$K$-causality and time functions in general relativity

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1. What is causality theory
2. Causal ladder
3. Causal continuity
4. Closed and transitive relations
5. The relation with time functions
6. Utility theory and Levin’s theorem
7. The existence of time
What is causality theory

- Lorentzian geometry: spacetime $(M, g)$ i.e. time oriented Lorentzian manifold.
- General Relativity: $G_{\mu\nu} = 8\pi k T_{\mu\nu}$.
- Conformal structure (causality): $[(M, g)]$, $g' \sim g \Leftrightarrow g' = \Omega^2 g$.

**Theorem**

Two Lorentzian metrics $g, g^*$ of a manifold of dimension $n > 2$ are pointwise conformal if and only if they have the same lightlike vectors (i.e. light cones).

Thus causality does not consider the Einstein equations, the metric structure, or the connection. It considers only the conformal structure namely the propagation of light.

Good aspects:

- It will probably survive changes in the equations for the gravitational field.
- It leads to very general statements, that the study of particular spacetime solutions can’t give.
- It has mathematical interest in its own right.
What is causality theory

Other conformal invariant concepts

- The concepts of *causal vector*, *timelike vector* or *lightlike vector*,
- Unparametrized *lightlike geodesic*,
- The presence of *conjugated points* on the lightlike geodesic is also conformally invariant.

Non conformal invariant concepts

- *proper time*,
- *timelike geodesic*,
- *geodesic completeness* property,
- *Lorentzian distance* concept,
- *(null) convergence condition*

\[ R_{ab} v^a v^b \geq 0 \quad \text{for all (lightlike) causal vectors } v \]

*Mathematical relativity* may use these concepts but causality theory tries to avoid them!
Two events $p, q \in (M, g)$ are related

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- chronologically, $p \ll q$, if there is a future directed timelike curve from $p$ to $q$,
- causally, $p \leq q$, if there is a future directed causal curve from $p$ to $q$ or $p = q$, 

horismotically, $p \rightarrow q$, if there is a maximizing lightlike geodesic segment connecting $p$ to $q$ or $p = q$.

They can be regarded as relations on $M$ i.e. as subsets of $M \times M$.

$J^+ = \{(p, q) \in M \times M : p \ll q\}$, chronology relation

$E^+ = \{(p, q) \in M \times M : p \leq q\}$, causal relation

$E^+ = J^+ \setminus I^+$, horismos relation

Theorem

If two points are connected by a causal curve then either it is an achronal lightlike geodesic or they are also connected by a timelike curve.
Two events \( p, q \in (M, g) \) are related

- **chronologically**, \( p \preccurlyeq q \), if there is a future directed timelike curve from \( p \) to \( q \),
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They can be regarded as relations on \( M \) i.e. as subsets of \( M \times M \)

\[
\begin{align*}
I^+ &= \{(p, q) \in M \times M : p \ll q\}, & \text{chronology relation} \\
J^+ &= \{(p, q) \in M \times M : p \leq q\}, & \text{causal relation} \\
E^+ &= \{(p, q) \in M \times M : p \rightarrow q\} = J^+ \setminus I^+, & \text{horismos relation}
\end{align*}
\]

**Theorem**

*If two points are connected by a causal curve then either it is an achronal lightlike geodesic or they are also connected by a timelike curve.*

As a consequence \( E^+ \) is the set of pairs connected by achronal lightlike geodesics.
Trapped surface: a spacelike 2-surface such that the two families of null geodesics orthogonal to it are converging.

Penrose’s (1965) singularity theorem

The spacetime cannot be null geodetically complete if
- $R_{ab}n^a n^b \geq 0$ for all lightlike vectors $n$ (null convergence condition)
- there is a non-compact Cauchy hypersurface
- there is a closed trapped surface
- Trapped surface: a spacelike 2-surface such that the two families of null geodesics orthogonal to it are converging.

### Penrose’s (1965) singularity theorem

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- there is a closed trapped surface

### Conformally invariant version

Let \((M, g)\) be a globally hyperbolic spacetime with a non-compact Cauchy hypersurface then there is no non-empty compact set \(S\) such that \(J^+(S)/I^+(S)\) is compact (i.e. there are no trapped sets).

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$I^+$ and $J^+$ are transitive. $I^+$ is open but $J^+$ and $E^+$ are not necessarily closed.
Remove a point from Minkowski spacetime
A more physical example: refocusing on plane waves

\[ ds^2 = dudv + h_{ij} x_i x_j du^2 - dx_i dx_i \]

\( n = \partial / \partial v \) is a null twist free Killing vector. Some plane waves are not causally simple.

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Figure: Figure taken from Penrose’s “A remarkable property of plane waves” (1965).
Causality conditions: the causal ladder

Below stable causality: protect against causality violation.
Above stable causality: protect against influence from infinity.
The transverse conformal ladder

- Compactness of causal diamonds: $\forall x, y \in M, J^+(x) \cap J^-(y)$ is compact.
- Closure of causal relation: $J^+ \subset M \times M$ is closed. Equivalently, $J^\pm(x)$ are closed for all $x \in M$.
- Reflectivity: $x \in \overline{I^-(y)} \iff y \in \overline{I^+(x)}$.
- Transitivity of $\overline{J^+}$: clear.

**Definition**

\[
D^+ = \{ (x, y) : y \in \overline{I^+(x)} \text{ and } x \in \overline{I^-(y)} \}
\]

$D^+$ is transitive (easy proof). Reflectivity is equivalent to $D^+ = \overline{J^+}$. 
Causality conditions: the definitions

**Chronology**  No closed timelike curves. Equivalently, $\ll$ is irreflexive.  
Equivalently, $\ll$ is antisymmetric.

**Causality**  No closed causal curves. Equivalently, $\leq$ is antisymmetric.

**Theorem**

*If the spacetime is chronological but non causal then there is a closed achronal lightlike geodesic.*

**Non-total imprisonment**  No future inextendible causal curve is future imprisoned in a compact set (replacing future with past gives the same property)

**Weak distinction**  $I^+(x) = I^+(y)$ and $I^-(x) = I^-(y) \Rightarrow x = y$. Equivalently, $D^+$ is antisymmetric.

**Distinction**  $I^+(x) = I^+(y)$ or $I^-(x) = I^-(y) \Rightarrow x = y$. Equivalently, every point admits an arbitrarily small open neighborhood such that every inextendible curve passing through the point intersects the neighborhood only once.

**Non-partial imprisonment**  No inextendible causal curve is future or past partially imprisoned in a compact.

**Strong causality**  Every point admits an arbitrarily small causally convex neighborhood. Equivalently, the Alexandrov topology is Hausdorff, or the Alexandrov topology is the manifold topology.

**Stable causality**  There exist some $g' > g$ with $(M, g')$ causal. In short: causality is preserved under small perturbations of the metric.
Causal continuity and outer continuity

Usually defined as: Distinction and reflectivity. Equivalently,

\[ D^+ \text{ is antisymmetric and } D^+ = \overline{J^+}. \]

<table>
<thead>
<tr>
<th>Inner continuity</th>
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<td>( I^- : M \to P(M) ), such that ( p \to I^-(p) ) is inner continuous if for every compact set ( K \subset I^-(p) ) there is a neighborhood ( U ) of ( p ) such that for all ( q \in U ), ( K \subset I^-(q) ).</td>
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\( I^- \) is inner continuous. Let \( \mu \) be a finite measure on \( M \)

\[ t^-(p) = \mu(I^-(p)) \quad t^+(p) = -\mu(I^+(p)) \quad \text{Geroch's times} \]

then \( t^- \) is lower semi-continuous and \( t^+ \) is upper semi-continuous.

<table>
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<th>Theorem</th>
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<td>A weakly distinguishing spacetime ((M, g)) is causally continuous iff ( I^\pm ) are outer continuous iff ( t^\pm ) are continuous.</td>
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Examples: Chronological non-causal

\[ \text{Diagram:} \]

- Square with arrows indicating direction.
- Cylinder with arrows indicating direction.

Identify
Examples: Causal but total imprisoning (Carter’s example)

Identify

Identify after shifting an irrational amount
Examples: Non-total imprisoning but non-weakly distinguishing (A), and weakly distinguishing but non-distinguishing (B)
Examples: Distinguishing non-strongly causal

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The $A$-causality subladder (Penrose)

Define $A^+ = \overline{J^+}$, the spacetime is $A^n$-causal if $A^{+n}$ is antisymmetric.

Figure: $A$-causality differs from strong causality

And all the $A^n$-causality levels differ.
The $A$-causality subladder II

$A^{\infty}$-causality: $A^n$-causality holds for every $n$, i.e. no finite closed chain of $A^+$-related events.
\( A^\infty \)-causality is still insufficient

This difficulties are at the origin of \( K \)-causality and stable causality.
Last examples

- Stably causal non-causally easy: take the previous example of distinguishing non-strongly causal and remove the identification.
- Causally easy non-causally continuous: 1+1 Minkowski spacetime with a timelike geodesic segment removed.
- Causally continuous non-causally simple: 1+1 Minkowski spacetime with a point removed.
- Causally simple non-globally hyperbolic: 1+1 Minkowski spacetime restricted to $x > 0$. 
Partially ordered sets

Let $\Delta = \{(p, p) : p \in M\}$

### Preorder

A preorder $R \subset M \times M$ is a (reflexive) *preorder* on $M$ if it is
- **reflexive**: $\Delta \subset R$,
- **transitive**: $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$,
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**Partial order**

$R$ is a (reflexive) **partial order** on $M$ if it is a preorder and it is

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**Total preorder**

A preorder which is

- **total:** $(x, y) \in R$ or $(y, x) \in R$

Every two elements are comparable.
Partially ordered sets

Let $\Delta = \{(p, p) : p \in M\}$

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$R \subseteq M \times M$ is a (reflexive) preorder on $M$ if it is
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**Total order**

A partial order which is total.
Closed and transitive relations

Stable Causality

(M, g) is stably causal if there is g' > g with (M, g') causal.

Here g' > g if the light cones of g are everywhere strictly larger than those of g.
Closed and transitive relations

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None of \(I^+, J^+\) or \(E^+\) are both closed and transitive

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The spacetime is stably causal iff \(J_S^+\) is antisymmetric.
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**Sorkin and Woolgar’s relation** $K^+$ (1996)

The smallest closed and transitive relation which contains $J^+$. A spacetime is $K$-causal if $K^+$ is antisymmetric. It is difficult to work with $K^+$. 

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By definition \(K^+ \subset J_S^+\); do they coincide?

No, but

- \(K\)-causality is equivalent to stable causality and in this case \(K^+ = J_S^+\).
It suffices to prove the equivalence

Suppose you prove that $K$-causality is equivalent to stable causality

then you prove that under $K$-causality $K^+ = J_S^+$. 
This proof is usually quite involved. Recall that causal continuity can be defined as $D^+$ is antisymmetric (weak distinction) and equals $J^+$ (reflectivity).

**Proof.**

As $D^+$ is transitive $\overline{J^+} = D^+$ is transitive and contains $J^+$, thus $K^+ = \overline{J^+} = D^+$. As $D^+$ is antisymmetric $K^+$ is antisymmetric ($K$-causality). \(\square\)
Time and stable causality

Time functions and temporal functions

Semi-time function: a continuous real function such that
\[ p \ll q \Rightarrow t(p) < t(q). \]

Time function: a continuous real function such that
\[ p < q \Rightarrow t(p) < t(q). \]

Temporal function: a \( C^1 \) time function with timelike gradient.

Relation with stable causality

Hawking 1968 Temporal function \( \Rightarrow \) stable causality

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So to prove "Time function \( \Rightarrow \) stable causality" you pass through a smooth time function.
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Hawking’s averaging method

Geroch’s time \( \mu(I^{-}(x)) \) is only lower semi-continuous.

- Stable causality \( \Rightarrow \) time function.

Let \( g_{\lambda} = (1 - \frac{\lambda}{2})g + \frac{\lambda}{2}\tilde{g} \) with \( \tilde{g} > g \), define

\[
t(x) = \int_{0}^{1} \mu(I^{-}_{(M,g_{\lambda})}(x))d\lambda
\]
Other route: prove directly

(i) The existence of a time function implies $K$-causality (skip smoothability),

(ii) $K$-causality implies the existence of a time function (skip Hawking’s averaging technique).

(i): Is possible and somewhat technical.
(ii): The idea behind (ii) is that the result holds because $K^+$ is closed.
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Concerning (i): Typical strategy to work with $K^+$

Theorem: If $(x, y) \in K^+$ and $\mathcal{I}$ then $\mathcal{T}$.
Proof: Define $B^+ = \{(x, y) \in K^+ : \text{if } \mathcal{I} \text{ then } \mathcal{T}\}$ and prove
- $J^+ \subset B^+ \subset K^+$,
- $B^+$ is transitive,
- $B^+$ is closed,
then $B^+ = K^+$ and the thesis follows.
Bernoulli models the preferences of an individual (an apple over an orange) as an abstract space of alternatives $A$ (prospects space) endowed with a total preorder $R$. Bernoulli introduces the concept of utility:

$$x \leq_R y \iff u(x) \leq u(y),$$

to quantify preference and solve S. Petersburg paradox.
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1944 Von Neumann and Morgenstern in “Theory of games and economic behavior” had the insight of considering the preference over lotteries as more fundamental. A lottery is a probability distribution on the (pure) prospects space. The idea is that often the agent is faced with some risky alternatives, in each alternative the final outcomes in the prospect space being given only with certain probabilities. **The total preorder is over the lotteries not over the prospects space.** They prove that the utility function $u$ exists on the basis of consistency conditions on the preorder (no mention of continuity).

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$$\mu_1 \leq \mu_2 \text{ iff } \int u \, d\mu_1 \leq \int u \, d\mu_2$$

**The formalisms**

The mathematics in Bernoulli’s approach is that of topological ordered spaces. The mathematics in Von Neumann and Morgenstern’s approach is that of (convex subsets of) topological ordered vector spaces.
1941,1954 Eilenberg and Debreu prove that the continuous utility exist provided $R^\pm(x)$ are closed (recall: for total preorders).

1954 Ward notes that for total preorders the condition "$R^\pm(x)$ are closed" is equivalent to $R$ is closed.

1962 Aumann removes the totality axiom from the model. The individual cannot always choose between alternatives (indecisiveness). The utility is defined by $x \sim R y \Rightarrow u(x) = u(y)$ and $x < R y \Rightarrow u(x) < u(y)$.

1970 Peleg proves his theorem for open strict partial orders. According to Peleg a strict partial order $S$ is separable if there is a countable subset $C$ of $X$ such that for any $(x, y) \in S$ the diamond $S^+ + (x)$ and $S^- - (y)$ contains some element of the subset $C$, and spacious if for $(x, y) \in S$, $S^- - (x) \subset S^- - (y)$.

Theorem (Peleg) Let $S$ be a strict partial order on a topological space $X$. Suppose that (a) $S^- - (x)$ is open for every $x \in X$, (b) $S$ is separable, and (c) $S$ is spacious, then there is a function $u: X \rightarrow \mathbb{R}$ such that $(x, y) \in S \Rightarrow u(x) < u(y)$.

1980 Sondermann tries to construct the utility from a finite measure on the space of alternatives (not being aware of Geroch's idea) but obtains only lower semi-continuity (as expected).

1983 Levin proves that the closure of $R$ suffices to prove the existence of a continuous utility, and that from the utilities $R$ can be recovered.
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Utility theory II: subsequent history of Bernoulli’s approach

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1941, 1954 Eilenberg and Debreu prove that the continuous utility exist provided \( R^\pm(x) \) are closed (recall: for total preorders).

1954 Ward notes that for total preorders the condition ”\( R^\pm(x) \) are closed” is equivalent to \( R \) is closed.

1962 Aumann removes the totality axiom from the model. The individual cannot always choose between alternatives (indecisiveness). The utility is defined by

"\( x \sim_R y \Rightarrow u(x) = u(y) \)" and "\( x <_R y \Rightarrow u(x) < u(y) \)."

Does a continuous utility exist?

1970 Peleg proves his theorem for open strict partial orders. According to Peleg a strict partial order \( S \) is separable if there is a countable subset \( C \) of \( X \) such that for any \( (x, y) \in S \) the diamond \( S^+(x) \cap S^-(y) \) contains some element of the subset \( C \), and spacious if for \( (x, y) \in S \), \( S^-(x) \subset S^-(y) \).

**Theorem (Peleg)**

Let \( S \) be a strict partial order on a topological space \( X \). Suppose that (a) \( S^-(x) \) is open for every \( x \in X \), (b) \( S \) is separable, and (c) \( S \) is spacious, then there is a function \( u : X \rightarrow \mathbb{R} \) such that \( (x, y) \in S \Rightarrow u(x) < u(y) \).

1980 Sondermann tries to construct the utility from a finite measure on the space of alternatives (not being aware of Geroch’s idea) but obtains only lower semi-continuity (as expected).

1983 Levin proves that the closure of \( R \) suffices to prove the existence of a continuous utility, and that from the utilities \( R \) can be recovered.
Utilities for $I^+$

In a chronological spacetime the utilities of the relation $I^+$ are the semi-time functions.

Utilities for $K^+$

In a $K$-causal spacetime the utilities of the relation $K^+$ are the time functions.
Utilities in spacetime

Utilities for $I^+$

In a chronological spacetime the utilities of the relation $I^+$ are the semi-time functions.

Utilities for $K^+$

In a $K$-causal spacetime the utilities of the relation $K^+$ are the time functions.

Given this correspondences Levin’s and Peleg’s theorems of utility theory lead to the following results

Theorem

A spacetime is $K$-causal if and only if it admits a time function. In this case, denoting with $\mathcal{A}$ the set of time functions we have that the partial order $K^+$ can be recovered from the time functions, that is

$$(x, y) \in K^+ \Leftrightarrow \forall t \in \mathcal{A}, \; t(x) \leq t(y).$$

Theorem

A chronological spacetime in which $\overline{J^+}$ is transitive admits a semi-time function.
From causality to time and back

Stable causality (antisymmetry of $J^+ S$) implies the existence of time. This is the analog of Szpilrajn order extension principle: every partial order can be extended to a total order. (But here continuity comes into play!)

From time to causality

In a stably causal spacetime the time functions on spacetime allow us to recover $J^+ S$ (whose antisymmetry is equivalent to stable causality). This is the analog of the result which states that: every partial order is the intersection of the total orders which extend it.

Considerations about time suggest to regard $J^+ S$ (or $K^+ S$) as more fundamental than $J^+ S$. 

Geodycos, Lyon 2010
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The existence of time: the physical content

The problem

Justify the existence of a time function on physical basis. Geroch and Horowitz (1979) identified this problem as one of the most important open problems concerning the global aspects of general relativity.

The physical argument (note how a conformal invariant statement is used to get a non-conformally invariant but more physical result)

Physical conditions:

Chronology

null completeness
null energy condition
null genericity condition

\[ R_{ab}v^av^b \geq 0, \ g(v, v) = 0. \quad n^c n^d n_{[a} R_{b]cd[e} n_{f]} \neq 0 \]

Every lightlike geodesic has conjugate points

Chronology and absence of lightlike lines ⇒ Stable causality

A lightlike line is an achronal inextendible causal curve (necessarily a geodesic).
The connection with $K$-causality

- If there are no lightlike lines then $\bar{J}^+$ is transitive.

Proof: take $(x, y) \in \bar{J}^+$ and $(y, z) \in \bar{J}^+$ and use a limit curve argument

Thus:
- If there are no lightlike lines then $K^+ = \bar{J}^+$.

It is easy to prove that under chronology the absence of lightlike lines implies that $\bar{J}^+$ is antisymmetric, then the spacetime is $K$-causal, and since $K$-causality and stable causality coincide, then it is stably causal and admits a time function.
That’s all
Thanks!
Lemma

Let \((M, g)\) be non-total imprisoning. Let \((p, q) \in K^+\) then either \((p, q) \in J^+\) or for every relatively compact open set \(B \ni p\) there is \(r \in \hat{B}\) such that \(p < r\) and \((r, q) \in K^+\).

Define the relation

\[
B^+ = \{(p, q) \in K^+: (p, q) \in J^+ \text{ or for every relatively compact open set } B \ni p \\
\text{there is } r \in \hat{B} \text{ such that } p < r \text{ and } (r, q) \in K^+\}.
\]

it is immediate \(J^+ \subset B^+ \subset K^+\), so we have only to prove closure and transitivity.