Spacelike slices from globally well-behaved simultaneity connections

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References:

- *Class. Quantum Grav.* **21**, 4123 (2004) gr-qc/0405139
- *Class. Quantum Grav.* **20**, 2443 (2003) gr-qc/0204063
- (with A. Macdonald) *Found. Phys. Lett.* **16**, 587 (2003) gr-qc/0211091

The GPS

- It aims at reproducing in practice the variable t of

$$
\begin{aligned}
ds^2 \;=\;& (1+2(\Phi-\Phi_0))\mathrm{d}t^2-(1-2V)[\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)] \\
&-2\omega_E r^2\sin^2\theta\,\mathrm{d}\phi\mathrm{d}t.
\end{aligned}
$$

  where V is the gravitational potential, Φ is the gravitational plus the centrifugal potential, and Φ_0 is the value of Φ at the surface of the Earth geoid.

- It uses a set of 24 satellites in orbit in such a way that each point on the Earth *sees* at least 4 satellites. Each satellite sends a signal with the information of its position and local time. From that information the pair $\{\mathbf{r},\mathbf{t}\}$ of the receiver is deduced (in Minkowski)

$$
c^2(t-t_j)^2=(\mathbf{r}-\mathbf{r_j})^{\mathbf{2}} \qquad \mathbf{j=1,2,3,4}
$$

The GPS II

- Common view: if the satellite sees two distant clocks on the Earth they can be synchronized with a signal from it
- The GPS takes into account the effect of time dilation
- The GPS assumes the existence of an Earth Centered Inertial(ECI) frame. It means that it is assumed that the metric is static, and in the ECI frame the Einstein convention is transitive (this assumption is not entirely correct as the Kerr metric tells us)
- The GPS relies on the model of the Earth geoid and derived corrections

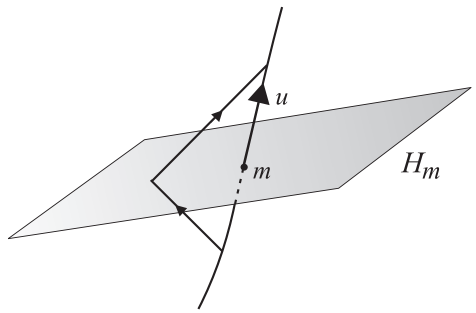
The frame concept in general relativity

As SR tells us the concept of simultaneity changes from inertial frame to inertial frame. What is a frame in general relativity?

- Frame: a congruence of timelike curves each defining the motion of a point of space. In other words the space S is defined only if the motion of the space points is assigned on spacetime. The space then is the manifold of curves.
- There is a natural projection $\pi : M \to S$ that assigns to each event $m \in M$ the worldline(i.e. the space point) $s(m)$ passing through it
- The frame has the local structure of a fibre bundle where the bundle has the fibre $\mathbb{R}$. The fibre of a space point is its worldline

Einstein's simultaneity as a connection

- The Einstein simultaneity convention is related to orthogonality in Minkowski spacetime. Locally the same holds in general relativity
- The planes orthogonal to the congruence of timelike curves of the frame define the Einstein simultaneity for that frame
- A distribution of spacelike planes defines a generalized connection $P^{\mu}_{\nu} = u^{\mu} u_{\nu}$ on the local fibre bundle $\pi : M \to S$



Local simultaneity connection: general expression

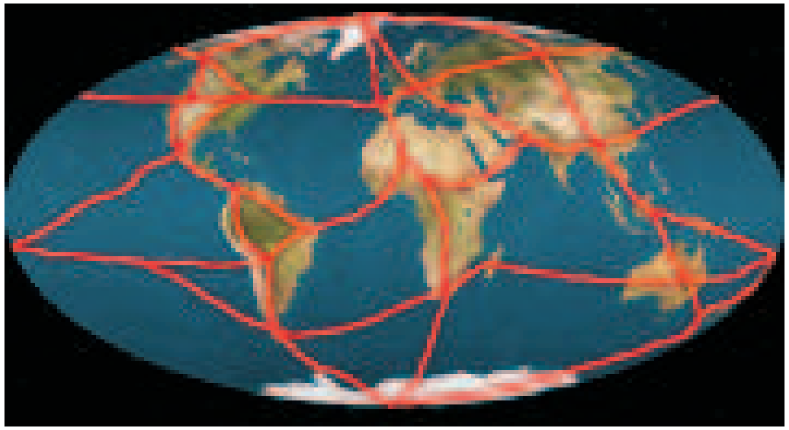
$$\omega_\alpha = u_\alpha + \psi^m(x) m_\alpha + \psi^a(x) a_\alpha + \psi^w(x) w_\alpha, \qquad m_\alpha = w_{\alpha\beta} a^\beta \neq 0$$

where ψ^m, ψ^a and ψ^w depends on x only through the dependence on the vorticity $w_{\alpha\beta}$, the acceleration a_α, the shear tensor $\sigma_{\alpha\beta}$, the expansion scalar θ and the Riemann tensor $R_{\alpha\beta\gamma\delta}$.

- Why are local simultaneity conventions important in practice?
- Are there integrable local simultaneity conventions?

Local simultaneity connection: the practical relevance

- If we find a local simultaneity convention that is integrable then we have solved the operative problem of determining a spacelike foliation in a way that, being local, does not privilege any space point
- It applies coherently to a Network of observers.
- It works in whatever spacetime, and does not rely on a supposed spacetime geometry



Are there integrable local simultaneity conventions?

We shall provide a partial answer. Our hypothesis

- The frame is generated by a timelike Killing vector field (thus the spacetime is stationary)
- The curvature of the connection does not vanish but it is proportional to the Riemann tensor (integrable in the weak field limit)
- We look only for connections independent of the Riemann tensor. The dependence on the shear $\sigma_{\alpha\beta}$ and the expansion θ is removed by the hypothesis of stationarity.

Thus our connection ω has functions ψ^m, ψ^a and ψ^w that depend only on the 4-velocity u_α, acceleration a_μ, angular velocity w_μ and angle between them θ.

A useful concept: C-Killing time

Let k be a Killing vector field and let x^i be coordinates on $S = M/T_1$. Let C be an integrable local simultaneity convention defined by a 1-form field ω.
The C-Killing time is that time function t such that $\mathrm{d}t[k] = 1$, $\mathrm{d}t \propto \omega$. In other words

- C-Killing time is that coordinate time whose rate is defined by k, and whose simultaneity is defined by ω

Let $\chi = \sqrt{k \cdot k}$ and define the 1-form field

$$\sigma = \chi^{-1}\omega.$$

If ω is integrable $\sigma = \mathrm{d}t$ with t C-Killing time.

Our strategy: Curvature proportional to the Riemann tensor

The curvature v^α of ω_α is proportional to the Riemann tensor $v^\alpha \propto R^\gamma{}_{\delta\eta\mu\nu}$. In the weak field limit the distribution of horizontal planes is integrable. The holonomy is proportional to the area enclosed by a closed curve and the curvature. Over a length scale L it is given by

$$\Delta \sim L^2 \mathcal{R} T,$$

Compare with the Einstein convention

$$\Delta \sim L^2 w.$$

In the weak field limit $\mathcal{R} \to 0$, our convention is much more well behaved than Einstein's. The lack of integrability here is caused by the spacetime geoemetry (Riemann tensor) and not by the kinematical properties of the reference frame (vorticity).

The theorem

Theorem. Let U be the open set $U = \{x : m(x) > 0$ and $a(x) \neq w(x)\}$. Consider in U the 1-form

$$\omega_\alpha = u_\alpha + \psi^m(x) m_\alpha + \psi^a(x) a_\alpha + \psi^w(x) w_\alpha .$$

Let ψ^m, ψ^a, ψ^w, be C^1 functions dependent only on a, w and θ. Then, regardless of the stationary spacetime considered, the connection is timelike in U (and hence it is a simultaneity connection in U) and has a curvature proportional to the Riemann tensor in U if and only if

$$\psi^m = \bar{\psi}^m \equiv \frac{a^2 + w^2 - \sqrt{(a^2+w^2)^2 - 4a^2 w^2 \sin^2\theta}}{2a^2 w^2 \sin^2\theta},$$

and ψ^a and ψ^w obey a certain constraint (satisfied by the choice $\psi^a = \psi^w = 0$).

$\bar{C}$-simultaneity

The $\bar{C}$-simultaneity convention is the convention

$$\bar{\omega}_\alpha = u_\alpha + \frac{a^2 + w^2 - \sqrt{(a^2+w^2)^2 - 4m^2}}{2m^2}\, m_\alpha$$

its curvature is

$$\begin{aligned}\bar{v}^\eta \;&=\; \frac{\bar{\psi}^m}{2} h^\eta{}_\nu \varepsilon^{\nu\beta\alpha\gamma} u_\beta R_{\gamma\delta\mu\sigma}\{ \frac{m_\alpha}{\sqrt{(a^2+w^2)^2-4m^2}} (u^\delta w^{\mu\sigma} + 2u^\delta u^\mu a^\sigma) \\ &+ m_\alpha \frac{2\bar{\psi}^m}{\sqrt{(a^2+w^2)^2-4m^2}} (u^\delta m^\mu a^\sigma + u^\mu u^\delta w^\sigma{}_\tau m^\tau) + u^\mu u^\delta w_\alpha{}^\sigma - \delta^\mu_\alpha u^\delta a^\sigma \}\end{aligned}$$

It is the simplest simultaneity connection with a curvature proportional to the Riemann tensor

Flat spacetime: rotating platform

Flat metric in cylindric coordinates,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\rho^2 - \rho^2 \mathrm{d}\theta^2 - \mathrm{d}z^2,$$

after the change of coordinates, $\theta \to \theta + \omega t$, where $c = 1$ and ω is a constant

$$\mathrm{d}s^2 = (1 - \rho^2\omega^2)\mathrm{d}t^2 - \mathrm{d}\rho^2 - \rho^2 \mathrm{d}\theta^2 - 2\rho^2\omega \mathrm{d}\theta \mathrm{d}t - \mathrm{d}z^2,$$

Worldlines of equation $x^i = const.$ belong to the congruence of timelike curves, and ∂_t is the timelike Killing vector field. We shall limit ourselves to the open set $\rho\omega < 1$, since in this set the Killing vector field is timelike.

Flat spacetime: rotating platform II

$$
\begin{aligned}
a_\alpha \mathrm{d}x^\alpha &= \frac{\omega^2 \rho}{1-\rho^2\omega^2}\mathrm{d}\rho, \\
w_{\alpha\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta &= \frac{\omega\rho}{(1-\rho^2\omega^2)^{3/2}}\mathrm{d}\rho \wedge \mathrm{d}\theta, \\
m_\alpha \mathrm{d}x^\alpha &= \frac{\omega^3\rho^2}{(1-\rho^2\omega^2)^{5/2}}\mathrm{d}\theta,
\end{aligned}
$$

and $\bar{\psi}^m = \frac{a^2}{m^2}$. Using these formulas we obtain finally,

$$
\bar{\sigma} = \chi^{-1}\bar{\omega}_\alpha \mathrm{d}x^\alpha = \chi^{-1}(u_\alpha \mathrm{d}x^\alpha + \bar{\psi}^m m_\alpha \mathrm{d}x^\alpha) = \mathrm{d}t,
$$

Thus $\bar{C}$-simultaneity coincides with Einstein simultaneity of the inertial frame we started with and the $\bar{C}$-Killing time is t.

Flat spacetime: rotating platform III

- The observers in the rotating platform succeed in recovering the simplest simultaneity convention for the rotating platform, that is the simultaneity in the inertial frame, using the $\bar{C}$-simultaneity convention
- All this even if they ignore that actually they are over a rotating platform
- The procedure used to define a global time variable is entirely local

Flat spacetime: rotating pseudocylindrical coordinates

The Minkowski metric in these coordinates is

$$\mathrm{d}s^2 = (\xi^2 - r^2\Omega^2)\mathrm{d}\tau^2 - \mathrm{d}\xi^2 - \mathrm{d}r^2 - r^2\mathrm{d}\phi^2 - 2r^2\Omega\mathrm{d}\phi\mathrm{d}\tau, \quad (1)$$

and $k = \partial_\tau$. The parameter t was chosen by Letaw and Pfautsch from a criteria of simplicity of the metric element. We repeat the calculations and

- The parameter τ is the $\bar{C}$-Killing time
- The $\bar{C}$-simultaneity convention is a natural one. It follows solely from the metric and the Killing vector field, so it is expected to simplify the metric in a coordinate system adapted to k

Since Letaw and Pfautsch looked for a gauge that simplifies the metric it is not surprising that they chose the $\bar{C}$ gauge.

The Earth case: an open problem

- Unfortunately the Riemann tensor is too strong at the Earth surface, so that $\bar{C}$-simultaneity is not better than Einstein there
- However, ψ^a and ψ^w have not been constrained, maybe we can give them a functional form that makes the curvature even weaker
- In any case this method of synchronization will hardly replace the one based on GPS which in some sense specialized to the Earth case.
- Instead, local simultaneity convention should be regarded as the correct answer to define simultaneity in whatever spacetime

Conclusions: Advantages of local simultaneity conventions

We believe that our approach shares a number of features that makes it preferable over other approaches

- It is coordinate independent as we always use a transparent covariant notation. Moreover, it is local and operationally clear as quantities like the acceleration or the vorticity have a transparent operational meaning
- It does not rely on Einstein equations or their exact solutions indeed, contrary to other approaches, we have no need to assume a special form of the metric in order to apply a local simultaneity convention, and we do not need to develop a different method of synchronization for any different spacetime considered. Moreover, the coordinate time found is universal as the operational definition of C-Killing time makes it possible to consider C-Killing time in different spacetimes

Conclusions: Advantages of local simultaneity conventions II

- It does not rely on experimental models of the distribution of matter (the geoid in the Earth case). This is very important since classical, relativistic or general relativistic effects are already taken into account in the local simultaneity convention. We do not have to change the method of synchronization if we discover, for instance, that the metric is not the one which was supposed to be. Once we have proved theoretically that a local simultaneity convention works, it works independently of the global information, like the metric, that we may or may not have
- Finally, it does not require particular spacetime symmetries apart from that of stationarity