An important and less trivial consequence of the rank formula is the following.

1.28 Theorem (Rank of the transpose). *Let* $\mathbf{A} \in M_{m,n}$. *Then we have*

(i) *the maximum number of linearly independent columns and the maximum number of linearly independent rows are equal, i.e.,*

$$\operatorname{Rank}\mathbf{A} = \operatorname{Rank}\mathbf{A}^T,$$

(ii) *let* $p := \operatorname{Rank}\mathbf{A}$. *Then there exists a nonsingular* $p \times p$ *square submatrix of* $\mathbf{A}$.

Proof. (i) Let $\mathbf{A} = [a^i_j]$, let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ be the columns of $\mathbf{A}$ and let $p := \operatorname{Rank}\mathbf{A}$. We assume without loss of generality that the first p columns of $\mathbf{A}$ are linearly independent and we define $\mathbf{B}$ as the $m \times p$ submatrix formed by these columns, $\mathbf{B} := [\mathbf{a}_1 \,|\, \mathbf{a}_2 \,|\, \dots \,|\, \mathbf{a}_p]$. Since the remaining columns of $\mathbf{A}$ depend linearly on the columns of $\mathbf{B}$, we have

$$a^k_j = \sum_{i=1}^{p} r^i_j a^k_i \qquad \forall k = 1, \dots, m,\ \forall j = p+1, \dots, n$$

for some $\mathbf{R} = [r^i_j] \in M_{p,n-p}(\mathbb{K})$. In terms of matrices,

$$\Big[\mathbf{a}_{p+1} \Big| \mathbf{a}_{p+2} \Big| \dots \Big| \mathbf{a}_n\Big] = \Big[\mathbf{a}_1 \Big| \dots \Big| \mathbf{a}_p\Big]\mathbf{R} = \mathbf{B}\mathbf{R},$$

hence

$$\mathbf{A} = \Big[\mathbf{B} \Big| \mathbf{B}\mathbf{R}\Big] = \mathbf{B}\Big[\operatorname{Id}_p \Big| \mathbf{R}\Big].$$

Taking the transposes, we have $\mathbf{A}^T \in M_{n,m}(\mathbb{K})$, $\mathbf{B}^T \in M_{p,m}(\mathbb{K})$ and

$$\mathbf{A}^T = \begin{pmatrix} \boxed{\operatorname{Id}_p} \\ \boxed{\mathbf{R}^T} \end{pmatrix} \mathbf{B}^T. \tag{1.6}$$

Since $[\operatorname{Id}_p \,|\, \mathbf{R}]^T$ is trivially injective, we infer that $\ker \mathbf{A}^T = \ker \mathbf{B}^T$, hence by the rank formula

$$\operatorname{Rank}\mathbf{A}^T = m - \dim\ker\mathbf{A}^T = m - \dim\ker\mathbf{B}^T = \operatorname{Rank}\mathbf{B}^T,$$

and we conclude that

$$\operatorname{Rank}\mathbf{A}^T = \operatorname{Rank}\mathbf{B}^T \le \min(m,p) = p = \operatorname{Rank}\mathbf{A}.$$

Finally, by applying the above to the matrix $\mathbf{A}^T$, we get the opposite inequality $\operatorname{Rank}\mathbf{A} = \operatorname{Rank}(\mathbf{A}^T)^T \le \operatorname{Rank}\mathbf{A}^T$, hence the conclusion.

(ii) With the previous notation, we have $\operatorname{Rank}\mathbf{B}^T = \operatorname{Rank}\mathbf{B} = p$. Thus $\mathbf{B}$ has a set of p independent rows. The submatrix $\mathbf{S}$ of $\mathbf{B}$ made by these rows is a square $p \times p$ matrix with $\operatorname{Rank}\mathbf{S} = \operatorname{Rank}\mathbf{S}^T = p$, hence nonsingular. □

1.29 ¶. Let $\mathbf{A} \in M_{m,n}(\mathbb{K})$, let $A(\mathbf{x}) := \mathbf{A}\mathbf{x}$ and let $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ be a basis of $\mathbb{K}^n$. Show the following:

(i) A is injective if and only if the vectors $A(\mathbf{v}_1), A(\mathbf{v}_2), \dots, A(\mathbf{v}_n)$ of $\mathbb{K}^m$ are linearly independent,

(ii) A is surjective if and only if $\{A(\mathbf{v}_1), A(\mathbf{v}_2), \dots, A(\mathbf{v}_n)\}$ spans $\mathbb{K}^m$,

(iii) A is bijective iff $\{A(\mathbf{v}_1), A(\mathbf{v}_2), \dots, A(\mathbf{v}_n)\}$ is a basis of $\mathbb{K}^m$.