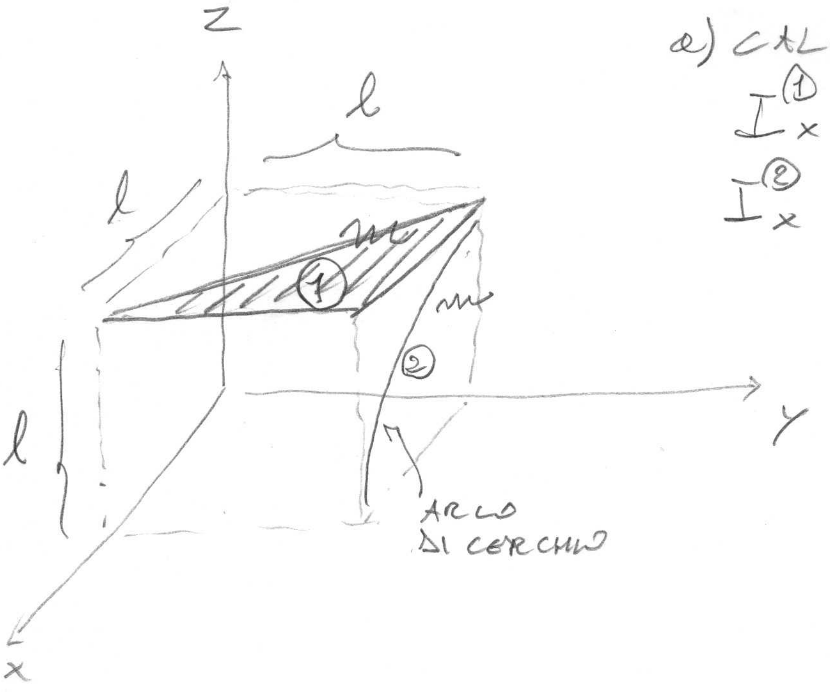
1)



a) CALCOLARE

$I_x^{(1)}, \; I_{xy}^{(1)}, \; I_{xz}^{(1)}$

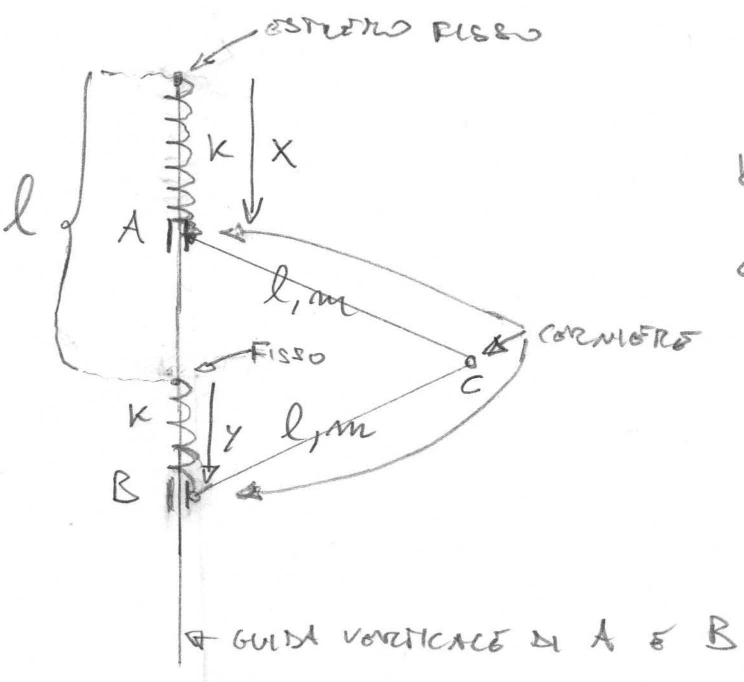
$I_x^{(2)}, \; I_{xy}^{(2)}$

2)

$\downarrow g$

$kl = mg$

ELIMINARE g IN FAVORE DI K.



a) CALCOLARE V.
PUNTI STAZIONARI
STABILITÀ

b) CALCOLARE T

c) PICCOLE OSCILLAZIONI

x, y: COORDINATE GENERALIZZATE.

1) (1) $\rho = \frac{m}{\ell^2/2} = \frac{2m}{\ell^2}$

$$I_x^{(1)} = \frac{2m}{\ell^2}\int_0^\ell (y^2+z^2)\int_{\ell-y}^{\ell} dx\; dy \qquad \text{ut} \quad z=\ell$$

$$= \frac{2m}{\ell^2}\int_0^\ell y\,(y^2+\ell^2)\,dy = \frac{2m}{\ell^2}\left(\frac{y^4}{4}+\frac{\ell^2 y^2}{2}\right)\Big|_0^\ell$$

$$= 2m\ell^2\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{3}{2}m\ell^2$$

$$I_{xy}^{(1)} = -\frac{2m}{\ell^2}\int_0^\ell dy\int_{\ell-y}^{\ell} dx\; xy = -\frac{2m}{\ell^2}\int_0^\ell dy\, y\left\{\frac{\ell^2}{2}-\frac{(\ell-y)^2}{2}\right\}$$

$$= -\frac{2m}{\ell^2}\int_0^\ell dy\, y\left\{-\frac{y^2}{2}+\ell y\right\} = -\frac{2m}{\ell^2}\left\{-\frac{y^4}{8}+\frac{\ell y^3}{3}\right\}$$

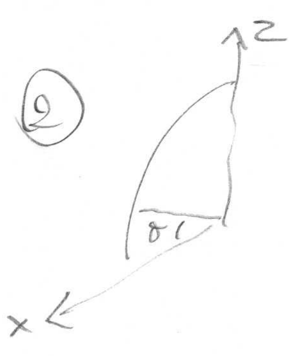
$$= -\frac{2m}{\ell^2}\left\{-\frac{\ell^4}{8}+\frac{\ell^4}{3}\right\} = -2m\ell^2\frac{5}{24} = -\frac{5}{12}m\ell^2$$

$$I_{xz}^{(1)} = -\frac{2m}{\ell^2}\int_0^\ell dy\int_{\ell-y}^{\ell} dx\; xz = -\frac{2m}{\ell}\int_0^\ell dy\int_{\ell-y}^{\ell} dx\; x$$

$$= -\frac{2m}{\ell}\int_0^\ell dy\left\{\frac{\ell^2}{2}-\frac{(\ell-y)^2}{2}\right\} = -\frac{2m}{\ell}\int_0^\ell dy\left\{-\frac{y^2}{2}+\ell y\right\}$$

$$= -\frac{2m}{\ell}\left(-\frac{\ell^3}{6}+\frac{\ell^3}{2}\right) = -\frac{2}{3}m\ell^2$$

(2)



$dm = \rho\, d\theta \qquad \rho = \frac{2m}{\pi} \qquad x = \ell\cos\theta,\; y = \ell,\; z = \ell\,\mathrm{sen}\,\theta$

$$I_x^{(2)} = \int (y^2+z^2)\,dm = \frac{2m}{\pi}\ell^2\int_0^{\pi/2}(1+\mathrm{sen}^2\theta)\,d\theta$$

$$= \frac{2m\ell^2}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = \frac{3}{2}m\ell^2$$

$$I_{xy}^{(2)} = -\int xy\,dm = -\frac{2m}{\pi}\ell^2\int_0^{\pi/2}\cos\theta\,d\theta = -\frac{2m}{\pi}\ell^2$$

a)

IL BARICENTRO HA LA STESSA QUOTA DI C OVVERO $-\dfrac{x+l+y}{2}$

QUINDI

$$V = -2mg\,\frac{x+l+y}{2} + \frac{1}{2}k\left(x^2+y^2\right) = -kl\left(x+l+y\right) + \frac{1}{2}k\left(x^2+y^2\right)$$

$$= k\left\{-l\left(x+y\right) + \frac{1}{2}\left(x^2+y^2\right)\right\} + \text{cost.}$$

$$\left.\begin{aligned}\frac{\partial V}{\partial x} &= k(-l+x) = 0\\ \frac{\partial V}{\partial y} &= k(-l+y) = 0\end{aligned}\right\} \Rightarrow \begin{aligned} x &= l \\ y &= l\end{aligned}$$

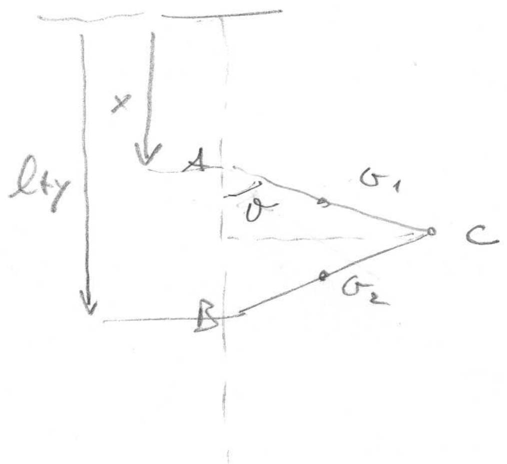
PUNTO STAZIONARIO (L'ANGOLO IN C È 60°)

$$\frac{\partial^2 V}{\partial x^2} = k = \frac{\partial^2 V}{\partial y^2} \qquad \frac{\partial^2 V}{\partial x \partial y} = 0$$

$$B = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

DEFINITA POSITIVA ⇒ EQ. STABILE

b)



$$l\cos\theta = (l+y-x)/2$$

IN OPPORTUNE COORDINATE CARTESIANE:

$$G_1 = \left(\frac{l}{2}\sin\theta,\ x + \frac{l+y-x}{4}\right)$$

$$\dot{G}_1 = \left(\frac{l}{2}\cos\theta\,\dot\theta,\ \frac{3\dot x}{4} + \frac{\dot y}{4}\right)$$

$$\dot{G}_1^{\,2} = \frac{l^2}{4}\cos^2\theta\,\dot\theta^2 + \left(\frac{3}{4}\dot x + \frac{1}{4}\dot y\right)^2$$

$$G_2 = \left(\frac{l}{2}\sin\theta,\ l+y - \frac{l+y-x}{4}\right)$$

$$\dot{G}_2 = \left(\frac{l}{2}\cos\theta\,\dot\theta,\ \frac{1}{4}\dot x + \frac{3}{4}\dot y\right)$$

$$\dot{G}_2^{\,2} = \frac{l^2}{4}\cos^2\theta\,\dot\theta^2 + \left(\frac{1}{4}\dot x + \frac{3}{4}\dot y\right)^2$$

$$T = \frac{1}{2} m \left(\dot{G}_1^2 + \dot{G}_2^2 \right) + 2 \, \frac{1}{2} \left(\frac{1}{12} m l^2 \right) \dot{\vartheta}^2$$

$$= \frac{1}{2} m \left\{ \frac{l^2}{2} \cos^2\vartheta \, \dot{\vartheta}^2 + \left(\frac{3}{4} \dot{x} + \frac{1}{4} \dot{y} \right)^2 + \left(\frac{1}{4} \dot{x} + \frac{3}{4} \dot{y} \right)^2 \right\}$$

$$+ \frac{1}{12} m l^2 \dot{\vartheta}^2 \qquad *$$

DERIVANDO RISPETTO AL TEMPO $2l \cos\vartheta = l + y - x$

$$-2l \sin\vartheta \, \dot{\vartheta} = \dot{y} - \dot{x}$$

MA $\quad \sin\vartheta = \sqrt{1 - \cos^2\vartheta} = \sqrt{1 - \left(\frac{l + y - x}{2l} \right)^2}$

QUINDI

$$\dot{\vartheta} = \frac{\dot{x} - \dot{y}}{2l} \left(1 - \left(\frac{l + y - x}{2l} \right)^2 \right)^{-1/2} \qquad **$$

E INFINE

$$T = \frac{1}{2} m \left\{ \frac{l^2}{2} \left(\frac{l + y - x}{2l} \right)^2 \frac{(\dot{x} - \dot{y})^2}{4l^2} \left(1 - \left(\frac{l + y - x}{2l} \right)^2 \right)^{-1} \right.$$

$$+ \frac{5}{8} \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{6}{8} \dot{x} \dot{y} +$$

$$\left. + \frac{1}{24} (\dot{x} - \dot{y})^2 \left(1 - \left(\frac{l + y - x}{2l} \right)^2 \right)^{-1} \right\}$$

c)

PER LE PICCOLE OSCILLAZIONI CALCOLIAMO $*$ NEL PUNTO STAZIONARIO USANDO $**$ E $\vartheta = 60°$, $x = y = l$

QUINDI $\quad \dot{\vartheta} = \frac{\dot{x} - \dot{y}}{\sqrt{3} \, l}$, $\cos\vartheta = 1/2$

$$T = \frac{1}{2} m \left\{ \frac{\cancel{l^2}}{2} \frac{1}{4} \frac{(\dot{x} - \dot{y})^2}{3 \cancel{l^2}} + \frac{5}{8} \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{6}{8} \dot{x} \dot{y} \right.$$

$$\left. + \frac{1}{6} \cancel{l^2} \frac{(\dot{x} - \dot{y})^2}{3 \cancel{l^2}} \right\}$$

$$= \frac{1}{2} m \left\{ \frac{13}{18} \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{10}{18} \dot{x} \dot{y} \right\}$$

$$\Rightarrow \quad A = \frac{m}{18} \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix} \quad \ldots$$