On the existence of time in non-singular spacetimes

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based on

K-causality coincides with stable causality - preprint 0809.1214
Chronological spacetimes without lightlike lines are stably causal - preprint 0806.0153
and
The causal ladder and the strength of K-causality: I, *Class. Quantum Grav.* **25** 015009 (2008)
The causal ladder and the strength of K-causality: II, *Class. Quantum Grav.* **25** 015010 (2008)

What is a time function?

Let (M, g) be a spacetime, that is time oriented Lorentzian manifold of signature $(-, +, \cdots, +)$.

- A *time function*, $t : M \to \mathbb{R}$, is a continuous function on spacetime (M, g) which increases on every future directed causal curve, that is: $x < y \Rightarrow t(x) < t(y)$.
- If it exists, it provides a (continuous) total ordering of the spacetime events which respects the notion of causal precedence.
- If it exists it is not unique, a function $\phi : M \to \mathbb{R}$ can be found so that $t' = t - \phi(x)$ is a new time function.
- A time function may not exist (for instance if there is a closed causal curve)

time function $\Leftrightarrow$ Stable causality $\Leftrightarrow$ Antisymmetry of J_S^+

Hawking (1968) and Bernal and Sánchez (2005) prove

Stable causality $\Leftrightarrow$ time function $\Leftrightarrow$ smooth time function with timelike gradient

Regard causal relations as subsets of $M \times M$, for instance

$$J^+ = \{(x, z) \in M \times M : x \leq z\}$$

Seifert (1971) defines: $J_S^+ = \bigcap_{g' > g} J_{g'}^+$
It has these remarkable properties

1. It is closed: $\overline{J_S^+} = J_S^+$.
2. It is transitive: $(x, y) \in J_S^+$ and $(y, z) \in J_S^+ \Rightarrow (x, z) \in J_S^+$.
3. It contains J^+.

It can be proved that

Stable causality $\Leftrightarrow$ J_S^+ is antisymmetric, that is $(x, z) \in J_S^+$ and $(z, x) \in J_S^+ \Rightarrow x = z$.

Sorkin and Woolgar (1996): definition of K-causality

- Define K^+ to be the smallest relation which satisfies 1,2 and 3.

It exists, just take the intersection (the set is empty i.e. $R^+ = M \times M$).

- Define a spacetime to be K-causal if K^+ is antisymmetric.
- J_S^+ satisfies 1,2 and 3, thus $K^+ \subset J_S^+$ and "stable causality $\Rightarrow$ K-causality."

Question: does K^+ coincide with J_S^+? NO, there are simple counterexamples.
Does K-causality coincide with stable causality? or more strongly
In a K-causal spacetime is $K^+ = J_S^+$?

Problems

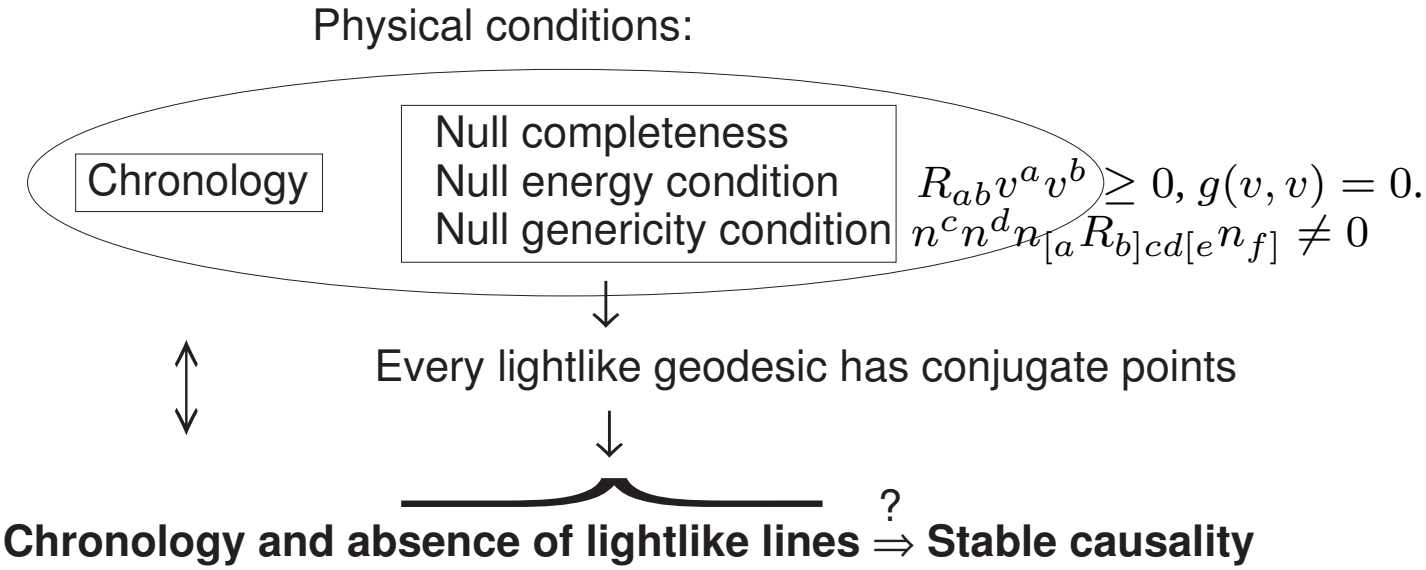
(a) Justify the existence of a time function on physical basis. Geroch and Horowitz (1979) identified this problem as one of the most important open problems concerning the global aspects of general relativity.

(b) K-causality coincides with stable causality? If so, in a K-causal (stably causal) spacetime, $K^+ = J_S^+$?

They are nicely related!

What does it mean *physical basis* in (a)? How to convert (a) in a mathematical statement?

Solution:

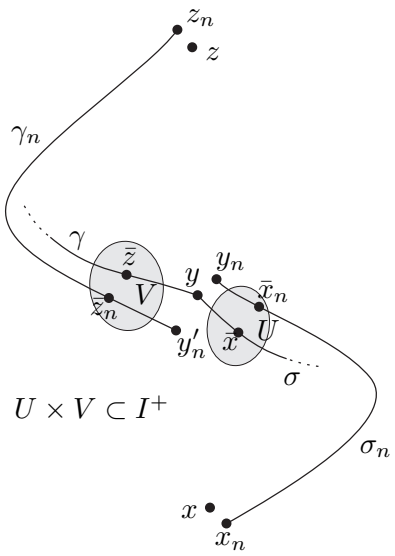


It relates only conformal invariant properties.

Connection between the two problems and strategy

- If there are no lightlike lines then $\bar{J}^+$ is transitive.

Proof: take $(x, y) \in \bar{J}^+$ and $(y, z) \in \bar{J}^+$ and use a limit curve argument



Thus:

- If there are no lightlike lines then $K^+ = \bar{J}^+$.

It is easy to prove that under chronology the absence of lightlike lines implies that $\bar{J}^+$ is antisymmetric, then the spacetime is K-causal, and if K-causality and stable causality coincide, then it is stably causal and admits a time function. It is indeed possible to prove that K-causality coincides with stable causality.

Conclusions

- K-causality coincides with stable causality and in a K-causal spacetime, $K^+ = J^+_S$.
- Chronological spacetimes without lightlike lines are stably causal and hence admit a smooth time function with timelike gradient.
- Physically this result states that under reasonable conditions a time function exists.
- Note that the assumption of null completeness is compatible with all the singularity theorems that do not assume a priori the existence of a time function.

The causal ladder