A connection between Lorentzian distance and mechanical least action

Ettore Minguzzi

Università Degli Studi Di Firenze

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Plato’s allegory of the cave

[...] they see only their own shadows, or the shadows of one another, which the fire throws on the opposite wall of the cave [...]. To them, I said, the truth would be literally nothing but the shadows of the images.

Plato, The allegory of the cave, Book VII of the Republic
Useful for clarifying

- the connection between relativistic and non-relativistic (classical) physics,
- the causality properties of gravitational waves,
- the problem of existence of solutions to the Hamilton-Jacobi equation and other problems in Lagrangian mechanics.

These apparently independent problems are strongly related as the study of lightlike dimensional reduction proves.
Let $E = T \times Q$, $T = \mathbb{R}$, be a classical $d + 1$-dimensional extended configuration space of coordinates $(t, q)$. Let $a_t$ be a positive definite time dependent metric on $S$, and $b_t$ a time dependent 1-form field on $S$. Let $V(t, q)$ be a time dependent scalar field on $S$.

In 1929 Eisenhart pointed out that the trajectories of a Lagrangian problem

$$L(t, q, \dot{q}) = \frac{1}{2} a_t(\dot{q}, \dot{q}) + b_t(\dot{q}) - V(t, q),$$

$$\delta \int_{t_0}^{t_1} L(t, q, \dot{q}) dt = 0, \quad q(t_0) = q_0, \quad q(t_1) = q_1$$

may be obtained as the projection of the spacelike geodesics of a $d + 2$-dimensional manifold $M = E \times Y$, $Y = \mathbb{R}$, of metric

$$ds^2 = a_t - dt \otimes (dy - b_t) - (dy - b_t) \otimes dt - 2V dt^2,$$

The considered Lagrangian is the most general which comes from Newtonian mechanics by considering holonomic constraints.

Eisenhart had in mind Jacobi’s metric $(E - V)a$, and Jacobi’s action principle which holds for time independent Lagrangians and $b = 0$. 
The Eisenhart metric takes its simplest and most symmetric form in the case of a free particle in Euclidean space $a_{bc} = \delta_{bc}, \ b_c = 0, \ V = \text{const.}$ Remarkably, in this case the Eisenhart metric becomes the Minkowski metric.

The Eisenhart metric is Lorentzian but Eisenhart did not give to this fact a particular meaning.
The vector field $n = \partial/\partial y$ is covariantly constant and lightlike.

It is better to proceed in steps starting from $(M, g)$

- Assume $n$ is Killing and lightlike in such a way that the quotient projection $\pi : M \rightarrow E$ gives a principal bundle over $\mathbb{R}$.
- $n$ is twist-free, $n \wedge dn = 0$, $\Rightarrow$ $E$ foliates into simultaneity slices, $n = -\psi dt$.
- $n$ is covariantly constant $\Rightarrow$ the foliation takes a natural ‘time parameter’ - the classical time $n = -dt$.
- Connection on principal bundles $\pi_t : N_t \rightarrow Q_t \Rightarrow$ Newtonian flow on $E$.
- Curvature $\Rightarrow$ Coriolis forces.
- If the curvature $\Omega_t$ vanishes the observers are non-rotating.

The Brinkmann-Eisenhart metric

$$ds^2 = a_t - dt \otimes (dy - b_t) - (dy - b_t) \otimes dt - 2Vdt^2,$$

on $M = E \times Y$, $Y = \mathbb{R}$, describes the most general spacetime with a covariantly constant lightlike field (Bargmann structure, generalized wave metric). Brinkmann proved this result locally. The space metric $a$ and 1-form field $b$ are fixed only if the coordinate system is fixed and this is done by choosing a Newtonian flow on $E$ which defines the space $Q$. 
A figure
Let $n$ be a lightlike Killing vector field on the spacetime $(M, g)$. In any spacetime dimension $n \wedge d n = 0$ if and only if $R_{\mu \nu} n^\mu n^\nu = 0$.

Define the *Newtonian flow* as a vector field $v$ on $E$ such that $d t[v] = 1$. The Newtonian flows and the connections $\omega_t$ on the bundles $\pi_t : N_t \rightarrow Q_t$ are in one-to-one relation through the formula $\omega_t(\cdot) = -g(\cdot, V)|_{N_t}$.

The metric $a_t$ on the space sections $Q_t$ is defined by $a_t(w, v) = g(W, V)$.

Given a 1-parameter family of sections $\sigma_t : Q \rightarrow N_t$ of the 1-parameter family of principal bundles $\pi_t : N_t \rightarrow Q_t$, the potential $b_t$ reads, $b_t = -\sigma_t^* \omega_t$. 
Assume from now on that $M = T \times Q \times Y$ with $T \simeq Y \simeq \mathbb{R}$ is given the metric

$$ds^2 = a_t - dt \otimes (dy - b_t) - (dy - b_t) \otimes dt - 2V dt^2.$$  

Every $C^1$ curve $(t, q(t))$ on $E = T \times Q$ is the projection of a lightlike curve on $(M, g)$, $\gamma(t) = (t, q(t), y(t))$ where

$$y(t) = y_0 + \int_{t_0}^{t} \left[ \frac{1}{2} a_t(\dot{q}, \dot{q}) + b_t(\dot{q}) - V(t, q) \right] dt = y_0 + S_{e_0, e(t)}[q|_{0,t}].$$

this result follows from

$$g(\dot{\gamma}, \dot{\gamma}) = a_t(\dot{q}, \dot{q}) - 2(\dot{y} - b_t[\dot{q}]) - 2V = 2(L - \dot{y}),$$

Conversely, every lightlike curve on $(M, g)$ with tangent vectors nowhere proportional to $n = \partial / \partial y$ projects on a $C^1$ curve on $E$. Also, given a timelike curve $\gamma$ we have necessarily

$$y(t) > y_0 + S_{e_0, e(t)}[q|_{0,t}].$$
A connection between Lorentzian distance...
The light lift III

Proposition

Every geodesic on \((M, g)\) not coincident with a flow line of \(n\) admits the function \(t\) as affine parameter and once so parametrized projects on a solution to the E-L equations. The light lift of a solution to the E-L equation is a lightlike geodesic.

It is based on

\[
\mathcal{I}[^\eta] = \frac{1}{2} \int_{\lambda_0}^{\lambda_1} g(\eta', \eta') \, d\lambda = \frac{1}{2} \int_{t_0}^{t_1} g(\dot{\eta}, \dot{\eta})(t') \, dt \\
= \int_{t_0}^{t_1} [L(t, q(t), \dot{q}(t)) - \dot{y}] (t') \, dt.
\]
The least action also called Hamilton’s principal function is $S : E \times E \rightarrow [-\infty, +\infty]$ given by

$$S(e_0, e_1) = \inf_{q \in C_{e_0, e_1}^1} S_{e_0, e_1}[q], \text{ for } t_0 < t_1,$$

$$S(e_0, e_1) = 0, \text{ for } t_0 = t_1 \text{ and } q_0 = q_1,$$

$$S(e_0, e_1) = +\infty, \text{ elsewhere.}$$
Proposition

Let \( x_0 = (e_0, y_0) \in M \), it holds

\[
I^+(x_0) = \{x_1 : y_1 - y_0 > S(e_0, e_1) \text{ and } t_0 < t_1\},
\]

\[
J^+(x_0) \subset \{x_1 : y_1 - y_0 \geq S(e_0, e_1)\},
\]

\[
E^+(x_0) \subset r_{x_0} \cup \{x_1 : y_1 - y_0 = S(e_0, e_1)\}.
\]

Analogous past versions hold.
The statement

Let \( x_1 \in J^+(x_0) \) and \( t_0 < t_1 \), thus in particular \( y_1 - y_0 \geq S(e_0, e_1) \). Let \( e(t) = (t, q(t)) \) be a \( C^1 \) curve which is the projection of some \( C^1 \) causal curve connecting \( x_0 \) to \( x_1 \) then \( y_1 - y_0 \geq S_{e_0,e_1}[q] \). Among all the \( C^1 \) causal curves \( x(t) = (t, q(t), y(t)) \), connecting \( x_0 \) to \( x_1 \), which project on \( e(t) \), the causal curve \( \gamma(t) = (t, q(t), y(t)) \) with

\[
y(t) = y_0 + S_{e_0,e(t)}[q|_{t_0,t}] + \frac{t-t_0}{t_1-t_0} (y_1 - y_0 - S_{e_0,e_1}[q])
\]

(1)

is the one and the only one that maximizes the Lorentzian length. The maximum is

\[
l(\gamma) = \{2(y_1 - y_0 - S_{e_0,e_1}[q])(t_1 - t_0)\}^{1/2}.
\]

(2)
The proof

Let \( \eta(t) = (t, q(t), w(t)) \) be a \( C^1 \) causal curve connecting \( x_0 \) to \( x_1 \), then since it is causal by Eq. \(-g(\dot{\eta}, \dot{\eta}) = 2(\dot{w} - L), \dot{w} \geq L \) and integrating \( y_1 - y_0 \geq S_{e_0,e_1} [q] \).

The curve \( \gamma \) is causal because (use Eq. \(-g(\dot{\gamma}, \dot{\gamma}) = 2(\dot{y} - L)\))

\[
-g(\dot{\gamma}, \dot{\gamma}) = \frac{2}{t_1 - t_0} (y_1 - y_0 - S_{e_0,e_1} [q]) \geq 0,
\]

taking the square root and integrating one gets

\[
l(\gamma) = \{2(y_1 - y_0 - S_{e_0,e_1} [q])(t_1 - t_0)\}^{1/2}.
\]

If \( \tilde{\gamma} = (t, q(t), \tilde{y}(t)) \) is another \( C^1 \) timelike curve connecting \( x_0 \) to \( x_1 \) and projecting on \( e(t) \)

\[
-g(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}) = \frac{2}{t_1 - t_0} (y_1 - y_0 - S_{e_0,e_1} [q]) = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} [-g(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}})] dt.
\]

Using the Cauchy-Schwartz inequality \( \int_{t_0}^{t_1} [-g(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}})] dt \geq (t_1 - t_0)^{-1} l(\tilde{\gamma})^2 \)

remplacig in the above equation, taking the square root and integrating

\( l(\gamma) \geq l(\tilde{\gamma}) \), thus \( \gamma \) is longer than \( \tilde{\gamma} \). In order to prove the uniqueness note that the equality sign in \( l(\gamma) \geq l(\tilde{\gamma}) \) holds iff it holds in the Cauchy-Schwarz inequality which is the case iff \( g(\dot{\gamma}, \dot{\gamma}) = const. \), that is iff \( \dot{y} - L = const. \) which integrated, once used the suitable boundary conditions, gives Eq. (1).
Corollary

Let $x_0, x_1 \in M$, $x_0 = (e_0, y_0)$, $x_1 = (e_1, y_1)$ then if $x_1 \in J^+(x_0)$,

$$d(x_0, x_1) = \sqrt{2[y_1 - y_0 - S(e_0, e_1)](t_1 - t_0)}.$$  \hfill (3)

In particular, $S(e_0, e_1) = -\infty$ iff $d(x_0, x_1) = +\infty$.

The triangle inequality

The function $S$ is upper semi-continuous everywhere but on the diagonal of $E \times E$ and satisfies the triangle inequality: for every $e_0, e_1, e_2 \in E$

$$S(e_0, e_2) \leq S(e_0, e_1) + S(e_1, e_2),$$

with the convention that $(+\infty) + (-\infty) = +\infty$. 
Relation between the triangle inequalities

What is the relation between the reverse triangle inequality satisfied by $d$ and the usual triangle inequality satisfied by $S$?

An abstract framework

Suppose on $X$ you are given a function $s : X \to (-\infty, +\infty]$ such that $s(x, x) = 0$. On the cartesian product $X \times \mathbb{R}$ define the relation

$$(x, a) \leq (y, b) \quad \text{if} \quad b - a \geq s(x, y)$$

Assume furthermore that $\leq$ is a total preorder on $X$ and that $t : X \to \mathbb{R}$ is an utility function, i.e. $x \leq y \iff t(x) \leq t(y)$. Finally, let $x \not\leq y \Rightarrow s(x, y) = +\infty$ be the compatibility condition of the total preorder with $s$. Define $d : (X \times \mathbb{R})^2 \to [0, +\infty]$ by

$$d((x, a), (y, b)) = \sqrt{2[b - a - s(x, y)](t(y) - t(x))} \quad (4)$$

if $(x, a) \leq (y, b)$ and 0 otherwise. Given $x_1, x_2, x_3 \in X$, the reverse triangle inequality

$$d((x_1, a_1), (x_2, a_2)) + d((x_2, a_2), (x_3, a_3)) \leq d((x_1, a_1), (x_3, a_3)) \quad (5)$$

holds for every triple $(x_1, a_1) \leq (x_2, a_2) \leq (x_3, a_3)$ if and only if for all $x, y, z \in X$, $s(x, z) \leq s(x, y) + s(y, z)$. 
The causal ladder of spacetimes

Global hyperbolicity
↓
Causal simplicity
↓
Causal continuity
↓
Stable causality
↓
Strong causality
↓
Distinction
↓
Causality
↓
Chronology

How does it change for our case? Can this properties be related with properties of the Lagrangian problem and in particular of the least action $S$?
Why should we expect a connection with the least action?

Because some causality properties can be expressed in terms of the Lorentzian distance $d$

**Global hyperbolicity**

A strongly causal spacetime is globally hyperbolic if and only if whatever the chosen conformal factor $d$ is finite.
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**Global hyperbolicity**
A strongly causal spacetime is globally hyperbolic if and only if whatever the chosen conformal factor $d$ is finite.

**Global hyperbolicity II**
A strongly causal spacetime is globally hyperbolic if and only if whatever the chosen conformal factor $d$ is continuous.
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**Global hyperbolicity**
A strongly causal spacetime is globally hyperbolic if and only if whatever the chosen conformal factor $d$ is finite.

**Global hyperbolicity II**
A strongly causal spacetime is globally hyperbolic if and only if whatever the chosen conformal factor $d$ is continuous.

**Causal simplicity**
A strongly causal spacetime is causally simple if and only if whatever the chosen conformal factor $d$ is continuous wherever it vanishes.

and $d$ and $S$ are related...
The answer

Global hyperbolicity:
(i) \( S(e_0, e_1), t_0 < t_1, \) is finite,
(ii) \( S_{e_0, e_1}, t_0 < t_1, \) is coercive.

Causal simplicity:
(a) \( S \) attains its infimum \( S \) whenever \( S \) is finite,
(b) \( S \) is lower semi-continuous.

Causal continuity:
\[
\liminf_{e \to e_1} S(e_0, e) = \liminf_{e \to e_0} S(e, e_1),
\]
and this quantity vanishes for \( e_0 = e_1 \).
The answer II

\[ \downarrow \]

Stable/Strong causality:

\[ S \] is lower semi-continuous on the diagonal.

\[ \downarrow \]

Distinction:

\[ \liminf_{e \to \tilde{e}} S(\tilde{e}, e) = \liminf_{e \to \tilde{e}} S(e, \tilde{e}) = 0. \]

\( (M, g) \) is always causal.
Locally the Lorentzian distance $d(x_0, x)$ satisfies the eikonal equation

$$g(\nabla d, \nabla d) + 1 = 0,$$

while $S(e_0, e)$ satisfies the Hamilton-Jacobi equation. They are related because, using the relation between $S$ and $d$

$$g(\nabla d, \nabla d) + 1 = \frac{2(t - t_0)^2}{d^2} \left[ \frac{\partial S}{\partial t} + \frac{1}{2} a_t^{-1}(dS - b_t, dS - b_t) + V \right].$$
The function $u : E \to \mathbb{R}$ is a viscosity solution of the Hamilton-Jacobi equation if

- It is a \textit{viscosity subsolution}: for every $(t, q) \in E$ there is a $C^1$ function $\varphi : E \to \mathbb{R}$ such that $u - \varphi$ has a local maximum at $e$ and at $e$

  \[ \partial_t \varphi + H(t, q, D_q \varphi) \leq 0. \]

- It is a \textit{viscosity supersolution}: for every $(t, q) \in E$ there is a $C^1$ function $\varphi : E \to \mathbb{R}$ such that $u - \varphi$ has a local minimum at $e$ and at $e$

  \[ \partial_t \varphi + H(t, q, D_q \varphi) \geq 0. \]
Viscosity solutions and the causal future

The slices of the causal future of the initial condition give a viscosity solution.

Smoothness properties follow from theorems in Lorentzian geometry.
This solution is that of the Lax-Oleinik semigroup

\[ u(t, q) = \inf_{q_0 \in Q, \alpha \in C^1} \left\{ u(t_0, q_0) + \int_{t_0}^{t} L(t, \alpha, \dot{\alpha}) dt \right\} \]
Conclusions

- The study of spacetimes admitting a parallel null vector is tightly related with the study of Lagrangian mechanical systems.
- In this framework there is a simple relation between the Lorentzian distance and the least action.
- The causality properties of the spacetime are connected with lower semi-continuity properties of the least action.
- Tonelli’s theorem on the existence of minimizers is basically the statement that global hyperbolicity implies causal simplicity.
- The Hamilton-Jacobi equation and the Lax-Oleinik semigroup are nothing but the causal relation in disguise.