

Simultaneity in special and general relativity*

E. Minguzzi

Departamento de Matemáticas, Universidad de Salamanca,
Plaza de la Merced 1-4, E-37008 Salamanca, Spain
and INFN, Piazza dei Caprettari 70, I-00186 Roma, Italy
minguzzi@usal.es

Abstract

We present some basic facts concerning simultaneity in both special and general relativity. We discuss Weyl's proof of the consistence of Einstein's synchronization convention and consider the difficult general relativistic problem of assigning a time function to a congruence of timelike curves.

1 Simultaneity in special relativity

In special relativity the possibility of synchronizing distant clocks so as to obtain a global coordinate time was proved by Weyl [15] in a fundamental and unfortunately overlooked proof. The Einstein convention states (in Weyl's version) that if a light beam is sent from the point A to the point B then the clock at B (say clock B for short) should be set so that the time of arrival according to clock B reads $t_B = t_A + \overline{AB}/c$ where \overline{AB} is the Euclidean distance between the two points. This synchronization convention should satisfy at least the following three properties in order to lead to a foliation of spacetime into (equal time) simultaneity slices

- (i) **Time homogeneity.** If clock B is set accordingly to the procedure above then repeating the same experiment the equation $t_B = t_A + \overline{AB}/c$ should hold for whatever value of t_A without the need of setting again clock B (i.e. it is meaningful to say that clock A is synchronized with clock B).

- (ii) **Symmetry.** If clock A is synchronized with clock B then clock B is synchronized with clock A (i.e. it is meaningful to say that the two clocks *are* synchronized).

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- (iii) **Transitivity.** If clocks A and B are synchronized and clocks B and C are synchronized then clocks A and C are synchronized.

One could be tempted to consider the equation $t_B = t_A + \overline{AB}/c$ as trivial since it is well known that the speed of light in special relativity is c . However, one should take into account that in order to make a statement regarding the value of the *one-way* (i.e. from one point to another) speed of light one needs first a global time variable to make sense of expressions such as $\Delta x/\Delta t$. In most special relativity textbook this important conceptual point is not explained, and the existence of a global time variable such that the one-way speed of light is c is assumed without further explanations. This approach was initiated by Einstein in his 1905 work “On the electrodynamics of moving bodies” [3] where he writes

Let a ray of light start at the “A time” t_A from A towards B, let it at the “B time” t_B be reflected at B in the direction of A, and arrive again at A at the “A time” t'_A .

In accordance with definition the two clocks synchronize if¹

$$t_B - t_A = t'_A - t_B.$$

We assume that this definition of synchronism is free from contradictions, and possible for any number of points; and that the following relations are universally valid:–

1. If the clock at B synchronizes with the clock at A, the clock at A synchronizes with the clock at B.
2. If the clock at A synchronizes with the clock at B and also with the clock at C, the clocks at B and C also synchronize with each other.

Thus with the help of certain imaginary physical experiments we have settled what is to be understood by synchronous stationary clocks located at different places, and have evidently obtained a definition of “simultaneous,” or “synchronous,” and of “time.” The “time” of an event is that which is given simultaneously with the event by a stationary clock located at the place of the event, this clock being synchronous, and indeed synchronous for all time determinations, with a specified stationary clock.

In agreement with experience we further assume the quantity

$$\frac{2\overline{AB}}{t'_A - t_A} = c.$$

to be a universal constant – the velocity of light in empty space.

¹This definition of clock synchronization was first given by Poincaré [10, 11] although at the time it was already applied by engineers in the calculation of longitudes by means of telegraphic signals [4].

Thus Einstein assumes without proof the validity of (ii) and (iii). On the contrary, Weyl maintains that in a correct conceptual development of the theory one should define a way to synchronize distant clocks, and also prove that the proposed procedure is coherent (i.e. symmetric and transitive) and in the end show that with respect to the global time variable provided by the clocks' readings the one-way speed of light is c .

Weyl succeeded in completing this program starting from the experimental fact that the speed of light is a constant c around any closed polygonal path (the light beam can be reflected over suitable mirrors so as to travel over the closed polygon). Contrary to the one-way speed of light, the average speed of light over a closed path is independent of the distant synchronization convention adopted and as such the statement that the speed of light over closed paths is a constant is indeed well defined prior to the construction of a global time variable. Under the reasonable assumption that the speed of light is a constant over closed polygons (it was reasonable since it could be tested) Weyl was able to prove (ii) and (iii) assuming (i) tacitly. The assumption (i) was recognized and removed in [9]. Since the complete proof is short we give it here; for more details the reader is referred to [9]

- (i) Emit a light beam at time t_A from point A . Let it arrive at B where clock B is set so that the time of arrival is $t_B = t_A + \overline{AB}/c$. We have to prove that if a second light beam is emitted at time $t'_A > t_A$ in direction of B , the time of arrival at B is given by the same equation $t'_B = t'_A + \overline{AB}/c$.

Consider a point C at a distance from B given by $\overline{BC} = \frac{c}{2}(t'_B - t_B)$ and assume that the second light beams once arrived at B is there reflected back to A that it reaches at time t''_A . Moreover, assume that the first light beam once reached B is there reflected to C , then again to B and finally A . We have three closed polygonal paths of interest, ABA , $ABCBA$ and BCB . Since the speed of light is c over any closed polygonal path we have the equations

$$2\overline{AB} = c(t''_A - t'_A) \quad (1)$$

$$2\overline{AB} + 2\overline{BC} = c(t''_A - t_A) \quad (2)$$

$$2\overline{BC} = c(t'_B - t_B) = c(t'_B - t_A) - \overline{AB} \quad (3)$$

Summing the first and the third equation and subtracting the second we obtain the thesis.

- (ii) Emit a light beam at time t_A from point A . Let it arrive at B where clock B is set so that the time of arrival is $t_B = t_A + \overline{AB}/c$. We have to prove that if a second light beam is emitted at time t'_B in direction of A , the time of arrival at A is given by the same equation $t'_A = t'_B + \overline{AB}/c$. Point (i) implies that the time needed by the light beam to reach A departing from B is independent of the instant of departure. We can therefore assume without loss of generality that $t'_B = t_B$. Then we can consider the two light beams as a single light beam that, reflected at B , covers the closed

path ABA . Thus

$$2\overline{AB} = c(t'_A - t_A) = c(t'_A - t_B) + \overline{AB} = c(t'_A - t'_B) + \overline{AB} \quad (4)$$

from which the thesis follows.

- (iii) Place two clocks at A that we denote with clock A and clock A' . From (i) and (ii) it follows that it makes sense to say that two distant clocks are synchronized. Consider the three points A , B and C with their respective clocks. Assume that A and B are synchronized, and that B and C are synchronized. We have to prove that A and C are synchronized. To this end synchronize A' with C and send a light beam all over the closed polygonal path $ABCA$. Let the sequence of time of arrivals/departures be given by t_A, t_B, t_C, t'_A and denote with $t'_{A'}$ the time at which the light beam returns at A according to clock A' . We have from the constancy of the speed of light over polygonal paths

$$c(t'_A - t_A) = \overline{AB} + \overline{BC} + \overline{AC} \quad (5)$$

but since the clock pairs $A - B$, $B - C$ and $C - A'$ are synchronized we have

$$t_B = t_A + \overline{AB}/c \quad (6)$$

$$t_C = t_B + \overline{BC}/c \quad (7)$$

$$t'_{A'} = t_C + \overline{AC}/c \quad (8)$$

$$(9)$$

and summing the three equations we obtain

$$c(t'_{A'} - t_A) = \overline{AB} + \overline{BC} + \overline{AC} \quad (10)$$

or $t'_{A'} = t'_A$, that is the measures of clock A coincide with those of clock A' and therefore clock A is synchronized with clock C .

Once the Einstein synchronization convention is proved to be coherent, from equation $t_B = t_A + \overline{AB}/c$ it follows that the one-way speed of light is c in the new global *Einstein* time. From this fact one recovers that, coherently with the hypothesis, the speed of light over closed paths is a constant c .

Unfortunately in a rotating frame it can be easily shown using special relativity that the speed of light over a closed path depends on the direction followed and therefore can not be a universal constant - this is the well known Sagnac effect [12, 1]. Thus Weyl's theorem can not be applied in a rotating frame and indeed the Einstein convention fails to be transitive in this case.

A natural question is whether one could find a small correction $\delta(A, B)$ such that defining $t_B = t_A + \overline{AB}/c + \delta(A, B)$ the transitivity is restored. The answer is affirmative but it requires some work to find its actual expression. In general δ is expected to vanish in an inertial frame as the Einstein convention works perfectly there. At the experimental level it should therefore vanish if the vorticity and acceleration of the frame vanish while it can be different from zero if these quantities do not vanish.

2 Simultaneity in curved spacetimes and non-inertial frames

We recall that a frame in special and general relativity is a congruence of timelike curves each one defining the motion of a point “at rest” in the frame. For instance, this paper is made of points that in spacetime are represented by a worldline. Each distinct point is represented by a different worldline and hence, in general, the motion of a body on spacetime is represented by a congruence of timelike curves defining the spacetime motion of its points. Mathematically there will be a projection $\pi : M \rightarrow S$ from the spacetime manifold M to the space S that to each event m associates the worldline (space point) $s = \pi(m)$ passing through it. Locally one has the structure of a fiber bundle where the fiber is diffeomorphic to \mathbb{R} . However we are not in a principle fiber bundle since no action of the group $(\mathbb{R}, +)$ on M has been defined.

This mathematical construction is known as the hydrodynamical formalism of general relativity as the motion of the body is regarded in a way analogous to the motion of a fluid in Euclidean space [6]. We shall try to associate to the timelike flow a foliation of spacetime in spacelike simultaneity slices. Our concept of simultaneity will be therefore related to a timelike flow, which can be regarded as the motion of a set of observers in spacetime [5]. Other authors consider instead simultaneity foliations associated to a privileged observer [2] (i.e. a privileged timelike worldline).

It is useful to choose coordinates on M as follows. First introduce space coordinates $\{x^i\}$ on S so that each space point corresponds to a different triplet x^i , and then complete the coordinate system with a time coordinate t , $dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \neq 0$, ∂_t timelike. There are many different ways in which t can be introduced and the surfaces $t = \text{const.}$ will be called simultaneity slices for the given time function choice.

Unfortunately, this way of defining a global time variable is not physically satisfactory since the time function is simply assumed to exist without giving a constructive procedure or a way to measure it. Moreover, it does not help in selecting a useful time function or simultaneity convention. For instance the Einstein time would be only one among many possible choices in Minkowski spacetime.

We propose, therefore, to construct the global time function from a local definition of simultaneity. A local definition of simultaneity is an assignment to each spacetime event of a spacelike hyperplane that, roughly speaking, determines the events that are locally simultaneous. These hyperplanes are called horizontal hyperplanes, and in the language of gauge theories they define a connection: the *simultaneity connection* [7]. If this connection is integrable, i.e. its curvature vanishes, the distribution of horizontal hyperplanes is integrable and gives rise to a spacetime foliation through spacelike hypersurfaces: the hypersurfaces of simultaneity. For instance, for the Einstein convention the horizontal hyperplanes are those perpendicular to the 4-velocity field $u(x)$ of the timelike flow and the curvature of the connection is proportional to the vorticity of the

flow.

By itself this constructive choice does not allow to reduce the large arbitrariness already seen in the previous approach through time functions. On the contrary it seems to make it even worse since each time function $t(x)$ has hypersurfaces $t = \text{const.}$ whose tangent hyperplanes determine a connection. However, we now require each allowed simultaneity connection to be a convention, i.e. the distribution of horizontal hyperplanes should depend on the spacetime point only through local measurable quantities related to the spacetime structure and frame. Examples of such quantities are the vorticity $w^\eta = \frac{1}{2}\varepsilon^{\eta\beta\alpha\gamma}u_\beta u_{\alpha;\gamma}$, acceleration $a_\mu = u_{\mu;\alpha}u^\alpha$, expansion and shear of the frame, and the metric or the curvature tensors.

This requirement implies that the observers can determine which events are simultaneous according to the simultaneity convention without the need of global information. For instance, in the Einstein convention one synchronizes its own clock with those of the few observers in its neighborhood: the observer does not need to be in contact with distant observers. Despite the local nature of the procedure the observer knows that the synchronization convention is coherent and that for this reason it will provide a global time variable.

At the mathematical level, let $u(x)$ be the normalized ($u^\mu u_\mu = 1$) 4-velocity field of the congruence of timelike curves. The distribution of horizontal hyperplanes $H_u(x)$ is uniquely determined as the Ker of a 1-form ω normalized so that $\omega(u) = 1$ and such that ω_μ is timelike (otherwise the horizontal hyperplane would not be spacelike). It is convenient to introduce the vector product between the vorticity vector and the acceleration, $m_\alpha = \epsilon_{\alpha\beta\gamma\delta}a^\beta u^\gamma w^\delta$ and limit our analysis only to those spacetime regions where $m_\alpha \neq 0$. We also define $a^2 = -a^\mu a_\mu$, $w^2 = -w^\mu w_\mu$ and $m^2 = -m^\mu m_\mu = a^2 w^2 \sin^2 \theta$ where θ is the angle between the vorticity vector and the acceleration for an observer moving at speed u . Since u^μ , a^μ , w^μ and m^μ are linearly independent any local simultaneity convention takes the form

$$\omega_\alpha = u_\alpha + \psi^m(x)m_\alpha + \psi^a(x)a_\alpha + \psi^w(x)w_\alpha, \quad (11)$$

for suitable functions ψ^m, ψ^a, ψ^w . From the definition of local simultaneity convention it follows that ψ^m, ψ^a, ψ^w , depend on the spacetime event x through the acceleration a , the vorticity w , the angle θ between them, and possibly on other scalars. Note that if the ψ functions are small the simultaneity connection may be considered as a perturbation of Einstein's for which we have $\omega_\alpha = u_\alpha$.

The remaining problem is that of finding a suitable local simultaneity convention that reduces to Einstein's in the Minkowskian-inertial frame case. We make some simplifying assumptions

- (a) The frame is generated by a Killing vector field k .
- (b) The functions ψ^m, ψ^w and ψ^a are constructed from the observable quantities a, w and θ (or equivalently a, w and m with $m = aw \sin \theta$).
- (c) The curvature of the 1-form connection ω_μ is proportional to the Riemann tensor (through contraction with a suitable tensor).

The first two conditions are natural simplifications that allow us to tackle the problem while keeping the calculations at a reasonable size. The last one is imposed since the requirement that the curvature of the corresponding gauge theory vanishes would be too restrictive and no simultaneity connection satisfying that requirement would be eventually found. With our condition (c), at least in the weak field limit, the distribution of horizontal planes becomes integrable providing a useful definition of simultaneity. In particular it becomes exactly integrable in Minkowski spacetime where the Riemann tensor vanishes.

The following theorem holds [8]

Theorem 2.1. *In a stationary spacetime let k be a timelike Killing vector field and set $u = k/\sqrt{k \cdot k}$. Let U be the open set $U = \{x : m(x) > 0 \text{ and } a(x) \neq w(x)\}$. Consider in U the connection*

$$\omega_\alpha = u_\alpha + \psi^m(x)m_\alpha + \psi^a(x)a_\alpha + \psi^w(x)w_\alpha. \quad (12)$$

Let ψ^m, ψ^a, ψ^w , be C^1 functions dependent only on a, w and θ . Then, regardless of the stationary spacetime considered, the connection is timelike in U (and hence it is a simultaneity connection in U) and has a curvature proportional to the Riemann tensor in U only if

$$\psi^m = \frac{a^2 + w^2 - \sqrt{(a^2 + w^2)^2 - 4m^2}}{2m^2}. \quad (13)$$

The theorem selects the simultaneity connection

$$\bar{\omega}_\alpha = u_\alpha + \frac{a^2 + w^2 - \sqrt{(a^2 + w^2)^2 - 4m^2}}{2m^2} m_\alpha, \quad (14)$$

that we call \bar{C} -simultaneity, as the most natural and useful in the weak field limit. It seems remarkable that it differs from the Einstein's simultaneity connection $\omega = u$. The relation of $\bar{\omega}$ with the function $\bar{\delta}$ used in practice can be found in [8]. The \bar{C} -simultaneity connection proves particularly useful in Minkowski spacetime since there it is exactly integrable. Contrary to the Einstein convention that does not provide an integrable foliation for observers in a rotating platform, the \bar{C} -simultaneity connection proves to be integrable in that case. The simultaneity slices turn out to be the same of the inertial observers at rest in the inertial frame i.e. those that observe the rotating platform from the outside. Thus, although the rotating observers could in principle ignore to be at rest in a rotating platform, the \bar{C} -simultaneity convention that they apply allow them to define a natural global time function.

3 Conclusions

In the first section we have presented the proof of the consistence of the Einstein synchronization convention starting from the constancy of the speed of light

over polygonal paths. As far as we know, despite the relevance of this proof for any conceptually rigorous development of special relativity, it could not be found in any special relativity textbook. The only relevant exception is the old 1923 book by Weyl which unfortunately has never been translated from the German (the English translation [14] is of the 1921 German edition which does not contain the proof; the relevant section on Einstein synchronization has recently been translated in [9]). We mention that sometimes the proof is, in our opinion incorrectly, attributed to Reichenbach. As a matter of fact Reichenbach introduced a round-trip axiom - *the speed of light over a closed path is the same in both directions* - which is weaker than Weyl's. Using Reichenbach postulate it is not possible to prove the consistence of Einstein's synchronization convention as Weyl did [9]. By the way, Weyl and Reichenbach were in contact in the early twenties when their books on special relativity appeared [13] so it is not surprising that they considered similar postulates.

At the end of the first section we have pointed out that in many circumstances, even in flat spacetime, the Weyl's theorem can not be applied since the speed of light over closed paths is not constant in rotating frames (the Sagnac effect). Therefore, in the second section we have considered the problem of finding a coordinate time in more general cases, i.e. in general relativity and for non-inertial reference frames. We have introduced the concept of local simultaneity connection and have shown that the requirement of being a convention strongly restricts its functional spacetime dependence. In the end we have stated without proof a theorem which represents a first step towards the search of simultaneity connections of wider applicability than Einstein's.

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