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JET INVOLUTION AND PROLONGATIONS OF CONNECTIONS

MARCO MODUGNO, Firenze

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Summary. This paper is concerned with the involution of the second anholonomic jet prolongation $J_1 J_1 E$ of a fibred manifold $p: E \to B$ and with some further useful jet techniques. As an application, jet prolongations of connections are obtained. These technical results turn out to be useful in several frameworks of differential geometry and mathematical physics.

Keywords: Jet involution, prolongation of connections, anholonomic jet prolongations.

It is well known [13] that the second tangent bundle TTM of a manifold M is endowed with two projections $\pi_{TM}: TTM \to TM$ and $T\pi_M: TTM \to TM$ and the canonical involution $s: TTM \to TTM$ which interchanges the two projections. On the other hand, if we consider the product bundle $p = pr_1: E = \mathbf{R} \times M \to B = \mathbf{R}$, then we have [10] the canonical isomorphisms $J_1 E \cong \mathbf{R} \times TM$ and $J_1 J_1 E \cong \mathbf{R} \times \times TTM$. So, we might expect that it is possible to generalize the involution s to $J_1 J_1 E$ for any fibred manifold $p: E \to B$. Unfortunately, such a canonical involution s does not exist in general. However, we can prove that the choice of a symmetric linear connection k of the base space B yields the result. Actually, the feature which turns out to be essential in the particular case of $p = pr_1: E = \mathbf{R} \times M \to B = \mathbf{R}$ is neither the product nor the dimension one of the base space, but just the canonical connection of $\mathbf{R}$. On the way we analyse several important affine and vector bundles by means of functorial jet techniques.

The above results yield an interesting application for the Ehresmann connections (briefly called connections), which constitute the most appropriate unifying framework of all standard connections [1, 2, 5, 6, 7, 8, 10, 11, 12]. Given a connection, i.e. a section $\gamma: E \to J_1 E$, it would be interesting to prolong it to a connection $\Gamma: J_1 E \to J_1 J_1 E$. An immediate idea could be to apply the functor J_1 to obtain $J_1 \gamma: J_1 E \to J_1 J_1 E$. Unfortunately, $J_1 \gamma$ is not a section, because it interchanges the two projections of $J_1 J_1 E$. However, the involution s_k is just what we need for setting the problem. In fact, $\Gamma_k = s_k \circ J_1 \gamma: J_1 E \to J_1 J_1 E$ turns out to be the jet prolongation of γ (which depends on the choice of k). Additionally, the curvature of Γ_k depends in a nice way on the curvatures of γ and k. Moreover, if we consider [12] a system (C, ξ) of connections of E and the system (K, ψ) of linear connections of T^*B, then we obtain the canonical prolonged system of connections of $J_1 E$. Following a similar line, we find the prolongation of a connection γ on E to a connection γ_k

on the vector bundle $J_1^*E = TB \otimes_E V^*E$. This result turns out to be useful for lagrangian theories [14].

For the basic notions and notation concerning jet spaces and connections we refer to [9, 10, 12]. — Thanks are due to I. Kolář for his critical reading of the manuscript.

1. INVOLUTION OF THE SECOND ANHOLONOMIC JET SPACE

Our consideration is in the category C^∞. Through the paper we are concerned with a fibred manifold $p: E \to B$. We denote by (x^λ, y^i) the generic fibred manifold chart of E. Roman indices $i, j, h, k, \ldots$ and Greek indices $\lambda, \mu, \varrho, \sigma, \ldots$ run on the coordinates of the fibres and of the base space, respectively. Let $\pi_E: VE \to E$ be the vertical prolongation of $p: E \to B$. We denote by $(x^\lambda, y^i, \dot{y}^i)$ the induced fibred manifold chart. Let $p_1: J_1E \to B$ be the first jet prolongation of $p: E \to B$ and let $p_{(01)}: J_1E \to E$ be the natural fibred epimorphism over B. We denote by $(x^\lambda, y^i, y^i_\lambda)$ the induced fibred manifold chart of J_1E. Then the coordinate expressions of p_1 and $p_{(01)}$ are $x^\lambda \circ p_1 = x^\lambda$ and $(x^\lambda, y^i) \circ p_{(01)} = (x^\lambda, y^i)$. The transition functions of J_1E are $\bar{y}^i_\lambda = A^\varrho_\lambda(\bar{A}^i_j y^j_\varrho + \bar{A}^i_\varrho)$ where $\bar{A}^i_j = \partial_j \bar{y}^i$, $\bar{A}^i_\varrho = \partial_\varrho \bar{y}^i$, $A^\varrho_\lambda = \partial_{\bar{\lambda}} x^\varrho$. Hence, $p_{(01)}: J_1E \to E$ turns out to be an affine bundle whose vector bundle is $T^*B \otimes_E \otimes_E VE \to E$.

We recall that, if $p: E \to B$ and $p': E \to B'$ are fibred manifolds and $f: E \to E'$ is a fibred morphism over B, then $J_1f: J_1E \to J_1E'$ is a fibred morphism over f. If $(x^\lambda, y^i) \circ f = (x^\lambda, f^i)$ is the coordinate expression of f, then the coordinate expression of J_1f is $(x^\lambda, y^i, y^i_\lambda) \circ J_1f = (x^\lambda, f^i, \partial_\lambda f^i + \partial_j f^i y^j_\lambda)$. By iteration, we obtain the first jet prolongation $p_{11}: J_1J_1E \to B$ of $p_1: J_1E \to B$ and the natural fibred epimorphism $p_{1(01)}: J_1J_1E \to J_1E$. Moreover, $p_{1(01)}: J_1J_1E \to J_1E$ is an affine bundle whose vector bundle is $T^*B \otimes_{J_1E} VJ_1E$. On the other hand, we have another fibred morphism $J_1p_{(01)}: J_1J_1E \to J_1E$ over B.

We denote by $(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda 0}, y^i_{\lambda\mu})$ the induced fibred manifold chart of J_1J_1E. Then the coordinate expressions of $p_{11}, p_{1(01)}$ and $J_1p_{(01)}$ are $x^\lambda \circ p_{11} = x^\lambda$, $(x^\lambda, y^i, y^i_\lambda) \circ p_{1(01)} = (x^\lambda, y^i, y^i_\lambda)$ and $(x^\lambda, y^i, y^i_\lambda) \circ J_1p_{(01)} = (x^\lambda, y^i, y^i_{\lambda 0})$.

The fibred morphism $J_1p_{(01)}: J_1J_1E \to J_1E$ over $p_{(01)}: J_1E \to E$ is affine. Hence

$$(p_{1(01)}, J_1p_{(01)}): J_1J_1E \to J_1E \times_E J_1E$$

turns out to be an affine bundle and its vector bundle is the sub-bundle $(J_1E \times_E J_1E) \times_E ((T^*B \otimes_B T^*B) \otimes_E VE) = J_1E \times_E (T^*B \otimes_{J_1E} V_EJ_1E) \subset \subset J_1E \times_E (T^*B \otimes_{J_1E} VJ_1E)$. Clearly, $J_1p_{(01)}$ provides another affine structure on J_1J_1E over J_1E.

Lemma 1. *Let $q: F \to B$ be a fibred manifold, $\pi: F \to E$ a fibred morphism over B such that $\pi: F \to E$ is an affine bundle whose vector bundle is $\bar{\pi}: \bar{F} \to E$. Then $J_1\pi: J_1F \to J_1E$ is an affine bundle whose vector bundle is $J_1\bar{\pi}: J_1\bar{F} \to J_1E$.*

Proof follows from a computation in local coordinates.

In particular, we can consider the fibred manifold $q = p_1 \colon F = J_1E \to B$ and the fibred morphism $\pi = p_{(01)} \colon J_1E \to E$.

Corollary 1. *$J_1 p_{(01)} \colon J_1J_1E \to J_1E$ is an affine bundle whose vector bundle is $J_1(T^*B \otimes VE)$.*

Summing up, we have obtained the following three affine bundles with their vector bundles:

i) $p_{1(01)} \colon J_1J_1E \to J_1E$ with $T^*B \otimes_E VJ_1E$,

ii) $J_1 p_{(01)} \colon J_1J_1E \to J_1E$ with $J_1(T^*B \otimes_E VE)$,

iii) $(p_{k(01)}, J_1 p_{(01)}) \colon J_1J_1E \to J_1E \times_E J_1E$ with $(J_1E \times_E J_1E) \times_E ((T^*B \otimes_B T^*B) \otimes_E VE)$.

The situation is clearly illustrated by the transition functions of J_1J_1E

$$\bar{y}^i_\mu = A^\varrho_\mu(\bar{A}^i_j y^j_\varrho + \bar{A}^i_\vartheta)\,,$$
$$\bar{y}^i_{\lambda 0} = A^\varrho_\lambda(\bar{A}^i_j y^j_{\varrho 0} + \bar{A}^i_\varrho)\,,$$
$$\bar{y}^i_{\lambda\mu} = A^\varrho_\lambda A^\sigma_\mu(\bar{A}^i_j y^j_{\varrho\sigma} + \bar{A}^i_{hk} y^k_{\varrho 0} y^k_\sigma + \bar{A}^i_{j\varrho} y^j_\sigma + \bar{A}^i_{j\sigma} y^j_{\varrho 0} + \bar{A}^i_{\varrho\sigma}) + A^\nu_{\lambda\mu}(\bar{A}^i_j y^j_\nu + \bar{A}^i_\nu)\,.$$

We need some further consequences of Lemma 1.

Corollary 2. *Let $q \colon F \to B$ be a fibred manifold, $\pi \colon F \to E$ a fibred morphism over B and $\pi \colon F \to E$ a vector bundle. Then $J_1\pi \colon J_1F \to J_1E$ is a vector bundle.*

Let $W \to B$ be a vector bundle. Let $F \to B$ be a fibred manifold, $\pi \colon F \to E$ a fibred morphism over B and $\pi \colon F \to E$ a vector bundle. Then the universal property of the tensor product induces the natural linear fibred morphism over J_1E

$$\tau \colon J_1W \otimes_{J_1E} J_1F \to J_1(W \otimes_E F)\,.$$

Its coordinate expression is

$$(x^\lambda, y^i, t^{ij}, y^i_\lambda, t^{ij}_\lambda) \circ \tau = (x^\lambda, y^i, w^i z^j, y^i_\lambda, w^i_\lambda z^j + w^i z^j_\lambda)\,.$$

In particular, we can consider the vector bundle $W = T^*B$, the fibred manifold $F = VE$ and the vector bundle $\pi = \pi_E \colon VE \to E$.

Corollary 3. *We have a natural linear fibred morphism over J_1E, $\tau \colon J_1T^*B \otimes_{J_1E} \otimes_{J_1E} J_1VE \to J_1(T^*B \otimes_E VE)$. Its coordinate expression is*

$$(x^\lambda, y^i, \dot{y}^i_\mu, y^i_\lambda, \dot{y}^i_{\lambda\mu}) \circ \tau = (x^\lambda, y^i, \dot{x}_\mu y^i, y^i_\lambda, \dot{x}_{\lambda\mu}\dot{y}^i + \dot{x}_\mu \dot{y}^i_\lambda)\,.$$

We recall [9] that there is a natural fibred isomorphism over $J_1E \times_E VE$,

$$i \colon VJE \to J_1VE\,,$$

which is linear over J_1E. Its coordinate expression is

$$(x^\lambda, y^i, \dot{y}^i, y^i_\lambda, \dot{y}^i_\lambda) \circ i = (x^\lambda, y^i, \dot{y}^i, y^i_\lambda, \dot{y}^i_\lambda) .$$

We recall [10] that a linear connection of B is a section

$$k\colon T^*B \to J_1T^*B ,$$

which is a linear fibred morphism over B. Moreover, k is symmetric if its torsion vanishes, i.e. if

$$0 = d \circ k\colon T^*B \to \Lambda^2 T^*B ,$$

where $d\colon J_1T^*B \to \Lambda^2T^*B$ is the natural fibred morphism over B.

Let k be a linear connection of B. Then we have the fibred morphism over $(T^*B \otimes_E VE) \times_E J_1E$

$$\mathbf{k}\colon T^*B \otimes_{J_1E} VJ_1E \to J_1(T^*B \otimes_E VE)$$

which is linear over J_1E, given by the composition

$$T^*B \otimes_{J_1E} VJ_1E \xrightarrow{k \otimes i} J_1T^*B \otimes_{J_1E} J_1VE \xrightarrow{\tau} J_1(T^*B \otimes_E VE) .$$

Its coordinate expression is

$$(x^\lambda, y^i, y^i_\lambda, \dot{y}^i_\mu, \dot{y}^i_{\lambda\mu}) \circ \mathbf{k} = (x^\lambda, y^i, y^i_\lambda, \dot{y}^i_\mu, \dot{y}^i_{\lambda\mu} + k^\nu_{\lambda\mu}\dot{y}^i_\nu) .$$

So, finally, we can state the main theorem.

Theorem 1. *Let k be a symmetric linear connection of B. Then there is a unique natural affine fibred morphism over J_1E*

$$s_k\colon J_1J_1E \to J_1J_1E$$

of the two affine bundles $p_{1(01)}\colon J_1J_1E \to J_1E$ and $J_1p_{(01)}\colon J_1J_1E \to J_1E$ such that

i) $s_k \circ s_k = \mathrm{id}$,

ii) $Ds_k = \mathbf{k}$.

The coordinate expression of s_k is

$$(x^\lambda, y^i, y^i_\mu, y^i_{\lambda 0}, y^i_{\lambda\mu}) \circ s_k = (x^\lambda, y^i, y^i_{\mu 0}, y^i_\lambda, y^i_{\mu\lambda} + k^\nu_{\mu\lambda}(y^i_\nu - y^i_{\nu 0})) .$$

Proof. Existence: The above coordinate expression does not depend on the choice of the chart (x^λ, y^i) of E, hence it yields a global and natural fibred morphism. In fact, the following relations hold

$$\bar{y}^i_\mu \circ s_k = \bar{y}^i_{\mu 0} = A^\varrho_\mu(\bar{A}^i_j y^j_{\varrho 0} + \bar{A}^i_\varrho) = A^\varrho_\mu(\bar{A}^i_j y^j_\varrho \circ s_k + \bar{A}^i_\varrho) ,$$

$$\bar{y}^i_{\lambda 0} \circ s_k = \bar{y}^i_\mu = A^\sigma_\mu(\bar{A}^i_j y^j_\sigma + \bar{A}^i_\sigma) = A^\varrho_\mu(\bar{A}^i_j y^j_{\varrho 0} \circ s_k + \bar{A}^i_\varrho) ,$$

$$\bar{y}^i_{\lambda\mu} \circ s_k = \bar{y}^i_{\mu\lambda} + \bar{k}^\nu_{\lambda\mu}(\bar{y}^i_\nu - \bar{y}^i_{\nu 0}) =$$

$$
\begin{aligned}
&= A^{\varrho}_{\lambda} A^{\sigma}_{\mu} (\bar{A}^i_j (y^j_{\sigma\varrho} + k^{\nu}_{\sigma\varrho}(y^j_{\nu} - y^j_{\nu 0})) + \bar{A}^i_{hk} y^h_{\varrho 0} y^k_{\sigma} + \bar{A}^i_{j\sigma} y^j_{\varrho 0} + \\
&+ \bar{A}^i_{j\varrho} y^j_{\sigma} + \bar{A}^i_{\varrho\sigma}) + A^{\nu}_{\lambda\mu} (\bar{A}^i_j y^j_{\nu 0} + \bar{A}^i_{\nu}) = \\
&= A^{\varrho}_{\lambda} A^{\sigma}_{\mu} (\bar{A}^i_j y^j_{\varrho\sigma} + \bar{A}^i_{hk} y^h_{\varrho 0} y^k_{\sigma} + \bar{A}^i_{j\sigma} y^j_{\varrho 0} + \bar{A}^i_{j\varrho} y^j_{\sigma} + \bar{A}^i_{\varrho\sigma}) + \\
&+ A^{\nu}_{\lambda\mu} (\bar{A}^i_j y^j_{\nu} + \bar{A}^i_{\nu})) \circ s_k .
\end{aligned}
$$

Moreover, this fibred morphism satisfies the conditions i) and ii).

Uniqueness. If s_k exists, then its coordinate expression is $(x^{\lambda}, y^i, y^i_{\mu}, y^i_{\lambda 0}, y^i_{\lambda\mu}) \circ s_k = (x^{\lambda}, y^i, y^i_{\mu 0}, y^i_{\lambda}, y^i_{\mu\lambda} + k^{\nu}_{\lambda\mu}(y^j_{\nu} - y^j_{\nu 0}) + t^i_{\lambda\mu})$, where t is a section $t: E \to \Lambda^2 T^*B \otimes_E VE$. Then the naturality yields $t = 0$, QED.

Clearly, $s_k: J_1 J_1 E \to J_1 J_1 E$ turns out to be also an affine fibred morphism over $J_1 E \times_E J_1 E$. Hence it makes the following diagram commutative

$$
\begin{array}{ccc}
J_1 J_1 E & \xrightarrow{s_k} & J_1 J_1 E \\
{\scriptstyle (p_{1(01)},\, J_1 p_{(01)})} \searrow & & \swarrow {\scriptstyle (J_1 p_{(01)},\, p_{1(01)})} \\
& J_1 E \times_E J_1 E &
\end{array}
$$

and its fibre derivative is

$$
\begin{aligned}
Ds_k &: (T^*B \otimes_B T^*B) \otimes_E VE \to (T^*B \otimes_B T^*B) \otimes_E VE , \\
&: u \otimes v \otimes w \mapsto v \otimes u \otimes w .
\end{aligned}
$$

Remark 1. In the particular case when $p = pr_1: E = \mathbf{R} \times M \to B = \mathbf{R}$ and k is the canonical connection of $\mathbf{R}$, we obtain [10] the canonical involution

$$
s = s_k: J_1 J_1 E = \mathbf{R} \times TTM \to J_1 J_1 E = \mathbf{R} \times TTM .
$$

We recall [10] that the sesquiholonomic second jet prolongation of $p: E \to B$ is

$$
\hat{J}_2 E = \operatorname{Ker}(p_{1(01)} - J_1 p_{(01)}): J_1 J_1 E \to T^*B \otimes_E VE .
$$

$\hat{J}_2 E$ turns out to be an affine sub-bundle of $J_1 J_1 E$ over $J_1 E$ (with respect to the both projections), whose vector bundle is

$$
J_1 E \times_E ((T^*B \otimes_B T^*B) \otimes_E VE) .
$$

In other words, $\hat{J}_2 E$ is the pull-back bundle induced by the commutative diagram

$$
\begin{array}{ccc}
\hat{J}_2 E & \longrightarrow & J_1 J_1 E \\
\downarrow & & \downarrow \\
J_1 E & \longrightarrow & J_1 E \times_E J_1 E
\end{array}
$$

Moreover, we recall that the holonomic second jet prolongation $J_2 E$ of $p: E \to B$

is an affine sub-bundle of $\hat{J}_2E$, whose vector bundle is

$$J_1E \times_E (S_2T^*B \otimes_E VE) \subset J_1E \times_E ((T^*B \otimes_B T^*B) \otimes_E VE).$$

Then we have [10] the canonical affine splitting over J_1E

$$\hat{J}_2E = J_2E \oplus_{J_1E} (\Lambda^2T^*B \otimes_E VE).$$

This splitting has an important role in the study of the curvature of connections [10, 12]. The coordinates induced on $\hat{J}_2E$ and J_2E are $(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu})$ and $(x^\lambda, y^i, y^i_\mu, y^i_{\lambda\mu})$, $\lambda \leqq \mu$. They turn out to be adapted to the above structures.

We remark that the restriction of s_k to $\hat{J}_2E \subset J_1J_1E$ does not depend on the choice of k and turns out to be the canonical involution

$$s\colon \hat{J}_2E = J_2E \oplus (\Lambda^2T^*B \otimes_E VE) \to \hat{J}_2E = J_2E \oplus (\Lambda^2T^*B \otimes_E VE)$$
$$: (\sigma, \alpha) \mapsto (\sigma, -\alpha).$$

2. THE FIRST JET PROLONGATION OF A CONNECTION

We recall [10, 11, 12] that a connection on $p\colon E \to B$ is a section $\gamma\colon E \to J_1E$. Its coordinate expression is $y^i_\lambda \circ \gamma = \gamma^i_\lambda$. In particular, γ is a fibred morphism over B. A connection induces the affine fibred translation over E

$$\nabla_\gamma\colon J_1E \to T^*B \otimes_B VE\colon \quad y_1 \mapsto y_1 - \gamma(y)$$

which characterizes γ itself. The coordinate expression of ∇_γ is

$$(x^\lambda, y^i, y^i_\lambda) \circ \nabla_\gamma = (x^\lambda, y^i, \nabla^i_\lambda) = (x^\lambda, y^i, y^i_\lambda - \gamma^i_\lambda).$$

Theorem 2. *Let k be a linear connection of B and $\gamma\colon E \to J_1E$ a connection of E. Then $\Gamma_k = s_k \circ J_1\gamma\colon J_1E \to J_1J_1E$ is a section of the bundle $p_{1(01)}\colon J_1J_1E \to J_1E$, hence a connection of $p_1\colon J_1E \to B$. Moreover, Γ_k is projectable over γ and its coordinate expression is*

$$(x^\lambda, y^i_\mu, y^i_{\lambda 0}, y^i_{\lambda\mu}) \circ \Gamma_k = (x^\lambda, y^i_\mu, \gamma^i_\lambda, \partial_\lambda\gamma^i_\mu + \partial_j\gamma^i_\lambda y^j_\mu - k^\nu_{\lambda\mu}\nabla^i_\nu).$$

Proof follows from the coordinate expression of $J_1\gamma$

$$(x^\lambda, y^i, y^i_\mu, y^i_{\lambda 0}, y^i_{\lambda\mu}) \circ J_1\gamma = (x^\lambda, y^i, \gamma^i_\mu, y^i_\lambda, \partial_\mu\gamma^i_\lambda + \partial_j\gamma^i_\mu y^j_\lambda).$$

We recall [10, 11, 12] that the curvature of the connection $\gamma\colon E \to J_1E$ is a fibred morphism over E

$$\varrho = \tfrac{1}{2}[\gamma, \gamma]\colon E \to \Lambda^2T^*B \otimes_E VE,$$

where [,] is the Frölicher-Nijenhuis bracket [3]. Its coordinate expression is

$$\varrho = (\partial_\lambda\gamma^i_\mu + \gamma^j_\lambda\,\partial_j\gamma^i_\mu)\,d^\lambda \wedge d^\mu \otimes \partial^i.$$

The curvature R_k of Γ_k is

$$R_k = \tfrac{1}{2}[\Gamma_k, \Gamma_k]: J_1E \to \Lambda^2 T^*B \otimes_{J_1E} VJ_1E .$$

Using direct evaluations, we find that R_k is projectable on the curvature $\varrho = \tfrac{1}{2}[\gamma, \gamma]: E \to \Lambda^2 T^*B \otimes VE$ of γ, i.e., the folliwing diagram commutes

$$\begin{array}{ccc} J_1E & \xrightarrow{R_k} & \Lambda^2 T^*B \otimes_{J_1E} VJ_1E \\ {\scriptstyle p(01)}\downarrow & & \downarrow \\ E & \xrightarrow{\varrho} & \Lambda^2 T^*B \otimes_E VE \end{array}$$

Furthermore, $R_k: J_1E \to \Lambda^2 T^*B \otimes_E VJ_1E$ is an affine fibred morphism over $\varrho: E \to \Lambda^2 T^*B \otimes_E VE$, whose fibre derivative

$$DR_k: T^*B \otimes_E VE \to (T^*B \otimes_E VE) \times_E ((\Lambda^2 T^*B \otimes_B T^*B) \otimes_E VE)$$

is yielded by the curvature $r: B \to \Lambda^2 T^*B \otimes_B T^*B \otimes_B TB$ of k by means of natural contractions. The coordinate expression of R_k is

$$R_k = \varrho^i_{\lambda\mu} d^\lambda \wedge d^\mu \otimes \partial_i + (J_\nu \varrho^i_{\lambda\mu} + \varrho^i_{\lambda\alpha} k^\alpha_{\mu\lambda} - \varrho^i_{\mu\alpha} k^\alpha_{\lambda\nu} + r^\alpha_{\lambda\mu\nu} \nabla^i_\alpha) \, d^\lambda \wedge d^\mu \otimes \partial^\nu_i .$$

An even more interesting application is obtained if we are concerned with connections γ belonging to a system of connections. We recall [12] that a system of connections is constituted by a fibred manifold $q: C \to B$ and a fibred morphism $\xi: C \times_B E \to J_1E$ over E. Hence, if $c: B \to C$ is a section then we obtain the connection $\gamma = \xi \circ \tilde{c}: E \to J_1E$, where $\tilde{c}: E \to C \times_B E$ is the natural extension of c. These connections γ are the distinguished connections of the system. The coordinate expression of $\gamma = \xi \circ \tilde{c}$ is $\gamma^i_\lambda = \xi^i_\lambda \circ \tilde{c}$ with $\xi^i_\lambda: C \times_B E \to \mathbf{R}$. We denote by (x^λ, v^a) the generic fibred manifold chart of C. For example, the system of linear connections of a vector bundle $p: E \to B$ is constituted by the fibred manifold $q: L \to B$, where $L \subset E^* \otimes_B J_1E$ is the affine sub-bundle which projects onto the identity section of $E^* \otimes_B E$, and the evaluation fibred morphism $\lambda: C \times_B E \to J_1E$ over E. In particular, we have the system (K, ψ) of symmetric linear connections of the vector bundle $T^*B \to B$, i.e. of the linear connections of B. We denote by $(x^\lambda, u^\nu_{\lambda\mu})$ the natural fibred manifold chart of K. Clearly, the jet involution can be expressed more intrinsically as follows.

Proposition 1. *We have a natural fibred morphism over B*

$$s: K \times_B J_1J_1E \to J_1J_1E ,$$

which yields s_k for each section $k: B \to K$.

Proposition 2. *Let (C, ξ) be a system of connections of $p: E \to B$. Then we have the system of connections $(K \times_B J_1C, \Xi)$ on $p_1: J_1E \to B$, where Ξ is the fibred* *morphism over J_1E given by the composition*

$$K \times_B (J_1C \times_B E) \xrightarrow{\mathrm{id} \times J_1\xi} K \times_B J_1J_1E \xrightarrow{s} J_1J_1E\,.$$

The system $(K \times_B J_1C, \Xi)$ prolongs (C, ξ) and its coordinate expression is

$$\Xi^i_\mu = \xi^i_\mu$$
$$\Xi^i_{\lambda\mu} = \partial_\mu \xi^i_\lambda + \partial^i \xi^i_\lambda v^a_\mu + \partial_j \xi^i_\lambda y^j_\mu + u^\nu_{\mu\lambda}(y^i_\nu - \xi^i_\nu)\,.$$

Remark 2. If $\gamma = \xi \circ \tilde{c}$ is a connection of the system, then its prolongation Γ is obtained by computing only the the derivatives $\partial_\mu c^a$ (with respect to the coordinates of the base space B), as the derivatives $\partial_j \gamma^i$ (with respect to the fibres of E) are atutomatically carried by Ξ in a way which does not depend on the particular γ but only on the system.

In particular, if $p: E \to B$ is a vector bundle, an affine bundle, a principal bundle, ..., then the previous results can be easily applied to linear, affine, principal, ... connections, respectively.

3. ANOTHER PROLONGATION OF A CONNECTION

We recall that $T^*B \otimes_E VE \to E$ is the vector bundle of $p_{(01)}: J_1E \to E$. Its dual is $J_1^*E = TB \otimes_E V^*E \to E$, which turns out to have an important role in lagrangian theories [44]. We are going to show that a connection $\gamma: E \to J_1E$ on $p: E \to B$ and a linear connection $k: TB \to J_1TB$ on the base space B induce a connection on $J_1^*E \to B$. To this end we are not concerned with the involution s_k but we use other jet techniques developed in the first section.

First, we analyse the vertical prolongation of the connection γ.

Proposition 3. *Let $\gamma: E \to J_1E$ be a connection. Then $\Gamma = i \circ V\gamma: VE \to J_1VE$ is a section, hence a connection of VE. Moreover, Γ is a linear fibred morphism over γ and its coordinate expression is*

$$(x^\lambda, y^i, \dot{y}^i, y^i_\lambda, \dot{y}^i_\lambda) \circ \Gamma = (x^\lambda, y^i, y^i_\lambda, \gamma^i_\lambda, \partial_j \gamma^i_\lambda \dot{y}^j)\,.$$

Proof. $V\gamma: VE \to VJ_1E$ is a section. Its coordinate expression is

$$(x^\lambda, y^i, y^i_\lambda, \dot{y}^i, \dot{y}^i_\lambda) \circ V\gamma = (x^\lambda, y^i, \gamma^i_\lambda, \dot{y}^i, \partial_j \gamma^i_\lambda \dot{y}^j)\,.$$

We observe that $J_1\pi_E: J_1VE \to J_1E$ is a vector bundle. The linearity of Γ is shown by its coordinate expression, QED.

Let $\Gamma^*: V^*E \to J_1V^*E$ be the dual connection of Γ in the sense of [6] whose coordinate expression is

$$(x^\lambda, y^i, \dot{y}_i, y^i_\lambda, \dot{y}_{\lambda i}) \circ \Gamma^* = (x^\lambda, y^i, \dot{y}_i, \gamma^i_\lambda, -\partial_i \gamma^j_\lambda \dot{y}_j)\,.$$

Theorem 3. *Let $\gamma: E \to J_1E$ be a connection and $k: TB \to J_1TB$ a linear connection on the space B. Then the fibred morphism*

$$\tau \circ (k \otimes \Gamma^*): TB \otimes_E V^*E \to J_1(TB \otimes_E V^*E)$$

which is given by the composition (see Section 1)

$$TB \otimes_E V^*E \xrightarrow{k \otimes \Gamma^*} J_1TB \otimes_{J_1E} J_1V^*E \xrightarrow{\tau} J_1(TB \otimes_E V^*E)$$

is a connection on $TB \otimes_E V^*E \to B$, which prolongs $\gamma: E \to J_1E$. Its coordinate expression is

$$(x^\lambda, y^i, \dot{y}^\alpha_i, y^i_\lambda, \dot{y}^\alpha_{\lambda i}) \circ \tau \circ (k \otimes \Gamma^*) = (x^\lambda, y^i, \dot{y}^\alpha_\lambda, \gamma^i_\lambda, k^\alpha_{\lambda\mu}\dot{y}^\mu_i - \partial_i\gamma^j_\lambda\dot{y}^\alpha_j)\,.$$

Proof follows from the fact that the coordinate expression of τ is

$$(x^\lambda, y^i, \dot{y}^\alpha_i, y^i_\lambda, \dot{y}^\alpha_{\lambda i}) \circ \tau = (x^\lambda, y^i, \dot{y}^\alpha_i, y^i_\lambda, \dot{x}^\alpha_\lambda y_i + \dot{x}^\alpha \dot{y}_{\lambda i})\,.$$

The curvature of the prolongation and the case of systems of connections can be studied in a way analogous to Section 2.

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Souhrn

INVOLUCE JETŮ A PRODLOUŽENÍ KONEXÍ

MARCO MODUGNO

Článek se zabývá involucí druhého anholonomického prodloužení jetů $J_1 J_1 E$ fibrované variety p: $E \to B$ a některými dalšími užitečnými jetovými technikami. Jejich aplikací se dostanou jetová prodloužení konexí. Výsledky technického rázu jsou užitečné v diferenciální geometrii a matematické fyzice.

Резюме

ИНВОЛЮЦИИ ДЖЕТОВ И ПРОДОЛЖЕНИЯ СВЯЗНОСТЕЙ

MARCO MODUGNO

В статье рассматриваются инволюции второго неголономического продолжения джетов $J_1 J_1 E$ расслоенного многообразия p: $E \to B$ и некоторые другие полезные джетовые техники. В качестве приложения получены продолжения связностей. Результаты технического характера полезны в дифференциальной геометрии и математической физике.

Author's address: Istituto di Matematica Applicata "G. Sansone", Via S. Marta 3, 50139 Firenze, Italia.