback of the bundle $w^k$ via the submanifold $N$ |
| $\pi: E \to B$ | fibred manifold |
| $\pi_k := \pi \circ \pi_{k,0}: J^k\pi \to B$ | fibred manifold of jets of sections |
| $\upsilon_\pi: V_\pi \to E$ | vertical subbundle of the tangent bundle |
| $\upsilon^k_\pi: V^{(k)}_\pi \to J^k\pi$ | pullback of the vertical bundle on $J^k$ |
| $w^k \simeq \pi^*_k(\upsilon_\pi)$ | canonical identification |
| $(x^\lambda, y^i_\sigma, z^\lambda)$ | chart on $c^k$ |
| $b_\lambda := \partial_\lambda + y^i_\lambda \partial_i$ | basis of $c^1$ |
| $\zeta_i \equiv \mathfrak{g}^0(\partial_i) \in \kappa^1 \subset \kappa^k$ | basis of $w^k$ |
| $D_\lambda = \partial_\lambda + \sum_{i,\sigma} y^i_{\sigma+\lambda} \dfrac{\partial}{\partial y^i_\sigma}$ | full derivatives |

| | |
|---|---|
| $\ni_\varphi = \sum_{\lambda,\sigma} D_\sigma(\varphi^i)\dfrac{\partial}{\partial y^i_\sigma}$ | evolutionary derivative |
| $X = \ni_\varphi + Y$ | decomp. of infinitesimal contact transformations |
| $\gamma\colon E \to J^1(E, m)$ | pre-connection |
| $h_\gamma\colon H_\gamma \to E$ | horizontal subbundle $\iota_\gamma\colon H_\gamma \hookrightarrow TE$ |
| $\upsilon_\gamma\colon V_\gamma \to E$ | vertical bundle $V_\gamma := TE/H_\gamma$ |
| $\nu_\gamma\colon TE \to V_\gamma$ | canonical projection |
| $\upsilon_\gamma \simeq \upsilon_\pi\ h_\gamma \simeq \pi^*(\tau_B)$ | canonical identifications |
| $\gamma^i_\lambda := y^i_\lambda \circ \gamma$ | components of $\gamma$ |
| $B_\lambda := \partial_\lambda + \gamma^i_\lambda \partial_i$ | basis of $\Gamma(h_\gamma)$ |
| $Q_i := \gamma^*(\zeta_i)$ | basis of $\Gamma(\upsilon_\gamma)$ |
| $h^k_\gamma\colon H^{(k)}_\gamma \to J^k$ | pullback of the horizontal bundle on $J^k$ |
| $\upsilon^k_\gamma\colon V^{(k)}_\gamma \to J^k$ | pullback of the vertical bundle on $J^k$ |
| $\nabla_\gamma := \mathrm{r}^1 \circ \iota^1_\gamma\colon H^{(1)}_\gamma \to W^1$ | covariant differential |
| $\nabla_\gamma\colon \Gamma(h^1_\gamma) \to \kappa^1$ | covariant differential |
| $\Box_\gamma := \nu^1 \circ д^1\colon C^1 \to V^{(1)}_\gamma$ | (alternative) covariant differential |
| $\delta_\gamma\colon \Gamma(h_\gamma) \times \Gamma(h_\gamma) \to \Gamma(\upsilon_\gamma)$ | deviation $(u, \upsilon) \mapsto \upsilon_\gamma([u, \upsilon])$ |
| $R_\gamma(X, Y) = \delta_\gamma(u, \upsilon)$ | curvature |
| $\boldsymbol{\delta}_\gamma\colon \Lambda^i \mathcal{H}^*_\gamma \otimes \mathcal{H}_\gamma \to \Lambda^{i+1} \mathcal{H}^*_\gamma \otimes \mathcal{V}_\gamma$ | graded extension of deviation |
| $\mathrm{п}\colon V_{\mathrm{п}} \to E$ | almost-fibring $V_{\mathrm{п}} \subset TE$ |
| $J^k \mathrm{п}$ | jet space of the almost-fibring п |
| $h_{\mathrm{п}}\colon H_{\mathrm{п}} \to E$ | horizontal bundle $H_{\mathrm{п}} := TE/V_{\mathrm{п}}$ |
| $\mathrm{п}^k\colon V^{(k)}_{\mathrm{п}} \to J^k$ | pullback of the almost-fibring on $J^k$ |
| $h^k_{\mathrm{п}}\colon H^{(k)}_{\mathrm{п}} \to J^k$ | pullback of the horizontal bundle on $J^k$ |
| $B_i = \partial_i + \mathrm{п}^\lambda_i \partial_\lambda$ | basis of $V_{\mathrm{п}}$ |
| $R^{\mathrm{п}}_\gamma$ | curvature of $\gamma$ with respect to п |
| $\omega_\gamma, \omega_{\mathrm{п}}\colon \mathcal{D}(E) \to \mathcal{D}(E)$ | projections induced by $\gamma$ and п |
| $\mathcal{N}(E) := \Lambda^*(E) \otimes \mathcal{D}(E)$ | module of tangent valued forms |
| $[\cdot,\cdot]\colon \mathcal{N}(E) \times \mathcal{N}(E) \to \mathcal{N}(E)$ | Frölicher-Nijenhuis bracket |
| $\Lambda^*(E) := \sum_{i \geq 0} \Lambda^i(E)$ | algebra of differential forms |
| $d_\omega\colon \mathcal{N}(E) \to \mathcal{N}(E)$ | differential $d_\omega(\rho) := [\omega, \rho]$ |
| $r_\omega := \dfrac{1}{2} d_\omega \omega$ | curvature of $\omega$ |
| $d_\gamma := d_{\omega_\gamma}\ d_{\mathrm{п}} := d_{\omega_{\mathrm{п}}}$ | covariant differentials |
| $\sigma \in \Gamma(h_\gamma)^* \underset{C^\infty(E)}{\otimes} \Gamma(\mathrm{п})$ | soldering form |
| $\tau_{\mathrm{п},\sigma}$ | torsion of $\gamma$ with respect to $\sigma$ |
| $\mathcal{D}(N; M, F) = \Gamma(F^*(\tau_N))$ | space of $M$-valued vector fields on $N$ along $F$ |
| $G^0\colon \mathcal{D}(N; M, F) \to \mathcal{D}(N; M', F \circ G)$ | map induced by $G\colon M' \to M$ |
| $F^0\colon \mathcal{D}(N) \to \mathcal{D}(N; M, F)$ | canonical map |
| $\delta^\infty_\gamma(X, Y) := \nu^\infty_\gamma([X, Y])$ | lifted deviation |
| $\mathcal{D}_V := \mathcal{D}_V(E; J^\infty, \pi_{\infty,0}) \subset \mathcal{D}_{\mathcal{C}}$ | submodule |
| $\kappa_V := \mathcal{D}_{\mathcal{C}} / \mathcal{D}_V$ | quotient module |
| $\mu\colon H_\mu \to J^k(E, m)$ | $k$-order pre-connection $H_\mu \subset T^{(k)}(E, m)$ |

| | |
|---|---|
| $\nabla_\gamma := \mathrm{r}^k \circ i_\gamma : H_\gamma \to W^k$ | covariant differential $\nabla_\gamma \in \Gamma^*(\gamma) \underset{C^\infty(J^k)}{\otimes} \kappa^k$ |
| $\nabla_{\gamma, Y}(s) := (j^k s)^* \nabla_{\gamma, Y}$ | covariant derivative |
| $\{\varphi, \psi\}^i = \ni_\varphi(\psi^i) - \ni_\psi(\varphi^i)$ | Lie algebra of generating functions |

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