5.2.5 Canonical Markov chains

Example 5.12 A typical example which may help intuition is that of *random walks*. A person is at a random position k, $k \in \mathbb{Z}$, and at each step moves either to the position $k-1$ or to the position $k+1$ according to a Bernoulli trial of parameter p, for example by tossing a coin. Let X_n be the position occupied at the nth step and let $\{\zeta_n\}$ be a sequence of *independent* random variables that follow $B(1,p)$ and such that ζ_n decides if the person moves backward or foward at step n. Then

$$X_{n+1} = X_n + 2\zeta_n - 1$$

so that $\{X_n\}$ is a Markov chain, see Theorem 5.13 below, whose transition matrix is

$$\mathbf{P} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & p & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & p & 0 & 0 & 0 & \dots \\ \dots & q & 0 & p & 0 & 0 & \dots \\ \dots & 0 & q & 0 & p & 0 & \dots \\ \dots & 0 & 0 & q & 0 & p & \dots \\ \dots & 0 & 0 & 0 & q & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & q & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Matrix $\mathbf{P}$ is represented as a graph in Figure 5.2.

Example 5.12 is indeed a standard way to construct Markov chains. The following holds.

Theorem 5.13 *Let S be a finite or denumerable set, and let $X_0 : \Omega \to S$ be a random variable on $(\Omega, \mathcal{E}, \mathbb{P})$. Let $\{\xi_n\}$, $\xi_n : \Omega \to \mathbb{R}^N$ be a sequence of independent, identically distributed random variables on $(\Omega, \mathcal{E}, \mathbb{P})$ that are also independent of X_0 and let $f : S \times \mathbb{R}^N \to S$ be a Borel function. Then the sequence $\{X_n\}$, $X_n : \Omega \to S$ defined by*

$$X_{n+1}(x) := f(X_n(x), \xi_n(x)), \qquad \forall n \geq 0, \tag{5.17}$$

is a homogeneous Markov chain with state space S and transition matrix

$$\mathbf{P}^i_j := \mathbb{P}(f(i, \xi_k) = j) \qquad \forall i, j \in S.$$

Proof. From the definition, it is clear that for any integer k, the random variable X_k is a function of X_0 and $\xi_1, \dots, \xi_{k-1}$. Consequently, for any integer n, the random variable ξ_n, which is by definition independent of $(X_0, \xi_1, \dots, \xi_{n-1})$, is also independent of $(X_0, \dots, X_n)$. Therefore,

$$\begin{aligned} &\mathbb{P}\left(X_{n+1} = j \,\middle|\, X_n = i, \ \dots, \ X_0 = i_0\right) \\ &\qquad = \mathbb{P}\left(f(i,\xi_n) = j \,\middle|\, X_n = i, \dots, \ X_0 = i_0\right) = \mathbb{P}(f(i,\xi_n) = j); \end{aligned} \tag{5.18}$$

the last equality holds since the random variable $(X_0, \dots, X_n)$ is independent of ξ_n, hence of $f(i, \xi_n)$. Since the right hand side of (5.18) is independent of the values of $X_{n-1}, \dots, X_0$, we conclude that

$$\mathbb{P}\left(X_{n+1} = j \,\Big|\, X_n = i,\ \dots,\ X_0 = i_0\right) = \mathbb{P}\left(X_{n+1} = j \,\Big|\, X_n = i\right),$$

i.e., $\{X_n\}$ is a Markov chain. The transition matrix is then $\mathbf{P} = (\mathbf{P}^i_j)$, $\mathbf{P}^i_j := \mathbb{P}(f(i, \xi_n) = j)$ and is independent of n since the random variables ξ_n are identically distributed.

Theorem 5.14 *Let S be a finite or denumerable set, and let $\mathbf{P}$ be a stochastic matrix. Let $X_0 : \Omega \to S$ be a random variable on $(\Omega, \mathcal{E}, \mathbb{P})$ and let $\{\xi_n\}$, $\xi_n : \Omega \to [0, 1]$ be a sequence of independent, uniformly distributed random variables that are also independent of X_0. Define*

$$f(i, s) := \min\left\{ j \,\Big|\, \sum_{h=1}^{j} \mathbf{P}^i_h \geq s \right\} \qquad \forall i > 0,\ s \in \mathbb{R}.$$

Then the sequence $\{X_n\}$, $X_n : \Omega \to S$ defined by

$$X_{n+1}(x) = f(X_n(x), \xi_n(x)), \quad x \in \Omega,\ n \geq 0,$$

is a homogeneous Markov chain with state-space S and transition matrix $\mathbf{P}$.

Proof. By Theorem 5.13, the sequence $\{X_n\}$ is a Markov chain. Moreover, $f(i, \xi_n(x)) = j$ if and only if

$$\sum_{h=1}^{j-1} \mathbf{P}^i_h < \xi_n(x) \leq \sum_{h=1}^{j} \mathbf{P}^i_h$$

hence

$$\mathbb{P}(f(i, \xi_n(x)) = j) = \sum_{h=1}^{j} \mathbf{P}^i_h - \sum_{h=1}^{j-1} \mathbf{P}^i_h = \mathbf{P}^i_j.$$

The previous theorem constructs in fact a homogeneous Markov chain $\{X_n\}$ with a given transition matrix and a given initial data X_0. In particular, Theorem 5.14 reduces the problem of the existence of a Markov chain with a given transition matrix $\mathbf{P}$ on a probability space $(\Omega, \mathcal{E}, \mathbb{P})$ and a given initial data X_0 to the existence of a sequence of independent random variables that are uniformly distributed on $[0, 1]$, see Section 2.2.5 and Section 4.4.4.