

# Torsions of connections on some natural bundles

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*Abstract:* The natural vector valued 1-forms  $Q$  on the natural bundles associated with product preserving functors (including the tangent bundle, the bundle of first order  $k$ -velocities, the bundle of second order 1-velocities and the bundle of linear frames) and on the cotangent bundle are classified. Then, these forms  $Q$  are used to study the torsion  $\tau = [\Gamma, Q]$  of connections  $\Gamma$  on the above bundles, where  $[\cdot, \cdot]$  is the Frölicher-Nijenhuis bracket.

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Roughly speaking, the theory of connections started with the study of a classical linear connection  $\Gamma$  on a manifold  $M$ , which yields the curvature  $C$  and the torsion  $\tau$ . The historical generalizing process gave first the concept of principal connection on an arbitrary principal fibre bundle, [2], and then the concept of general connection on an arbitrary fibred manifold, [19]. In both cases the curvature can be introduced in a standard way. But the definition of torsion requires an additional structure. For the classical linear connection  $\Gamma$  on  $M$  we have two approaches to torsion, which give the same result. On one hand, if we consider  $\Gamma$  as a linear connection on the vector bundle  $TM \rightarrow M$ , we can define  $\tau$  as the covariant exterior differential, in the sense of J.L. Koszul [17], of the identity tensor on  $M$ . In particular, this yields the classical formula

$$\tau(X, Y) = \frac{1}{2}(\nabla_X Y - \nabla_Y X - [X, Y])$$

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for every two vector fields  $X$  and  $Y$  on  $M$ . On the other hand, if we interpret  $\Gamma$  as a principal connection on the frame bundle  $LM$  of  $M$ , we can take the canonical  $\mathbb{R}^m$ -valued form  $\Theta : TLM \rightarrow \mathbb{R}^m$  of  $LM$ ,  $m = \dim M$  and introduce  $\tau$  as the standard covariant exterior differential of  $\Theta$ , [9].

It should be underlined at the very beginning that torsion is also a subject of interest for mathematical physics. For example, the role played by torsion in the Einstein-Cartan theory of gravitation is well known. An increasing role of torsion can further be expected in modern development of field theories, see [23].

The first geometrical approaches to generalize the concept of torsion were strictly based on the standard techniques from the theory of linear and principal connections. However, in the theory of general connections it has been clarified recently that the Frölicher-Nijenhuis (in short: F-N) bracket [4], is a very powerful tool [21]. If a connection  $\Gamma$  on an arbitrary fibred manifold  $E \rightarrow M$  is identified with its horizontal projection, which is a vector-valued 1-form on  $E$ , then the curvature of  $\Gamma$  coincides with the F-N bracket  $\frac{1}{2}[\Gamma, \Gamma]$ . Moreover, the Bianchi identity for a general connection is a direct consequence of the basic properties of the F-N bracket.

Using such a point of view, if we have a fibred manifold  $E \rightarrow M$  endowed with a canonical vector-valued 1-form  $Q$ , then we can say that the F-N bracket  $[\Gamma, Q]$  is the torsion of a given connection  $\Gamma$  on  $E$ . The case when the values of  $Q$  lie in the vertical tangent bundle of  $E$  is studied in [23] and [25]. In particular, a bracket formula characterizing torsion and a generalized Bianchi identity is deduced there. Interesting results were also obtained for the polynomial connections on affine bundles, [24].

In the present paper we start from the fact that we are able to determine all natural vector-valued 1-forms (or, which is the same, all natural affinors) on some natural bundles. This means that we know all reasonable candidates to be used in the latter approach to the theory of torsions on the natural bundle in question. First we study the natural bundles determined by the product preserving functors on the category of all manifolds. Recently it has been clarified, [1, 7, 20], that these bundles coincide with the Weil bundles [28]. We prove that all natural affinors on a Weil bundle  $T^A M$  correspond to the multiplication by the elements of the Weil algebra  $A$ . Then we discuss the special cases of the tangent bundle  $TM$ , of the bundle  $T_k^1 M$  of first order velocities of an arbitrary dimension  $k$  and of the frame bundle  $LM \subset T_m^1 M$ . It is remarkable that, for a nonprincipal connection on the frame bundle  $LM$ , our approach to torsion gives much more information than the covariant exterior differential of the canonical form  $\Theta$ . Another special case we discuss in detail is the bundle  $T_1^2 M$  of all 1-dimensional velocities of second order (called the second order tangent bundle of  $M$  in some branches of analytical mechanics).

Finally we study the torsion of a connection on the cotangent bundle, which is not a Weil bundle. Using a classification list of [16] we determine all natural affinors on  $T^*M$  as well. Analogously to the frame bundle  $LM$ , besides the torsion  $\tau$  in the sense of the F-N bracket, we can consider the covariant exterior differential  $D\lambda$  of the Liouville form  $\lambda$  (which can also be characterized in the framework of the theory of sesquiholonomic 2-jets [22]). For a linear connection on  $T^*M$  we deduce that  $\tau$  and  $D\lambda$  coincide, but

for a non-linear connection they have different character.

All manifolds and maps are assumed to be infinitely differentiable and all manifolds are paracompact.

## 1. Torsions of connections on natural bundles

We recall that the concept of a natural bundle over  $m$ -manifolds was introduced by A. Nijenhuis as a modern reformulation of the classical idea of bundle of geometric objects [27]. Let  $\mathfrak{Mf}$  be the category of all manifolds and all smooth maps,  $\mathfrak{Mf}_m$  be the category of all  $m$ -dimensional manifolds and their local diffeomorphisms,  $\mathfrak{FM}$  be the category of fibred manifolds and  $B : \mathfrak{FM} \rightarrow \mathfrak{Mf}$  be the base functor.

**Definition 1.** A *natural bundle* over  $m$ -manifolds is a functor  $F : \mathfrak{Mf}_m \rightarrow \mathfrak{FM}$  satisfying  $B \circ F = \text{id}$  and the localization condition: for every inclusion of an open subset  $i_U : U \rightarrow M$ ,  $FU$  is the restriction  $p_M^{-1}(U)$  of  $p_M : FM \rightarrow M$  over  $U$  and  $F i_U$  is the inclusion  $p_M^{-1}(U) \hookrightarrow FM$ .

If we replace the category  $\mathfrak{Mf}_m$  by the category  $\mathfrak{Mf}$  in Definition 1, we obtain the concept of a *bundle functor* on the category of all manifolds.

In general, by an *affinor*  $Q$  on a manifold  $M$  we mean a tensor of type  $(1, 1)$ , i.e. a linear morphism  $Q : TM \rightarrow TM$  over  $\text{id}_M$ .

**Definition 2.** A *natural affinor* on a natural bundle  $F$  over  $m$ -manifolds is a system of affinors  $Q_M : TFM \rightarrow TFM$  for every  $m$ -manifold  $M$  satisfying

$$TFf \circ Q_M = Q_N \circ TFf$$

for every local diffeomorphism  $f : M \rightarrow N$ .

The identities  $1_{FM} = \text{id}_{TFM}$  constitute a trivial natural affinor on  $F$ . Clearly the natural affinors on  $FM$  constitute a linear subspace in the space of all tensors of type  $(1, 1)$  on  $FM$ .

Given any fibred manifold  $E \rightarrow M$ , a (*general*) *connection* on  $E$  means any section  $\Gamma : E \rightarrow J^1E$ . Such a connection can be identified with the associated horizontal projection (denoted by the same symbol  $\Gamma$ ), which is a special tensor of type  $(1, 1)$  on  $E$ .

Affinors coincide with vector valued 1-forms in the sense of the Frölicher-Nijenhuis theory [4]. Hence the Frölicher-Nijenhuis (in short F-N) bracket of any two affinors is a vector-valued 2-form.

**Definition 3.** Let  $\Gamma : FM \rightarrow J^1(FM \rightarrow M)$  be a general connection on a natural bundle  $FM$  and  $Q$  be a natural affinor on  $FM$ . Then the F-N bracket  $[\Gamma, Q]$  is called the *torsion* of  $\Gamma$  of type  $Q$ .

For the identity affinor  $1_{FM}$ , as well as for its constant multiples, we have  $[\Gamma, 1_{FM}] = 0$  for every connection  $\Gamma$ . That is why we shall consider only the case  $Q \neq k1_{FM}$ ,  $k \in \mathbb{R}$ , in the sequel.

## 2. Natural affinors on Weil bundles

We introduce the concept of a *Weil bundle* in a form generalizing the classical definition of the bundle  $T_k^r$  of  $k$ -dimensional velocities of order  $r$  by C. Ehresmann [3]. Let  $E(k)$  be the algebra of all germs of smooth functions on  $\mathbb{R}^k$  at zero,  $\mathfrak{m}(k)$  be the ideal of all germs vanishing at zero and  $\mathfrak{m}(k)^{r+1}$  be its  $(r+1)$ st power, i.e. the ideal of all germs vanishing at zero up to order  $r$ . Clearly, two maps  $g, h : \mathbb{R}^k \rightarrow M$  determine the same  $r$ -jet at zero if and only if  $\varphi \circ g - \varphi \circ h \in \mathfrak{m}(k)^{r+1}$  for every germ  $\varphi$  of a smooth function on  $M$  at  $x$ .

More generally, take any ideal  $\mathfrak{A}$  in  $E(k)$  satisfying  $\mathfrak{m}(k)^{r+1} \subset \mathfrak{A} \subset \mathfrak{m}(k)$ . Such an ideal is said to be a *Weil ideal* and the factor algebra  $A = E(k)/\mathfrak{A}$  is called a *Weil algebra*. Since  $\mathfrak{m}^{r+1}(k) \subset \mathfrak{A}$ ,  $\mathfrak{A}$  can be generated by some polynomials. Hence there is an ideal  $\tilde{\mathfrak{A}} \subset \mathbb{R}[x_1, \dots, x_k]$  in the algebra of all polynomials with  $k$  variables corresponding to  $\mathfrak{A}$ . Then the formula  $A = \mathbb{R}[x_1, \dots, x_k]/\tilde{\mathfrak{A}}$  gives a purely algebraic description of a Weil algebra. We have a canonical decomposition  $A = \mathbb{R} \oplus N$ , where  $N = \mathfrak{m}(k)/\mathfrak{A}$  is the ideal of all nilpotent elements of  $A$ .

**Definition 4** [12]. Let  $A = E(k)/\mathfrak{A}$  be a Weil algebra. Two maps  $g, h : \mathbb{R}^k \rightarrow M$ ,  $g(0) = h(0) = x$  are said to be *A-equivalent*, if

$$\varphi \circ g - \varphi \circ h \in \mathfrak{A}$$

for every germ  $\varphi$  of a smooth function on  $M$  at  $x$ . Such an equivalence class will be denoted by  $j^A g$  and called an *A-velocity* on  $M$ . The point  $g(0)$  is said to be the *target* of  $j^A g$ .

Denote by  $T^A M$  the set of all *A-velocities* on  $M$ . It is easy to see  $T^A \mathbb{R} = A$ . The target map is a bundle projection  $T^A M \rightarrow M$ . Further, for every  $f : M \rightarrow N$  we define  $T^A f : T^A M \rightarrow T^A N$  by  $T^A f(j^A g) = j^A(f \circ g)$ . Obviously,  $T^A$  is a bundle functor on the category of all manifolds, see also [14], which is called the *Weil functor* corresponding to  $A$ . In particular, for  $\mathfrak{A} = \mathfrak{m}(k)^{r+1}$  we obtain the functor  $T_k^r M$  of  $k$ -dimensional velocities of order  $r$ .

There is another idea how to define the functor  $T^A$ , which was presented by A. Weil in his original paper [28]. Let  $C^\infty M$  denote the algebra of all smooth functions on a manifold  $M$ . Then we can consider the space of all algebra homomorphisms  $\text{hom}(C^\infty M, A)$  of  $C^\infty M$  into  $A$ . Morimoto proved that  $\text{hom}(C^\infty M, A)$  can be identified with  $T^A M$  in a canonical way [26]. In this setting, every smooth map  $f : M \rightarrow N$  yields an algebra homomorphism  $C^\infty N \rightarrow C^\infty M$ ,  $\varphi \mapsto \varphi \circ f$  and the induced map  $\text{hom}(C^\infty M, A) \rightarrow \text{hom}(C^\infty N, A)$  coincides with  $T^A f$ .

Let  $B$  be another Weil algebra and  $h : A \rightarrow B$  an algebra homomorphism. If we compose  $h$  with elements of  $\text{hom}(C^\infty M, A)$ , we obtain a map

$$T(h)_M : \text{hom}(C^\infty M, A) \rightarrow \text{hom}(C^\infty M, B),$$

i.e.

$$T(h)_M : T^A M \rightarrow T^B M.$$

One easily verifies that  $T(h) : T^A \rightarrow T^B$  is a natural transformation of the functor  $T^A$  into the functor  $T^B$ .

The important role of the Weil functors in differential geometry has been clarified in recent papers by G. Kainz and P.W. Michor [7], D.J. Eck [1], and O.O. Luciano [20]. They have proved that every product preserving bundle functor  $F$  on the category of all manifolds is a Weil functor. The related Weil algebra is  $A = F\mathbb{R}$  endowed with the extensions  $Fa : F\mathbb{R} \times F\mathbb{R} \rightarrow F\mathbb{R}$  and  $Fm : F\mathbb{R} \times F\mathbb{R} \rightarrow F\mathbb{R}$  of the addition  $a : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and the multiplication  $m : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  of reals. Moreover, the above mentioned authors deduced that all natural transformations between any two Weil functors are induced by the homomorphisms of the related Weil algebras.

Koszul remarked that there is a canonical action of the elements of  $A$  on the tangent vectors of  $T^A M$ . This can be introduced as follows. The multiplication of the tangent vectors of  $M$  by reals is a map  $\mu : \mathbb{R} \times TM \rightarrow TM$ . Applying the functor  $T^A$ , we obtain  $T^A \mu : A \times T^A TM \rightarrow T^A TM$ . But there is a canonical exchange map  $T^A TM \rightarrow TT^A M$ . Hence  $T^A \mu$  can be interpreted as a map  $A \times TT^A M \rightarrow TT^A M$ . Since the algebra multiplication in  $A$  is the prolongation of the multiplication of reals, the action of  $a \in A$  on  $(b_1, \dots, b_m, c_1, \dots, c_m) \in TT^A \mathbb{R}^m = A^{2m}$  has the form

$$a(b_1, \dots, b_m, c_1, \dots, c_m) = (b_1, \dots, b_m, ac_1, \dots, ac_m). \quad (1)$$

This implies that every  $a \in A$  defines an affiner  $Q(a)_M : TT^A M$  on every manifold  $M$ .

Clearly the restriction of every Weil functor  $T^A : \mathfrak{Mf} \rightarrow \mathfrak{FM}$  to the subcategory  $\mathfrak{Mf}_m \in \mathfrak{Mf}$  is a natural bundle over  $m$ -manifolds. In [13] it is proved that even all natural transformations between two restricted Weil functors  $T^A : \mathfrak{Mf}_m \rightarrow \mathfrak{FM}$  are in bijection with the algebra homomorphisms  $A \rightarrow B$ .

**Proposition 1.** *All natural affiners on  $T^A M$  are of the form  $Q = Q(a)$  for all elements  $a \in A$ .*

**Proof.** According to the general theory, the Weil algebra corresponding to the functor  $TT^A$  is  $A \times A$  endowed with the following multiplication

$$(a, b)(c, d) = (ac, ad + bc)$$

the products of the components being in  $A$ . In [13] the first author deduced that all algebra homomorphisms  $A \times A \rightarrow A \times A$  over the identity on the first factor have the form

$$(a, b) \mapsto (a, cb + D(a)), \quad c \in A, D \in \text{der } A, \quad (2)$$

where  $cb$  is the product in  $A$  and  $\text{der } A$  means the space of all algebra derivations of  $A$ . The requirement for (2) to be linear implies  $D = 0$ . This proves our assertion.  $\square$

The unit of the algebra  $A$  determines the identity tensor  $1_{T^A M}$  and the F-N bracket  $[\Gamma, 1_{T^A M}]$  vanishes for every connection  $\Gamma$  on  $T^A M$ . Hence the torsions of  $\Gamma$ , in the sense of Definition 3, form a  $(\dim A - 1)$ -parameter family corresponding to the nilpotent part  $N \subset A$ . Formula (1) implies directly that for every  $a \in N$  the values of the affiner  $Q(a)$  lie in the vertical bundle  $VT^A M$ , i.e.  $Q(a)$  is a section of  $VT^A M \otimes T^* T^A M$ .

### 3. F-N bracket in certain special cases

Before discussing the most interesting special Weil bundles, we present some useful formulas for F-N bracket on fibred manifolds due to the second author and L. Mangiarroti [23, 25].

Consider an arbitrary fibred manifold  $p : E \rightarrow M$ , a vertical valued 1-form  $\varphi : E \rightarrow VE \otimes T^*E$  and a general connection  $\Gamma$  on  $E$ . Then the F-N bracket  $[\Gamma, \varphi]$  is characterized by

$$\begin{aligned} [\Gamma, \varphi](U, V) &= \frac{1}{2} \{ [\Gamma U, \varphi V] - [\Gamma V, \varphi U] - \varphi[U, \Gamma V] + \varphi[V, \Gamma U] + \varphi\Gamma[U, V] \} \end{aligned} \quad (3)$$

for every two vector fields  $U$  and  $V$  on  $E$ . In the special case  $\varphi : E \rightarrow VE \otimes T^*M$  (with the inclusion  $T^*M \subset T^*E$ ),  $[\Gamma, \varphi]$  is a section of  $VE \otimes \Lambda^2 T^*M$  characterized by

$$[\Gamma, \varphi](X, Y) = \frac{1}{2} ([\Gamma X, \varphi Y] - [\Gamma Y, \varphi X] - \varphi[X, Y]) \quad (4)$$

for every two vector fields  $X$  and  $Y$  on  $M$ .

Let  $(x^i, y^p), i, j, \dots = 1, \dots, \dim M, p, q, \dots = 1, \dots, \dim E - \dim M$ , be some local fibre coordinates on  $E$ . If

$$dy^p = F_i^p(x, y) dx^i \quad (5)$$

are the equations of a section  $\Gamma : E \rightarrow J^1 E$ , then the coordinate form of the horizontal projection of  $\Gamma$  is

$$\delta_j^i \frac{\partial}{\partial x^i} \otimes dx^j + F_i^p \frac{\partial}{\partial y^p} \otimes dx^i. \quad (6)$$

A section  $\varphi : E \rightarrow VE \otimes T^*E$  has the coordinate expression

$$\varphi_i^p(x, y) \frac{\partial}{\partial y^p} \otimes dx^i + \varphi_q^p(x, y) \frac{\partial}{\partial y^p} \otimes dy^q \quad (7)$$

and for  $[\Gamma, \varphi]$  we have

$$\begin{aligned} & \left( \frac{\partial \varphi_j^p}{\partial x^i} + F_i^q \frac{\partial \varphi_j^p}{\partial y^q} - \varphi_j^q \frac{\partial F_i^p}{\partial y^q} - \varphi_q^p \frac{\partial F_j^q}{\partial x^i} \right) \frac{\partial}{\partial y^p} \otimes dx^i \wedge dx^j \\ & + \left( \frac{\partial \varphi_q^p}{\partial x^i} + F_i^r \frac{\partial \varphi_q^p}{\partial y^r} - \varphi_q^r \frac{\partial F_i^p}{\partial y^r} + \varphi_r^p \frac{\partial F_i^r}{\partial y^q} \right) \frac{\partial}{\partial y^p} \otimes dx^i \wedge dy^q. \end{aligned} \quad (8)$$

In the special case of a section  $\varphi : E \rightarrow VE \otimes T^*M$  with coordinate expression

$$\varphi_i^p(x, y) \frac{\partial}{\partial y^p} \otimes dx^i \quad (9)$$

(8) reduces to

$$\left( \frac{\partial \varphi_j^p}{\partial x^i} + F_i^q \frac{\partial \varphi_j^p}{\partial y^q} - \varphi_j^q \frac{\partial F_i^p}{\partial y^q} \right) \frac{\partial}{\partial y^p} \otimes dx^i \wedge dx^j. \quad (10)$$

#### 4. Torsions on $TM$ , $T_k^1 M$ and $LM$

Consider now the simplest case of the tangent bundle  $p : TM \rightarrow M$ . The Weil algebra associated with the functor  $T$  is  $\mathbb{D} := \mathbb{R}[t]/\langle t^2 \rangle$ , where  $\langle t^2 \rangle$  denotes the ideal generated by  $t^2$ . Usually  $\mathbb{D}$  is called the algebra of dual numbers. The elements of  $\mathbb{D}$  have the form  $a + bt$  and the multiplication is given by  $(a + bt)(c + dt) = ac + (ad + bc)t$ . The element  $t \in \mathbb{D}$  determines an affinor  $Q(t) =: Q$  on every  $TM$ . If  $\Gamma$  is a connection on  $TM$ , then the torsions of  $\Gamma$  in the sense of Definition 3 are the constant multiples of  $[\Gamma, Q]$ . Hence it is natural to say that  $\tau = [\Gamma, Q]$  is the torsion of  $\Gamma$ .

It is instructive to present the coordinate expressions, even though we shall not use them actually. Let  $x^i$  be some local coordinates on  $M$  and  $y^i = dx^i$  be the induced coordinates on  $TM$ . One can easily find that the coordinate expression of  $Q$  is

$$(dx^i, dy^i) \mapsto (0, dx^i), \quad \text{i.e.} \quad \delta_j^i \frac{\partial}{\partial y^i} \otimes dx^j. \quad (11)$$

Consider a connection  $\Gamma : TM \rightarrow J^1 TM$  with equations

$$dy^i = F_j^i(x, y) dx^j. \quad (12)$$

Then (10) yields the following coordinate expression of  $\tau$

$$\frac{\partial F_k^i}{\partial y^j} \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k. \quad (13)$$

To obtain a geometrical interpretation of  $\tau$ , we first give another description of  $Q$ . The vertical tangent bundle  $VTM$  coincides with the pullback  $p^*TM$  of  $TM \rightarrow M$  over  $p : TM \rightarrow M$ . If we denote by  $1_M$  the identity tensor of  $TM$ , then (11) yields that  $Q$  is the pullback  $Q = p^*1_M$ . For a vector field  $X$  on  $M$  we denote by  $p^* : TM \rightarrow VTM$  its pullback to  $p^*TM$ . Using (4) we can characterize  $\tau$  by a bracket expression

$$2\tau(X, Y) = [\Gamma X, p^*Y] - [\Gamma Y, p^*X] - p^*([X, Y]) \quad (14)$$

for every vector fields  $X$  and  $Y$  on  $M$ . By [23], (14) can be written in terms of covariant derivatives

$$2\tau(X, Y) = \nabla_X p^*Y - \nabla_Y p^*X - p^*([X, Y]) \quad (15)$$

because the covariant derivative  $\nabla_X p^*Y$  with respect to  $\Gamma$  coincides with the bracket  $[\Gamma X, p^*Y]$ .

In the special case of a linear connection  $\Gamma$  on  $TM$ , whose equations are

$$dy^i = \Gamma_{jk}^i(x) y^j dx^k, \quad (16)$$

we find easily

$$\nabla_X p^*Y = p^*(\nabla_X Y) \quad (17)$$

where  $\nabla_X Y$  on the right-hand side has the classical meaning from the theory of linear connections. Hence, for a linear connection, (15) and (17) yield

$$2\tau(X, Y) = p^*(\nabla_X Y - \nabla_Y X - [X, Y]). \quad (18)$$

In other words, the general torsion of a linear connection  $\Gamma$  is just the pullback of the classical torsion of  $\Gamma$ .

The bundle  $T_k^1 M$  of all  $k$ -dimensional velocities of first order on  $M$  coincides with the Whitney sum of  $k$  copies of  $TM$  and the corresponding Weil algebra is the sum of  $k$  copies of  $\mathbb{D}$ . This implies that all natural affinors on  $T_k^1 M$  form a  $(k+1)$ -parameter family generated by the identity and by the tensors  $Q_M^\alpha$ ,  $\alpha = 1, \dots, k$  defined as follows. The vertical bundle  $VT_k^1 M$  of  $p : T_k^1 M \rightarrow M$  is the pullback  $p^* T_k^1 M$ . Hence we can take the tensor  $1_M^\alpha : M \rightarrow T_k^1 M \otimes T^* M$  generated by the identity on the  $\alpha$ th factor and by the zero map on the remaining ones, and set

$$Q_M^\alpha = p^* 1_M^\alpha. \quad (19)$$

Then we define the  $\alpha$ th torsion of a connection  $\Gamma : T_k^1 M \rightarrow J^1 T_k^1 M$  by  $\tau^\alpha = [\Gamma, Q^\alpha]$ . Since  $Q^\alpha$  is a section on  $T_k^1 M \rightarrow VT_k^1 M \otimes T^* M$ ,  $\tau^\alpha$  is a section  $T_k^1 M \rightarrow VT_k^1 M \otimes \Lambda^2 T^* M$  and is geometrically characterized by

$$2\tau^\alpha(X, Y) = [\Gamma X, Q^\alpha Y] - [\Gamma Y, Q^\alpha X] - Q^\alpha([X, Y]) \quad (20)$$

for every vector fields  $X$  and  $Y$  on  $M$ . By (19),  $Q^\alpha(Y)$  means the pullback of  $Y$  with respect to the  $\alpha$ th component of the Whitney sum in question.

Let  $x^i$  be some local coordinates on  $M$  and  $y_\alpha^i$ ,  $\alpha = 1, \dots, k$ , be the induced coordinates on  $T_k^1 M$ . Then the coordinate expression of  $Q^\alpha$  is

$$\delta_j^i \frac{\partial}{\partial y_\alpha^i} \otimes dx^j. \quad (21)$$

The equations of a connection  $\Gamma : T_k^1 M \rightarrow J^1 T_k^1 M$  are

$$dy_\alpha^i = F_{\alpha j}^i(x^k, y_\beta^l) dx^j. \quad (22)$$

Using (10), we evaluate  $\tau^\alpha$  in the form

$$\sum_{\beta=1}^k \frac{\partial F_{\beta k}^i}{\partial y_\alpha^j} \frac{\partial}{\partial y_\beta^i} \otimes dx^j \wedge dx^k. \quad (23)$$

Since  $\tau^\alpha : T_k^1 M \rightarrow VT_k^1 M \otimes \Lambda^2 T^* M$  and  $VT_k^1 M$  is a Whitney sum, we can decompose  $\tau^\alpha$  into the sum  $\tau^\alpha = \sum_{\beta=1}^k \tau_\beta^\alpha$ . The coordinate expression of  $\tau_\beta^\alpha$  is

$$\frac{\partial F_{\beta k}^i}{\partial y_\alpha^j} \frac{\partial}{\partial y_\beta^i} \otimes dx^j \wedge dx^k \quad (24)$$

where we do not use the summation convention with respect to  $\beta$ .

The frame bundle  $LM$  of an  $m$ -dimensional manifold  $M$  is an open dense subset of  $T_m^1 M$  natural with respect to local diffeomorphisms. Hence the restrictions of the above affinors  $Q^i$ ,  $i = 1, \dots, m$ , to  $LM$ , which will be denoted by the same symbol, are natural affinors of  $LM$ . Thus, for a general connection  $\Gamma : LM \rightarrow J^1 LM$  our approach gives  $m$  torsions

$$\tau^i = [\Gamma, Q^i],$$



each of them with  $m$  components (24).

It is interesting to compare the two approaches based on the F-N bracket and on the canonical form  $\Theta : TLM \rightarrow \mathbb{R}^m$  [9]. The covariant exterior differential

$$D\Theta(X, Y) = d\Theta(hX, hY) \quad (25)$$

where  $h$  means the horizontal projection, can be defined for a general connection  $\Gamma$  on  $LM$ . The 2-form  $D\Theta$  is also called the *torsion* of  $\Gamma$ . The coordinate expression of  $\Theta$  is

$$\tilde{y}_j^i dx^j \quad (26)$$

where  $\tilde{y}_j^i$  means the inverse matrix of  $y_j^i$ . If

$$dy_j^i = F_{jk}^i(x^l, y_n^m) dx^k \quad (27)$$

are the equations of  $\Gamma$ , then a direct evaluation gives the following coordinate expression of  $D\Theta$ :

$$-\tilde{y}_k^i \tilde{y}_j^l F_{lm}^k dx^m \wedge dx^j. \quad (28)$$

Clearly, in the case of a non-principal connection the one object (28) and the  $m^2$  objects (24) give different information.

However, consider a principal connection  $\Gamma$  on  $LM$

$$dy_j^i = \Gamma_{lk}^i(x) y_j^l dx^k. \quad (29)$$

By (23), the coordinate expression of  $\tau^i$  is

$$\Gamma_{kj}^l \frac{\partial}{\partial y_i^l} \otimes dx^j \wedge dx^k. \quad (30)$$

On the other hand, (28) has the form

$$\tilde{y}_l^i \Gamma_{jk}^l dx^j \wedge dx^k. \quad (31)$$

Thus, for a principal connection both approaches lead to the classical torsion.

**Remark 1.** Even the  $r$ th frame bundle  $L^r M$  of an  $m$ -manifold  $M$  is an open dense subset in  $T_m^r M$ . There is a canonical  $\mathbb{R}^m \oplus \mathfrak{g}_m^{r-1}$ -valued form  $\Theta_r$  on  $L^r M$  introduced by Kobayashi [8], provided  $\mathfrak{g}_m^{r-1}$  denotes the Lie algebra of the structure group  $G_m^{r-1}$  of  $L^{r-1} M$ . The absolute differential  $D\Theta_r$  of  $\Theta_r$  with respect to a connection  $\Gamma$  on  $L^r M$  is called the *torsion* of  $\Gamma$  by Yuen [29], see also [10]. Taking into account the inclusion  $L^r M \subset T_m^r M$ , our approach by means of the F-N bracket gives a multiparameter family of torsions of  $\Gamma$ . This situation is similar to the first order case, but we do not go into details here.

## 5. Torsions on $T_1^2 M$

We are going to consider the bundle  $p_1 : T_1^2 M \rightarrow M$  of one-dimensional velocities of second order on  $M$  (called the *second order tangent bundle* of  $M$  in some branches of analytical mechanics). Write  $p_2 : T_1^2 M \rightarrow TM$  and  $p : TM \rightarrow M$  for the underlying projections. The Weil algebra associated with the functor  $T_1^2$  is  $A = \mathbb{R}[t]/\langle t^3 \rangle$ . The elements  $t$  and  $t^2$  of  $A$  determine two natural affinors  $Q_1 = Q(t)$  and  $Q_2 = Q(t^2)$  on every  $T_1^2 M$ . We first present their coordinate expressions and then we give a direct geometrical construction of both  $Q_1$  and  $Q_2$ .

Let  $x^i$  be some coordinates on  $M$  and  $y^i = dx^i/dt$ ,  $z^i = d^2 x^i/dt^2$  be the induced coordinates on  $T_1^2 M$ . One finds easily the following coordinate expressions of  $Q_1$ :

$$(dx^i, dy^i, dz^i) \mapsto (0, dx^i, 2 dy^i), \quad \text{i.e.} \quad \delta_j^i \frac{\partial}{\partial z^i} \otimes dx^j + 2\delta_j^i \frac{\partial}{\partial z^i} \otimes dy^j, \quad (32)$$

and of  $Q_2$ :

$$(dx^i, dy^i, dz^i) \mapsto (0, 0, dx^i), \quad \text{i.e.} \quad \delta_j^i \frac{\partial}{\partial z^i} \otimes dx^j. \quad (33)$$

It is well known that  $p_2 : T_1^2 M \rightarrow TM$  is an affine bundle with associated vector bundle  $p^*TM$ . Hence  $V(T_1^2 M \rightarrow TM) = p_1^*TM$  is the pullback of  $TM$  over  $T_1^2 M$  and one sees directly  $Q_2 = p_1^*1_M$ .

To construct  $Q_1$ , we first define a natural affinor on  $TTM$  and then we shall use the inclusion  $T_1^2 M \subset TTM$ . Let  $x^i, y^i, X^i = dx^i, Y^i = dy^i$  be the usual coordinates on  $TTM$ . Since  $TTM = T(TM)$ , we have the natural affinor of Section 4

$$(dx^i, dy^i, dX^i, dY^i) \mapsto (0, 0, dx^i, dy^i) \quad (34)$$

which we denote by  $W_1$ . Taking into account the canonical involution  $i_M : TTM \rightarrow TTM$ , we define  $W_2 = i_M \circ W_1 \circ i_M$ . Its coordinate expression is

$$(dx^i, dy^i, dX^i, dY^i) \mapsto (0, dx^i, 0, dX^i). \quad (35)$$

Then the sum  $W_1 + W_2$  is

$$(dx^i, dy^i, dX^i, dY^i) \mapsto (0, dx^i, dx^i, dy^i + dX^i). \quad (36)$$

The inclusion  $T_1^2 M \subset TTM$  is characterized by  $y^i = X^i$  and  $z^i = Y^i$ . Hence (36) can be restricted to  $T_1^2 M$  and the restriction has the form  $(dx^i, dy^i, dz^i) \mapsto (0, dx^i, 2 dy^i)$ . This is just  $Q_1$ . (We remark that another construction of  $Q_1$  is given in [18].)

Consider a connection  $\Gamma$  on  $T_1^2 M$

$$\begin{aligned} dy^i &= F_j^i(x, y, z) dx^j, \\ dz^i &= G_j^i(x, y, z) dx^j. \end{aligned} \quad (37)$$

According to our general theory, all torsions of  $\Gamma$  form a 2-parameter family linearly generated by

$$\tau_1 := [\Gamma, Q_1] \quad \text{and} \quad \tau_2 := [\Gamma, Q_2].$$

We call  $\tau_1$  and  $\tau_2$  the *first* and *second torsion* of  $\Gamma$ , respectively.

Using (10) we find the following coordinate expression of  $\tau_2$

$$\frac{\partial F_j^k}{\partial z^i} \frac{\partial}{\partial y^k} \otimes dx^i \wedge dx^j + \frac{\partial G_j^k}{\partial z^i} \frac{\partial}{\partial z^k} \otimes dx^i \wedge dx^j. \quad (38)$$

Since  $Q_2 = p_1^* 1_M$ , the geometrical interpretation of  $\tau_2$  has a form similar to Section 4.

By (8), the coordinate form of  $\tau_1$  is

$$\begin{aligned} & \frac{\partial F_j^k}{\partial y^i} \frac{\partial}{\partial y^k} \otimes dx^i \wedge dx^j + \left( \frac{\partial G_j^k}{\partial y^i} - 2 \frac{\partial F_j^k}{\partial x^i} \right) \frac{\partial}{\partial z^k} \otimes dx^i \wedge dx^j \\ & - 2 \frac{\partial F_j^k}{\partial z^i} \frac{\partial}{\partial y^k} \otimes dx^i \wedge dy^j + 2 \left( \frac{\partial F_i^k}{\partial y^j} - \frac{\partial G_i^k}{\partial z^j} \right) \frac{\partial}{\partial z^k} \otimes dx^i \wedge dy^j \\ & + 2 \frac{\partial F_i^k}{\partial z^j} \frac{\partial}{\partial y^k} \otimes dx^i \wedge dz^j. \end{aligned} \quad (39)$$

This is a section of  $VT_1^2 M \otimes \Lambda^2 T^* T_1^2 M$ . It is interesting to discuss the geometrical meaning of different kinds of projectability of  $\tau_1$ . The projections  $p_2 : T_1^2 M \rightarrow TM$  and  $p_1 : T_1^2 M \rightarrow M$  determine the inclusions  $T^* TM \subset T^* T_1^2 M$  and  $T^* M \subset T^* T_1^2 M$ .

(I)  $\tau_1$  is projectable to  $VT_1^2 M \otimes \Lambda^2 T^* TM$  if and only if  $\partial F_i^k / \partial z^j = 0$ . The fact  $F_i^k$  is independent on  $z$  means the connection  $\Gamma$  is projectable, i.e. there exists a connection  $\Delta$  on  $TM$ , whose coordinate expression is

$$dy^i = F_j^i(x, y) dx^j, \quad (40)$$

such that  $(J^1 p_1) \circ \Gamma = \Delta \circ p_1$ .

(II) The requirement  $\tau_1$  to be projectable to  $VT_1^2 M \otimes \Lambda^2 T^* M$  gives an additional condition

$$\frac{\partial G_i^k}{\partial z^j} = \frac{\partial F_i^k(x, y)}{\partial y^j}. \quad (41)$$

Hence  $G_i^k$  is of the form

$$G_i^k = \frac{\partial F_i^k}{\partial y^j} z^j + H_i^k(x, y). \quad (42)$$

To geometrize (42), we take into account that  $T_1^2 M \rightarrow TM$  is an affine bundle with associated vector bundle  $p^* TM$ . In general, if we have a fibred manifold  $E \rightarrow M$  and an affine bundle  $Z \rightarrow E$ , then  $J^1(Z \rightarrow M)$  is an affine bundle over  $J^1 E$ . A projectable connection  $\Gamma : Z \rightarrow J^1 Z$  over  $\Delta : E \rightarrow J^1 E$  is said to be *semi-affine*, if  $\Gamma$  is an affine bundle morphism over  $\Delta$ . The fact (42) is linear in  $z^i$  means  $\Gamma$  is semi-affine. Moreover, every semi-affine connection induces a linear connection on the associated vector bundle. In our case, the equations of the associated linear connection are

$$dy^i = F_j^i(x, y) dx^j, \quad du^i = \frac{\partial F_j^i}{\partial y^k} u^k dx^j \quad (43)$$

where  $u^i$  are the coordinates derived from  $z^i$ . The associated vector bundle of  $T_1^2 M$  is  $p^*TM = VTM$  and (43) are just the equations of the vertical prolongation  $\mathcal{V}\Delta$  of (40) [11]. (We recall that the vertical prolongation  $\mathcal{V}\Lambda : VE \rightarrow J^1VE$  of a connection  $\Lambda : E \rightarrow J^1E$  is defined as follows. We construct the vertical map  $V\Lambda : VE \rightarrow VJ^1E$  and apply the canonical identification  $VJ^1E \simeq J^1VE$  [6].) Thus,  $\tau_1$  is projectable to  $VT_1^2 M \otimes T^*M$  if and only if  $\Gamma$  is a *semi-affine connection over a connection  $\Delta$  on  $TM$  and the associated linear connection of  $\Gamma$  coincides with  $\mathcal{V}\Delta$ .*

(III) Finally, we interpret geometrically the vanishing of  $\tau_1$ . The condition

$$\frac{\partial F_j^k}{\partial y^i} \frac{\partial}{\partial y^k} \otimes dx^i \wedge dx^j = 0 \quad (44)$$

means  $\Delta$  is *without torsion*. Now we are going to find a geometrical formula for the remaining term  $N$  with coordinate expression

$$N = \left( \frac{\partial G_j^k}{\partial y^i} - 2 \frac{\partial F_j^k}{\partial x^i} \right) \frac{\partial}{\partial y^k} \otimes dx^i \wedge dx^j. \quad (45)$$

In our situation, we have a connection  $\Gamma$  on  $T_1^2 M$  with the equations (40) and

$$dz^i = \left( \frac{\partial F_j^i}{\partial y^k} z^k + H_j^i(x, y) \right) dx^j \quad (46)$$

satisfying

$$\frac{\partial F_j^i}{\partial y^k} = \frac{\partial F_k^i}{\partial y^j}. \quad (47)$$

The underlying connection  $\Delta$  induces an identification  $i_\Delta : T_2^1 M \simeq TM \oplus TM$  defined by the restriction of the vertical projection  $TTM \rightarrow VTM$  to  $T_1^2 M \subset TTM$ . The equations of  $i_\Delta$  are  $x^i = x^i, y^i = y^i$  and

$$w^i = z^i - F_j^i(x, y) y^j. \quad (48)$$

In particular, (48) implies  $\partial/\partial z^i = \partial/\partial w^i$ . Then  $\Gamma$  is identified with a connection on  $TM \oplus TM$ , whose equations we obtain by differentiating (48). This yields

$$\begin{aligned} dw^i &= \left( \frac{\partial F_j^i}{\partial y^k} w^k + H_j^i + \frac{\partial F_j^i}{\partial y^k} F_l^k y^l - \frac{\partial F_k^i}{\partial x^j} y^k - \frac{\partial F_l^i}{\partial y^k} y^l F_j^k - F_k^i F_j^k \right) dx^j \\ &=: E_j^i dx^j. \end{aligned} \quad (49)$$

If we consider the torsions of  $\Gamma$  as a connection on  $TM \oplus TM = T_2^1 M$  in the sense of Section 4, we find easily that the only non-zero one is  $\tau_2^1$  with coordinate expression

$$\frac{\partial E_j^i}{\partial y^k} \frac{\partial}{\partial w^i} \otimes dx^k \wedge dx^j. \quad (50)$$

Consider the curvature  $C_\Delta$  of  $\Delta$ , whose coordinate form is

$$\frac{1}{2} \left( \frac{\partial F_j^i}{\partial x^k} + \frac{\partial F_j^i}{\partial y^l} F_k^l \right) \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k. \quad (51)$$

Replacing  $\partial/\partial y^i$  by  $\partial/\partial w^i$ , we obtain a modification  $\bar{C}_\Delta$  of  $C_\Delta$ . Moreover, the vertical differential  $VC_\Delta : VTM \oplus \Lambda^2 T^*M \rightarrow V(TM \oplus TM)$  of  $C_\Delta : TM \oplus \Lambda^2 T^*M \rightarrow VTM$  with respect to the first factor has the coordinate expression

$$\frac{1}{2} \left( \frac{\partial^2 F_j^i}{\partial x^k \partial y^l} Y^l + \frac{\partial^2 F_j^i}{\partial y^l \partial y^m} Y^m F_k^l + \frac{\partial F_k^l}{\partial y^l} \frac{\partial F_j^i}{\partial y^m} Y^m \right) \frac{\partial}{\partial x^i} \otimes dx^j \wedge dx^k. \quad (52)$$

If we restrict (52) to the diagonal in  $TM \oplus TM$ , we obtain  $\bar{V}C_\Delta : TM \oplus \Lambda^2 T^*M \rightarrow V(TM \oplus TM)$ . The coordinate form of  $\bar{V}C_\Delta$  is derived from (52) when replacing  $Y^i$  by  $y^i$ . Then a direct evaluation of (50) yields the following formula

$$-\tau_2^1 = N + 3\bar{C}_\Delta + \bar{V}C_\Delta. \quad (53)$$

This gives a geometrical interpretation of  $N$ .

**Remark 2.** In [15] we clarified that the values of the functor  $T_1^2$  lie in the category of linear 2-tower bundles. A *linear tower connection* on  $T_1^2 M$  can be defined by the property that the flows of the  $\Gamma$ -lifts of all vector fields on  $M$  are constituted by local linear 2-tower morphisms. The linear 2-tower connections represent the simplest class of special connections on  $T_1^2 M$  (there are no linear connections because  $T_1^2 M$  is not a vector bundle). By [15], the equations of a linear tower connection on  $T_1^2 M$  have the form

$$\begin{aligned} dy^i &= \Delta_{kj}^i(x) y^k dx^j, \\ dz^i &= (\Gamma_{klj}^i(x) y^k y^l + \Gamma_{kj}^i(x) z^k) dx^j, \end{aligned} \quad (54)$$

with arbitrary smooth functions  $\Delta_{kj}^i, \Gamma_{kj}^i, \Gamma_{klj}^i = \Gamma_{lkj}^i$  on  $M$ . We leave to the reader the interesting task to evaluate the torsions of a linear tower connection on  $T_1^2 M$  and to discuss their geometrical properties.

## 6. The case of the cotangent bundle

The cotangent functor  $T^*$  is a natural bundle over  $m$ -manifolds, provided for every local diffeomorphisms  $f : M \rightarrow N$  we define  $T^*f : T^*M \rightarrow T^*N$  by taking pointwisely the inverse map to the dual map  $(T_x f)^* : T_{f(x)}^* N \rightarrow T_x^* M, x \in M$ .

We are going to determine all natural affinors on  $T^*M$ . First we construct one natural affinator  $Q_M : TT^*M \rightarrow TT^*M$  as follows. Let  $\lambda_M : TT^*M \rightarrow \mathbb{R}$  be the Liouville form, [5], i.e.

$$\lambda_M(X) = \langle q(X), T\pi(X) \rangle, \quad X \in TT^*M$$

where  $q : TT^*M \rightarrow T^*M$  is the bundle projection of  $T(T^*M)$  and  $T\pi : TT^*M \rightarrow TM$  is the tangent map of the bundle projection  $\pi : T^*M \rightarrow M$ . Since  $T^*M \rightarrow M$  is a vector bundle, we have  $VT^*M \simeq T^*M \oplus T^*M$ . Then the rule  $X \mapsto (q(X), \lambda(X)q(X))$  composed with the latter identification gives a natural affiner  $Q_M : TT^*M \rightarrow VT^*M$ . If  $x^i$  are some local coordinates on  $M$  and  $p_i$  are the induced coordinates on  $T^*M$ , the coordinate expression of  $Q_M$  is

$$p_i p_j \frac{\partial}{\partial p_i} \otimes dx^j. \quad (55)$$

**Proposition 2.** *All natural affiners on  $T^*M$  constitute a 2-parameter family linearly generated by the identity of  $TT^*M$  and by  $Q_M$ .*

**Proof.** Write  $\xi^i = dx^i$ ,  $\pi_i = dp_i$ . Proposition 1 from [16] implies easily that all natural transformations  $S : TT^*\mathbb{R}^m \rightarrow TT^*\mathbb{R}^m$  are of the form

$$\bar{\xi}^i = F(\lambda)\xi^i, \quad \bar{p}_i = H(\lambda)p_i, \quad \bar{\pi}_i = F(\lambda)H(\lambda)\pi_i + G(\lambda)p_i \quad (56)$$

where  $\lambda = p_i \xi^i$  is the Liouville form and  $F, G, H : \mathbb{R} \rightarrow \mathbb{R}$  are smooth functions of one variable. We have to distinguish those elements of (56) which correspond to the natural affiners in  $T^*M$ .

The requirement  $S$  is over the identity of  $T^*M$  implies  $H = 1$ . Then we use the linearity condition for  $S$ . The homogeneity with respect to  $0 \neq k \in \mathbb{R}$  yields

$$kF(\lambda)\xi^i = F(k\lambda)k\xi^i.$$

For  $k \rightarrow 0$ , we obtain  $F(\lambda) = F(0)$ , i.e.  $F = a$  is constant. Hence

$$\bar{\xi}^i = a\xi^i, \quad (57)$$

$$\bar{\pi}_i = a\pi_i + G(\lambda)p_i. \quad (58)$$

Fix  $p = (1, 0, \dots, 0)$ , so that  $\bar{\pi}_1 = a\pi_1 + G(\xi^1)$ . Then the linearity with respect to  $\xi$  yields  $G(\xi^1 + \bar{\xi}^1) = G(\xi^1) + G(\bar{\xi}^1)$ . Hence  $G(\lambda)$  is a linear function  $b\lambda$ , so that

$$\bar{\pi}_i = a\pi_i + bp_i p_j \xi^j. \quad (59)$$

But (57) and (59) is the coordinate form of our assertion.  $\square$

Let  $\Gamma$  be a connection on  $T^*M$ . Having a situation similar to the case of the tangent bundle, we say that the F-N bracket  $\tau = [\Gamma, Q]$  is the *torsion* of  $\Gamma$ .

The equations of  $\Gamma$  are

$$dp_i = F_{ij}(x, p)dx^j. \quad (60)$$

By (10) and (55), the coordinate form of  $\tau$  is

$$\left( p_k F_{ij} + p_i F_{kj} - \frac{\partial F_{ij}}{\partial p_l} p_l p_k \right) \frac{\partial}{\partial p_i} \otimes dx^j \wedge dx^k. \quad (61)$$

If  $\Gamma$  is a linear connection, i.e.

$$dp_i = \Gamma_{ij}^k(x) p_k dx^j \quad (62)$$

then (61) has the form

$$p_i \Gamma_{kj}^l p_l \frac{\partial}{\partial p_i} \otimes dx^j \wedge dx^k. \quad (63)$$

It is interesting that, analogously to the case of the frame bundle  $LM$ , there is another approach to the concept of torsion on  $T^*M$ . Since the Liouville form  $\lambda : TT^*M \rightarrow \mathbb{R}$  is natural, the covariant exterior differential  $D\lambda$  with respect to a connection  $\Gamma$  on  $T^*M$  is a geometric object determined by  $\Gamma$ , which can be said to be the  $\lambda$ -torsion of  $\Gamma$ . Clearly, the coordinate expression of  $D\lambda$  is

$$F_{ij} dx^i \wedge dx^j. \quad (64)$$

Hence for a linear connection (62) both  $\tau$  and  $D\lambda$  coincide, in fact. However, for a non-linear connection on  $T^*M$ , the torsion of type  $Q$  and the  $\lambda$ -torsion give different information.

**Remark 2.** There is another definition of the  $\lambda$ -torsion of a connection  $\Gamma : T^*M \rightarrow J^1T^*M$  [22]. The space  $J^1T^*M$  is identified with the bundle  $\hat{T}^{2*}M$  of all sesquiholonomic 2-jets of  $M$  into  $\mathbb{R}$  with target 0. By the theory of sesquiholonomic 2-jets, we have a natural map  $\sigma : J^1T^*M \rightarrow \Lambda^2T^*M$  (which coincides with the formal version of the exterior differential of 1-forms). One verifies easily that the composed map  $\sigma \circ \Gamma$  is equal to  $D\lambda$ .

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