Chronological null complete spacetimes admit a time function

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based on

K-causality coincides with stable causality - preprint 0809.1214
Chronological spacetimes without lightlike lines are stably causal - preprint 0806.0153
and
The causal ladder and the strength of K-causality: I, *Class. Quantum Grav.* **25** 015009 (2008)
The causal ladder and the strength of K-causality: II, *Class. Quantum Grav.* **25** 015010 (2008)

What is a time function?

Let (M, g) be a spacetime, that is time oriented Lorentzian manifold of signature $(-, +, \cdots, +)$.

- A *time function*, $t : M \to \mathbb{R}$, is a continuous function on spacetime (M, g) which increases on every future directed causal curve, that is: $x < y \Rightarrow t(x) < t(y)$.
- If it exists, it provides a (continuous) total ordering of the spacetime events which respects the notion of causal precedence.
- If it exists it is not unique, a function $\phi : M \to \mathbb{R}$ can be found so that $t' = t - \phi(x)$ is a new time function. Problem: how can we assign a time function for any congruence of timelike curves? This is a further problem which has to do with the definition of simultaneity (not treated here).
- A time function may not exist (for instance if there is a closed causal curve)

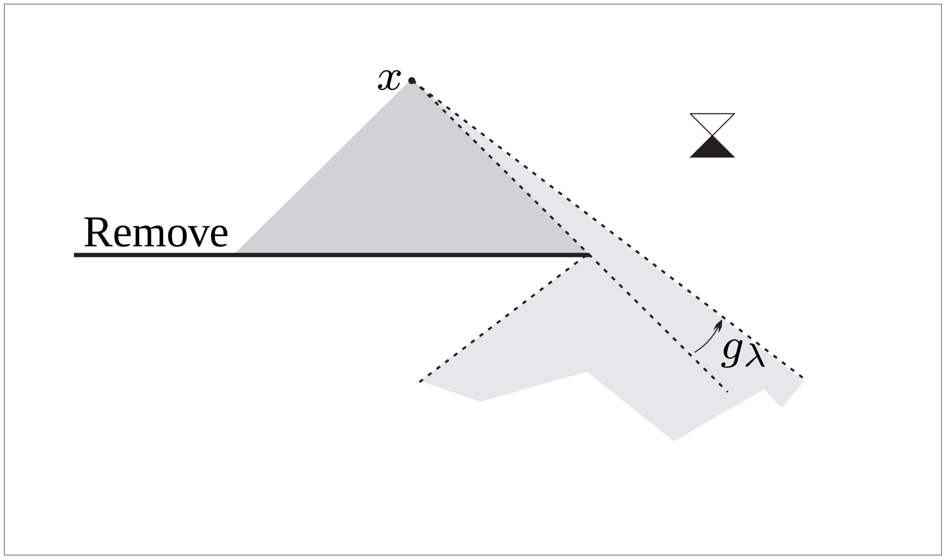
Stable causality $\Leftrightarrow$ time function

Hawking (1968) and Bernal and Sánchez (2005) prove

Stable causality $\Leftrightarrow$ time function $\Leftrightarrow$ smooth time function with timelike gradient

- Stable causality $\Rightarrow$ time function. Let $g_\lambda = (1-\frac{\lambda}{2}) + \frac{\lambda}{2}\tilde{g}$ with $\tilde{g} > g$, define

$$t(x) = \int_0^1 \mu(I^-_{(M,g_\lambda)}) \mathrm{d}\lambda$$



Stable causality $\Leftrightarrow$ Antisymmetry of J_S^+

Regard causal relations as subsets of $M \times M$, for instance

$$J^+ = \{(x,z) \in M \times M : x \le z\}$$

Seifert (1971) defines: $J_S^+ = \bigcap_{g'>g} J_{g'}^+$
It has these remarkable properties

1. It is closed: $\overline{J_S^+} = J_S^+$.
2. It is transitive: $(x,y) \in J_S^+$ and $(y,z) \in J_S^+ \Rightarrow (x,z) \in J_S^+$.
3. It contains J^+.

It can be proved that

Stable causality $\Leftrightarrow$ J_S^+ is antisymmetric, that is $(x,z) \in J_S^+$ and $(z,x) \in J_S^+ \Rightarrow x = z$.

Seifert (1971) claimed

- If there is a *smallest* partial order with properties 1,2 and 3 above, then it coincides with J_S^+,

but *the argument was incorrect*.

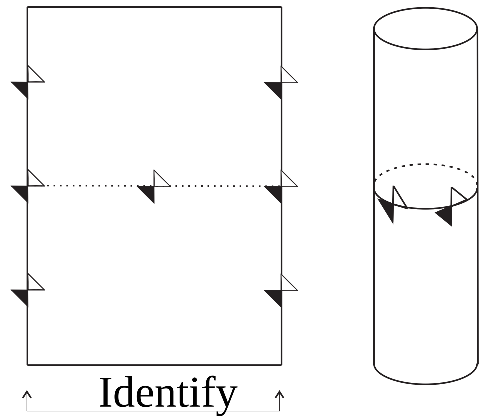
Sorkin and Woolgar (1996): definition of K-causality

- Define K^+ to be the smallest relation which satisfies 1,2 and 3.

It exists, just take the intersection (the set is empty i.e. $R^+ = M \times M$).

- Define a spacetime to be K-causal if K^+ is antisymmetric.

K^+ was originally conceived to relax differentiability conditions on the metric.
J_S^+ satisfies 1,2 and 3, thus $K^+ \subset J_S^+$ and “stable causality $\Rightarrow$ K-causality.”
Question: does K^+ coincide with J_S^+? NO



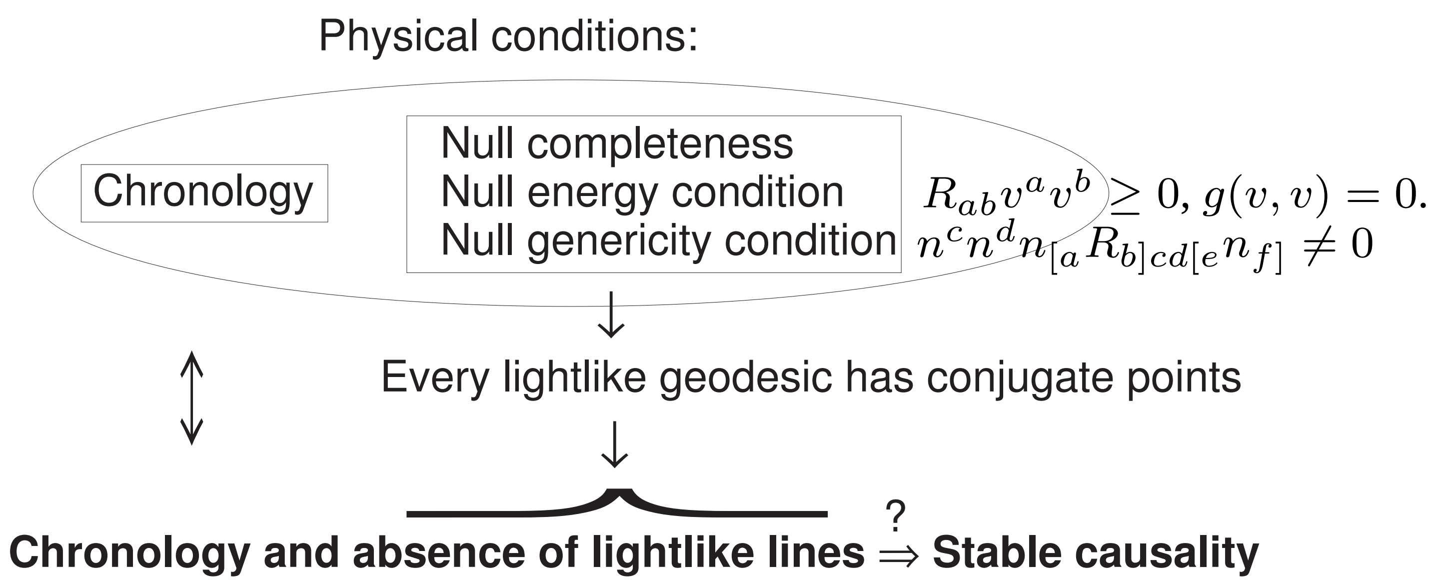
Here $J_S^+ = M \times M$, while $K^+ = J^+$.
Does K-causality coincide with stable causality? or more strongly
In a K-causal spacetime is $K^+ = J_S^+$? This is actually equivalent to Seifert’s original claim.

Problems

(a) Justify the existence of a time function on physical basis. Geroch and Horowitz (1979) identified this problem as one of the most important open problems concerning the global aspects of general relativity.

(b) K-causality coincides with stable causality? If so, in a K-causal (stably causal) spacetime, $K^+ = J_S^+$?

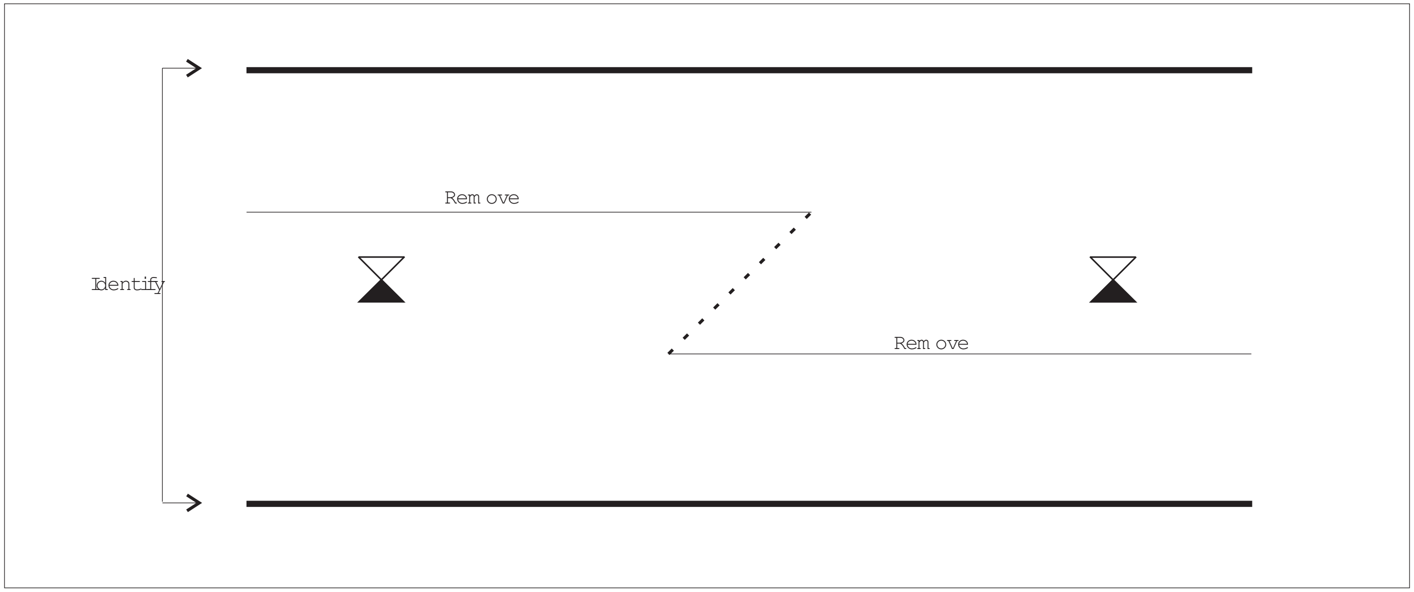
They are nicely related!
What does it mean *physical basis* in (a)? How to convert (a) in a mathematical statement?
Solution:



It relates only conformal invariant properties.

Examples

This implication can be checked with examples. All known chronological spacetime which are not stably causal have a lighlike line somewhere



Hawking (1966) proves

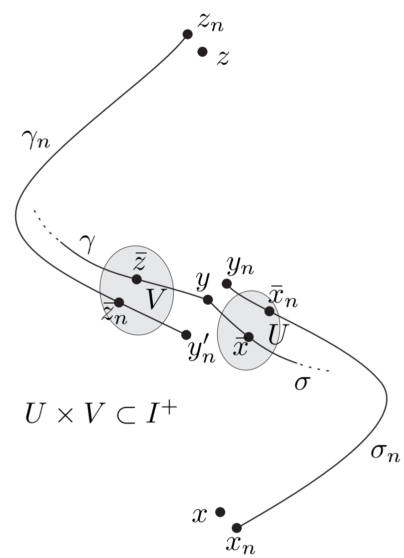
- A chronological spacetime without lightlike lines is strongly causal

But it is not enough, indeed there are infinite causality levels between strong causality and stable causality (Carter, Penrose).

Connection between the two problems and strategy

- If there are no lightlike lines then $\bar{J}^+$ is transitive.

Proof: take $(x, y) \in \bar{J}^+$ and $(y, z) \in \bar{J}^+$ and use a limit curve argument



Thus:

- If there are no lightlike lines then $K^+ = \bar{J}^+$.

It is easy to prove that under chronology the absence of lightlike lines implies that $\bar{J}^+$ is antisymmetric, then the spacetime is K-causal, and if K-causality and stable causality coincide, then it is stably causal and admits a time function.

Last and more complex step

- Yes, K-causality coincides with stable causality and in a K-causal spacetime, $K^+ = J_S^+$.

The proof is lengthy, but it is also possible to avoid it and give a shorter proof that *chronological spacetime without lightlike lines are stably causal*.
In any case it is necessary to pass through a new concept, that of

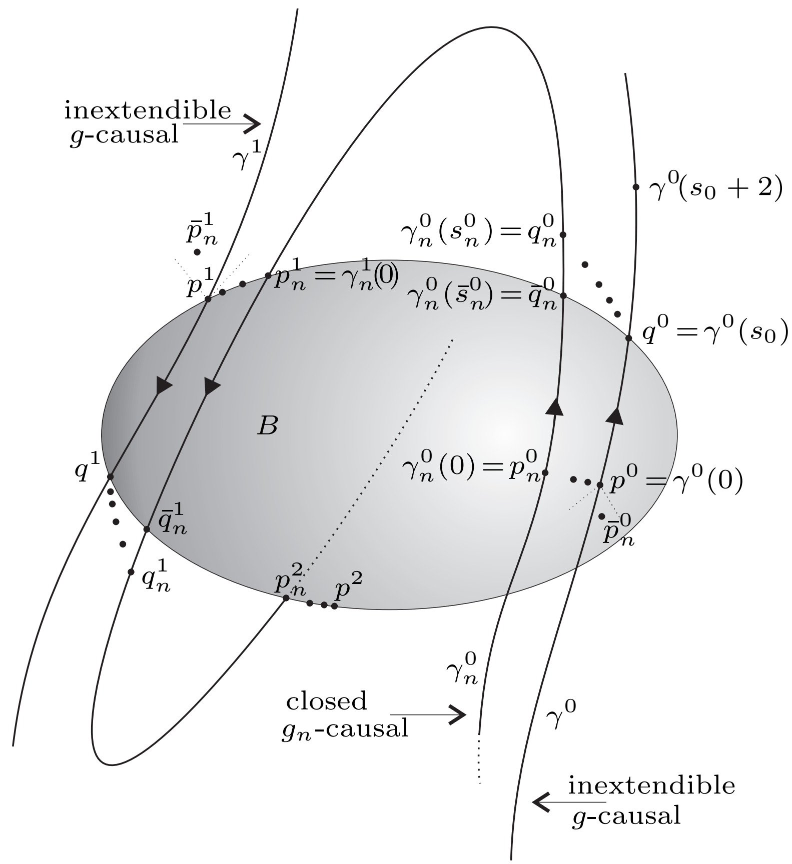
- Compact stable causality: A spacetime is compactly stably causal if whatever the compact the cones can be strictly widened on the compact without introducing closed causal curves.

The first step is then to prove that

- K-causality $\Rightarrow$ compact stable causality

Idea of the proof of K-causality $\Rightarrow$ compact stable causality

Assume compact stable causality does not hold, then there is a compact C such that for every enlarged metric $g_n > g$ on C there is a closed causal curve (necessarily passing through C), take $g_n \to g$...



Conclusions

- K-causality coincides with stable causality and in a K-causal spacetime, $K^+ = J_S^+$.
- Chronological spacetimes without lightlike lines are stably causal and hence admit a smooth time function with timelike gradient.
- Physically this result states that under reasonable conditions a time function exists.
- Note that the assumption of null completeness is compatible with all the singularity theorems that do not assume a priori the existence of a time function.
- The result can be regarded as a singularity theorem: If a spacetime does not admit a time function (and weak energy and null genericity conditions hold) then either chronology is violated or there is a singularity.

The causal ladder