Causality of gravitational waves and Weak KAM theory

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Plato’s allegory of the cave

[...] they see only their own shadows, or the shadows of one another, which the fire throws on the opposite wall of the cave [...]. To them, I said, the truth would be literally nothing but the shadows of the images.

Plato, The allegory of the cave, Book VII of the Republic
Useful for clarifying

- the connection between relativistic and non-relativistic (classical) physics,
- the causality properties of gravitational waves,
- the problem of existence of solutions to the Hamilton-Jacobi equation and other problems in Lagrangian mechanics.

These apparently independent problems are strongly related as the study of lightlike dimensional reduction proves.
A Lorentzian manifold is a Hausdorff, connected manifold $M$, of dimension $n \geq 2$, endowed with a Lorentzian metric, that is a section $g : M \to T^* M \otimes T^* M$ with signature $(-, +, \ldots, +)$. 

A tangent vector $v \in T M$ is:
- timelike, if $g(v, v) < 0$.
- lightlike, if $g(v, v) = 0$.
- causale, if timelike or lightlike, i.e. $g(v, v) \leq 0$.
- spacelike, if $g(v, v) > 0$. 
Lorentzian manifold

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Light cone

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- *lightlike*, if $g(v, v) = 0$.
- *causale*, if timelike or lightlike, i.e. $g(v, v) \leq 0$.
- *spacelike*, if $g(v, v) > 0$. 
A Lorentzian manifold is *time orientable* if it is possible to make a continuous choice of one of the two timelike cones (called *future*) on the tangent bundle over the manifold.
Spacetime and causal relations

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**Spaziotempo**

A *spacetime* $(M, g)$ is a time oriented Lorentzian manifold. Its points are called *events*.
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Two events \(p, q \in (M, g)\) are related:

- *chronologically*, \(p \ll q\), if there is a future directed timelike curve from \(p\) to \(q\);
Spacetime and causal relations

A Lorentzian manifold is \textit{time orientable} if it is possible to make a continuous choice of one of the two timelike cones (called \textit{future}) on the tangent bundle over the manifold.

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**Causal relations**

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- *causally*, \(p \leq q\), if there is a future directed causal curve from \(p\) to \(q\), or \(p = q\);

\[
I^+(p) = \{ q \in M : p \ll q \}, \quad \text{chronological future } p \\
J^+(p) = \{ q \in M : p \leq q \}, \quad \text{causal future } p
\]

\[
I^+ = \{ (p, q) \in M \times M : p \ll q \}, \quad \text{chronology relation} \\
J^+ = \{ (p, q) \in M \times M : p \leq q \}, \quad \text{causal relation}
\]
Fields on the configuration space

Let $E = T \times Q$, $T = \mathbb{R}$, be a classical $d + 1$-dimensional extended configuration space of coordinates $(t, q)$. Let $a_t$ be a time dependent Riemannian metric on $S$, and $b_t$ a time dependent 1-form field on $S$. Let $V(t, q)$ be a time dependent scalar field on $S$. Define

$$L(t, q, \dot{q}) = \frac{1}{2} a_t(\dot{q}, \dot{q}) + b_t(\dot{q}) - V(t, q),$$

The considered Lagrangian is the most general which comes from Newtonian mechanics by considering holonomic constraints.
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The original idea

In 1929 Eisenhart pointed out that the trajectories of a Lagrangian problem

$$\delta \int_{t_0}^{t_1} L(t, q, \dot{q}) dt = 0, \quad q(t_0) = q_0, \quad q(t_1) = q_1$$

may be obtained as the projection of the spacelike geodesics of a $d + 2$-dimensional manifold $M = E \times Y$, $Y = \mathbb{R}$, of metric

$$g = a_t - dt \otimes (dy - b_t) - (dy - b_t) \otimes dt - 2V dt^2,$$

Eisenhart had in mind Jacobi’s metric $(E - V)a$, and Jacobi’s action principle which holds for time independent Lagrangians and $b = 0$. 
The Eisenhart metric takes its simplest and most symmetric form in the case of a free particle in Euclidean space $a_{bc} = \delta_{bc}$, $b_c = 0$, $V = const.$ Remarkably, in this case the Eisenhart metric becomes the Minkowski metric.

The Eisenhart metric is Lorentzian but Eisenhart did not give to this fact a particular meaning.
The vector field \( n = \partial / \partial y \) is covariantly constant and lightlike.

It is better to proceed in steps starting from \((M, g)\):

- Assume \( n \) is Killing and lightlike in such a way that the quotient projection \( \pi : M \rightarrow E \) gives a principal bundle over \( \mathbb{R} \).
- \( n \) is twist-free, \( n \wedge dn = 0 \), \( \Rightarrow \) \( E \) foliates into simultaneity slices, \( n = -\psi dt \).
- \( n \) is covariantly constant \( \Rightarrow \) the foliation takes a natural ‘time parameter’ - the classical time \( n = -dt \).
- Connection on principal bundles \( \pi_t : N_t \rightarrow Q_t \Rightarrow \) Newtonian flow on \( E \).
- Curvature \( \Rightarrow \) Coriolis forces.
- If the curvature \( \Omega_t \) vanishes the observers are non-rotating.

The Brinkmann-Eisenhart metric

\[
g = a_t - dt \otimes (dy - b_t) - (dy - b_t) \otimes dt - 2V dt^2,
\]

on \( M = E \times Y, \ Y = \mathbb{R} \), describes the most general spacetime with a covariantly constant lightlike field (Bargmann structure, generalized wave metric). Brinkmann proved this result locally. The space metric \( a \) and 1-form field \( b \) are fixed only if the coordinate system is fixed and this is done by choosing a Newtonian flow on \( E \) which defines the space \( Q \).
Let $n$ be a lightlike Killing vector field on the spacetime $(M, g)$. In any spacetime dimension $n \wedge dn = 0$ if and only if $R_{\mu\nu} n^\mu n^\nu = 0$.

Define the \textit{Newtonian flow} as a vector field $v$ on $E$ such that $dt[v] = 1$. The Newtonian flows and the connections $\omega_t$ on the bundles $\pi_t : N_t \to Q_t$ are in one-to-one relation through the formula $\omega_t(\cdot) = -g(\cdot, V)|_{N_t}$.

The metric $a_t$ on the space sections $Q_t$ is defined by $a_t(w, v) = g(W, V)$.

Given a 1-parameter family of sections $\sigma_t : Q \to N_t$ of the 1-parameter family of principal bundles $\pi_t : N_t \to Q_t$, the potential $b_t$ reads, $b_t = -\sigma_t^* \omega_t$. 
The spacetime

Assume from now on that $M = T \times Q \times Y$ with $T \simeq Y \simeq \mathbb{R}$ is given the metric

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The action functional

Given $e_0 = (t_0, q_0), e_1 = (t_1, q_1), q : [t_0, t_1] \to Q, q(t_0) = q_0, q(t_1) = q_1$

$$S_{e_0, e_1}[q] = \int_{t_0}^{t_1} \left[ \frac{1}{2} a_t(\dot{q}, \dot{q}) + b_t(\dot{q}) - V(t, q) \right] dt$$
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\[
S_{e_0, e_1}[q] = \int_{t_0}^{t_1} \left[ \frac{1}{2}a_t(\dot{q}, \dot{q}) + b_t(\dot{q}) - V(t, q) \right] dt
\]

The light lift

Every \( C^1 \) curve \( (t, q(t)) \) on \( E = T \times Q \) is the projection of a lightlike curve on \( (M, g) \), \( \gamma(t) = (t, q(t), y(t)) \) where
\[
y(t) = y_0 + S_{e_0, e'(t)}[q]|_{[0, t]}.
\]

Conversely, every lightlike curve on \( (M, g) \) with tangent vectors nowhere proportional to \( n = \partial/\partial y \) projects on a \( C^1 \) curve on \( E \).

Hint:
\[
g(\dot{\gamma}, \dot{\gamma}) = a_t(\dot{q}, \dot{q}) - 2(\dot{y} - b_t[\dot{q}]) - 2V = 2(L - \dot{y}),
\]
The light lift II

\[ n = \partial y \]

Causality of gravitational waves and Weak KAM
The light lift III

Proposition

Every geodesic on \((M, g)\) not coincident with a flow line of \(n\) admits the function \(t\) as affine parameter and once so parametrized projects on a solution to the E-L equations. The light lift of a solution to the E-L equation is a lightlike geodesic.

It is based on

\[
\mathcal{I}[\eta] = \frac{1}{2} \int_{\lambda_0}^{\lambda_1} g(\eta', \eta') \, d\lambda = \frac{1}{2} \int_{t_0}^{t_1} g(\dot{\eta}, \dot{\eta})(t') \, dt
\]

\[
= \int_{t_0}^{t_1} [L(t, q(t), \dot{q}(t)) - \dot{y}] (t') \, dt.
\]
The least action also called Hamilton’s principal function is $S : E \times E \to [-\infty, +\infty]$ given by

\[
S(e_0, e_1) = \inf_{q \in C^{1}_{e_0, e_1}} S_{e_0, e_1}[q], \quad \text{for } t_0 < t_1,
\]

\[
S(e_0, e_1) = 0, \quad \text{for } t_0 = t_1 \text{ and } q_0 = q_1,
\]

\[
S(e_0, e_1) = +\infty, \quad \text{elsewhere}.
\]
Proposition

Let $x_0 = (e_0, y_0) \in M$, it holds

$$I^+(x_0) = \{x_1 : y_1 - y_0 > S(e_0, e_1) \text{ and } t_0 < t_1\},$$

$$J^+(x_0) \subset \{x_1 : y_1 - y_0 \geq S(e_0, e_1)\}.$$ 

Analogous past versions hold.
Recall that $d(x, y) := \sup_{\gamma} L(\gamma)$ with $\gamma$ causal curve connecting $x$ to $y$. $L(\gamma) = \int \sqrt{-g(\dot{\gamma}, \dot{\gamma})} \, ds$ is the Lorentzian length functional. If $x \leq y \leq z$

$$d(x, y) + d(y, x) \leq d(x, z) \quad reverse \ triangle \ inequality$$
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**Theorem**

Let $x_0, x_1 \in M$, $x_0 = (e_0, y_0)$, $x_1 = (e_1, y_1)$ then if $x_1 \in J^+(x_0)$,

$$d(x_0, x_1) = \sqrt{2[y_1 - y_0 - S(e_0, e_1)](t_1 - t_0)}. \quad (1)$$

In particular, $S(e_0, e_1) = -\infty$ iff $d(x_0, x_1) = +\infty$. 
Lorentzian distance vs mechanical least action

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**Theorem**

Let $x_0, x_1 \in M$, $x_0 = (e_0, y_0)$, $x_1 = (e_1, y_1)$ then if $x_1 \in J^+(x_0)$,

$$d(x_0, x_1) = \sqrt{2[y_1 - y_0 - S(e_0, e_1)](t_1 - t_0)}.$$ (1)

In particular, $S(e_0, e_1) = -\infty$ iff $d(x_0, x_1) = +\infty$.

**The triangle inequality**

The function $S$ is upper semi-continuous everywhere but on the diagonal of $E \times E$ and satisfies the triangle inequality: for every $e_0, e_1, e_2 \in E$

$$S(e_0, e_2) \leq S(e_0, e_1) + S(e_1, e_2),$$

with the convention that $(+\infty) + (-\infty) = +\infty$. 
Causality violations

- Non-chronological
- Non-causal
- Non-future distinguishing
- Non-strongly causal
The causal ladder of spacetimes

Global hyperbolicity
⇓
Causal simplicity
⇓
Causal continuity
⇓
Stable causality
⇓
Strong causality
⇓
Distinction
⇓
Causality
⇓
Chronology

How does it change for our case? Can this properties be related with properties of the Lagrangian problem and in particular of the least action $S$?
The causality of these spacetimes is not trivial

Figure from Penrose’s 1965 “A remarkable property of plane waves in general relativity”. Plane waves are not causally simple.

\[ g = [dw^2 + dz^2] - dt \otimes dy - dy \otimes dt - \{f(u)[w^2 - z^2] + 2h(u)wz\}dt^2. \]

Coordinate \(w\) is the amplitude of an harmonic oscillator.
Why should we expect a connection with the least action?

Because some causality properties can be expressed in terms of the Lorentzian distance $d$

**Global hyperbolicity**

A strongly causal spacetime is globally hyperbolic if and only if whatever the chosen conformal factor $d$ is finite.
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A strongly causal spacetime is globally hyperbolic if and only if whatever the chosen conformal factor $d$ is continuous.

**Causal simplicity**

A strongly causal spacetime is causally simple if and only if whatever the chosen conformal factor $d$ is continuous wherever it vanishes.

and $d$ and $S$ are related...
The answer

Global hyperbolicity:

(i) \( S(e_0, e_1), t_0 < t_1, \) is finite,

(ii) \( S_{e_0, e_1}, t_0 < t_1, \) is coercive.

Causal simplicity:

(a) \( S \) attains its infimum \( S \) wherever \( S \) is finite,

(b) \( S \) is lower semi-continuous.

Causal continuity:

\[
\lim_{e \to e_1} \inf S(e_0, e) = \lim_{e \to e_0} \inf S(e, e_1),
\]

and this quantity vanishes for \( e_0 = e_1. \)
The answer II

\[
\begin{align*}
\text{Stable/Strong causality:} \\
S \text{ is lower semi-continuous on the diagonal.}
\end{align*}
\]

\[
\begin{align*}
\text{Distinction:} \\
\liminf_{e \to \tilde{e}} S(\tilde{e}, e) = \liminf_{e \to \tilde{e}} S(e, \tilde{e}) = 0.
\end{align*}
\]

\((M, g)\) is always causal (in fact non-total imprisoning). If \(Q\) is compact then \(M\) is globally hyperbolic.
Marchal’s theorem and causal simplicity

Tonelli’s theorem is just the known result that global hyperbolicity implies casual simplicity.
Marchal’s theorem and causal simplicity

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Do these levels differ? Yes

Let $T \times Q, Q = \mathbb{R}^{3N}$, be the configuration space of the Newtonian $N$-body problem. The spacetime $M = T \times Q \times \mathbb{R}$ endowed with

$$g = \sum_{j}^{N} m_j (d\vec{r}_j)^2 - 2dt dy + 2 \sum_{i<j} \frac{m_i m_j}{\|\vec{r}_i - \vec{r}_j\|} dt^2,$$

is a causally simple but not globally hyperbolic. It also solves the vacuum Einstein equations. Uses a deep theorem by Marchal and Chenciner: The minimizer of the fixed endpoints variational problem exists and is collision free.
Locally the Lorentzian distance $d(x_0, x)$ satisfies the eikonal equation

$$g(\nabla d, \nabla d) + 1 = 0,$$

while $S(e_0, e)$ satisfies the Hamilton-Jacobi equation. They are related because, using the relation between $S$ and $d$

$$g(\nabla d, \nabla d) + 1 = \frac{2(t - t_0)^2}{d^2} [ \frac{\partial S}{\partial t} + \frac{1}{2} \tilde{a}_t^{-1} (dS - b_t, dS - b_t) + V ].$$
Future set $F$: if $x \in F$, then $I^+(x) \in F$.

Let $U$ be a closed future set on $(M, g)$ then there is a lower semi-continuous function $u : E \to [-\infty, +\infty]$ such that

$$U = \{(e, y) : u(e) \leq y\}. \quad (2)$$

Furthermore, for every $e_0, e_1 \in \pi(U)$ with $t_0 < t_1$, the function $u$ satisfies

$$u(e_1) - u(e_0) \leq S(e_0, e_1). \quad (3)$$

whenever $u(e_1)$ or $u(e_0)$ differ from $+\infty$. Conversely, given a lower semi-continuous function $u : E \to [-\infty, +\infty]$ which satisfies Eq. (3) whenever $u(e_1)$ or $u(e_0)$ differ from $+\infty$, the set given by Eq. (2) is a closed future set.
Lax-Oleinik evolution map as the boundary of a future set

Definition

The Lax-Oleinik evolution map $T_{t_0,t}^{-}$ for $t \geq t_0$ is defined by the expression

$$(T_{t_0,t_1}^{-}u_{t_0})(q_1) = \inf_{e_0, t(e_0)=t_0} [u(e_0)+S(e_0, e_1)] = \inf_{q} [u(q_0, t_0) + \int_{t_1}^{t} L(s, q(s), \dot{q}(s)) ds],$$

where the last infimum is with respect to all the $C^1$ curves $q : [t_0, t] \to Q$ which end at $q_1$ starting from any point $q_0$.

It extends the function $u_{t_0}$ to the function $u(t, q) := (T_{t_0,t}^{-}u_{t_0})(q)$ which is a map $u : [0, +\infty) \times Q \to [-\infty, +\infty)$. 
The Lax-Oleinik evolution map $T_{t_0,t}$ for $t \geq t_0$ is defined by the expression

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It extends the function $u_{t_0}$ to the function $u(t, q) := (T_{t_0,t} u_{t_0})(q)$ which is a map $u : [0, +\infty) \times Q \to [-\infty, +\infty)$.

Let $A = \{(t_0, q, u_{t_0}(q)) : q \in Q\}$ be the graph of the initial condition $u_{t_0}$. Then

$$d(A, x) = \sqrt{2(y - u(t, q))(t - t_0)}, \quad \text{where} \quad x = (t, q, y).$$
The achronal generating set and the initial condition

\[
\mathcal{Q} \{ q^a \} \quad \pi \quad \mathcal{E}
\]

\[
u(t_0, q)
\]

\[
y = 0
\]

\[
A
\]
Weak KAM solution and the causal future

The slices of the causal future of the initial condition give a weak KAM solution.
Generators of the future set and $u$-calibrated curves

**Weak KAM solution $u(t, q)$ of negative type**

- $u(t_1, q_1) - u(t_0, q_0) \leq S(e_0, e_1)$
  (domination: means that $u$ defines a future set $F$).

- there is a $u$-calibrated curve $q : (t - \epsilon, t] \to Q$ which ends at $q_1$, where
  $u$-calibrated means that for every $t, t'$ in the interval of definition
  
  \[
  u(e(t')) - u(e(t)) = \int_t^{t'} L(s, q, \dot{q}) \, ds.
  \]

  (the light lift of some curve is contained in the (achronal) boundary of the future set $F$, i.e. every point of $F$ is the endpoint of some generator.)

Some future sets are generated by lightlike geodesics. This is the case for $F = J^+(A)$. The generators of the boundary $B = \partial F$ are the light lift of the so called $u$-calibrated curves.
In Weak KAM theory (autonomous case)

**Differentiability and uniqueness of calibration**

A weak KAM solution $u_-$ has a derivative at $q$ if and only if there is a unique $(L, c, u_-)$-calibrated path $\gamma^q_-$ which ends at $q$. 
In Weak KAM theory (autonomous case)

Differentiability and uniqueness of calibration

A weak KAM solution $u_-$ has a derivative at $q$ if and only if there is a unique $(L, c, u_-)$-calibrated path $\gamma^q_-$ which ends at $q$.

This result can be obtained as the projection to $Q$ of the next result in Lorentzian geometry

Differentiability of lightlike hypersurfaces

Boundaries of achronal sets are locally Lipschitz. Smoothness results on null hypersurfaces (those generated by lightlike geodesics) are due to Beem, Chruściel, Galloway, Królak: smoothness holds everywhere but at the points where two generators meet.
Suppose the Lagrangian does not depend on time. We look for a Weak KAM solution of H.-J. of the form for $t \geq t_0$

$$u(t, q) = \nu(q) - ct.$$ 

If $Q$ is compact there is a critical value of $c$ for which it exists.
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In any case $V$ must be bounded from above

Suppose to redefine $y' = y + ct$, $V' = V - c$, $L' = L - c$, then $c \to 0$ and $u(t, q) \to \nu(q)$. The vector $\partial_t$ must be achronal under this choice thus $g(\partial_t, \partial_t) = -2V \geq 0$. Thus the original $V$ had to be bounded from above, indeed if $M$ is compact $c = \max_Q V$. In this case $y - ct$ is non-decreasing...
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**In any case \( V \) must be bounded from above**

Suppose to redefine \( y' = y + ct, V' = V - c, L' = L - c \), then \( c \to 0 \) and \( u(t, q) \to \nu(q) \). The vector \( \partial_t \) must be achronal under this choice thus \( g(\partial_t, \partial_t) = -2V \geq 0 \). Thus the original \( V \) had to be bounded from above, indeed if \( M \) is compact \( c = \max_Q V \). In this case \( y - ct \) is non-decreasing...

**Global Rosen coordinates**

Let us suppose that the \( E - L \) flow is complete (e.g. \( Q \) is compact) and that there is a smooth solution \( u(t, q) \) to H.-J. then we can redefine \( y' = y - u(t, q), q' = f(q, t) \), so that \( b_t \to 0 \) and \( V \to 0 \).

\[
g = a_t - 2dt dy.
\]
The study of spacetimes admitting a parallel null vector is tightly related with the study of Lagrangian mechanical systems.

In this framework there is a simple relation between the Lorentzian distance and the least action.

The causality properties of the spacetime are connected with lower semi-continuity properties of the least action.

Tonelli’s theorem on the existence of minimizers is basically the statement that global hyperbolicity implies causal simplicity.

The Hamilton-Jacobi equation and the Lax-Oleinik semigroup are nothing but the causal relation in disguise.

The concept of domination is a way of speaking of future sets without mentioning them. The concept of $u$-calibrated curve is connected with the lightlike geodesics generating an achronal boundary.