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Ph.D. Thesis

Teleparallelism and Simultaneity in Relativity

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Chapter 1

Introduction

In this thesis I present the work I did during the last three years under the supervision of my advisor, Prof. C. Destri. It is based on two publications and a preprint [55, 57, 56] and on some unpublished material.

I worked on two distinct topics: teleparallelism and simultaneity in relativity. In both cases I applied the same guiding principles thus arriving to a research that has shared the same fruitful ideas in different areas of physics. In this introduction I outline my work, emphasizing its underlying unity of methods and ideas. I also introduce some notations and concepts from gauge theory.

Throughout the thesis the conventions are those of "Gravitation" [58] except for the Minkowski metric which is fixed by $\eta_{00} = 1$ (the timelike convention).

We use the Greek alphabet (μ, ν, ρ,... =0,1,2,3) to denote indices related to spacetime, and the Latin alphabet (a, b, c,... =0,1,2,3) to denote indices related to the internal space of local orthonormal frames . The totally antisymmetric tensor is normalized by $\epsilon_{0123} = 1$. The differential geometry conventions are those of [44] except for the exterior product which is fixed by $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ (α and β are 1-forms) in order to be consistent with the conventions of [58]. The rules involving exterior product and exterior differentiation are therefore those of Spivak [84].

1.1 Invariance and conventionality

The invariance of the laws of physics under a Lie group has emerged several times in twentieth century physics. We can roughly distinguish between two kinds of invariance: internal space invariance, where no reference to the spacetime coordinates used by the observer is made, and external space invariance where each transformation connects the outcome of the observations of different observers. To the latter case belongs the Lorentz group which connects the observations of inertial observers (frames), or the group of cosmological isometries, that connects observers at rest in the cosmological fluid but placed in different points of the universe.

Let us consider two observer K and K' that label the spacetime events using the same operative conventions (for instance, they construct polar coordinates following the instructions of the same physics book). Suppose, moreover, that the laws of physics in the shared conventions, turn out to be the same. This means that looking at their local laboratory, they are not able to characterize their relative position and motion. Let $x'^{\mu} = f^{\mu}_{K'K}(x)$ be the transformations between the outcomes of their observations (in this case the labelling of spacetime events). Let us define $f_{KK'} = f^{-1}_{K'K}$. Given a third observer K'', we have two more functions $f_{K''K}$ and $f_{K''K'}$. Given the outcome x^{μ} of K, x''^{μ} can be obtained passing from x to x' and then to x'' or directly from x to x''. This implies that

$$f_{K''K} = f_{K''K'} \circ f_{K'K}. \tag{1.1}$$

Moreover, the function $f_{K'K}$ should depend on a set of continuous parameters $\alpha^i_{K'K}$ (like the Euler angles, the relative speed and so on) which identify unambiguously the reciprocal motion and position. One arrives at

$$f(\alpha^i_{K''K}) = f(\alpha^i_{K''K'}) \circ f(\alpha^i_{K'K}) \quad \Rightarrow \quad \alpha^i_{K''K} = \alpha^i(\alpha^j_{K''K'}, \alpha^k_{K'K}). \tag{1.2}$$

Thus the observational invariance (given a shared convention) of the laws of physics implies the existence of a Lie group of transformations between those observers that share that invariance.

When the set of observer considered is that of inertial frames, the invariance of the laws of physics is termed "relativity principle". The previous analysis shows that this implies a group of transformations. The experience has told us that, in flat spacetime, such group is not the Galilei group but the Poincaré group.

When the set of observer consists of those at rest with respect to the cosmological fluid, the invariance of the laws of physics (in which we include the appearance of the universe) is termed "cosmological principle". In this case, the experience has not yet determined the group of isometries, since there are at least three (Friedmann-Robertson-Walker) isotropic models with different isometry groups.

What I wish to emphasize here is that the invariance group does not depend on the way the observers label the events, and not even on the experimental outcome that they have chosen to compare. We can change the way the observers label the events but the Lorentz group will always emerge. What changes is solely its representation $f(\Lambda)$. There is nothing special in the usual matrix form that the Lorentz group has in every textbook. It simply depends on the convention we have used to label events (and hence on a particular simultaneity convention, see II) but it is only one of many different representations. For instance, I want to cite here that, if two inertial observers at the same place take a photograph of an object, the resulting images are related by a rational linear transformation where complex coordinates have been introduced on the pictures [71, 89]

$$z' = \frac{az+b}{cz+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}). \tag{1.3}$$

A result of this kind could have been expected from the previous reasoning as long as one admits that all the information on the group transformation can be recovered from the comparison of two pictures.

1.1.1 Simultaneity

The considerations of the previous section introduce the main ideas underlying my work. Like the Lorentz group in the previous example, many concepts of physics are independent from the conventions adopted. To point out such independence can be very useful for a deeper understandings of physics. Moreover, such independent concepts, like the causal structure, the Lorentz group, or the constancy of the speed of light over closed paths, can be deduced from each other without relying on conventions that hide, under implicit assumptions, the basic physical facts.

This idea was applied in my work [57], where I showed that the causal structure of Minkowski spacetime follows solely from the constancy of the speed of light over closed paths. In this way I circumvented an old problem regarding the conventional nature of the isotropy of the speed of light. I proved that the convention-free concept of causal structure follows from the convention-free concept of two-way speed of light, thus providing new reliability on the local Minkowski nature of the spacetime manifold. In that paper I studied in detail the constraint imposed by the constancy of the speed of light over closed paths (L/c law) on the allowed conventions. Non-isotropic conventions turned out to be possible. Moreover I showed that after a suitable clock resynchronization they become isotropic. Thus, apart from time coordinate changes, I concluded that the speed of light is isotropic. Stated in another way, I proved, making use of the L/c law, the existence of a time coordinate that makes the speed of light isotropic, a feature always assumed without proofs in previous works on special relativity.

The need for an approach independent of the convention chosen (synchronization convention in the present case) led also to my second work on this topic [56]. Instead of expressing each quantity in a convention-free notation it is possible to write it in a given synchronization convention and then to prescribe definite rules to change it when passing from one convention to another. In this context, I showed that the concept of simultaneity in general relativity can be treated mathematically as a gauge theory. I proved that since simultaneity is definable in different ways, many objects, under a resynchronization of clocks, transform as in gauge theory. There is the analogue of the gauge potential, of the strength tensor, of the Bianchi identities and of the holonomy. In order to present these aspects, it is however necessary to introduce a generalized mathematical setting for gauge theories. This setting, already studied by mathematicians, makes use of the Frölicher-Nijenhuis bracket, and of some other geometrical concepts that have not yet found a wide audience in the physics community.

1.1.2 Teleparallelism

Every physicist should agree on the necessity of formulating theoretical models in a way independent of conventions. Nevertheless very often it happens to find in literature physical theories which are not invariant under, for instance, coordinate changes. In general relativity, where the metric is a dynamical entity this turns out to be a serious drawback. For instance, earlier attempts to quantize gravity [40] involved a splitting of the metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1.4}$$

Even if this splitting is well defined in classical physics, its interpretation is difficult in a quantum theory. In quantum theory the perturbation $h_{\mu\nu}$ receives a particle interpretation and thus should be well defined i.e. independent of the particular coordinate system chosen*. It is trivial to realize that, however, it does not transform as a tensor under coordinate transformation. It is hard to imagine how a coordinate independent quantum gravity can arise from objects like that. Moreover, the background $g_B = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ depends on the coordinate system chosen. One can try to avoid this problem by saying that the correct background has the previous expression but only in the right system of coordinates. In that case however, one has to determine that system of coordinates in some ways, that is, some prescription must be given to find a privileged labelling $\{x^\mu\}$. No such prescription is present in general relativity. This problem, also known as the background structure problem, has received little attention in literature. Nevertheless, most calculations on quantum gravity [96] start from Eq. (1.4); it is then of primary importance to investigate this issue.

In the paper [55] I addressed this problem making use of gauge theories and teleparallel theories. I showed that a covariant splitting of the metric can be re-

*In the Unruh effect the relation between the usual Minkowski coordinates and Rindler coordinates leads to a different particle content. Thus there, our qualitative reasoning seems not to work. However, in that case the coordinate change can be seen as an active transformation that relates a static detector to an accelerated detector and hence one can expect a different particle content. On the contrary, most of the coordinates that we are considering here cannot receive similar active interpretations because they are obtained by generic coordinate changes.

alized by relating the vierbein of general relativity with the soldering form of a four dimensional gauge group. As a byproduct I proved that general relativity can be considered as a gauge theory over the group $U(2)$. The teleparallel theory was used only as a tool useful to transform the Einstein-Hilbert action in a form explicitly invariant under gauge transformations. The Lagrangian was proved invariant under $U(2)$ gauge transformation and quadratic in the strength tensor. The first term of Yang-Mills type is followed by two more terms. In this work I thus was able to show the formal similarities between gravity and Yang-Mills theory.

1.2 Some notations of gauge theories

Gauge theories were axiomatized in the fifties by Ehresmann [44] as connections over principal bundles. Since I need to fix the notation, here I shortly review that setting. A principal bundle is given by a differentiable manifold (the bundle) P, a differentiable manifold (the base) M, a projection

$$\pi : P \to M \tag{1.5}$$

a Lie group G, and a right action of G on P

$$p \to pg \qquad p \in P, \quad g \in G \tag{1.6}$$

such that $M = P/G$, i.e. M is the orbit space. Moreover, the fiber bundle P is locally the product $P = M \times G$. To be more precise, given a point $m \in M$ there is an open set U of m, such that $\pi^{-1}(U)$ is diffeomorphic to $U \times G$ and the diffeomorphism preserves the right action. If this property holds also globally the principal bundle is called trivial. The set $\pi^{-1}(m)$ is the fiber of m and it is diffeomorphic to G. Let $\mathcal{G}$ be the Lie algebra of G, and let τ_a be a base of generators

$$[\tau_a, \tau_b] = f^c_{ab}\tau_c. \tag{1.7}$$

Let $p \in P$ be a point of the principal bundle; it can be considered as an application $p : G \to P$ which acts as $g \to pg$. The fundamental fields† τ^*_a over P are defined in p as the pull-forward of the group generators: $\tau^*_a = p_*\tau_a$. They are vertical fields in the sense that they are in the ker of π: $\pi_*(\tau^*_a) = 0$. They form a base of the vertical tangent space at p.

A connection over P is a 1-form $\omega : P \to \mathcal{G}$ with the following properties

(a) $\omega(X^*) = X \qquad X \in \mathcal{G}$

†We follow mostly the conventions of Kobayashi-Nomizu. The upper star * indicates the pull-back when applied to a function, the fundamental field when applied to a generator, and the horizontal lift when applied to a curve or a tangent vector on the base.

(b) $R_g^* \omega = g^{-1} \omega g$

The tangent space at p is split into the sum of two subspaces: the vertical space, that is the ker of π, and the horizontal space, that is the ker of ω

$$T_p P = H_p \oplus V_p. \tag{1.8}$$

Let U be an open set of M. A section σ is a function $\sigma : U \to \pi^{-1}(U)$ such that $\pi \circ \sigma = I_U$. The gauge potential depends on the section and is defined by

$$A = \tau_a A_\mu^a \mathrm{d}x^\mu = \sigma^* \omega \tag{1.9}$$

where $\{x^\mu\}$ are coordinates on the base. A change of section is a gauge transformation. The curvature is defined by[‡]

$$\Omega = d\omega h = \mathrm{d}\omega + \omega \wedge \omega \tag{1.10}$$

where h projects the vector arguments to the horizontal space[44]. The field strength is defined by $F = \tau_a F_{\mu\nu}^a \mathrm{d}x^\mu \mathrm{d}x^\nu = \sigma^* \Omega$. In other words

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c. \tag{1.11}$$

Given a section one can construct a system of coordinates over P in a canonical way. Simply let (x, g) be the coordinates of the point $p = \sigma(x) g$. In this coordinates the connection can be rewritten

$$\omega = g^{-1} \mathrm{d}g + g^{-1} A g, \tag{1.12}$$

and the curvature can be rewritten

$$\Omega = g^{-1} F g, \tag{1.13}$$

indeed the form of the connection given here satisfies both the requirements above and $A = \sigma^* \omega$. From these last equations one easily recovers the gauge transformation roles after a change of section $\sigma' = \sigma u(x)$ $(g' = u^{-1}(x) g)$, that is

$$A'_\mu = u^{-1} A_\mu u + u^{-1} \partial_\mu u \tag{1.14}$$
$$F'_{\mu\nu} = u^{-1} F_{\mu\nu} u \tag{1.15}$$

The horizontal lift $v^* \in TP_p$ of a vector $v \in TM_m$ in $p \in \pi^{-1}(m)$ is the only horizontal vector which projects to v: $\pi_* v^* = v$. In general

$$v^* = \sigma_* v - A^a(v) \tau_a^* \tag{1.16}$$

[‡]In our notation the exterior product is defined through $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ where α and β are 1-forms. As a consequence $\omega \wedge \omega = [\omega, \omega]$

as it can be realized applying ω and π_* to both sides. The horizontal lift τ^* of a curve τ on M is a curve on P whose tangent vectors are horizontal lifts of tangent vectors of τ. It is determined uniquely once its starting point is given on the fiber of the starting point of τ.

Given an anholonomic basis e_i, $[e_i, e_j] = C^k_{ij} e_k$ on M, its horizontal lift to $p \in P$ gives a basis for the horizontal space at p. Hence, the fields $\{\tau^*_a, e^*_i\}$ form a basis for the tangent space of any point p. It can be shown that their commutation coefficients are

$$[\tau^*_a, \tau^*_b] = f^c_{ab}\tau^*_c \tag{1.17}$$
$$[\tau^*_a, e^*_i] = 0 \tag{1.18}$$
$$[e^*_i, e^*_j] = C^k_{ij} e^*_k - \Omega^a_{ij}\tau^*_a \tag{1.19}$$

Finally a matter field is a global section $\psi : M \to E$ where $E = P \times_G \mathbb{F}^m$ is an associated fiber bundle with fiber $\mathbb{F}^m$, $\mathbb{F} = \mathbb{C}, \mathbb{R}$, and G acts on the left of $\mathbb{F}^m$ with the matrix representation ρ. The matrix representation induces a matrix representation ρ_* of the Lie algebra. I shall abbreviate $\rho_*(X) = X^\rho$ The points of E are obtained from $P \times \mathbb{F}^m$ through the identification $(p, f) \sim (pg, \rho(g)^{-1} f)$. Given a section $\sigma : U \to P$ one can choose, for each fiber, the representative $p = \sigma(x)$, so that the section ψ becomes $\psi = (\sigma(x), \psi_\sigma(x))$. In physics one usually uses $\psi_\sigma(x) : U \to \mathbb{F}^m$ instead of ψ, and, for short, removes the subscript σ which reminds the dependence on the section σ. Herein I shall follow the same convention. The covariant derivative of a section $\psi : U \to \mathbb{F}^m$ gives a new section $\nabla_X \psi$. The operator ∇_X is defined by

$$\nabla_X = \partial_X + A^a(X)\tau^\rho_a. \tag{1.20}$$

It transforms covariantly under gauge transformations and satisfies

$$[\nabla_i, \nabla_j] = C^k_{ij}\nabla_k - F^a_{ij}\tau^\rho_a \tag{1.21}$$

This is the usual trick through which the field strength is obtained in physics textbooks.

1.3 Generalized gauge theories

Remarkably the structure of gauge theories can be generalized [80, 54, 59] to the case of a fibred manifold $\pi : F \to M$ without the need of the right action of a Lie group on F. This not widely known generalization, maintains many properties of the original mathematical structure. The application on simultaneity, that I shall present in chapter 6, is probably one of the few available.

1.3.1 The Frölicher-Nijenhuis algebra

Let F be a n-dimensional manifold. The Lie bracket acts on a pair of vectors fields v_1 and v_2, giving a third vector field v_3. In a coordinate system $\{z^\mu\}$

$$v_3^\alpha = [v_1, v_2]^\alpha = v_1^\beta\, v_{2,\beta}^\alpha - v_2^\beta\, v_{1,\beta}^\alpha. \tag{1.22}$$

It makes the space $\mathcal{T}(F)$ of vector fields over F, a Lie algebra. Let $\bigwedge \mathcal{T}^*(F) \otimes \mathcal{T}(F) = \sum_r \oplus \overset{r}{\wedge}\ \mathcal{T}^*(F) \otimes \mathcal{T}(F)$ be the space of vector valued forms

$$K = K^\mu_{\lambda_1 \ldots \lambda_k} \frac{1}{k!}\, dz^{\lambda_1} \wedge \ldots \wedge dz^{\lambda_k} \otimes \partial_\mu. \tag{1.23}$$

The Frölicher-Nijenhuis bracket is defined on $\bigwedge \mathcal{T}^*(F) \otimes \mathcal{T}(F)$ that, with this bracket, becomes a graded Lie algebra,

$$[\,,]_{FN} : \overset{k}{\wedge}\, \mathcal{T}^*(F) \otimes \mathcal{T}(F) \times \overset{l}{\wedge} \mathcal{T}^*(F) \otimes \mathcal{T}(F) \to \overset{k+l}{\wedge}\, \mathcal{T}^*(F) \otimes \mathcal{T}(F) \tag{1.24}$$

through

$$\begin{aligned}
[K, L]_{FN}(X_1, \ldots, X_{k+l}) \;=& \sum_\sigma \mathrm{sign}\sigma \{ \frac{1}{k!l!} [K(X_{\sigma 1}, \ldots, X_{\sigma k}), L(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})] \\
&+ \frac{-1}{k!(l-1)!} L([K(X_{\sigma 1}, \ldots, X_{\sigma k}), X_{\sigma(k+1)}], X_{\sigma(k+2)}, \ldots) \\
&+ \frac{(-1)^{kl}}{l!(k-1)!} K([L(X_{\sigma 1}, \ldots, X_{\sigma l}), X_{\sigma(l+1)}], X_{\sigma(l+2)}, \ldots) \\
&+ \frac{(-1)^{k-1}}{(k-1)!(l-1)!2!} L(K([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \ldots), X_{\sigma(k+2)}, \ldots) \\
&+ \frac{(-1)^{(k-1)l}}{(k-1)!(l-1)!2!} K(L([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \ldots), X_{\sigma(l+2)}, \ldots) \}.
\end{aligned}$$

It reduces to the Lie bracket for vector valued 0-forms (vector fields). Its coordinate expression reads,

$$\begin{aligned}
[K, L]_{FN} \;=& (K^\beta_{\lambda_1 \ldots \lambda_k} \partial_\beta L^\mu_{\lambda_{k+1} \ldots \lambda_{k+l}} - (-1)^{kl} L^\beta_{\lambda_1 \ldots \lambda_l} \partial_\beta K^\mu_{\lambda_{l+1} \ldots \lambda_{k+l}} - k K^\mu_{\lambda_1 \ldots \lambda_{k-1}\beta} \partial_{\lambda_k} L^\beta_{\lambda_{k+1} \ldots \lambda_{k+l}} \\
+ & \; (-1)^{kl} l L^\mu_{\lambda_1 \ldots \lambda_{l-1}\beta} \partial_{\lambda_l} K^\beta_{\lambda_{l+1} \ldots \lambda_{k+l}}) \frac{1}{(k+l)!}\, dz^{\lambda_1} \wedge \ldots \wedge dz^{\lambda_{k+l}} \otimes \partial_\mu
\end{aligned}$$

Moreover, it satisfies,

$$\begin{gathered}
[\phi, \sigma]_{FN} = -(-1)^{|\phi||\sigma|} [\sigma, \phi]_{FN} \\
(-1)^{|\theta||\sigma|} [\theta, [\phi, \sigma]_{FN}]_{FN} + (-1)^{|\sigma||\phi|} [\sigma, [\theta, \phi]_{FN}]_{FN} + (-1)^{|\phi||\theta|} [\phi, [\sigma, \theta]_{FN}]_{FN} = 0
\end{gathered} \tag{1.25}$$

where $\phi, \sigma, \theta \in \bigwedge \mathcal{T}^*(F) \otimes \mathcal{T}(F)$ and by $|\phi|$ is meant the degree of the form ϕ. If ϕ and σ have the factorized form $\phi = \alpha \otimes u$ and $\sigma = \beta \otimes v$, the Frölicher-Nijenhuis bracket becomes

$$\begin{aligned}[][\phi, \sigma]_{FN} &= [\alpha \otimes u, \beta \otimes v]_{FN} = \frac{|\alpha|!\,|\beta|!}{(|\alpha| + |\beta|)!} \big(\alpha \wedge \beta \otimes [u, v] + \alpha \wedge L_u \beta \otimes v \\ &-(-1)^{|\alpha|\,|\beta|} \beta \wedge L_v \alpha \otimes u + (-1)^{|\alpha|} (v \rfloor \alpha) \wedge d\beta \otimes u - (-1)^{|\alpha|\,|\beta| + |\beta|} (u \rfloor \beta) \wedge d\alpha \otimes v\big),\end{aligned}$$

where $v \rfloor \alpha$ is the contraction

$$v \rfloor \alpha = v^\mu \alpha_{\mu\lambda_2 \ldots \lambda_{|\alpha|}} \frac{1}{(|\alpha| - 1)!} dz^{\lambda_2} \wedge \ldots \wedge dz^{\lambda_{|\alpha|}}. \tag{1.26}$$

and L_u is the Lie derivative with respect to u.

1.3.2 Generalized connection

Let now F be a fibred manifold $\pi : F \to M$. A generalized connection over F is a distribution of "horizontal" planes H_f, $f \in F$, of dimension $\dim M$ whose projection satisfies $\pi_* H_f = T_{\pi(f)} M$. The tangent space at $f \in F$ splits into the sum of $V_f = \ker \pi_*$ and H_f,

$$TF_f = H_f \oplus V_f. \tag{1.27}$$

There are many ways to characterize H_f. For instance, given the distribution H_f, one has a projection P, $P \circ P = P$, on the vertical space. P is thus a vector valued one form over F. Conversely if one has a vector valued one form P which is a projection on the vertical space, one has a distribution of planes $H_f = \ker P(f)$. Thus P can be called the connection of F [54]. The curvature is then a vector valued two-form defined through[§].

$$2R = -[P, P]_{FN}. \tag{1.28}$$

The Bianchi identity follows from Eq. (1.25),

$$[P, R]_{FN} = 0. \tag{1.29}$$

If F is a principal bundle, its vertical space at $f \in F$ can be identified with the Lie algebra of the structure group. For this reason, in that case, the connection and the curvature are Lie algebra valued forms.

[§]Here the sign differs from that of [54] because we want to be consistent with the one for the principal bundle case (see chapter 6)

Part I

Teleparallelism

Chapter 2

Teleparallelism and gauge gravity theories

In this chapter I review some basic facts about teleparallel gravity theories. Throughout part I, $c = \hbar = 1$.

2.1 A lesson from spinors

In Minkowski spacetime the Dirac Lagrangian is given by:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{2.1}$$

where γ^μ are the gamma matrices: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. How can this Lagrangian be generalized to the curved spacetime of general relativity? The answer came to Cartan and Weyl [66]. They introduced a vector basis e_a^μ in a covering of open sets $U \subset M$. The spacetime metric took the form $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$. That basis was nothing but a section of the bundle of orthonormal linear frames $L(M)$. Spinors transforms with the linear representations of the double covering $SL(2, C)$ of $SO(1, 3)$. Since the bundle of linear frames is already a principal bundle with structure group $SO(1, 3)$, neglecting for a moment the difference between the Lorentz group ad its double covering, and recalling the definition of matter field given in paragraph 1.2, one suspects that we have all the ingredients to introduce a generalization of the Dirac Lagrangian to curved spacetimes. It is:

$$\mathcal{L}_{\text{Dirac}} = \det(e_\mu^a) \left\{ \frac{i}{2}(\bar{\psi}\gamma^a e_a^\mu \nabla_\mu \psi - e_a^\mu \overline{\nabla_\mu \psi} \gamma^a \psi) - m\bar{\psi}\psi \right\} \tag{2.2}$$

where

$$\nabla_\mu = \partial_\mu + \frac{1}{2}\Sigma^{ab} e_a^\nu \frac{\partial}{x^\mu} e_{b\nu} = \partial_\mu + \rho_*({\Gamma^c}_{d\mu}) \tag{2.3}$$

and $\Sigma^{ab} = \rho_*(J^{ab})$ are the generators of the Lorentz group in the ρ representation. We recognize, in the last expression, the covariant derivative of gauge theories.

When the difference between the Lorentz group and its double covering is taken into account, some surprises arise. Let us consider an n-dimensional manifold M with metric $g_{\mu\nu}$ of signature (p, q) $(p + q = n)$. Speaking rigorously, a spinor structure over (M, g) is a pair (P, Λ), where P is a principal bundle with base M and structure group $Spin(p, q)$, and $\Lambda : P \to L(M)$ is a local isomorphism that commutes with the right action of the respective structure groups. We recall that the group $Spin(p, q)$ is that generated by the even invertible elements of the Clifford algebra defined through

$$e_a e_b + e_b e_a = 2\eta_{ab} \tag{2.4}$$

where η has p plus signs and q minus signs.

If n is even, a spinor is a global sections of the associated bundle $P\times\mathbb{C}^{n/2}/Spin(p, q)$, where $Spin(p, q)$ acts on $\mathbb{C}^{n/2}$ with a representation ρ that cannot be recovered from a representation ρ' of $SO(p, q)$ through the composition with the double covering $d : Spin(p, q) \to SO(p, q)$. That is, for any ρ', $\rho \neq \rho' \circ d$.

Since mathematically spinor particles, as far as we know, can be described only with this mathematical setting, the existence of a spinor structure becomes essential in order to account for observations. Haefligern, Milnor, Greub and Petry proved the following theorem [27]

Existence of spinor structures: A manifold M allows spin structures of signature (p, q) if and only if it is orientable, it has a metric g with signature (p, q) and satisfies a topological condition which amounts to require that the second Stiefel-Whitney class vanishes.

However, in four-dimensional spacetime there is an even stronger theorem due to Geroch [29]

Geroch's theorem: In dimension four and signature $(1, 3)$ a non compact manifold M admits a spin structure if and only if it is parallelizable.

Since compact spacetime are forbidden by causality, Geroch's theorem states that *a spacetime admits a spinor structure if and only if it is parallelizable.*

Previously, in order to introduce a Lagrangian for spinors, we saw that a vierbein field e^μ_a was needed in each open set of a covering of M. They were patched in the intersections with spacetime dependent transition functions that took values in the Lorentz group. Now, we realize that that was a too weaker requirement: there must be a single open set, M itself, with a well defined frame $e^\mu_a(x)$ throughout it.

To summarize, from the study of spinors we learn that the very existence of the electron implies (until we find a different mathematical description for spinor particles) that spacetime is parallelizable, and that the vierbein should be one of the protagonist of the field description of gravity. This observation is very important for our purposes , since, as we shall see, the teleparallel formulation of gravity works only in parallelizable spacetimes.

2.2 The torsion and the generalized Poincaré algebra

In this section we introduce the torsion from the point of view of differential geometry. From a more elementary level, the torsion can be introduced as the asymmetric part of an affine connection ∇ over M [58]

$$T(u,v) = \nabla_u v - \nabla_v u - [u,v], \tag{2.5}$$

or, with respect to a basis $e_a = e^\mu_a \partial/\partial x^\mu$,

$$T^a{}_{cb} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} - C^a{}_{cb} \tag{2.6}$$

where the commutation coefficients are defined by $[e_a, e_b] = C^c{}_{ab} e_c$,

$$C^c{}_{ab} = e^c_\alpha(\partial_a e^\alpha_b - \partial_b e^\alpha_a) = e^\mu_a e^\nu_b(\partial_\nu e^c_\mu - \partial_\mu e^c_\nu). \tag{2.7}$$

From Eq. (2.6) it follows that (lowering upper indices on the left, e.g. $C_{acb} = g_{ad} C^d_{cb}$

$$\Gamma_{abc} = \frac{1}{2}(C_{acb} + C_{bac} - C_{cba} + T_{acb} + T_{bac} - T_{cba}). \tag{2.8}$$

Cartan considered the torsion as the generalized exterior derivative of the vierbein field $e^a = e^a_\mu \mathrm{d}x^\mu$. In this way it appears as a $\mathbb{R}^n$-valued two-form

$$T^a = de^a + \Gamma^a{}_b \wedge e^b, \tag{2.9}$$

where $\Gamma^a{}_b = \Gamma^a_{b\mu} \mathrm{d}x^\mu$ is the connection one-form. Its expression resembles that of the curvature (Riemann tensor) two-form

$$R^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b \tag{2.10}$$

Both the connection and the curvature have values in the Lie algebra of the Lorentz group. They transform covariantly under local lorentz transformation of the vierbein field, and are invariant under coordinate transformations. For these reasons, general relativity can be considered as a gauge theory over the Lorentz group. We shall return on this point in the next section.

It is convenient to introduce the torsion in the context of the fiber bundle of orthonormal frames $L(M)$, since there it becomes evident its geometrical meaning. Each point in the space $L(M)$ represents a particular tetrad in a particular event, hence an observer at a given instant. Let $\xi \in \mathbb{R}^n$, the *standard horizontal vector field* $B(\xi)$ *corresponding to* ξ is the unique horizontal field in L(M), whose value in $p \in L(M)$ is such that $\pi(B(\xi)_p) = p(\xi)$, where $p(\xi)$ is the tangent vector of M, that in the tetrad p has components ξ . Roughly speaking $B(\xi)$ connects two tetrads, where the second one is obtained displacing the first one in the direction ξ as seen from the first one. Let e_a be a standard basis on $\mathbb{R}^n$, e.g. $e_0 = (1, 0, \ldots)$, then $B_a = B(e_a)$ generate the action of "go in direction e_a". As an example, $B(e_2)$ can have the meaning of "translate to the right", and $B(e_3)$ can have the meaning of "go straight". I stress that B_a is not the horizontal lift of a field over M.

What is the effect of going one step forward, one step to the right, one step backward and one step to the left? In a flat spacetime one simply returns to the initial position, but in a space with curvature one finds himself rotated and, in a space with torsion, translated from the initial position. To see this, notice that in $L(M)$ the fundamental vector fields J^{ab*}, generators of the Lorentz group, and the standard horizontal fields B_a, form a basis in each point of the principal bundle $L(M)$. One can therefore calculate their Lie bracket. The bracket $[B_a, B_b]$ will answer the previous question.

The answer is* $(B^a = \eta^{ab} B_b)$

$$[J^{ab*}, J^{cd*}] = \eta^{bc} J^{ad*} + \eta^{ad} J^{bc*} - \eta^{ac} J^{bd*} - \eta^{bd} J^{ac*} \tag{2.11}$$

$$[J^{cd*}, B^a] = \eta^{ad} B^c - \eta^{ac} B^d \tag{2.12}$$

$$[B_a, B_b] = -\frac{1}{2} R_{cdab}(x) J^{ab*} - T^c_{ab}(x) B_c \tag{2.13}$$

It is the generalization of the Poincaré algebra to curved spacetimes. The last commutation rule states that after the movement described in our previous question, the observer is rotated of an amount given by the Riemann tensor and displaced of an amount given by the torsion tensor.

It can be shown that the Jacobi identity of the Lie bracket gives the Bianchi identity. Indeed

$$\begin{aligned} 0 &= [B_c, [B_a, B_b]] + [B_b, [B_c, B_a]] + [B_a, [B_b, B_c]] \\ &= -\frac{3}{2}(R_{mnd[c} T^d_{ab]} + R_{mn[ab;c]}) J^{mn*} + 3(R^m_{\ [cab]} + T^d_{[ab} T^m_{c]d} - T^m_{[ab;c]}) B_m \end{aligned}$$

Using the linear independence of the fields J^{mn*}, B_m, one recovers the Bianchi identities.

*I have a proof of my own, but as far as I know this algebra appeared for the first time in [39]

Contrary to the case of flat spacetime, the generalized Poincarè algebra is not a Lie algebra. This is at the origin of the well known difficulties in defining the concept of particle in curved spacetime.

Another interpretation of the torsion comes from the bundle of affine frames $A(M)$. The points in this bundle are the planes tangent to the manifold M. Each plane has its own orientation and the origin of the plane considered can be displaced from the point of contact with the manifold. One can define the parallel transport of a plane along the curve $\tau : \mathbb{R} \to M$ as the rolling without slippering of the plane over τ, in such a way that the orientation of the plane is parallel transported over the manifold. When the curve τ is closed, and the enclosed area

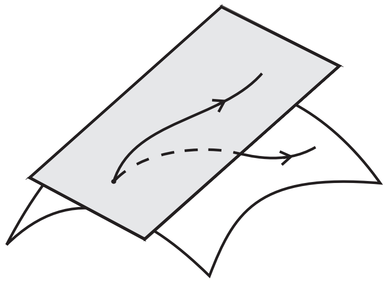

Figure 2.1: Rolling of the plane over M.

is small, the final position of the plane is related to the initial position of the plane through an affine transformation of the form $z^a \to R^a_b z^b + T^a$, where the curvature and the torsion are evaluated on the oriented area enclosed by the path. Keep in mind, however, that even in spaces where $T^a = 0$ the translational part can differ from zero for paths τ that enclose a large area. This behavior differs from that of the curvature. If $R^a_b = 0$ the affine transformation has no rotational part even for a large enclosed area.

2.3 Classification of gravity theories

Despite the success of general relativity many authors felt uneasy with its apparent rigid structure. With the growing interest on gauge theory many attempts were made in order to recast GR in that new setting. The original approach, based on the metric tensor as field variable did not exhibit such a structure. The Palatini method of variation introduced the affine connection $\Gamma^\mu_{\alpha\beta}$ as an independent field variable and the Cartan formalism, with the introduction of the vierbein field e^a_μ, allowed to interpret the affine connection as a connection ${\Gamma^a}_b$ (in the principal bundle sense) with values in the Lie algebra of the structure group $SO(1,3)$. In this formalism the compatibility with the metric was obtained as a field equation of the theory [92, 93].

With the discovery of the Yang-Mills theory for the electroweak model the gauge approach to gravity became even more relevant. A renewed interest pushed many authors to try to recast GR in a formalism which exhibits a gauge invariance. The old approach revealed that gravity could be considered as a gauge theory over the group $SO(1,3)$ but this was obtained with the introduction of an auxiliary field e^a_μ which had no analogue in the Yang-Mills case. Here came the idea of a Poincaré gravity theory. It starts from the observation that the Poincaré group is the semidirect product $ISO(1,3) = SO(1,3) \otimes_s T_4$. As a consequence the Lie algebra can be split in the semidirect sum of the Lie algebras of the Lorentz and the translation group. Furthermore, one requires a connection of the form $\pi^A = (\omega^a_b, e^c)$, that is the same connection that appears in the bundle of affine frames $A(M)$ for a parallel transport defined through the rolling of planes. One of the consequence of this approach was the introduction of torsion as a new dynamical variable aside with the vierbein field [39]. The idea was that, in this way, both the connection and the vierbein appeared as a single entity: the components of a connection in an higher dimensional bundle. On the contrary, in my opinion, things became even more complicated, and in any case the natural physical content of gauge theories was lost. The key point is that Poincaré gauge theories must share the following features

- The dynamics is determined by a Lagrangian invariant under Poincaré gauge transformations.
- Observables must be a gauge invariant.

These features are absent in general relativity and in its "Poincaré" competitors. This is evident if we look at the Einstein Lagrangian

$$\mathcal{L}_E = \frac{1}{32\pi G} R^{ab} \wedge e^c \wedge e^d \epsilon_{abcd}. \tag{2.14}$$

It depends on the gauge component e^a and hence it is not invariant under gauge translation. Moreover in the"Poincarè" gravity candidates the torsion turns out to be related with the spin density. In a Poincaré gauge theory this would be a nonsense since the torsion is not invariant under gauge translations [11]. One concludes that these theories (from now on called Einstein-Cartan theories) are not exactly Poincarè gauge theories but rather gravitational theories with torsion and curvature like Poincaré gauge theories.

The non invariance of the torsion under Poincaré gauge transformation is easily seen in matrix form. In a matrix notation an affine transformation takes the form

$$\begin{pmatrix} z'^a \\ 1 \end{pmatrix} = \begin{pmatrix} O^a_{\ b} & \xi^a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^b \\ 1 \end{pmatrix}$$

Hence the connection and the curvature are (they are Lie-algebra valued)

$$\pi = \begin{pmatrix} \Gamma^a{}_b & e^a \\ 0 & 0 \end{pmatrix} \qquad ; \qquad \Pi = \begin{pmatrix} R^a{}_b & T^a \\ 0 & 0 \end{pmatrix}$$

After a gauge transformation they transform as in formulas (1.14)

$$\pi' = \begin{pmatrix} O^{-1a}{}_c \Gamma^c{}_d O^d{}_b + O^{-1a}{}_c \mathrm{d}O^c{}_b & O^{-1a}{}_c(e^c + \mathrm{d}\xi^c + \Gamma^c{}_d \xi^d) \\ 0 & 0 \end{pmatrix}$$

$$\Pi' = \begin{pmatrix} O^{-1a}{}_c R^c{}_d O^d{}_b & O^{-1a}{}_c(T^c + R^c{}_d \xi^d) \\ 0 & 0 \end{pmatrix}$$

From these formulas we see two basic facts. After a gauge translation the gauge potential is no more a vierbein since it is not related to the metric in the standard way (hence the potential, in general, needs another letter, usually ϕ^a). In the same way the torsion in not invariant under the same transformation (thus one has to use a different letter for the translational part of the curvature, for instance Φ^a). The standard formalism of Einstein-Cartan theories starts from a Poicaré gauge theory but then admits the existence of a peculiar section where the potential field ϕ^a becomes a vierbein. Moreover the value of the curvature Φ in this section, the torsion, is related to the spin. Of course this alters the spirit of gauge theory since the physical interpretation depends on a particular section and on the values of fields in that section.

Not only this. Suppose we want to construct a Poincaré invariant Lagrangian combining R^a_b and Φ^a and some other fields that do not transform under a gauge translation. It is quite clear that because of its transformation rule $\Phi'^a = \Phi^a + R^a{}_d \xi^d$ that translates its value, one cannot find an invariant involving Φ^a. But without a Lagrangian involving Φ it is impossible to deduce a dynamical field equation for ϕ. One can try to reduce the degrees of freedom in such a way that there is no need of another field equation. In general relativity one has the constraint $\Phi = 0$ in a suitable gauge, that however is not allowed in the context of Poincaré gauge theories because it is gauge dependent. There is however a way out. Impose $R^a{}_b = 0$. This condition is meaningful in Poincaré gauge theories since it is gauge independent. What about the Lagrangian? The transformation under gauge translation because of that condition is simplified to $\Phi'^a = \Phi^a$ and one can easily construct invariants that involve contraction of the form $\Phi^a_{\mu\nu} \Phi_{a\alpha\beta} Q^{\mu\nu\alpha\beta}$ with Q non gauge field. Now, these theories are nothing but teleparallel theories, where the curvature vanishes but the torsion is different from zero. Thus one can conclude that if a Poincaré gauge theory for gravity is possible it must be a teleparallel theory. We shall return on teleparallel theories later. In that context they will be considered as gauge theories over the group T_4. The reason is that since $R^a_b = 0$ we can find and work in a section where $\Gamma = 0$. Now, since we do not want to bother about Γ, we always stay in that section and thus forbid non translational gauge transformations that would alter its value. In the same section $\Phi^a = \mathrm{d}\phi^a$ and the residual gauge translational invariance is $\phi'^a = \phi^a + \mathrm{d}\xi^a$. This however do not mean that the theory will be section dependent but only that the rotational degree o freedom of Poincaré gauge theories can be easily eliminated with the Poincaré gauge invariant condition $R^a_b = 0$.

Let us leave Poincaré gauge invariance and return to the most famous candidates for a description of gravity. Given a differential manifold M, a metric $g_{\mu\nu}$ and an affine connection $\Gamma^\mu_{\alpha\beta}$ we define the nonmetricity

$$Q_{\mu\nu} = -\nabla g_{\mu\nu} \tag{2.15}$$

and distinguish gravity theories through the values of the tensors $R^\mu_{\nu\alpha\beta}$, $T^\mu_{\alpha\beta}$ and $Q_{\mu\nu}$ [32]. Historically the most relevant attempt to introduce a non metric compatible connection was that of Weyl [66]. Nowadays most gravity theories set $Q_{\alpha\beta} = 0$. If, moreover, both the Riemann and the torsion tensor are different from zero we are working in Einstein-Cartan space U_4, if the torsion vanishes we recover Riemann spacetime V_4 of general relativity, and if the curvature tensor vanishes but the torsion tensor is different from zero we are in the Weitzenböck space W_4 of teleparallel gravity theories. Finally, if both the curvature and the torsion vanish we recover the trivial flat Minkowski spacetime. Figure 2.2 summarizes this classification. As we shall see the teleparallel geometry offers the possibility of expressing general relativity in a different geometrical language. In that way GR can be interpreted as a gauge theory over the group T_4, but, this time, without the problems just mentioned for Poincaré gravity theories.

2.4 The teleparallel equivalent of general relativity

Teleparallel theories of gravity have a venerable history. In 1928 Einstein† [24] introduced the notion of absolute parallelism in his attempt of unifying gravity and electromagnetism. That project failed but later [60, 62, 69, 70] the idea of a teleparallel geometry was revived as a geometrical alternative to the Riemannian approach of general relativity. Those investigations were motivated by the search of a gravitational theory with a good behavior of the gravitational stress-energy tensor. In particular Møller was looking for a stress-energy tensor covariant under global Lorentz transformations and coordinate transformations. Indeed the study

†An English translation of Einstein's papers can be found in Alexander Unzicker's page at http://www.lrz-muenchen.de/ aunzicker/dg.html.

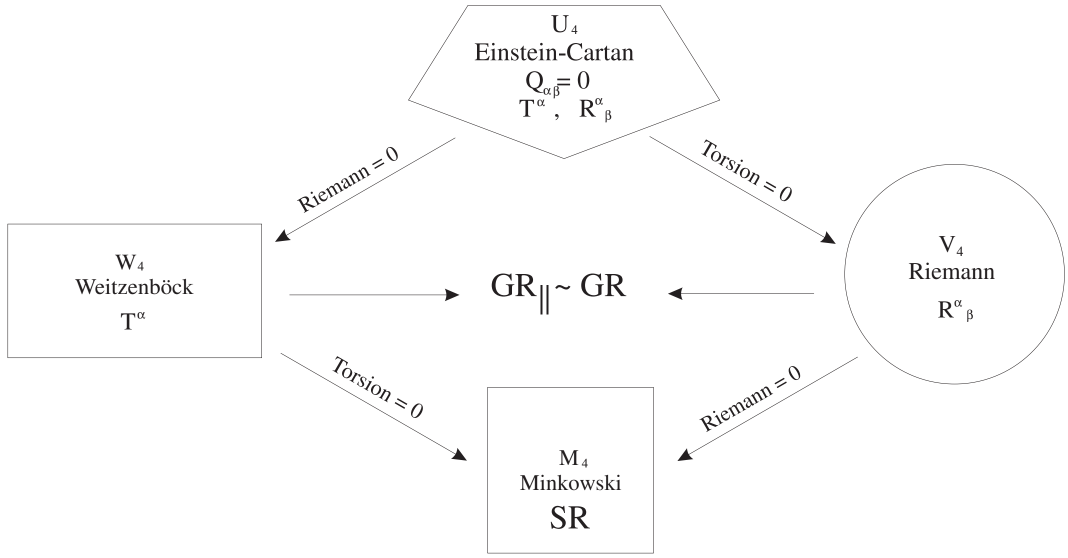

Figure 2.2: Distinction between different geometrical settings in gravity theories. The underlying gravity theories can turn out to be the same $\mathrm{GR}_{\|} \sim \mathrm{GR}$.

of energy-momentum in the teleparallel equivalent of general relativity [18] led to remarkable results, particularly in connection with its covariance, positivity and localization. In particular it led to a pure tensorial proof [65](alternative to Witten's [101]) of the positivity of total energy for Einstein gravity. In this section I present the teleparallel equivalent of general relativity.

A Weitzenböck spacetime W_4 is given by a manifold M, a metric $g_{\mu\nu}$ and a compatible connection $\Gamma^{\mu}_{\alpha\beta}$. Moreover the curvature tensor vanishes but the torsion is, in general, different from zero. Since the Riemann tensor vanishes the parallel transport of vectors does not depend on the path followed, but only on the initial and final points p and q (I assume that M is simply connected). The parallel transport defines a linear isomorphism $\phi_{qp} : TM_p \rightarrow TM_q$ that is associative $\phi_{qr} = \phi_{qp} \circ \phi_{pr}$ so that one can think that the tangent spaces over M are all copies of a single vector space W.

Let us consider an orthonormal tetrad $e_a = e^{\mu}_a \partial/\partial x^{\mu}$ in p; we can transport it unambiguously in every point of spacetime thus obtaining a tetrad field $\hat{e}_a$. We call it "OT frame", from "orthoteleparallel", and we write it in brackets $\{\hat{e}_a\}$. In a teleparallel geometry the OT frame is defined up to global Lorentz rotations because of the initial choice of the tetrad in p. The brackets recall that the OT frame is an equivalence class of tetrad fields. Notice that an OT frame exist only if the manifold is parallelizable, thus a teleparallel geometry is possible only in that case. We have proved, however, that such a condition is not very stringent since it is required also for the existence of spinor fields (see section 2.1).

From a teleparallel geometry follows an OT frame, conversely, given a metric $g_{\mu\nu}$ and an orthonormal tetrad field e_a there is only one curvature-less connection $\Gamma^{\mu}_{\alpha\beta}$ that has OT frame $\{e_a\}$. It is the connection for which a vector is transported in such a way that its components with respect to the orthonormal field remain constant. It is called "Weitzenböck connection" Γ. Its components Γ^a_b in the OT basis vanish, $\hat{\Gamma}^a_b = 0$, since the equation of parallel transport of a vector $v = v^a \hat{e}_a$ in this basis becomes

$$\frac{\mathrm{d}v^a}{\mathrm{d}\tau} = 0. \tag{2.16}$$

In a holonomic system the connection takes the form

$$\Gamma^{\rho}_{\mu\nu} = \hat{e}^{\rho}_{a} \partial_\nu e^a_\mu. \tag{2.17}$$

In a general frame the torsion takes the form

$$T^a = \mathrm{d}e^a + \Gamma^a{}_b \wedge e^b, \tag{2.18}$$

so that the torsion can be written $T^a = \mathrm{d}\hat{e}^a$ or

$$T^{\rho}{}_{\mu\nu} = \hat{e}^{\rho}_{a}(\partial_\mu \hat{e}^a_\nu - \partial_\nu \hat{e}^a_\mu). \tag{2.19}$$

In a Weitzenböck spacetime one has not only the Weitzenböck connection but also the Levi-Civita connection

$$\tilde{\Gamma}^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{2.20}$$

The Levi-Civita connection is torsion-less $\tilde{T}^a = 0$. The difference of the two connections is a tensor, it is called "contorsion tensor"

$$K^{\rho}{}_{\mu\nu} = \Gamma^{\sigma}{}_{\mu\nu} - \tilde{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{2}(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu}). \tag{2.21}$$

The teleparallel geometry is reacher than the Riemannian one, indeed one can define different teleparallel geometry starting form the same metric. One has simply to take a tetrad field and declare that it is an OT frame for a new Weitzenböck connection. In this way one has a Weitzenböck connection (teleparallel geometry) aside with the Levi-Civita connection (Riemannian geometry). Different choices of tetrad field lead to different teleparallel geometries.

From Eq. (2.8) one finds that, in a generic frame, the Levi-Civita connection reads

$$\tilde{\Gamma}_{abc} = \frac{1}{2}(C_{acb} + C_{bac} - C_{cba}) \tag{2.22}$$

Plugging this into the expression for the curvature scalar $\tilde{R}$ one finds, after some algebra [47],

$$\hat{R} = -2C^{a\ \ c}_{\ a\ ,c} - \left\{ \frac{1}{4} C^{abc} C_{abc} + \frac{1}{2} C^{abc} C_{bac} + C^{a}_{\ ac} C^{b\ c}_{\ b} \right\} \tag{2.23}$$

Moreover,

$$\sqrt{-g} C^{a\ \ c}_{\ a\ ,c} = (\sqrt{-g} C^{a\ c}_{\ a} e^{\nu}_{c})_{,\nu} - C^{a\ c}_{\ a} (\sqrt{-g} e^{\nu}_{c})_{,\nu} = \tag{2.24}$$

$$= (\sqrt{-g} C^{a\ c}_{\ a} e^{\nu}_{c})_{,\nu} - \sqrt{-g} C^{a\ c}_{\ a} e^{\nu}_{c;\nu} = v^{\nu}_{\ ,\nu} - \sqrt{-g} C^{a\ c}_{\ a} C^{b}_{\ bc} \tag{2.25}$$

with $v^{\nu} = \sqrt{-g} C^{a\ c}_{\ a} e^{\nu}_{c}$. The Einstein-Hilbert Lagrangian is finally

$$-\frac{\sqrt{-g}}{16\pi} \tilde{R} = \frac{1}{8\pi} v^{\nu}_{\ ,\nu} + \frac{\sqrt{-g}}{16\pi} \left\{ \frac{1}{4} C^{abc} C_{abc} - C^{a}_{\ ac} C^{b\ c}_{\ b} + \frac{1}{2} C^{abc} C_{bac} \right\}. \tag{2.26}$$

Now take as a frame the OT frame. In this frame one has (see Eq. (2.6)) $T_{abc} = -C_{abc}$. Since the previous expression is valid in any frame, apart from a total divergence, the Einstein-Hilbert Lagrangian becomes

$$\mathcal{L}_E = \frac{\sqrt{-g}}{16\pi G} \left\{ \frac{1}{4} T^{abc} T_{abc} - T^{a}_{\ ac} T^{b\ c}_{\ b} + \frac{1}{2} T^{abc} T_{bac} \right\}. \tag{2.27}$$

Since T is a tensor one can substitute internal indices with spacetime indices. This is the teleparallel equivalent ($\mathrm{GR}_{\|}$) of general relativity. The equivalence is based on the following observation. If one starts from a teleparallel geometry, and constructs the Lagrangian (2.27), quadratic in the torsion, one obtains field equations for $g_{\mu\nu}$ that coincide with those of general relativity. This follows from the fact that Eq. (2.27), when recast in terms of the Levi-Civita connection corresponding to $g_{\mu\nu}$ gives exactly the Einstein-Hilbert Lagrangian.

Notice that the field equations cannot determine the OT frame. Indeed the Einstein-Hilbert Lagrangian is invariant under local Lorentz transformations of the tetrad field $e'^{a}_{\nu} = \Lambda^{a}_{b}(x) e^{b}_{\nu}$ (since the Einstein-Hilbert Lagrangian depends only on the invariant quantity $g_{\mu\nu}$). This fact remains in the new Lagrangian. There we have invariance under local transformations of the OT frame. If $\hat{e}'^{a}_{\nu} = \Lambda^{a}_{b}(x) \hat{e}^{b}_{\nu}$

$$\frac{1}{4} T'^{abc} T'_{abc} - T'^{a}_{\ ac} T'^{b\ c}_{\ b} + \frac{1}{2} T'^{abc} T'_{bac} = \frac{1}{4} T^{abc} T_{abc} - T^{a}_{\ ac} T^{b\ c}_{\ b} + \frac{1}{2} T^{abc} T_{bac} + v(\Lambda(x))^{\nu}_{,\nu} \tag{2.28}$$

Since the difference is a total divergence, a fact that follows easily from Eq. (2.26), the field equations cannot determine $\hat{e}^{b}_{\nu}$ but only the invariant quantity $g_{\mu\nu} = \eta_{ab} \hat{e}^{a}_{\mu} \hat{e}^{b}_{\nu}$. It can also be proved that this is the only Lagrangian quadratic in the torsion that has this property [14].

In some sense, in the teleparallel equivalent of general relativity, the use of a teleparallel geometry is somewhat an artifact. Actually the field equations do not determine the teleparallel geometry but only the Riemannian one, so one can feel that the teleparallel formulation is deceptive because it uses non observable concepts like the OT frame and the torsion. On the contrary it has many advantages. It is quadratic in the torsion, a tensor that will be related to the curvature of a gauge theory (but we have already seen this in the discussion of Poincaré gauge theories) and it does not involve second derivatives of the dynamical fields.

2.5 The one-parameter family of viable teleparallel Lagrangians

The teleparallel geometries allow many different dynamics. Contrary to general relativity, through the torsion tensor, it is possible to construct many different Lagrangian quadratic in the first derivatives of the dynamical field $\hat{e}^a_\mu$. For instance, one can change the coefficients in Eq. (2.27) obtaining a number of new candidates as gravitational theories. This procedure was believed to solve the problem of the non deterministic behavior of the vierbein field, indeed, changing the coefficients spoils the symmetry under local Lorentz transformation but maintains the global one,

$$\mathcal{L} = \frac{\sqrt{-g}}{16\pi G}\left\{A\,T^{abc}T_{abc} + B\,T^{abc}T_{bac} + C\,T^a{}_{ac}T^b{}_b{}^c\right\} \tag{2.29}$$

$$\mathrm{GR} \Leftrightarrow A = \frac{1}{4} \quad B = \frac{1}{2} \quad C = -1$$

The values of the coefficients were then constrained using several arguments. Møller [62] discovered a one-parameter family of Lagrangians requiring the same linearized equations of general relativity. It was defined by

$$2A + B = 1 \Leftrightarrow A = \frac{1+2\alpha}{4} \quad B = \frac{1-2\alpha}{2} \quad C = -1. \tag{2.30}$$

Hayashi and Shirafuji [37] investigated the Newtonian and post-Newtonian approximation of teleparallel gravity theories comparing their predictions with the data from solar experiments. In particular they introduced the functions $a(r)$ and $b(r)$ in the static metric of a massive spherical body of mass M ($r^2 = x^i x_i$)

$$\mathrm{d}s^2 = a(r)\mathrm{d}t^2 - b(r)\mathrm{d}x^i\mathrm{d}x_i, \tag{2.31}$$

and expanded in a small parameter (GM/r)

$$a(r) = 1 - 2\frac{GM}{r} + 2\beta(\frac{GM}{r})^2 + \ldots, \tag{2.32}$$

$$b(r) = 1 + 2\gamma\frac{GM}{r} + 2\delta(\frac{GM}{r})^2 + \ldots. \tag{2.33}$$

Solar system experiments prove that both β (perihelion advances, lunar laser ranging) and γ (retardation of radio waves, solar deflection) are 1 up to errors of order 10^{-3}. Once computed the relation between β, γ and A, B, C, Hayashi and Shirafuji found

$$|2A + B + C| \leq O(10^{-3}) \quad , \quad |C + 1| \leq O(10^{-3}). \tag{2.34}$$

Thus one is left essentially with the one-parameter family of Møller. A further theoretical analysis was performed by Hayashi, Shirafuji and Müller-Hoissen and Nitsch looking at the presence of ghosts in the teleparallel linearized theory. They concluded that for a ghost-free theory it must be

$$2A + B + C = 0 \qquad \text{and if } C = -1, \quad \alpha \geq 0 \tag{2.35}$$

moreover with the same argument they discarded the possibility of a parity violating term proportional to $T^{a}{}_{ab}\epsilon^{bcde}T_{cde}$. Finally, one is left with Møller one-parameter family of Lagrangians with the further requirement that $\alpha \geq 0$,

$$\begin{aligned} \mathcal{L} &= \frac{\sqrt{-g}}{16\pi G}\{\frac{1}{4}T_{\alpha\mu\nu}T^{\alpha\mu\nu} + \frac{1}{2}T_{\alpha\mu\nu}T^{\mu\alpha\nu} + \\ &- T^{\mu}{}_{\mu\alpha}T^{\nu}{}_{\nu}{}^{\alpha} + \alpha\frac{3}{2}T_{[\alpha\mu\nu]}T^{[\alpha\mu\nu]}\} \qquad \alpha \geq 0, \end{aligned} \tag{2.36}$$

2.6 An initial value problem in teleparallel gravity

We have seen, in the last section, that one of the reasons that led to teleparallel theories distinguished from $GR_{\|}$ was the search of a cure for the unpredictable behavior of the vierbein field. It was soon recognized that the one-parameter family of Lagrangians compatible with experiments does not spoil completely the Lorentz gauge invariance of the teleparallel equivalent of general relativity. This can be seen as follows [38, 45]. Let me introduce

$$a^{\beta} = T_{\mu\nu\alpha}\epsilon^{\mu\nu\alpha\beta}, \tag{2.37}$$

and let me consider the term proportional to α in (2.36),

$$T_{[\gamma\delta\eta]}T^{[\gamma\delta\eta]} = -\frac{1}{6}a_{\mu}a^{\mu} \tag{2.38}$$

If we can find a local Lorentz transformation that leaves a^{μ} invariant we have that the vierbein is undetermined. Indeed the other part of the Lagrangian coincides, up to a total divergence, to the Einstein -Hilbert Lagrangian and therefore is invariant under general local Lorentz transformations. Under a local Lorentz transformation $\hat{e}'^{a}_{\nu} = \Lambda^{a}{}_{b} e^{b}_{\nu}$, $a^{\mu} = a'^{\mu}$ if and only if

$$\hat{e}^{\mu}_{a} \epsilon^{abcd} \hat{e}^{\nu}_{b} \Lambda_{ec} \partial_{\nu} \Lambda^{c}{}_{d} = 0 \tag{2.39}$$

where $\Lambda_{ec} = \eta_{eb} \Lambda^{b}{}_{c}$. This freedom is called "restricted local Lorentz invariance". Kopczynski [45] constructed an explicit example where the previous equation holds. Hayashi and Shirafuji proposed to interpret the restricted Lorentz invariance as a gauge freedom: two vierbein fields related by such a gauge transformation represent the same physical configuration. This interpretation was criticized by Kopczynski by showing that it runs into problem when introducing matter fields.

After this analysis the fortune of teleparallel gravity theories declined but then Nester [64] proved that the non-predictable behavior of the torsion was non-generic occurring only in very special solutions. He realized that Eq. (2.39) has solutions only if the vierbein $\hat{e}^{\mu}_{a}$ satisfy a restrictive integrability condition. Unfortunately that condition holds for spherically symmetric metrics that, however, could be considered as idealizations of non-symmetric ones. After these not easy reading papers the Cauchy problem was not further analyzed. The feeling however is that it is still worth studying and that a final clear word on this topic has yet to come. This sensation justifies the strategy of Müller-Hoissen and Nitsch [63] who avoided directly the Cauchy problem by adding forth order terms in the torsion to the Lagrangian.

2.7 Matter fields

The teleparallel geometry involves two connections: the Weitzenböck connection and the Levi-Civita connection, and therefore two kinds of geodesics: the curve that maintains the same direction with respect to the OT frame, and the curve of minimum length between two points. In a flat spacetime both curves coincide with straight lines.

When one adds matter to the picture considered so far, one has to decide whether the free fall is represented by Levi-Civita or Weitzenböck geodesics. Analogously, when adding spinor fields to the Lagrangian one has to choose a replacement for the derivatives. In the OT frame ($\Gamma = 0$)

$$\partial_{\mu} \psi^{i} \quad \rightarrow \quad D_{\mu} \psi^{i} = \partial_{\mu} \psi^{i} \tag{2.40}$$

$$\partial_{\mu} \psi^{i} \quad \rightarrow \quad \tilde{D}_{\mu} \psi^{i} = \partial_{\mu} \psi^{i} + \rho_{*}(\tilde{\Gamma})^{i}{}_{j\mu} \psi^{j} \tag{2.41}$$

The former implies that a spinor ψ^{i} has a vanishing derivative when its components with respect‡ to the OT frame do not change. The latter is the usual minimal

‡Keep in mind that whenever the local frame transforms locally even the spinor transforms

correspondence of general relativity. In the first case the Dirac Lagrangian

$$\mathcal{L}_D = \frac{i}{2}(\bar{\psi}\gamma^a\psi_{,a} - \bar{\psi}_{,a}\gamma^a\psi) - m\bar{\psi}\psi \tag{2.42}$$

becomes (in the OT frame)

$$\mathcal{L}_D = \det(\hat{e}^a_\mu)\,\{\frac{i}{2}\,\hat{e}^\mu_a(\bar{\psi}\gamma^a\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^a\psi) - m\bar{\psi}\psi\} \tag{2.43}$$

while in the second case it becomes

$$\tilde{\mathcal{L}}_D = \det(e^a_\mu)\,\{\frac{i}{2}\,e^\mu_a(\bar{\psi}\gamma^a\tilde{D}_\mu\psi - \overline{\tilde{D}_\mu\psi}\,\gamma^a\psi) - m\bar{\psi}\psi\} \tag{2.44}$$

Both Lagrangian were studied; in particular earlier works involved the first choice [37, 38] while later the second one was preferred [45, 19]. A combination of them was also considered [15]. Eq. (2.44) is invariant under local Lorentz transformation of the OT frame while Eq. (2.43) is not. Let me show why Eq. (2.44) is preferred in $GR_\parallel$.

Let $\mathcal{L}(e^a_\mu, \psi^i)$ be a Lagrangian that depends on the vierbein and on a matter field ψ^i that transforms with a representation of the Lorentz group. Assume, moreover, that $\mathcal{L}$ is invariant under local Lorentz transformation of both e^a_μ and ψ^i. For example, $\mathcal{L}$ can be the Einstein Lagrangian $\mathcal{L}_E$ (in this case there is no dependence on ψ) or the Dirac Lagrangian $\tilde{\mathcal{L}}_D$. A variation of the action $S = \int \mathrm{d}^4x\mathcal{L}$ after a local Lorentz transformation $\Lambda^a{}_b(x) = \delta^a_b + \omega^a{}_b(x)$ gives

$$\delta S = \int \mathrm{d}^4x\,(\frac{\delta\mathcal{L}}{\delta e^a_\mu}\,\delta e^a_\mu + \frac{\delta\mathcal{L}}{\delta\psi^i}\,\delta\psi^i) = 0 \tag{2.45}$$

where $\delta e^a_\mu = \omega^a{}_b(x)e^b_\mu$ and $\delta\psi^i = (\Sigma^b{}_a)^i{}_j\omega^a{}_b(x)\psi^j$. If $\mathcal{L}$ does not depend on ψ^i the second term vanishes, while if $\mathcal{L}$ depends on it we go on-shell. The equation of motion for ψ^i are

$$\frac{\delta\mathcal{L}}{\delta\psi^i} = 0, \tag{2.46}$$

so that even in this case the second term is null. We are left with

$$\delta S = \int \mathrm{d}^4x\,\frac{\delta\mathcal{L}}{\delta e^a_\mu}\,\omega^a{}_b(x)e^b_\mu = 0 \tag{2.47}$$

but $\omega^{ab}(x)$ is an arbitrary spacetime dependent skew-symmetric tensor so, after a little algebra,

$$\frac{\delta\mathcal{L}}{\delta e^a_{[\alpha}}\,e^{a\beta]} = 0. \tag{2.48}$$

locally, so it is meaningful to relate the values of the spinor components to those of the local frame.

Now we recall that by definition the Einstein tensor and the stress energy-tensor of matter[§] are

$$G^{\mu\nu} = \frac{8\pi}{\sqrt{-g}} \frac{\delta \mathcal{L}_G}{\delta e^a_\nu}\, e^{a\mu} \qquad T^{\mu\nu} = \frac{-1}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta e^a_\nu}\, e^{a\mu} \tag{2.49}$$

where $\mathcal{L}_M$ is a gravitational Lagrangian and $\mathcal{L}_M$ is a matter field Lagrangian. Hence, to summarize

(a) If $S_G = \int \mathrm{d}^4x \mathcal{L}_G$ is invariant under local Lorentz transformations then $G_{\mu\nu}$ is symmetric.

(b) If $S_M = \int \mathrm{d}^4x \mathcal{L}_M$ is invariant under local Lorentz transformations, and the matter fields are on shell, then $T_{\mu\nu}$ is symmetric.

In $GR_\parallel$ the gravitational Lagrangian fulfills (a) and therefore the spinor Lagrangian must be Eq. (2.44). Otherwise the field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ would equate a symmetric tensor to a generic tensor thus leading to the further equation $T_{[\mu\nu]} = 0$. Now, this equation would be an equation for ψ^i not for e^a_μ (indeed the local Lorentz invariance of the Lagrangian implies that all the information inside the tetrad is fixed once $g_{\mu\nu}$ is given and $g_{\mu\nu}$ can be found from the first set of equation $G_{(\mu\nu)} = 8\pi T_{(\mu\nu)}$ like in general relativity). In general they will not be soluble because the degrees of freedom of ψ are already reduced by the on-shell condition.

[§]For the relation between the stress-energy tensor as source for gravity and the canonical stress-energy tensor of quantum field theory see [35]

Chapter 3

Gauge invariance of teleparallel theories and solution to the background problem

In the last years there has been a revived interest in teleparallel theories especially in connection with their gauge interpretation. Unfortunately, the gauge invariance is usually exhibited in a non-rigorous way. In this chapter I fill this gap defining, since the beginning, the principal bundle we are working in. The group acting on the bundle is determined and a generalization of the gauge interpretation to new groups is developed. As a byproduct a solution of the background structure problem shows up. I identify unambiguously a background structure and its perturbation, an issue never addressed in tetrad theories of gravity. When the curvature vanishes the metric reduces to its background form which coincides with Minkowski flat metric for the T_4 case and with the Einstein static universe metric for the $U(2)$ case. The perturbation has a coordinate independent definition and allows the introduction of observables distinguished from those obtained from the metric alone. Finally, I show that any teleparallel theory of gravity, and hence general relativity, can be considered as a gauge theory over the groups introduced.

3.1 Introduction

In 1967 Hayashi and Nakano [36] showed that teleparallel gravity can be seen as a gauge theory over the translation group [14, 20] with a Lagrangian quadratic in the strength tensor. Some authors started to consider modifications of the teleparallel equivalent of general relativity discovering a one parameter family of quadratic Lagrangians which reproduce the correct low energy behavior [37]. Such teleparallel theories are experimentally indistinguishable from Einstein's gravity even if, from the theoretical side, they are not invariant under local Lorentz transformations of the tetrad field. Moreover, the one parameter family cannot be altered with a slight modification of the coefficients without introducing ghosts [63], and the parameter, for the same reason, should be taken to be positive. Therefore, Kopczyński's [45] proof that such Lagrangians were unable to determine uniquely the evolution of the teleparallel geometry was taken as a serious drawback even if, as shown later by Nester [64] and Cheng et al. [13], it was less severe than expected and limited to special solutions. Those who take this problem seriously, have to consider the teleparallel geometry only as a tool useful to write down Lagrangians. The matter Lagrangian should be invariant under the hidden symmetries of the gravitational Lagrangian. In particular the coupling of spinor matter fields with gravity should be written through the Levi-Civita connection, not Weitzenböck's one [45, 38, 19]. This solution is not completely satisfying because even if we can introduce a stress-energy tensor of gravity which is covariant under coordinate transformations, global Lorentz transformations and gauge transformations [18], we cannot establish, for instance, the value of the energy density as seen in a local Lorentz frame. This is because the stress-energy tensors so far proposed are not invariant under local Lorentz transformations of the orthoteleparallel (OT) frame, and moreover the dynamics is not able to define a privileged OT frame because of the local hidden Lorentz symmetry of the family of viable Lagrangians [45, 38]. Fortunately, this problem arises only in those special solutions studied by Nester. Alternatively, one can remove altogether the lack of determinism adding to the Lagrangian higher order terms in the torsion [63]. This, however, not only means a more radical departure from general relativity (at least from the theoretical side), but also from the Yang-Mills theory that the gauge interpretation of teleparallel theory so strongly resembles.

Here we study the teleparallel theories as Yang-Mills theories, even if with a dependence of the Lagrangian on the curvature somewhat more complicated. Our task is to show that the gauge formulation of teleparallel gravity theories allows a solution of the background structure problem. Ultimately this can be considered as the true reason for the success of these theories in dealing with the stress-energy tensor. We shall deal only with the one parameter family of experimentally viable teleparallel theories, relying on the fact that an unpredictable behavior, if present, can be removed with a slight perturbation of the initial value data [64]. One can also get rid of it by assuming that a higher order term in the curvature was implicitly added to the Lagrangian.

We use the natural units: $\hbar = 1$, $c = 1$.

3.2 Background identification

The identification of a background structure in theories invariant under diffeomorphisms is notoriously difficult. This is an important issue because, in turn, in quantum field theory, the excitations with respect to a fixed background receive a particle interpretation [50]. Hence the problem of identifying a background structure is central in quantum gravity. One can also avoid this problem taking unaltered the spirit of general relativity. A program of quantization of this kind can be performed and leads to canonical gravity [8]. Before we abandon the idea of a privileged background structure let us look more closely to the problem of its definition.

In metric theories of gravity one can try to introduce by hand a background structure rewriting the metric in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.1}$$

and identifying $\eta_{\mu\nu}$ with the background and $h_{\mu\nu}$ with the perturbation. This can be justified if we are studying a spacetime which is Minkowskian at spatial infinity but is still not satisfying. The splitting of the metric in background and perturbation parts turns out to be dependent on the coordinate system chosen. Let two observers label events in different ways with systems of coordinates $\{x^\mu\}$ and $\{x'^\mu\}$ respectively. Even if the two systems of coordinates coincide at spatial infinity, the associated backgrounds $\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ and $\eta_{\mu\nu}\mathrm{d}x'^\mu\mathrm{d}x'^\nu$ differ. The prescription (3.1) introduces a background in a non-covariant way with respect to coordinate changes. It cannot be used in many circumstances: for instance in quantum gravity where we cannot arbitrarily privilege a system of coordinates *. We can reach the same conclusion even if we look for scalars that can be constructed from η and h. It is not difficult to show that the only scalars that can be constructed this way are exactly those that can be constructed using the full metric $g_{\mu\nu}$ alone. This means, because any observable should be coordinate independent, that the splitting proposed is artificial. Any observable expresses properties of the full geometry, nothing can be said about the background metric and its perturbation separately. A better definition would be

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu}, & (3.2)\\ \bar{R}^{\alpha}{}_{\beta\mu\nu} &= 0. & (3.3) \end{aligned}$$

It says that a coordinate system where (3.1) holds exists without, however, fixing it from the beginning. The dynamics should determine that coordinate system

*To privilege a coordinate system means to privilege a way of labelling events. For instance, an observer can take minus the redshift as time variable for the light emission from distant stars, whereas another observer can take, as time variable, the distance obtained with the distance-redshift relation.

starting from the initial value data. Unfortunately, this prescription is not well suited for practical calculations. In what follows, the introduction of new variables will greatly simply this problem. Before, however, we need some mathematical preliminaries.

Let us consider a principal fiber bundle P with structure group G and let the spacetime M be the base. Over P let us consider a flat connection $\tilde{\omega}$ and a second connection ω. Then, clearly, if M is simply connected, the fiber bundle is trivial. A structure like this is familiar in teleparallel theories where P is the bundle of linear frames and the two connections are given by Weitzenböck's and Levi-Civita's respectively. Here, however, this structure acquires a different role. We require G to be a 4-dimensional Lie group and we define the metric of M with

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2 I(\tilde{A}-A, \tilde{A}-A). \tag{3.4}$$

I is a symmetric, bilinear and ad-invariant function defined on the Lie algebra of the structure group, $I : \mathcal{G}\times\mathcal{G} \to \mathbb{R}$, and $\tilde{A}_\mu$, A_μ, are the potentials of the connections in a given section $\sigma(x)$, e.g.: $A = \tau_a A^a_\mu \mathrm{d}x^\mu = \sigma^*\omega$. The constant a has the dimension of a length; its presence gives to A^a_μ the dimension of a mass. Under gauge transformations the difference of two potentials transforms with the adjoint representation. Since I is ad-invariant the metric so defined is gauge invariant.
The function I is a metric for the Lie algebra $\mathcal{G}$. It must be Minkowskian because $g_{\mu\nu}$ is Minkowskian too. We can chose a base of generators τ_a such that

$$I(\tau_a, \tau_b) = \eta_{ab}. \tag{3.5}$$

The requirement of ad-invariance for I leads us to the classification of the allowed four-dimensional groups

$$I([\tau_a,\tau_b],\tau_c) + I(\tau_b,[\tau_a,\tau_c]) = 0 \quad \Rightarrow \quad f_{abc} = f_{[abc]}, \tag{3.6}$$

where $[\tau_a, \tau_b] = f^c_{ab}\tau_c$ and $f_{abc} = \eta_{ad} f^d_{bc}$. Let us rewrite $f_{abc} = v^e \epsilon_{eabc}$. With a Lorentz transformation of the generators we can recast v^a in a canonical form [99], moreover the residual parameter can be eliminated with a suitable rescaling of τ_a, I, a^2, in such a way that the product $a^2 I$ and Eq. (3.5) are left unchanged. Finally, the Lie algebras involved are

a) $v^a = 0$: Lie algebra of the translation group T_4

$$f^c_{ab} = 0. \tag{3.7}$$

b) $v^2 > 0$: Lie algebra of the group $U(2)$

$$f^c_{ab} = 2\epsilon_{0abd}\,\eta^{dc}. \tag{3.8}$$

c) $v^2 < 0$: Lie algebra of the group $GL(2, \mathbb{R})$

$$f^c_{ab} = 2\epsilon_{3abd}\,\eta^{dc}. \tag{3.9}$$

d) $v^2 = 0$: Lie algebra of the two-dimensional Euclidean group with central charge

$$f^c_{ab} = \epsilon_{0abd}\,\eta^{dc} + \epsilon_{3abd}\,\eta^{dc}. \tag{3.10}$$

Notice that the generator $v^a \tau_a$ belongs to the center of the group. In the last case it plays the role of central charge for the Lie algebra of the Euclidean group in two dimensions E_2.

From Eq. (3.5) it follows that the adjoint representation acts as a subgroup H of the Lorentz group. Moreover if $v^a \neq 0$, H is the little group of the four-vector v^a, since the structure coefficients are left unchanged after an adjoint transformation. For $G = T_4$ this subgroup is the trivial group $H = e$, for $G = U(2)$ it is $SO(3)$, for $G = GL(2, \mathbb{R})$ it is $SO(2, 1)$ and in the last case it is E_2.

Let $\tilde{\sigma}(x)$ be a horizontal section with respect to the flat connection $\tilde{\omega}$ and let $\phi(x)$ be the transition function between the two sections $\sigma(x)\phi^{-1}(x) = \tilde{\sigma}(x)$ then $\tilde{A} = \sigma^* \tilde{\omega} = \phi^{-1}(x)\mathrm{d}\phi(x)$. Finally,

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2 I(\phi^{-1}\mathrm{d}\phi(x) - A,\, \phi^{-1}\mathrm{d}\phi(x) - A) \tag{3.11}$$

and the tetrad field can be defined to be

$$\tau_a e^a_\mu \mathrm{d}x^\mu = a(\tau_a A^a_\mu \mathrm{d}x^\mu - \phi^{-1}\mathrm{d}\phi(x)). \tag{3.12}$$

Notice that $\phi(x)$ is determined only up to global left multiplications $\phi(x) \to u\phi(x)$ because of the arbitrariness of $\tilde{\sigma}(x)$. This does not effect Eq. (3.12) which depends only on the combination $\phi^{-1}\mathrm{d}\phi$.

The degrees of freedom given by the field ϕ are easily removed with a gauge transformation that sends ϕ to the identity. Let us call such gauge "OT gauge" because of its connection, that we shall exploit, with the OT frame. In the OT gauge the tetrad field and the potential A^a_μ are proportional. The potential in the OT gauge is our dynamical variable; in what follows we shall assume that the dynamics determines this field completely. Clearly, this is not the case if the Lagrangian is constructed from the metric alone, indeed the tetrad field is determined only up to a local Lorentz transformation. Like in teleparallel theories, we have to consider modifications to the general relativistic Lagrangian.

Let us return to the main question to show that (3.11) splits the metric in background and perturbation parts in a coordinate and gauge independent way. We identify the perturbation with the potential A and the background metric with

$$g_B = a^2 I(\theta,\, \theta), \tag{3.13}$$

where θ is the canonical 1-form of the group G [44]. When the perturbation is a pure gauge, $A = U^{-1}\mathrm{d}U(x)$, the potential can be sent to zero through a gauge transformation. Then the metric reduces to $\phi(x)^* g_B$, that is, apart from a pullback which amounts simply to a coordinate transformation † , the metric takes its background form. In other words, if F is the curvature tensor,

$$F = 0 \;\Rightarrow\; g = \phi^* g_B. \tag{3.14}$$

The coordinate system in which $g = g_B$ is not fixed by the requirement $F = 0$, indeed it depends on the map $\phi(x)$ that, in turn, depends on the value of $U(x)$. This shows that the equation $F = 0$ is a coordinate independent way to state the equivalence of the metric with its background form. Here the splitting is not artificial because we can construct quantities like $F^a_{\mu\nu} F^b_{\alpha\beta} \eta_{ab} g^{\mu\alpha} g^{\nu\beta}$ that are scalars and gauge invariant. Moreover they cannot be recovered from the metric alone. Their gauge invariance is assured because the tetrad field (3.12) and the curvature transform, under gauge transformations, with the adjoint representation which, as we have seen, acts as a subgroup of the Lorentz group.

Before we start studying the dynamics let us compute the background metric for the structure groups T_4 and $U(2)$.

a) T_4: With the parameterization $\phi = \mathrm{e}^{-\tau_a \phi^a}$ the canonical 1-form becomes $\theta = \phi^{-1}\mathrm{d}\phi = -\tau_a \mathrm{d}\phi^a$ and the background metric

$$g_B = a^2 \eta_{ab}\, \mathrm{d}\phi^a \mathrm{d}\phi^b$$

coincides with the Minkowski metric.

b) $U(2)$: With the parameterization

$$\phi = \mathrm{e}^{\lambda\tau_0}\, \mathrm{e}^{\chi(\tau_1 \sin\theta\cos\varphi + \tau_2 \sin\theta \sin\varphi + \tau_3 \cos\theta)} \;,$$

and the representation (the metric is independent from the representation of the Lie algebra chosen) $\tau_\mu = i\sigma_\mu$, $\sigma_0 = \mathrm{I}$, we find $I(\alpha,\alpha) = -\det(\alpha)$, $\alpha \in \mathcal{G}$ and

$$g_B = a^2\{\mathrm{d}\lambda^2 - \mathrm{d}\chi^2 - \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2)\}.$$

Hence, for $G = U(2)$ the background metric coincides with that of Einstein's static universe.

†The coordinate transformation can be singular. In that case the metric turns out to be degenerate. This does not happen if the condition $F \approx 0$ in some region of spacetime follows from the dynamics and if the dynamics itself guarantees the metric to be not degenerate.

Analogous calculations lead to the background metric for the cases c) and d). They, however, do not have maximal symmetric spacelike slices and hence do not enjoy the cosmological principle.

To summarize, we have identified the perturbation with the potential of a gauge theory. Its transformations under gauge and coordinate change are well known. From that variable one can construct scalars that express properties of the perturbation with respect to a background structure. The scalars to be considered depend on the supposed background structure, for instance, they are invariant under $U(2)$ gauge transformations in an Einstein's static universe background.

3.3 Dynamics and teleparallel theories

Now we have to construct a dynamics for the gauge potential. Let $\hat{A}^a_\mu$ be the potential in the OT gauge and let $\hat{e}^a_\mu$ be the tetrad field (3.12) in the same gauge: $\hat{e}^a_\mu = a\hat{A}^a_\mu$. We introduce a teleparallel geometry on M of OT frame $\{\hat{e}^a_\mu\}$, where the brackets recall that the OT frame is defined up to a global Lorentz transformation. We can rewrite the usual expression of the torsion in terms of the OT frame

$$T^\rho_{\mu\nu} = \hat{e}^\rho_a(\partial_\mu \hat{e}^a_\nu - \partial_\nu \hat{e}^a_\mu), \tag{3.15}$$

in a gauge invariant way

$$T^\rho_{\mu\nu} = a\, e^\rho_c F^c_{\mu\nu} - \frac{1}{a}\, e^\rho_c f^c_{ab}\, e^a_\mu\, e^b_\nu, \tag{3.16}$$

where the tetrad field is given by Eq. (3.12) and where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc}\, A^b_\mu A^c_\nu. \tag{3.17}$$

In order to shorten the notation let $F_{abc} = \eta_{ad} F^d_{\mu\nu} e^\mu_b e^\nu_c$ and $T_{abc} = e_{c\rho} T^\rho_{\mu\nu} e^\mu_b e^\nu_c$. Eq. (3.16) becomes

$$T_{abc} = aF_{abc} - \frac{1}{a}\, f_{abc}, \tag{3.18}$$

and the one parameter family of viable teleparallel Lagrangians [37, 63]

$$\begin{aligned} \mathcal{L} &= \frac{\sqrt{-g}}{16\pi G}\, \{\frac{1}{4} T_{\alpha\mu\nu} T^{\alpha\mu\nu} + \frac{1}{2} T_{\alpha\mu\nu} T^{\mu\alpha\nu} + \\ &- T^\mu{}_{\mu\alpha} T^\nu{}_\nu{}^\alpha + \alpha \frac{3}{2} T_{[\alpha\mu\nu]} T^{[\alpha\mu\nu]}\}, \end{aligned} \tag{3.19}$$

becomes

$$\begin{aligned} \mathcal{L} &= \frac{\sqrt{-g}}{16\pi\sigma^2} \left\{ \frac{1+2\alpha}{4} F_{abc} F^{abc} + \frac{1-2\alpha}{2} F_{abc} F^{bac} + \right. \\ &\left. - F^a{}_{ab} F^c{}_c{}^b + \frac{1-6\alpha}{2a^2} F_{abc} f^{abc} - \frac{1-6\alpha}{4a^4} f_{abc} f^{abc} \right\}. \end{aligned} \tag{3.20}$$

The linearized theory is ghost-free when the parameter α is positive [37]: $\alpha \geq 0$. General relativity corresponds to $\alpha = 0$. The first term in Eq. (3.20) is the well known Yang-Mills Lagrangian; its dimensionless coupling constant is given by

$$\sigma = \frac{L_P}{a}. \tag{3.21}$$

The reader should keep in mind that this similarity is only apparent: here even the metric and the tetrad field depend on the potential A; this leaves us with a theory that is still not renormalizable. The role of Eq. (3.20) is to exhibit the gauge invariance of teleparallel gravity theories, and hence of general relativity, under any of the four dimensional group studied in the previous section. Actually, we have obtained this gauge invariance with the introduction of a new degree of freedom given by the field $\phi(x)$. One can correctly suspect that the need for the introduction of a new field is a signal that the symmetries developed are not genuine new symmetries of the original Lagrangian. Moreover, one can be concerned about the fact that our dynamical variable is the potential in the OT gauge: if in practical calculation we have to return to the usual tetrad formulation, what is the advantage of introducing such dependence of the metric? To answer these questions we need to introduce the internal coordinate representation.

So far, we have used the spacetime coordinates. Let us choose a gauge such that $\phi : M \to G$ is injective. In such a gauge we can perform a coordinate transformation from the spacetime coordinates $\{x^\mu\}$ to given internal coordinates $\{\phi^a\}$ of the group manifold G. For example the Lagrangian in the internal coordinates becomes $\phi^{-1*}\mathcal{L}(x)$ with a tetrad field given by ‡

$$\tau_a e^a_b(\phi^a)\, \mathrm{d}\phi^a = \tau_a A^a_b(\phi^a)\, \mathrm{d}\phi^b - \theta. \tag{3.22}$$

From this equation we see that the field $\phi(x)$ has been completely removed by the coordinate change. In the internal coordinate representation the gauge symmetry turns out to be an alternative way of recasting the invariance under coordinate transformations. To any coordinate change in the internal formalism corresponds a gauge transformation in the spacetime formalism; the converse, however, is not true. For instance the OT gauge can not be accomplished in the internal formalism because the field $\phi(x)$, in that case, is not injective. In the internal representation the invariance under coordinate transformation and the invariance under gauge transformation are linked: the transformation law for the potential becomes ($\sigma' = \sigma\, u$)

$$\tau_a {A'}^a_c = \{u^{-1}\tau_a A^a_b u + u^{-1}\partial_b u\}\frac{\partial \phi^b}{\partial {\phi'}^c}\,, \tag{3.23}$$

‡For clearness, in internal coordinates the reciprocal of e^a_b should be written ${e^{-1}}^b_c$.

and the transformation law for the curvature becomes

$$F'_{ab} = u^{-1} F_{cd}\, u \, \frac{\partial \phi^c}{\partial \phi'^a} \frac{\partial \phi^d}{\partial \phi'^b} \,, \tag{3.24}$$

where the matrix $u(\phi)$ is related to the transformation $\phi'^a(\phi^b)$ by the product $\phi' = \phi\, u(\phi)$. In the same way it can be shown, for example, that the metric given by (3.11) transforms as a tensor under (3.23). Notice that in the internal formalism there is no distinction between internal and spacetime indices because in the spacetime manifold we have introduced internal coordinates.

Let us investigate more closely the case $G = T_4$. In the previous section we defined the coordinates $\{\phi^a\}$ on the group manifold: $\phi = \mathrm{e}^{-\tau_a \phi^a}$. The tetrad field in the spacetime formalism is [36]

$$e^a_\mu = \partial_\mu \phi^a(x) + A^a_\mu(x), \tag{3.25}$$

whereas in the internal formalism is (Eq. (3.22))

$$e^a_b(\phi) = \delta^a_b + A^a_b(\phi). \tag{3.26}$$

Apart from changes due to the notation, this is exactly the dependence of the tetrad field on the potential given in [14]. Indeed, our formalism in the $G = T_4$ case gives rise to the widely studied translational gauge symmetry of teleparallel theories. In literature it has been studied both in internal coordinates [14] and in spacetime coordinates [36, 18].

Now we see the advantage of these gauge formulations. In the internal coordinate representation, contrary to what happens in the spacetime representation, the Lagrangian does not reduce to its usual tetrad form because we are not working (and we cannot work) in the OT gauge. Moreover, the dynamical variable is given by the potential $A^a_b(\phi)$ and there is no further degree of freedom apart from that of making gauge-coordinate transformations. The metric depends on the potential through the tetrad field (3.22). The identification of the potential $A(\phi)$ with the perturbation is coordinate independent, its transformation law being Eq. (3.23).

As a final point let us investigate the invariance under global Lorentz transformations. We should expect that the formulation in terms of a perturbation spoils this explicit invariance see (Eq. (3.19)) if the background structure does not share the same symmetry. This is indeed the case. Only the structure coefficients of the group $G = T_4$ are left unchanged under the replacement $\tau_a = \Lambda^{b'}{}_a \tau_{b'}$. For this group one can accomplish the global Lorentz invariance with $\phi^a = \Lambda^a{}_{b'} \phi^{b'}$ and $A^a_\mu(x) = \Lambda^a{}_{b'} A^{b'}_\mu(x)$. Of course, in the internal representation even the coordinates transform and we have

$$A^a_b(\phi) = \Lambda^a{}_{c'}\, A^{c'}_{d'}(\phi)\, \Lambda^{d'}{}_b. \tag{3.27}$$

Surprisingly, in the previous literature, despite the properties of the gauge formulation, most of the calculations were performed in the usual tetrad formalism.

3.4 Conclusions

In order to solve the background structure problem of gravity theories, I proposed a metric of the form (3.11) where two connections were introduced. I showed that, if the dynamics determines the potential of the gauge theory completely (up to gauge or coordinate changes), the splitting of the metric in background and perturbation parts is physical since new observables, not dependent on the full metric alone, can be constructed. I classified the allowed four-dimensional structure groups finding suitable backgrounds for the T_4 and $U(2)$ cases. I went to exploit the dynamics with the introduction of a teleparallel geometry on the manifold. Taking advantage of teleparallel gravity results, that geometry was used to construct Lagrangians compatible with the low energy limit of general relativity. It should be mentioned, however, that the gauge approach allows a more large variety of gravitational Lagrangians than the teleparallel approach. For instance the Lagrangian $F^a_{\mu\nu}F^b_{\alpha\beta}\delta_{ab}g^{\mu\alpha}g^{\nu\beta}$ is coordinate and gauge independent but has no teleparallel analogue. Teleparallel Lagrangians have the advantage of being invariant under global Lorentz transformation. This still seems a necessary condition for the introduction of spinor matter fields. Finally, teleparallel theories were interpreted as gauge theories over different backgrounds in dependence of the structure group chosen. Most interesting cases were the Minkowski flat metric and the Einstein static universe. As a final comment, Eq. (3.21) suggests that renormalization can effect the 'radius' of the universe. Of course, the theory is not renormalizable and this is only a speculation, anyway it seems to deserve further investigations.

Part II

Simultaneity

Chapter 4

Simultaneity: historical introduction

In this chapter I give an historical account of simultaneity, causality and related problems. It is only a summary, mainly focused on mathematical results rather than on their philosophical interpretations.

4.1 The conventionality thesis

As it was emphasized by Lakatos [46] the act of measurement and the interpretation of experiments depend deeply on the accepted theoretical model. The way we perceive reality is filtered through a number of conventions without which it would be impossible to start any interpretation. Sometimes, in order to distinguish clearly what are the physical facts and what are not, it may be useful to make explicit some of the underlying conventions.

Since its birth [72, 23], there has been a long debate to establish to what extent special relativity, and the hypothesis of constancy of the speed of light, in a given inertial frame, could be considered conventional [75, 33]. It was soon realized by Einstein [23] that experimentally one can establish the speed of light only by measuring the time of flight of a light beam over a closed path. Indeed, in order to measure the *one-way* speed of light, the time of departure from a starting point O_1 and the time of arrival to an ending point O_2 are needed. A convention to synchronize two distant clocks would be useful. The Einstein convention, was conceived with that purpose. Every inertial observer has, along its worldline a proper time parameterization. It is given by the time of a clock at rest with respect to the observer. In order to extend this parameterization to events outside the observer worldline, Einstein noticed that each event can be thought as an event of reflection of a straight light beam that, starting form the observer and being reflected, comes back. He proposed (but before him Poincaré made the same proposal [73, 74]) to assign to the event of reflection the mean of the proper time of departure and arrival of the light beam from and to the observer. Einstein's synchronization (simultaneity) convention, however, cannot be used to measure the one-way speed of light. Implicitly in its definition there is the assumption that the speed of light can depend on direction but not on its verse. The proof of the isotropy, when given with such synchronization convention, would be partially tautological. The isotropy of the speed of light, then, seems to turn out as a convention and not as a fact of experience. This is the conventionality thesis.

Already in 1898 Poincaré [72], seven years before the birth of special relativity, discussed the independence of the speed of light of direction with these words:

This is a postulate without which it would be impossible to start any measurement of this velocity. It will always be impossible to verify experimentally the said postulate.

Similarly, in 1916, Einstein [23] wrote:

. . . that light requires the same time to traverse the path AM . . . as the path BM [M being the midpoint of the line AB] is in reality neither a supposition nor a hypothesis about the physical nature of light, but a stipulation which I can make of my own free will.

The situation is often illustrated, in the one dimensional case, in the following way [75](see figure 4.1). Let us consider a light beam: it leaves the observer O_1, reaches

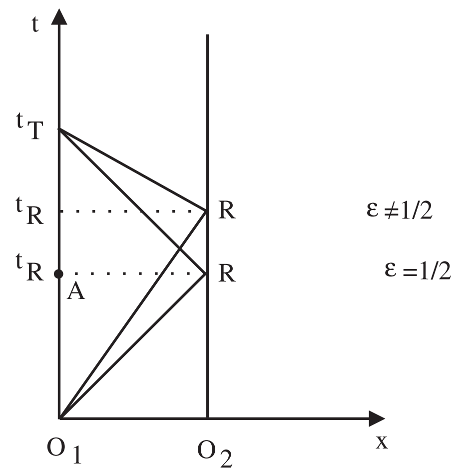

Figure 4.1: Different values of ϵ give different definitions of simultaneity.

O_2 and, being reflected, it comes back. O_1, with his clock, measures the total time of flight, t_T, and verifies the relation $2\overline{O_1O_2} = ct_T$. If the speed of light is the same in both directions, the beam is reflected by O_2 at the time $t_R = t_T/2$. This data, once communicated from O_1 to O_2, can be used by O_2 to synchronize his clock with O_1's (the Einstein procedure of synchronization). In other words the assumption of isotropy leads to the conclusion that the events A and R are simultaneous, so that the definition of simultaneity suffers from the same conventionality content of the isotropic assumption.

It is often noticed (for an historical account of these positions see [6] that this conclusion cannot be drawn essentially for two reasons.

- There may be some way to synchronize distant clocks without using the isotropic assumption, for instance, with a slowly transport of a third clock from O_1 to O_2.

- In the above argument we used only one experimental fact, that the speed of light as measured over a closed path is always c, (hereafter this law* is referred to, in short, as the "L/c law"). Other experimental facts could restrict the allowed values of the speed of light in one direction, eventually leaving us with only the isotropic possibility.

However, the solution suggested in the first point simply replaces the isotropic convention with other equivalent assumptions [22, 52, 6]. It does not exclude the possibility of alternative anisotropic choices. As we shall see, the second argument does not work as well, because there are anisotropic conventions compatible with every physical law, the only price to be paid being a change in their mathematical expression. Moreover, conventions different from the isotropic one can prove to be natural in some contexts such as when the observers live over the surface of a spinning planet (Section 5.5).

As a first example of an alternative convention [75] let us return to the one dimensional case. Following the supporters of the conventionality of simultaneity, we are able to fix the time reflection, $t_R = \epsilon t_T$, where ϵ is the Reichenbach coefficient usually taken in the range $\epsilon \in (0, 1)$. Once the choice has been made, the speed of light directed right becomes $c/2\epsilon$ and the one directed left becomes $c/2(1-\epsilon)$. Whatever the choice of ϵ is, the L/c law is satisfied. The restriction to the one-dimensional case however does not clarify the problem, nor exhibits the richness of the possible conventions. In the next chapter we pass to the three-dimensional case.

*Elsewhere [6] it is called "Weyl's *Erfahrungstatsache*", and it is distinguished from the Reichenbach round-trip axiom which states the independence of the round-trip time from the direction of the journey.

4.2 Fallacious isotropy proofs

Since Reichenbach renewed the attention on the conventionality thesis, many authors, who disagreed, tried to prove the isotropy of the speed of light by means of theoretical arguments and thought experiments. As we shall argue every proof of this kind is fallacious, often for very subtle reasons. Here we analyze a variant of the original Fizeau's experiment which can be naively used to "prove" the isotropy of the speed of light. In his original experiment in 1849 Fizeau used a single toothed wheel. The light beam passed through a gap between the teeth, reached a mirror, and were reflected back through the adjoining gap. To this hand the toothed wheel should rotate at a suitable angular velocity. From the measurement of that angular velocity he was able to recover the two-way speed of light. In that experiment Fizeau measured the two-way speed of light since the rotation of the toothed wheel served as an indirect method to measure the round-trip time needed by the light beam to leave the wheel, reach the mirror and come back. In

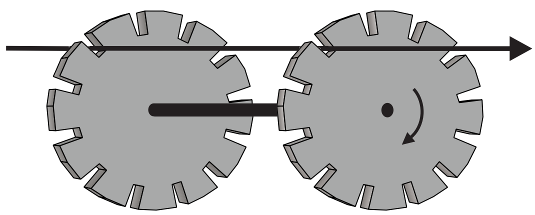

Figure 4.2: The toothed double wheel.

order to measure the one way speed of light one can modify the original experiment using two toothed wheels placed at a distance $2l$ and centered on the same axis. The teeth of the wheels are parallel in the sense that a straight line parallel to the axis pass through a gap of the first wheel if and only if it pass through a gap of the second wheel. The toothed wheels are such that the distance d between gaps is much greater than that between teeth. The double wheel rotates rigidly. Let us consider a light beam that traverses the system parallel to the axis. It passes through a gap on the first wheel and then through the next gap on the second wheel (the definition of what is the next gap in given when the apparatus is at rest). This happens, for the first time, when the angular velocity, accelerating from zero, reaches the value

$$\omega = \frac{d\,c}{2lR} \tag{4.1}$$

where R is the radius of the wheels. Hence, from a measurement of ω one can recover the one-way speed of light. It seems impossible to reconcile this thought experiment with an anisotropic speed of light. However, as we now show, any anisotropic value for the speed of light would give the same formula for ω. Thus this experiment does not measure the one way speed of light but, as it will be clear, it measure the two-way speed of light.

To begin with, observe that the hypothesis of rigidity is not physical if the fastest signal has an upper bound. The teeth of the wheels are parallel (see above) when the system is at rest but there is no reason to believe that they are still parallel when the system begins to rotate. This is because if the torque on the system is applied, as we shall suppose, in the middle of the axis then the two wheels begin to rotate at the same time only if the speed of light is isotropic. Indeed, it requires some time, for the information on the rotation applied on the middle point of the axis to reach the wheels. Let us call $c_+ = c/2\epsilon$ the speed of light directed right and $c_- = c/2(1-\epsilon)$ the one directed left. If the torque is applied at time $t = 0$, the right wheel begins to rotate at time $t = l/c_+$ whereas the left one at time $t = l/c_-$. When a beam which passes through a gap on the left wheel, just before it begins to rotate, reaches the right wheel, it finds the second wheel rotated of an angle

$$\theta = \omega(2l/c_+ + l/c_- - l/c_+) = \omega 2l/c \tag{4.2}$$

In order to pass through the right wheel the angular velocity must be again (4.1). Now, however, it is clear that the parameter that appears in that expression should be interpreted as the two-way speed of light. It is thus impossible to measure the one-way speed of light, at least with the apparatus suggested in this thought experiment.

It is not always easy to find the implicit assumptions in a fallacious proof of the isotropy of the speed of light. Another "proof" is based on Rømer's method. Since he deduced the value of the speed of light from the seasonal variation of the apparent transit time of Jupiter's moon Io, only the one way light speed from Jupiter to the Earth seems to be involved. However, the presence of a speed upper bound implies the well known effect of time dilation which, if the speed of light is anisotropic, is different depending on whether the Earth moves towards Jupiter or recedes from it. This effect compensates the anisotropic speed of light and finally what is really measured is the two-way speed of light in the Earth frame [6].

4.3 Lorentz group from causal structure

Since my work concerns, not only simultaneity but also the consequences of the L/c law on the causal structure, I recall here Alexandrov-Zeeman's theorem which relates the Minkowskian causal structure of spacetime and the Lorentz group.

4.3.1 Alexandrov-Zeeman's theorem

As we said in section 1.1 the explicit matrix representation of the Lorentz group depends on the way we label events. Thus, it is conventional. The abstract Lorentz group, however, does not depend on its representation and hence is independent of the way the observers label events. It is not surprising that it can be related to the causal structure [2, 3, 4, 88, 103]. The concept of causal structure does not depend on coordinates as it becomes clear with the following construction. We shall follow Zeeman's notations. Zeeman considers Minkowski spacetime as a partial ordered set defined by

$$x < y \Leftrightarrow y - x \text{ is a timelike vector oriented towards the future} \tag{4.3}$$

When x and y are spacelike separated nor $x < y$ nor $y < x$ hold. The relation $<$ is transitive. Then Zeeman removes the metric structure of the spacetime manifold and defines the causal structure of spacetime as the pair $(M, <)$. A one-to-one mapping $f : M \to M$ is a causal automorphism if both f and f^{-1} preserve the partial ordering. In other words

$$x < y \Leftrightarrow f(x) < f(y) \quad \text{all} \quad x, y \in M \tag{4.4}$$

The causal automorphisms form a group: the causality group.

Alexandrov-Zeeman's theorem. Let G be the group generated by (i) the orthochronous Lorentz transformations (plus space reflections) (ii) the translations (iii) the dilations. G is also known as "Weyl group". The causality group coincides with G, in the sense that they act in the same way on M. Their matrix representation is the usual one when over M are defined the usual orthogonal coordinates $\{x^\mu\}$.

4.3.2 Liouville's theorem

If the partial ordering is deduced from a Lorentz metric over M there is also another route to define the causal structure and its automorphism. In a Lorentz manifold the causal structure is determined, locally, by the light cone. Two metrics related by a conformal factor $g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$ induce the same light cones and hence the same causal structure over M. Conversely, if two metrics induce the same light cones, they are related by a conformal factor. This is proved e.g. in appendix D of Wald's book [97]. We repeat that proof in section 4.4.

Let us consider a smooth transformation $f : M \to M$. The vectors v and w in $m \in M$ are sent into vectors $f_* v$ and $f_* w$ in $f(m)$. The application f preserves the causal structure if and only if $(f^*g)_{\mu\nu}$ has the same casual structure of $g_{\mu\nu}$, that is, if and only if there is a function Ω_f such that

$$f^*g = \Omega_f^2\, g \tag{4.5}$$

The causal structure of $(M, g_{\mu\nu})$ is the pair $(M, [g_{\mu\nu}])$ where $[g_{\mu\nu}]$ is the equivalence class of metrics which differ by a (positive) conformal factor. The previous equation then says $f^*[g] = [g]$. Those functions f which share this property form the conformal group. One can distinguish the causal structure from the light cone structure, by saying that in the latter case the conformal factor is not required to be positive. The equivalence class of such metric will be indicated with $\{g\}$. Again an automorphism preserving the light cone structure is characterized by $f^*\{g\} = \{g\}$. The following theorem holds [21]

Liouville's theorem: Let M be a n-dimensional,$n \geq 3$, manifold with a diagonal metric $g_{\mu\nu}$ of eigenvalues ± 1. The C^4 automorphisms of M that preserve the light cone structure are generated by

- Lorentz transformations and spacetime reflections: $x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \Lambda \in O(n-s, s)$
- translations: $x'^\mu = x^\mu + a^\mu$
- dilations: $x'^\mu = \lambda\, x^\mu \qquad \lambda > 0$
- inversions: $x'^\mu = \frac{x^\mu}{x^\alpha g_{\mu\nu} x^\nu}$

where s is the number of negative eigenvalues of $g_{\mu\nu}$.

The Lie algebra study shows that the group generated is $O(n-s+1, s+1)$. Notice that for $0 < s < n$ the inversions are not defined in the submanifold $x^\mu g_{\mu\nu} x^\nu = 0$. This singular behavior cannot be corrected with the addition of a point at infinity like in the $s = 0$ and $s = n$ cases. For $n = 3$ the group of non-singular orthochronous conformal transformation is generated by Lorentz transformations (plus space reflections), translations and dilations. We thus recover Alexandrov-Zeeman's theorem which appears here as a particular case of Liouville's theorem.

4.4 Simultaneity from causal structure: Malament's theorem

Since any proof of the isotropy of the light speed appeared, after a careful analysis, to be fallacious, Grünbaum [33] came to the conclusion that the only relationship between two events that appears to be non-conventional is that of spacelike separation. He defined the notion of "topological simultaneity" by saying that two event are topological simultaneous if their distance is spacelike. Then he went ahead claiming essentially that what really matters is the causal structure and that no simultaneity relationship, which is also an equivalence relation (Grünbaum's topological simultaneity was not transitive), can be deduced from the causal structure of Minkowski spacetime. Hence, he maintained that for these reasons, every equivalence relation which plays the role of simultaneity relationship should be considered as conventional since it is not related to the factual character of the causal structure. In this last statement, however, he was wrong as it was proved by Malament [49]. The original Malament's proof is rather cumbersome, however the statement of the theorem is simple. He starts with the causal structure of Minkowski spacetime (M, η) and considers a straight timelike worldline C which represents an inertial observer O. He then looks for an equivalence relation $\mathcal{R}(p, q)$ to be termed "simultaneity relation with respect to O "

$$\mathcal{R}(p, q) \Leftrightarrow p \text{ and } q \text{ are simultaneous with respect to } O \tag{4.6}$$

with the property of being invariant under the group of causal automorphisms $f : M \to M$ that send the worldline C into itself $f(C) = C$,

$$\mathcal{R}(p, q) \Leftrightarrow \mathcal{R}(f(p), f(q)) \tag{4.7}$$

He finally proves that such an equivalence relation exists, it is unique, and coincides with Einstein's definition of simultaneity. The reason why the first proof was so complicated relies on the fact that he used the definition of casual structure as a partial ordered set. We have shown, however, that in a Lorentz manifold it coincides with that deduced from the light cone structure of the Lorentz metric. It is then easiest to work directly with the metric. Malament's theorem follows quickly and easily from the observation, made in the previous section, that two metrics with the same light cones coincide up to a conformal factor. We provide a proof here because of its importance [97].

Theorem. Let $(M, g_{\mu\nu})$ and $(M, \tilde{g}_{\mu\nu})$ be two Lorentz manifold with identical light cones. Then, there is a non vanishing function $\lambda : M \to \mathbb{R}$ such that $\tilde{g}_{\mu\nu} = \lambda g_{\mu\nu}$. If both manifolds share also the same time orientation (and hence the causal structure) λ is positive.

Proof. Let u^μ, $e_1^\mu, \ldots, e_1^\mu$, be an orthonormal basis of $g_{\mu\nu}$ at $p \in M$. Then $u^\mu \pm e_i^\mu$ is null with respect to $g_{\mu\nu}$ and hence with respect to $\tilde{g}_{\mu\nu}$, which implies that, with respect to $\tilde{g}_{\mu\nu}$, u and e_i are orthogonal and have the same magnitude (up to a sign). Moreover since for $i \neq j$, $u + 2^{-1/2}(e_i + e_j)$ is null with respect to $g_{\mu\nu}$

it is also null with respect to $\tilde{g}_{\mu\nu}$ and hence e_i and e_j are orthogonal with respect to $\tilde{g}_{\mu\nu}$. Thus u^μ, $e_1^\mu, \ldots, e_1^\mu$ are orthogonal and of the same norm with respect to $\tilde{g}_{\mu\nu}$, therefore there is a function λ such that $\tilde{g}_{\mu\nu} = \lambda g_{\mu\nu}$. If $(M, g_{\mu\nu})$ and $(M, \tilde{g}_{\mu\nu})$ not only share the light cones, but even the direction of time, then the constant λ is positive $\lambda = \Omega^2$.

This theorem allows one to define unambiguously the concept of "orthogonality of two vectors with respect to a conformal structure $[g]$". Let us indeed consider two vectors v^μ and w^μ in $p \in M$. If they are orthogonal with respect to $g_{\mu\nu}$: $w^\mu g_{\mu\nu} v^\nu = 0$, they are clearly orthogonal even with respect to $\Omega^2 g_{\mu\nu}$. Hence, the relation of orthogonality of two vectors does not depend on the specific representative of $[g]$ but only on the causal structure $[g]$ itself. One can use, for instance, the notation $w \perp_{[g]} v$ to denote the orthogonality with respect to $[g]$.

The next step comes from an old observation by Robb [77] who noticed that, in special relativity, the Einstein simultaneity convention selects as simultaneous those events which lie on the same surface orthogonal to the worldline of the inertial observer. That is, p and q are Einstein simultaneous with respect to O (inertial observer) of four-velocity u if and only if $u \perp (p-q)$. Since the relation of orthogonality follows solely from the causal structure we can replace the previous relation with

p and q are Einstein simultaneous with respect to O
(inertial observer) of four-velocity u if and only if $u \perp_{[\eta]} (p-q)$.

This is another way of stating Malament characterization of Einstein simultaneity through the causal structure. Of course, it turns to be invariant under causal automorphisms that send u (i.e. the worldline of the inertial observer) into itself.

Malament's theorem had a great impact on the philosophical interpretation of simultaneity. However we have to keep in mind that Malament's theorem simply characterizes Einstein simultaneity with the concept of causal structure. It does not imply that the only reliable simultaneity convention is Einstein's. We shall later return on this issue. We anticipate that Malament's argument has no analogue when one skips to general relativistic spacetimes (see chapter 6).

4.5 Slow transport of clocks

The method of distant synchronization through the slow transport of clocks is recurrent in the debate on the conventional nature of simultaneity. It was proposed, for the first time, by Eddington [22]. In this section we review it briefly.

Let us consider two observers O_1 and O_2 at distance d. A third observer O_3 synchronizes its clock with that of O_1, travels along the straight line connecting O_1 and O_2 and finally synchronizes O_2's clock with its own. The key observation is that, in special relativity, this method of distant synchronization coincide with Einstein synchronization if O_3 moves very slowly. To see this let t be the Einstein global time, and let τ be the proper time of O_3's clock. The proper time needed by O_3 to leave O_1 and reach O_2 is

$$\Delta\tau = \int_{O_1}^{O_2} \sqrt{1 - v^2/c^2}\mathrm{d}t \tag{4.8}$$

Now, if O_3 moves slowly the previous expression becomes

$$\Delta\tau = t_{O_{23}} - t_{O_{13}} + O(v/c) \tag{4.9}$$

where $t_{O_{23}}$ is the Einstein time when observers O_2 and O_3 meet and analogously for $t_{O_{13}}$. Since $\Delta\tau = \Delta t$ the method of synchronization with a slowly moving clock coincides with Einstein's method of synchronization. For more on this method and for its role in the debate on the conventional nature of simultaneity see [6].

4.6 Simultaneity and reduction postulate in quantum mechanics

The asymmetric role between time and space in quantum mechanics is well known. In first quantization it is possible to define three operators X^i which stand for position observables. The same cannot be done for time. An old, and controversial [28], theorem by Pauli shows that there cannot be a self-adjoint time operator T having a canonical commutation relation with energy

$$[T, H] = i\hbar. \tag{4.10}$$

He proved that if an operator of that kind exists then the Hamiltonian has a spectrum unbounded from below (the whole real line to be precise). This of course is unacceptable since there would be no ground state and the stability of matter would be impossible. Many attempts to avoid this spacetime asymmetry were tried but none seems convincing. It has however emerged that a good definition for a time-of-arrival operator can be given [42].

At first sight it seems impossible to reconcile this asymmetry with the unification of time and space inherent in special relativity. However quantum field theory, through the process of second quantization, has done that. Historically no time operator has been found but the spacetime asymmetry has been partially removed by treating coordinates, for instance in the path integral approach, as parameters. So, instead of making the time variable an operator, the position operators were transformed in parameters like time in first quantization. Of course this does not solve the problem of finding a suitable time operator. In the QFT approach the measurement apparatus must be mathematically modelled from the available quantum fields, a procedure less direct and more complicated than in first quantization. The success of the procedure is however guaranteed once one keeps in mind that many different "time operators" or "position operators" can emerge in dependence on how the actual apparatus interacts on the system. Unfortunately no one has filled the gap from the principles of QFT to a useful definition of time operator. We have to notice that the need for a time operator, or at least of an amplitude "in time", is still present since an experimenter can ask, for example, when a particle will decay, or when a particle will strike a panel, and how a rotating toothed wheel in the particle trajectory alters the time-of-arrival expectation value. QFT has simply circumvented these problems reconciling quantum mechanics with the spacetime unification of special relativity. Here I what to investigate briefly, in a naive first quantization picture, whether simultaneity can be characterized using quantum mechanical expectation values. As we shall see the answer is negative. Einstein's simultaneity seems not to play a fundamental role at the quantum level.

In first quantization time acts as an external parameter. It appears, for example in the wave dependence $\psi(x,t)$ and in Schrödinger equation. What kind of global time is it? It is implicitly the global time deduced from Einstein simultaneity convention, since that was the simultaneity convention used in the early days of quantum mechanics. Some postulates of quantum mechanics, like the reduction postulate (or Von Neumann postulate) relies deeply on the time parameterization. That postulate states that if a measure of projector P is performed at a time t_0 the ket is given, after an affirmative outcome, and in a first kind measurement (a measurement that does not destroy the state), by $P|\psi(t)\rangle$. One who plans to make a measurement after the first one, has to use the new ket for his or her predictions. The prescription, then, distinguishes carefully between the temporal order of measurements. When the ket describes an entangled system of particles like in the EPR thought experiment, two measures can act on the same ket even if they are performed at large distance from one another. Apparently, for the prescription of the reduction postulate, it is important to distinguish the time order of two measures. This implies that distant simultaneity and hence Einstein's convention seems to play a role in the predictions of such a naive quantum mechanics. Actually, what really happens, is that the predictions of an observer are not altered by the measurement, on the same system, of another, spacelike separated, observer. The predictions made with quantum mechanics are always the best predictions that an observer can make on the basis of the information available [12]. Whether a distant observer is making a measurement does not influence the expectations of the other. This can be seen in two ways: the first observer can use the unaltered ket $|\psi\rangle$, or both $P|\psi\rangle$ and $(I-P)|\psi\rangle$ with their respective probabilities, where P is the measurement made by the second observer. In both methods the expectations for the first observer turn out to be the same. This becomes also evident if we consider that spacelike-separated operators commute (this fact is however formally realized only in QFT) and hence two observers cannot operationally distinguish who has made the measurement as first. We conclude that even if Einstein distant simultaneity appears explicitly in quantum mechanics axioms it does not seems to have a fundamental role. I, finally, notice that Einstein simultaneity seems also to be fundamental in the Hamiltonian approach (even in second quantization) to quantum mechanics. There, indeed, the canonical commutation relations, are equal time commutation relation where by equal time is meant Einstein's time. Again even if it appears explicitly, it does not seem to be fundamental. In order to see a spacetime symmetric quantization one can skip from Hamiltonian to path integral quantization.

Chapter 5

Constancy of two-way speed of light and its consequences

In this chapter, starting from the experimental fact that light propagates over a closed path at speed c (L/c law), I show to what extent the isotropy of the speed of light can be considered a matter of convention. I prove the consistence of anisotropic and inhomogeneous conventions, limiting the allowed possibilities. All conventions lead to the same physical theory even if its formulation can change in form. The mathematics involved is that of gauge theories and the choice of a simultaneity convention is interpreted as a gauge choice (I shall return on this aspect in the next chapter) .

Moreover, I prove that a Euclidean space where the L/c law holds, gives rise to a spacetime with a Minkowskian causal structure, and I exploit the consequences for the causal approach to the conventionality of simultaneity.

5.1 Causal structure from constancy of two-way light speed

Let us consider a Euclidean space E^3 where light propagates on straight lines. A beam of light leaves its starting point O_1 and through the reflection over suitable mirrors covers a closed path γ ending in O_1. If we use a large number of mirrors the path can approximate, as much as we want, a smooth closed curve of arbitrary shape, so that we can assume γ to be an arbitrary differentiable closed curve. If L is the length of the curve, by the L/c law, the total travelling time is $\tau = L/c$. Let us introduce a field $\vec{A}(\vec{x})$ so that $\nabla{\times}\vec{A} = 0$ (or, which is the same, $\vec{A} = \nabla\phi$ for

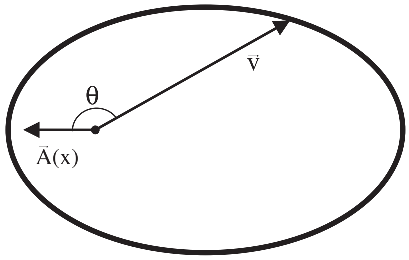

Figure 5.1: The anisotropy of the speed of light at the point $\vec{x}$ is elliptical.

a suitable scalar function $\phi(\vec{x})$), then

$$\tau = L/c + \oint_\gamma \vec{A} \cdot \mathrm{d}\vec{l}. \tag{5.1}$$

The previous expression can be rewritten

$$\tau = \oint_\gamma \mathrm{d}l(\frac{1}{c} + A\cos\theta) = \oint_\gamma \frac{\mathrm{d}l}{v(\vec{x},\theta)}, \tag{5.2}$$

where θ is the angle between $\vec{A}$ and $\mathrm{d}\vec{l}$ and where

$$\vec{v} = \frac{c\hat{v}}{1 + c\hat{v} \cdot \nabla\phi(\vec{x})} \tag{5.3}$$

is a new modified speed of light. It is anisotropic, in fact its absolute value depends on the direction $\hat{v}$ (see figure 5.1). Now it is clear that, if the speed of light has the anisotropic and inhomogeneous value given by Eq. (5.3), then the L/c law is fulfilled. Therefore, the arbitrariness of the speed of light amounts at least to an entire field $\phi(\vec{x})$.

There are more general expressions which verify the L/c law for any closed path starting at O_1. However, if the L/c law is verified by any observer, that is for any choice of O_1, then the most general expression for the velocity is given by Eq. (5.3). This is proved in paragraph 5.7, and is correctly stated by the following

Theorem. Let $M = E^3 \times \mathbb{R}$ be a spacetime consisting of a Euclidean space E^3 endowed with a global time $\mathbb{R}$. Suppose that any observer at rest measures with his clock, and along his worldline, a time which differs from the global time t only by an additive constant. Suppose moreover that light propagates along straight lines with a finite speed of norm f,

$$\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = f(\hat{v},\, \vec{x},\, t)\, \hat{v}. \tag{5.4}$$

Finally, assume that two light worldlines coincide if they pass through the same event with the same direction. Then, if the L/c law holds:

(a) In M there is a global time variable η that makes the speed of light isotropic

$$\vec{w} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}\eta} = c\hat{w}. \tag{5.5}$$

The function $\eta(\vec{x}, t)$ is unique up to an additive constant, and of the form

$$\eta = t - \phi(\vec{x}). \tag{5.6}$$

(b) If f is continuous in its arguments, the function $\phi(\vec{x})$ is differentiable, with $|\nabla\phi(\vec{x})| < 1/c$, and the speed of light is given by Eq. (5.3).

Moreover, if light is, for any given direction, the fastest signal* that carries information then:

(c) M has a Minkowskian causal structure.

In the proof I construct, using the Einstein synchronization convention, a new global time variable η which makes the speed of light isotropic. Then I prove that η, like t, measures the time of clocks at rest (Eq. 5.6). The other statements follow (a).

Notice that the first part of the theorem (firsts two points), applies to whatever signal propagates over straight lines such as, for example, the sound. In that case, with c I mean the *two-way* velocity of the signal.

Statement (a) of the theorem is in some respect opposite to statement (b). The latter implies the conventionality of the isotropic assumption whereas the former shows that all the freedom in choosing the convention can be eliminated with a time coordinate change. This last circumstance makes the one to one correspondence between the convention in the velocity distribution and the convention in the concept of simultaneity, clear. Two different time variables η and t, related by Eq. (5.6), have indeed, different simultaneity slices.

The theorem, moreover, suggests to replace the postulate of constancy of the speed of light with the L/c law which is independent from the synchronization convention used.

Anderson and Stedman [7], starting from the isotropic convention and making the substitution (5.6), obtained the conventions expressed by (5.3). I have shown that even the converse is true: every convention for the velocity of light, allowed by the L/c law, takes that form and, hence, every allowed convention can be seen

*Experimentally it is possible, for any given direction, to establish unambiguously which of two signals is the fastest. Let the two signals start at the same instant from O_1 and reach O_2. Clearly, the first of the two signal that reach O_2, as measured by O_2's clock, is the fastest. This conclusion does not depend on the synchronization of the clocks of O_1 and O_2.

as arising from the isotropic convention by a change in the clocks synchronization. In particular the theorem shows that there exists a time coordinate that makes the speed of light isotropic, a fact that should be deduced from experimental results, like the L/c law, rather that assumed since the beginning, like in standard textbooks.

A step in this direction was already made by Anderson et al. [6, p. 132] who proved that such a coordinate η exists, provided the L/c law holds, and the anisotropy is elliptical [6, p. 127]. However, the strong requirement of ellipticity was made *a priori*, so that a proof of our theorem was still lacking.

I have derived the causal structure of M without the need of a spacetime metric. Let, now, M be a Lorentzian manifold where light rays propagate over null geodesics. By the theorem, its metric $\mathrm{d}s^2$, in the coordinates $\{\vec{x}, \eta\}$, must equal Minkowski's up to a conformal factor (see, for instance, [34]). If additionally, the metric is required to measure the proper time of clocks at rest ($\mathrm{d}s = \mathrm{d}\eta$), then the conformal factor is fixed to unity and the Minkowski metric is completely recovered.

Finally, the reader can verify, following the proofs of paragraph 5.7, that the conditions of the theorem can be considerably weakened allowing a generalization of the first part. The space E^3 may be replaced with a Riemannian space of $\mathbb{R}^3$ topology where light is no longer required to propagate along the geodesics of the space, but needs only to move in trajectories invariant under inversion of direction. Then in Eq. (5.3) with $\vec{v}$ it is meant $\mathrm{d}x^i/\mathrm{d}t$ and with $\hat{v}$ it is meant $\mathrm{d}x^i/\mathrm{d}l$ where $\mathrm{d}l^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the metric of the Riemannian space. Here, we are mainly concerned with the theoretical and experimental foundations of Minkowski spacetime, so as not to discuss this generalization further.

5.2 Consistence of anisotropic conventions

We have seen that any possible convention is related to the isotropic one by the coordinate transformation of Eq. (5.6). This is the fundamental ingredient which allows us to prove our inability to find some physical phenomena ruling out one convention instead of another, as shown in great detail by Anderson et al. [6] . Let us shortly review their findings; this will justify the use of the word “convention” which, in this thesis, is referred to any scheme allowed by experience.

We can express all the known laws of physics in a conventional global time coordinate obtaining a set of physical laws coherent with experience. The set is the one we had developed if, in our history of science, instead of using the isotropic convention we had chosen an anisotropic one. This set is recovered simply by performing a coordinate transformation from the coordinates of the isotropic convention, $\{\vec{x}, \eta\}$, to the coordinates of an anisotropic convention, $\{\vec{x}, t\}$. For instance, the Gauss law of electromagnetism reads, in the anisotropic coordinates,

$$\nabla \cdot \vec{E} + \nabla\phi \cdot \frac{\partial \vec{E}}{\partial t} = 4\pi\rho. \tag{5.7}$$

As another example, the velocity of a particle of worldline $\vec{x}(t)$ is given by

$$\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \frac{\vec{w}}{1 + \vec{w} \cdot \nabla\phi}, \qquad \text{where } \vec{w} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}\eta}, \tag{5.8}$$

and the proper time of the particle is

$$\mathrm{d}\tau = \sqrt{1 - \frac{\vec{w}^2}{c^2}}\,\mathrm{d}\eta = \sqrt{(1 - \vec{v} \cdot \nabla\phi)^2 - \frac{\vec{v}^2}{c^2}}\,\mathrm{d}t. \tag{5.9}$$

Not every physical law requires a time variable for its formulation. When it is possible, a convention-free formulation clearly reveals the physical meaning of the law. We have already shown that the causal structure is independent from the chosen convention. In $M = E^3 \times \mathbb{R}$ we can even define, in a convention-free manner, the "light cone" of an event A,

An event B is said inside the light cone of an event A if there is a path γ which allows a light beam starting in A, and travelling over γ (with suitable mirrors), to end in B.

As a consequence, the law which states that information cannot propagate faster than light, can be expressed in a convention-free way, too.

In the literature there are theoretical proofs [41] of the isotropy of the speed of light but all of them use a non-modified law of physics in the process, and no one deals with the problem of velocity inhomogeneity. It is an easy task to "prove" the isotropy of the speed of light if we implicitly use an assumption or a law that holds only in the isotropic convention. For instance, it is easy to prove the isotropy of the speed of light if we use the Gauss law in its standard form. Indeed, its predictions agree with physical phenomena, Eq. (5.7), only if the velocity of light is isotropic. In the same way, contrary to the claims of Will [100], it is impossible to experimentally test the isotropy of the speed of light. Many of the experiments he had analyzed have some value as tests of special relativity, but as tests of the isotropy of the speed of light, they are inconsistent [6, p. 148].

5.3 Simultaneity from constancy of two-way light speed

In the previous section, we have shown that the laws of physics are simplified in the Einstein convention whereas, in anisotropic conventions, they loose their symmetries. The requirement of some symmetry becomes a way to restrict the allowed conventions to the isotropic one. This is seen even in the causal approach to the conventionality of simultaneity [33] whose cornerstone is the theorem of Malament [49]. This theorem (see [95] for a readable account of it) essentially proves that, if the causal structure of spacetime is that of Minkowski, and C is the worldline of an observer at rest, the only non trivial equivalence relation (simultaneity relation) invariant under C-causal automorphisms (diffeomorphisms of spacetime that preserve causal relations and map C onto itself) is that of Einstein. Our theorem enforces that of Malament in the following sense. The causal structure of spacetime is non-conventional because it can be tested experimentally. However, Malament takes it for granted that it is Minkowskian, that is, derived from a pseudo-Riemannian manifold of $\mathbb{R}^4$ topology, where the metric

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}\vec{x}^2 \tag{5.10}$$

vanishes on light worldlines. Now, one may wonder if this belief is well founded. After all, from (5.10), there also follows that the speed of light is isotropic (by dividing by $\mathrm{d}t^2$). In other words we cannot rely on equation (5.10) to state the causal structure of spacetime because it is only compatible with the isotropic assumption. We need experimental evidence that the causal structure of spacetime is Minkowskian and to do this we cannot rely on speed of light conventions. Alternatively, we need a proof of the independence of the causal structure from the convention chosen. At this point, the last part of our theorem enters. It states that, because of the L/c law, the causal structure of spacetime is Minkowskian even if our spacetime $M = E^3 \times \mathbb{R}$ is not a Lorentzian manifold†. Malament's argument then works. We summarize the entire deduction in a scheme

$$\boxed{M = E^3 \times \mathbb{R} \xrightarrow{L/c \text{ law}} \begin{array}{c}\text{Minkowskian}\\ \text{causal structure}\end{array} \xrightarrow{\substack{\text{Malament's}\\ \text{argument}}} \text{Einstein convention.}}$$

Although attractive, in what follows we leave this approach to the conventionality of simultaneity essentially for one reason. Malament, in order to recover the Einstein convention, requires an invariance principle, namely the invariance of the

†Remember that the causal structure of spacetime follows our theorem without the need of a spacetime metric. Moreover, Malament's argument uses only the causal structure of Minkowski spacetime.

simultaneity relation under C-automorphisms; but we have seen that a number of physical laws have the same effect if we require some symmetry. There is no physical reason for such a requirement; after all, the concept of simultaneity has to do with clocks and not with light (causal structure) and in this regard clocks show that there are a number of viable conventions, those given by Eq. (5.3). Moreover, Malament's argument is hard to generalize to the case of observers in generic motion [10], or to generic spacetimes, because in such contexts C-automorphisms may be absent.

5.4 The choice of a good convention

Once one agrees on the conventional nature of simultaneity, the problem becomes how to find a good convention for the physical context at hand. I suggest three criteria

- Simplicity of the laws of physics.
- Invariance of the convention used under change of the observer.
- Existence of a global time variable.

The first criterion is clear, one has to choose the simplest convention whenever the last two points are satisfied. The second criterion has the following meaning: a convention is good if it is the same for every observer, in such a way that communication among them is possible without referring each time to a subjective choice. This implies that the function ϕ must be the same for all observers: indeed, if the observer O_1 uses the time variable $t_1 = \eta_1 - \phi_1$ and the observer O_2 uses the time variable $t_2 = \eta_2 - \phi_2$, a communication among them is useless, unless each observer knows the function ϕ used by the other. To meet the second point, the set of observers must agree that function ϕ be used. Depending on the physical situation, we have to restrict and define the set of observers under which the invariance of the convention holds. This happens in the following example.

So far I have considered only observers at rest, here I look for conventions well suited for moving observers. In this example, the set of observers, under which invariance of the convention must be met, is given by inertial observers. Let us consider the Galilean principle of relativity

(*) *A reference frame in uniform rectilinear motion with respect to an inertial frame is inertial by itself.*

Here, for "inertial frame", we mean any observer who does not feel inertial forces. This definition, based on detectable forces, avoids any convention and is ideal for our purpose. The function ϕ, common to all inertial observers, is taken in such a way that the Galilean relativity principle (*) remains unchanged passing from the time variable η to the time variable t. This restricts the allowed conventions to a subset where the relativity principle holds and where the laws of physics happen to be particularly simple. Let $\vec{w}$ be the uniform velocity of an inertial observer, from Eq. (5.8), we see that in the time variable t the inertial observer has a uniform velocity only if $\nabla_{\vec{w}}\nabla_{\vec{w}}\phi = 0$ which implies[‡] $\phi = \vec{a}\cdot\vec{x} + \text{const.}$. With this choice, the coordinate transformations from one inertial observer to another form a group. If $U(a)$ is the coordinate transformation from $\{\vec{x}, t\}$ to $\{\vec{x}, \eta\}$ then the coordinate transformation to a second observer of velocity $\vec{v}$ is given by [6, 94]

$$G(\vec{a}, \vec{v}) = U(a)^{-1}\Lambda(\vec{w})U(a) \tag{5.11}$$

where $\vec{v}$ and $\vec{w}$ are related by Eq. (5.8). The group of coordinate transformations can be shortly written $G = U(a)^{-1}\Lambda\, U(a)$ where Λ is the Lorentz group in the usual representation. We take, $\vec{a} = \frac{\alpha}{c}\hat{i}$, where α is a dimensionless constant. In one spatial dimension the modified Lorentz transformation becomes

$$\begin{aligned} x' &= \gamma(w(v))\{(1+\alpha\beta)x - \beta ct\} && (5.12)\\ ct' &= \gamma(w(v))\{(1-\alpha)\beta ct + (1+\alpha-\alpha\beta+\alpha^2\beta)x\}, && (5.13) \end{aligned}$$

where

$$\begin{aligned} \beta &= \frac{w}{c} = \frac{v/c}{1-\alpha v/c}, && (5.14)\\ \gamma(w(v)) &= \frac{1-\alpha v/c}{\sqrt{(1-\alpha v/c)^2 - (v/c)^2}}. && (5.15) \end{aligned}$$

With the simplest choice, $\alpha = 0$, we recover the Lorentz transformation.

We mention another interesting convention. Starting from a realistic viewpoint, and with purposes very different from ours [82]-[76], Selleri [81] renewed some interest in the absolute synchronization convention [52], that is on the proposal $\phi_{\vec{v}} = (\vec{v}\cdot\vec{x})/c^2$, where $\vec{v}$ is the velocity of the observer $O_{\vec{v}}$ with respect to a privileged frame O_0. This convention, depending on the velocity $\vec{v}$ of the observer, is not invariant under change of inertial frame. As a consequence, the laws of physics are not invariant either and the modified Lorentz transformations do not form a group. However, the Galilean relativity principle (*) remains true because $\phi_{\vec{v}}$ is linear in $\vec{x}$. Implicitly, in the previous section, much in the spirit of Mansouri

[‡]Torretti [90, 91], already considered this convention for a single observer requiring that Newton's first law be satisfied.

and Sexl [52], a problem raised by Selleri in his paper [81] is solved, that of finding how the laws of physics must be written in the absolute synchronization convention and if there is an experiment capable of ruling it out [17]. Nevertheless, I stress, in contrast to him [82], that the possibility of anisotropic conventions does not imply the inconsistency of special relativity.

The third point requires a wider discussion; I devote the following section to it.

5.5 The existence of a global time variable

So far, we have considered only the case in which the L/c law holds everywhere; to generalize our treatment the field $\vec{A}(\vec{x})$ is now taken arbitrarily, that is, we remove the condition $\vec{A} = \nabla\phi(\vec{x})$. If light has the velocity

$$\vec{v} = \frac{c\hat{v}}{1 + c\hat{v}\cdot\vec{A}(\vec{x})}, \tag{5.16}$$

then the time taken by a light beam to travel round trip over the path γ is given by

$$\tau = L/c + \oint_\gamma \vec{A}\cdot \mathrm{d}\vec{l}, \tag{5.17}$$

and the difference from the case in which light travels in the opposite sense is

$$\delta\tau = 2\oint_\gamma \vec{A}\cdot \mathrm{d}\vec{l}. \tag{5.18}$$

This is a generalized Sagnac effect due to the distribution of velocities, Eq. (5.16). Being a measurable quantity, if a Sagnac effect is present, every allowed convention must account for it. Hence, in the presence of a Sagnac effect, it is impossible to find a new global time variable which allows the speed of light to be everywhere c. The existence of a global time variable is very useful, and must be considered one of the main tasks of a good convention, so this excludes the isotropic convention in a number of physical situations. Moreover, if two conventions on the velocity of light are allowed, because they lead to the same correct predictions (Sagnac effect), then their fields are linked to one another by the gauge transformation

$$\vec{A}(\vec{x}) \longrightarrow \vec{A}(\vec{x}) + \nabla\phi(\vec{x}). \tag{5.19}$$

In order to see this, take the difference of two $\vec{A}$ fields, then obtain a new field that is rotation-free because its integral over an arbitrary closed path is zero.

Measurable quantities, being convention-free quantities, must be gauge invariant. For instance, the rotation $\vec{B}(\vec{x}) = \nabla \times \vec{A}$ is gauge independent and plays the role of the field strength of the gauge theory. It can be measured revealing the Sagnac effect for a closed path in the neighborhood of $\vec{x}$. Again, the gauge transformation (5.19) follows from a resynchronization of clocks, that is from a transformation in the global time coordinate $t \to t + \phi(\vec{x})$.

Our treatment generalizes Anderson and Stedman's [7] who presented the choice of simultaneity, within an inertial frame, as a gauge choice.

In our formalism, the usual non-relativistic Sagnac effect [78, 85] is obtained once one takes

$$\vec{A}(\vec{x}) = \frac{\vec{\omega} \times \vec{x}}{c^2}, \tag{5.20}$$

where ω is the angular velocity of the rotating platform. Indeed, with this choice, one recovers the well known formula

$$\delta\tau = \frac{4S\omega}{c^2}, \tag{5.21}$$

where S is the area of the surface subtended by the curve γ. Eq. (5.20) is even the best convention that people living on the Earth surface can take, where $\vec{x}$ is the displacement from the Earth's axis.

Eq. (5.20) can be recovered from the general relativistic work of Møller [61] who proved that an expression like (5.16) with[§] $A_i = -g_{0i}$ holds in whatever stationary frame in which $g_{00} = 1$, where $x^i = const.$ is the worldline of a generic observer at rest in the frame. In our case [61], $g_{00} = (1 - \frac{r^2\omega^2}{c^2})$, and the previous condition holds, for a given radius, only in the non-relativistic limit. In more general relativistic circumstances, $g_{00} \neq 1$, the chosen global time variable no longer measures the rate of clocks at rest, and Eq. (5.16) no longer holds. However, using fiber bundle techniques one can still reveal the gauge nature of simultaneity, as I shall show in the next chapter.

In the present chapter we are mainly concerned with the case of a Euclidean space in which the L/c law holds, at least locally. In the theorem I proved that $\vec{A} = \nabla\phi$. However, if the L/c law does not hold globally, we can only conclude that $\nabla \times \vec{A} = 0$. In a space not simply connected this does not leave out the possibility of a Sagnac effect, and, as a consequence, this proves that the isotropic convention can be unsuitable even when the L/c law locally holds.

Think, as an example, of a large cylindrical spaceship of radius R spinning along its axis at angular velocity ω. People live on the internal cylindrical surface of the spaceship and ω is chosen to reproduce the gravitational acceleration

[§]Indices are lowered and raised with the positive definite space metric $\mathrm{d}l^2 = (-g_{ij} + g_{0i}g_{0j}/g_{00})\mathrm{d}x^i\mathrm{d}x^j$: $A_i = (-g_{ij} + g_{0i}g_{0j}/g_{00})A^j \Rightarrow \vec{A}\cdot\vec{v} = A_i dx^i/dt$ where the dot stands for the scalar product with the space metric.

$g = \omega^2 R$. Light and electric signals propagate along the surface. In such a situation the L/c law holds locally but a Sagnac effect is present when a light beam travels all around the spaceship.

5.6 Simultaneity, gauge theories, and causal structure: a summary

In the first section of this chapter I developed the consequences of the L/c law. I found that the allowed distributions for the velocity of light are given by Eq. (5.3), and that each of them can be recovered from the isotropic value via a time coordinate transformation. The relation with a coordinate transformation enabled us to rewrite the laws of physics coherently with the anisotropic convention adopted. This change in their expression clearly does not alter the physical content of the laws, so that a physics based on anisotropic conventions appears to be feasible. Moreover, anisotropic conventions can prove extremely useful, as I showed in the last part of the chapter. The approach to the conventionality of simultaneity as a gauge theory is very attractive. Conventional quantities, like the one-way speed of light turn out to depend on the potential whereas measurable quantities, like the Sagnac effect, depend on its curvature.

My analysis of the L/c was proved useful even in the causal approach to the conventionality of simultaneity. I showed that, if the L/c law holds in a Euclidean space, then the associated spacetime is causally the same as Minkowski spacetime. This result relates convention-free concepts, namely the L/c law and the causal structure of spacetime. It justifies Malament's argument if one wants to base the simultaneity concept on the causal structure. This last approach, however, appears untenable when one passes from Minkowski spacetime to more realistic spacetimes where causal automorphisms are absent.

5.7 Proof of the theorem

The theorem requires a lemma.

Lemma. Under the hypothesis of the theorem, let O_1 be an observer at rest. Let t_1 be the time measured by the clock of O_1 along its own worldline $\vec{x}_{O_1} = const$. From the hypothesis, on the worldline of the observer, we have $t_1 = t + const.$ where t is the global time. Outside the worldline of the observer O_1 the parameter t_1 is not yet defined. If O_1 labels events outside his worldline using the Einstein synchronization convention, then the variable t_1 becomes a global function: $t_1 = \phi_1(\vec{x}, t)$. Moreover,

ϕ_1 turns out to be increasing and continuous in t. For a given $\vec{x}$, t_1 takes any real value and the speed of light, in this new variable, becomes c: $\mathrm{d}\vec{x}/\mathrm{d}t_1 = c\hat{v}$.

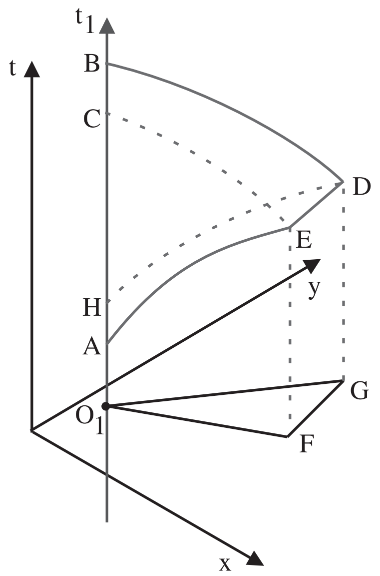

Figure 5.2: Note: the worldline of the light beam is not necessarily straight.

Proof of the lemma[¶]. Look at figure 5.2 where the coordinate z is omitted for the sake of clarity. Starting from O_1, and being reflected in F and G, a light beam travels all along the path O_1FGO_1. FG is an infinitesimal displacement and our task is to show that the speed of light over FG is c if the time used is t_1. We added to the picture the trajectories of the light rays CE and HD useful to define $t_1(D)$ and $t_1(E)$. From our hypothesis

$$
\begin{aligned}
c(t_B - t_A) &= \overline{FO_1} + \overline{FG} + \overline{O_1G} && (5.22)\\
c(t_C - t_A) &= 2\overline{FO_1} && (5.23)\\
c(t_B - t_H) &= 2\overline{O_1G} && (5.24)\\
\Rightarrow\ \overline{FG} &= \frac{c}{2}(t_B + t_H - t_A - t_C) = c(t_1(D) - t_1(E)). && (5.25)
\end{aligned}
$$

Or, more explicitly

$$\frac{\mathrm{d}t_1}{\mathrm{d}l} = \frac{1}{c}\,. \tag{5.26}$$

For a given $\vec{x}$, $t_1 = \phi_1(\vec{x}, t)$ takes any real value. To see this, integrate the previous equation over closed paths of arbitrary length starting from $\vec{x}$.

¶This lemma can be generalized in the case of curved spaces. You have simply to read $\overline{AB}$ as the length of the trajectory of the light beam from A to B.

The definition of ϕ_1 implies that t_1 increases with t (because, by hypothesis, two worldlines with the same direction cannot intersect; see figure 5.3); hence $\phi_1(\vec{x}, t)$ is invertible and t_1 can be used to label events. Finally ϕ_1 is continuous in t because, for a given $\vec{x}$, it is increasing in t and its image is $\mathbb{R}$. The lemma is proved.

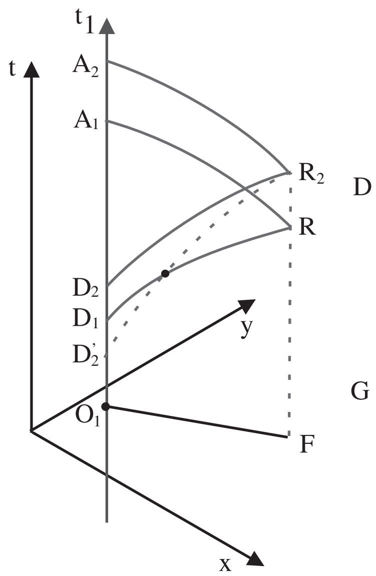

Figure 5.3: The function $t_1 = \phi_1$ increases with t, indeed if $t_1(R_2) \leq t_1(R_2)$ then, by definition of t_1, $t_1(D_2) \leq t_1(D_1)$ or $t_1(A_2) \leq t_1(A_2)$. The figure represents the former possibility with $D_2 \to D_2'$. In any case there would be an event common to two distinct light worldlines, contrary to the hypothesis.

Notice that we have not yet shown that the "time" function $t_1 = \phi_1(\vec{x}, t)$ measures the flow of time for observers different from O_1. We are ready to prove the theorem.
Proof of the theorem. Let us prove that a global time variable η which makes the speed of light isotropic must be unique up to an additive constant. Let γ be the worldline of a light beam of direction $\hat{v}$. If η_1 makes the speed of light isotropic

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}l} = \frac{1}{c}, \tag{5.27}$$

where l is the natural parameter of the projection of γ on E^3. Subtracting this equation with the same equation for η_2 we find that $\eta_1 = \eta_2 + c(\gamma)$ for all the events that lie on the worldline γ of the light beam considered. The constants of two different light beams must be equal if their worldlines intersect. In the coordinates $\{\vec{x}, \eta_1\}$ the light cone of an event $a = (\vec{x}(a), \eta_1(a))$ has the usual equation $|\vec{x} - \vec{x}(a)| = |\eta_1 - \eta_1(a)|$, therefore the light cones of two events a and b, intersects. This implies that a light worldline passing through a intersects a light worldline passing through b and hence that

$$\eta_1(a) - \eta_2(a) = \eta_1(b) - \eta_2(b) \tag{5.28}$$

Making b arbitrary, we see that a constant c exists such that $\eta_1 = \eta_2 + c$.

Let $t_1 = \phi_1(\vec{x}, t)$ and $t_2 = \phi_2(\vec{x}, t)$ be the functions of the lemma related to the observers O_1 and O_2. These functions are global time variables which make the speed of light isotropic. From our previous result t_1 and t_2 differ by an additive constant. But t_2, over the worldline $\vec{x} = \vec{x}_{O_2}$, equals the global time variable t up to an additive constant and hence, because O_2 is arbitrary,

$$\phi_1(\vec{x}, t) = t - \tilde{\phi}_1(\vec{x}) \tag{5.29}$$

for a suitable scalar function $\tilde{\phi}_1(\vec{x})$. This proves[‖] (a).

Let now f be continuous in its arguments. By hypothesis the worldline $\vec{x}(t)$ of a light beam of direction $\hat{v}$ is derivable (see Eq. (5.4)). This implies that $t(l)$ is derivable in the open set $A \subset S^2 \times M$ where[**] $f(\hat{v}, \vec{x}, t) > 0$. The lemma shows that $t_1(l)$ is derivable too, therefore the same is true for $\tilde{\phi}_1(\vec{x}(l))$ and we obtain

$$\nabla_{\hat{v}} \tilde{\phi}_1(\vec{x}) = \frac{1}{f(\hat{v},\, \vec{x},\, t)} - \frac{1}{c} \quad \text{in } A. \tag{5.30}$$

This proves that f is independent of time in A. Considering only the t dependence, f takes two values: zero outside A and $c/[1 + c\nabla_{\hat{v}}\tilde{\phi}_1(\vec{x})]$ inside A. However, f is continuous in t and hence independent of time throughout $S^2 \times M$.
If $f(\hat{v}, \vec{x}) > 0$ there is a neighborhood of $\hat{v} \in S^2$ where we can find three linear independent vectors $\hat{v}_1, \hat{v}_2, \hat{v}_3$ which verify $f(\hat{v}_i, \vec{x}) > 0$. The partial derivatives $\nabla_{\hat{v}_i}\tilde{\phi}_1(\vec{x})$ are continuous in $\vec{x}$ (see Eq. (5.30)) and therefore $\tilde{\phi}_1$ is differentiable: $\nabla_{\hat{v}}\tilde{\phi}_1(\vec{x}) = \hat{v} \cdot \nabla\tilde{\phi}_1(\vec{x})$.
Let $B \subset S^2 \times E^3$ be the open subset where $f(\hat{v}, \vec{x}) > 0$. From Eq. (5.30),

$$f(\hat{v}, \vec{x}) = \frac{c}{1 + c\hat{v} \cdot \nabla\tilde{\phi}_1(\vec{x})} \quad \text{in } B, \tag{5.31}$$

therefore $f \geq c/[1 + c|\nabla\tilde{\phi}_1(\vec{x})|]$ in B. If $(\hat{v}_1, \vec{x}_1) \in B$ then $\forall\ \hat{v}\ (\hat{v}, \vec{x}_1) \in B$, indeed, for a given $\vec{x}_1$, there is no continuous function $f : S^2 \to \mathbb{R}$ which is greater than

[‖] Another proof of (a) can be given making use of earlier results by Macdonald [48] on the consistency of the Einstein synchronization method.

[**] One can believe that the value $f = 0$ is incompatible with the L/c law because how can, for instance, a light beam follow a closed path if it passes through a point where the speed of light vanishes? In the same way, how can it follow its straight line if it stops where the speed of light vanishes? Thus, there seems to be no reason to bother with the case in which f vanishes in some points. Actually a light beam can pass through a point where the speed of light vanishes. Indeed, for instance, the equation $\frac{dx}{dt} = \sqrt{|x|}$ has the solution $x = \mathrm{sg}(t)t^2/4$.

$c/[1 + c|\nabla\tilde{\phi}_1(\vec{x}_1)|]$ in an open subset of S^2 and zero elsewhere.
Let $C \subset E^3$ be the open subset where $f(\hat{v}, \vec{x}) > 0$. From Eq. (5.31), because f is finite, $|\nabla\tilde{\phi}_1(\vec{x})| < 1/c$ or $f(\hat{v}, \vec{x}) > c/2$ in C. But f is continuous in $\vec{x}$ throughout E^3 therefore $C = E^3$, $B = S^2 \times E^3$ and $A = S^2 \times M$. In other words f is positive. Eq. (5.31) proves statement (b).

In order to prove (c) notice that the causal structure does not depend on the coordinates used to label events. From a) and using the coordinates $\{\vec{x}, t_1\}$ we recover the light cone structure of Minkowski spacetime, and since, by hypothesis, light is the fastest signal, the causal structure of M coincides with that of Minkowski spacetime.

Chapter 6

The gauge nature of simultaneity

In this chapter I consider stationary extended frames in general relativity. The requirement of stationarity allows to treat the spacetime as a principal fiber bundle over the one-dimensional group of time translations. Over this bundle a connection form establishes the simultaneity between neighboring events accordantly with the Einstein synchronization convention. The mathematics involved is that of gauge theories where a gauge choice is interpreted as a global simultaneity convention. Then, simultaneity in non-stationary frames is investigated: it turns to be described by a gauge theory in a fiber bundle without structure group, the curvature being given by the Frölicher-Nijenhuis bracket of the connection. This chapter is a rather technical continuation of the previous one, where the gauge nature of simultaneity was only sketched.

6.1 Introduction

Since the early days of special relativity, the isotropy of the speed of light was considered as conventional, and related to the synchronization convention of distant clocks. As we have seen in chapter 4, the discussion was revived in the last decades by a controversial experimental test of the isotropy of the speed of light [100, 6] and by arguments which showed the privileged status of the Einstein synchronization convention over other conventions [49, 6]. On the other side, some authors [6, 57] reconsidered those results proving the independence of the laws of physics from the synchronization convention (simultaneity convention) adopted, and the existence of a set of conventions alternative to the isotropic one. Along these lines, I have shown that the causal structure of Minkowski spacetime follows solely from the constancy of the speed of light over closed paths [57]. This provides new confidence on the local Minkowski nature of the spacetime manifold, and suggests how to make statements without relying on synchronization conventions. The strategy is to connect convention-free quantities (i.e. independent of the synchronization convention), like the two-way speed of light, to convention-free quantities, like the causal structure. This scheme will prove to be useful not only in special relativity, and not only for inertial frames.

Leaving inertial frames in special relativity things become far more clear. In general it becomes impossible even to synchronize distant clocks using the Einstein synchronization convention and hence no question arises on its predominance over other conventions. One cannot avoid the arbitrariness of the choice of a global synchronization. A formalism would be welcome if able to provide quantities independent of the synchronization convention adopted. This formalism is that of gauge theories. The aim of this chapter is to fix completely the parallelism between conventionality of simultaneity and gauge theories. The need for a gauge formalism was already noticed [7, 57]. Here, a coherent exposition in light of Ehresmann's theory of connections [44] is given. It has the great advantage of unifying old and new results (see Eq. (6.29) and its interpretation as Bianchi identity of the gauge theory) in a common formalism which provides new insights on the geometry of spacetime. To this extent, we start with stationary frames in general relativity. Our conclusions will have straightforward specializations in usual contexts like the rotating platform in special relativity or the Kerr metric in general relativity. Then we study simultaneity in non-stationary frames showing that it is still described by a gauge theory in a generalized sense already developed by mathematicians [54, 51, 59]. The derivation is short but needs at least some notions of differential geometry of principal fiber bundles [44, 102].

6.2 The geometry of extended reference frames

In everyday experience we deal with reference frames extended over large scales. On the Earth's surface, physics laboratories communicate to each other considering themselves at rest with respect to the same global reference frame. Such a situation is very different from the local inertial reference frame that can be constructed in the neighborhood of a given free falling, non rotating, observer. There, locally, the usual Minkowski spacetime is recovered and the Euclidean spatial metric (that experienced by rods at rest with respect to the observer) is derived as the projection of the spacetime metric over the simultaneity slice, where the simultaneity of events is defined by the Einstein convention. Things change when a large number of observers are brought together to form an extended reference frame. For instance, the Einstein simultaneity convention is unsuited for rotating systems because clocks cannot be synchronized in that way all around a closed path. The spatial metric can no more be defined as the projection of the spacetime metric over the simultaneity slice because the Einstein synchronization convention, in general, does not work in the large and, therefore, such a slice does not exist. Hence, the spatial metric needs a more general definition (see section 6.3).

Let M be the spacetime manifold of metric $g_{\mu\nu}$, and let k be a time-like Killing vector field. We shall say that a body is at rest with respect to the stationary frame if its worldline is an integral line of the Killing vector field. Its four-velocity is (we shall use the symbol u also to denote the 1-form u_α; its meaning will be clear from the context)

$$u = \frac{k}{\sqrt{k \cdot k}}. \tag{6.1}$$

Let $G = T_1$ be the one-parameter group of diffeomorphisms generated by the Killing vector field and let t be the corresponding parameter, $k = \partial_t$. The previous definition identifies a point of space of the stationary frame with an integral curve of the Killing vector field, hence the "space" is defined by

$$S = M/G, \tag{6.2}$$

that is, as the quotient space of M under the action of the Lie group G. Here, we recognize, at least locally, all the ingredients of a principal fiber bundle: M is the principal bundle, G is the structure group and S is the base. We need only to define the right action of $g = e^t \in G$ on $m \in M$ as $mg \equiv \phi_t(m)$ where ϕ_t sends an event to its evolution after a Killing time t. The fiber F_s over $s \in S$ is given by the integral line identified with s, and we shall refer to it also as the worldline of s. For definiteness it is assumed that the projection $\pi : M \to S$ of the bundle onto the base is differentiable. Finally, the principal bundle is trivial because its structure group, the group of translations in one-dimension, is contractible [86].

The systematic construction of the gauge theory would be complete if we could provide a 1-form ω (the connection) on M, with the following properties [44]

$$(a) \qquad \omega(k) = 1, \tag{6.3}$$

$$(b) \qquad \phi_t^* \omega = \omega. \tag{6.4}$$

where ϕ_t^* is the pullback of ϕ_t. This is accomplished by the choice

$$\omega = \frac{k_\mu \mathrm{d}x^\mu}{k \cdot k}. \tag{6.5}$$

Statement (a) follows trivially from substitution and statement (b) from the invariance of (6.5) under translations: $L_k k = L_k g = 0$. In a given event m, the tangent space T_m splits in two parts, the vertical space V_m generated by the Killing vector field, and the horizontal space H_m orthogonal to it: $\omega(X) = 0 \leftrightarrow X \in H$.

Let us investigate more closely the physical meaning of this gauge theory. An observer at rest in the stationary frame can define, in his neighborhood, a simultaneity slice in compliance with the Einstein synchronization convention. The

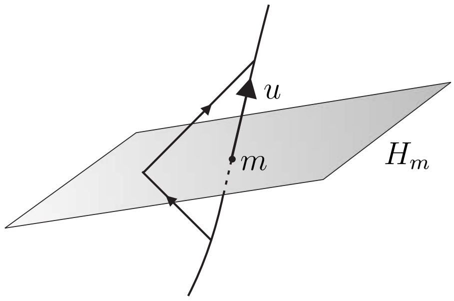

Figure 6.1: The horizontal plane and its relation with Einstein's simultaneity.

coordinates obtained following the Einstein convention, even for events far away from the worldline, are known as Märzke-Wheeler coordinates [53, 68]. Here we are interested only in the trivial result that the worldline is indeed orthogonal to the simultaneity slices. This result should be expected because it is a local statement that holds in special relativity. At least locally this means, as it was emphasized by Robb [77, 6], that the Einstein convention corresponds of taking as simultaneous those events which reside in the plane perpendicular to u, the four-velocity of the observer. From this we see that the horizontal space H_m identifies those events which are simultaneous with m with respect to an observer at rest in the stationary frame and placed in $s = \pi(m)$. In the following we shall refer to ω as the Poincaré connection, because Poincaré [73, 74] was the first who defined the synchronization convention that, later, was named after Einstein.

Let $\tau(\lambda)$ be a curve on S. The observers, all along τ, synchronize the neighboring clocks using the Einstein synchronization convention. Given an event $\tau^*(0)$, $\pi(\tau^*(0)) = \tau(0)$, they find a curve $\tau^*(\lambda)$ of simultaneous events. The procedure that we have sketched is the natural one that should be followed in synchronizing clocks, since it is simply obtained by using the prescriptions of the Einstein convention in each point of space. The reader should distinguish between this procedure and the one that leads to the Märzke-Wheeler coordinates since in the latter case an observer is privileged. In order to avoid confusions, in what follows we shall refer to the Einstein synchronization convention as a local procedure to be applied in each point of space. As we shall see this procedure cannot be applied in every circumstance [6]. In gauge theory the curve τ^* is called the horizontal lift of τ; it is the only curve which starts from $\tau^*(0)$, has horizontal tangent vectors, and projects onto τ.

Before we start studying the curvature and its implications, we recall some notions regarding local observers. An observer is identified by a tetrad of orthonormal vectors $\{u, e_1, e_2, e_3\}$. A picture of e_i may be the following: imagine the local laboratory of the observer to be a cube. The observer, at rest in the cube, stands by one of the corners and the normal vectors e_i point towards the

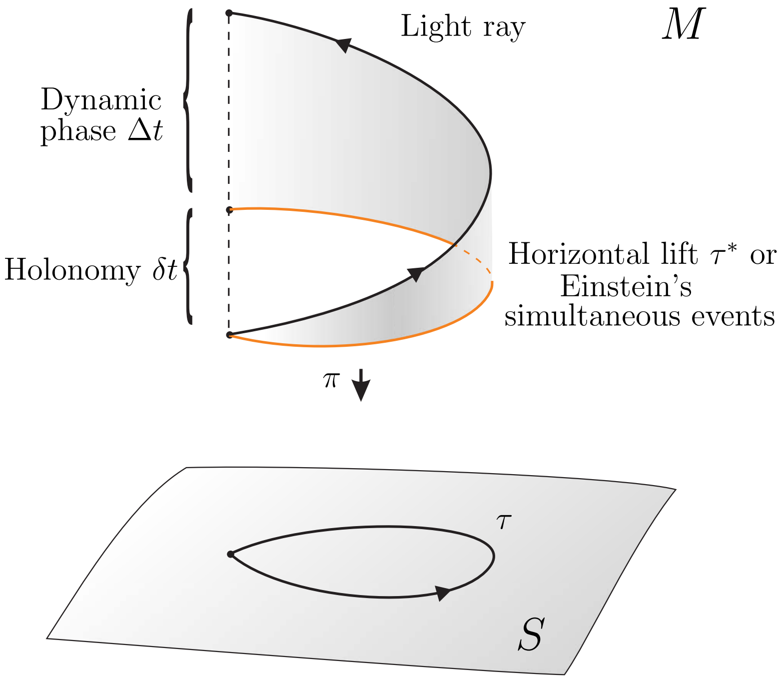

Figure 6.2: The horizontal lift and its relation with the pointwise Einstein synchronization.

next three corners. Since e_i points always towards the same corner it must satisfy

$$L_k\, e_i = 0. \tag{6.6}$$

Let $h^{\mu}_{\nu} = \delta^{\mu}_{\nu} - u^{\mu} u_{\nu}$ be the projector on the horizontal space of the observer. The evolution of the components of e_i with respect to a Fermi transported tetrad (non-rotating reference frame) $\hat{e}_{\mu}$ is well known [34], the relevant quantities are the vorticity tensor

$$w_{\mu\nu} = h^{\alpha}_{\mu} h^{\beta}_{\nu}\, u_{[\alpha;\,\beta]}\ , \tag{6.7}$$

which represents the angular velocity of the triad e_i with respect to gyroscopes $(\hat{e}_i)$, and the expansion tensor

$$\theta_{\mu\nu} = h^{\alpha}_{\mu} h^{\beta}_{\nu}\, u_{(\alpha;\,\beta)}\ , \tag{6.8}$$

which represents the rate of separation of neighboring points from the worldline. From Eq. (6.1), after some algebra, we find that the expansion tensor vanishes thus allowing a local rigidity interpretation. The vorticity can be rewritten

$$w_{\mu\nu} = \sqrt{k\cdot k}\ \omega_{[\mu;\,\nu]} = -\frac{\sqrt{k\cdot k}}{2}\, \Omega_{\mu\nu}, \tag{6.9}$$

where we have introduced the curvature

$$\Omega = D\omega = (\mathrm{d}\omega)h = \mathrm{d}\omega. \tag{6.10}$$

The translational invariance of spacetime implies that the vorticity, the curvature and the norm of the Killing vector field have a vanishing Lie derivative. As a consequence the angular velocity of an observer at rest in the stationary frame points always in the same direction with the same norm. Its components with respect to e_i do not change in time.

Gauge theory tells us that if S is paracompact and simply connected the curvature vanishes if and only if the horizontal lifts of closed curves in S are closed in M. In order to make use of the Einstein convention the horizontal lift should be closed, elsewhere there would be two simultaneous events placed in the same time-like worldline. In that case, the Einstein convention would be of little use. If the topological requirements are satisfied: the Einstein synchronization convention is applicable if and only if, in every place of the stationary frame, observers at rest do not rotate [30]. Moreover, if the vorticity is different from zero in a point s of space, an observer at rest in that point feels inertial forces because its local reference frame rotates with respect to Fermi transported gyroscopes. These observations assure that the rotating platform in special relativity is indeed quite representative of the general interrelationship between the Einstein convention, inertial forces, and rotation.

Let us study the simple case $\Omega = 0$. Gauge theory tells us that, when the curvature vanishes, there exists a foliation of the principal bundle in three-dimensional hypersurfaces with horizontal tangent spaces. They are the hypersurfaces of simultaneity accordantly with the Einstein convention. Those hypersurfaces are orthogonal to the Killing vector field because so are their tangent spaces. The existence of hypersurfaces orthogonal to a vector field is the object of Frobenius theorem: the necessary and sufficient condition for the vector field v^μ to be hypersurface orthogonal is $v_{[\alpha}\nabla_\beta v_{\gamma]} = 0$. This condition is indeed coherent with $\Omega = 0$ because the equations

$$\Omega = 0, \tag{6.11}$$

$$k \wedge \mathrm{d}k = 0, \tag{6.12}$$

are algebraically the same.

When $\Omega \neq 0$ the Einstein convention is no longer applicable. Here gauge theory suggests a standard procedure to extend coordinates $\{x^i\}$ on S to the whole principal bundle. First choose a section $\sigma : S \to M$ and then associate to the event $\sigma(x^i)e^t$ the coordinates $\{x^i, t\}$. This procedure depends on the section (gauge) chosen and, from the physical side, corresponds of taking the events on the hypersurface $\sigma(S)$ as simultaneous. A number of different simultaneity conventions can be adopted, each one in correspondence with a particular section. In geometry the change of section is exactly what in physics is a gauge transformation. Hence we pass from one convention to another via gauge transformations. Usually in physics one works with the potential, that is with the pullback* $A_\sigma(x^i) = \sigma^*\omega$. In the coordinates $\{x^i, t\}$, the Poincaré connection becomes

$$\omega = \mathrm{d}t + A_\sigma(x^i). \tag{6.13}$$

It is invariant under the change of section (gauge transformation)

$$t' = t - \alpha(x^i), \tag{6.14}$$
$$A_{\sigma'} = A_\sigma + \mathrm{d}\alpha(x^i). \tag{6.15}$$

It is also convenient to introduce the field strength $F(x^i) = \sigma^*\Omega = \mathrm{d}A_\sigma$. Let us come to a frequent question in gauge theories. If $\tau(\lambda) : [0,1] \to S$ is a closed curve, how much is δt such that $\tau^*(1) = \tau^*(0)e^{\delta t}$? In order to answer this question we have to recall that τ^* has horizontal tangent vectors. Integrating ω over τ^* we find

$$\delta t = -\oint_\tau A_\sigma. \tag{6.16}$$

The physical interpretation of this result is connected with the Sagnac effect [78, 5].

6.3 Space metric and Sagnac effect

Let us define over S a metric through the formula [61, 31, 47, 98]

$$\mathrm{d}l^2 = -h_{ij}\mathrm{d}x^i\mathrm{d}x^j \qquad i,j = 1,2,3 \tag{6.17}$$

in the coordinates $\{x^i, t\}$ introduced above. From a geometrical perspective this formula for the space metric $\mathrm{d}l^2$ can be rewritten

$$\mathrm{d}l^2(X,Y) = -g(X^*, Y^*)\,, \tag{6.18}$$

where X^*, Y^*, are the horizontal lifts of the space vectors X, Y. The vectors X^*, Y^* belong to the horizontal space of the observer which passes through their origin, and that horizontal space is nothing but the local space of an inertial observer. Hence, from a physical point of view the previous formula says that the space metric is induced from the measurements made by local inertial observers. This statement relies on the fact that, for an inertial observer, the spacetime metric

*This definition of the potential differs for a sign with respect to the one given in chapter 5. There, in a coordinate system, $A_i = -g_{0i}/g_{00}$, while here $A_i = g_{0i}/g_{00}$

serves also to define the space metric through the formula $\eta(X^*, Y^*)$ where X^*, Y^* belong to its space. However it is an assumption that for accelerating or rotating observers, space measurements should coincide with those performed by inertial observers. This postulate referred also as the "surrogate frames postulate" [43] is waiting an experimental verification. Actually in our stationary frame, observers at rest in the frame are not necessarily inertial. To state that equation (6.18) describes accurately the space metric as experimented by observers at rest in the frame, goes beyond our knowledge. A number of alternatives, especially for the rotating platform have been suggested [87, 43]. A similar debate raised in the contest of Kaluza-Klein extensions of gravity where two different interpretations of the five dimensional metric were natural: those of Weyl and Pauli [67]. This is not accidental as the mathematics involved in the spacetime splitting in time plus space is that of Kaluza -Klein theories. Indeed the spacetime metric can be rewritten

$$g(A, B) = (k \cdot k)\omega(A)\omega(B) - \mathrm{d}l^2(\pi_* A, \pi_* B)\,, \tag{6.19}$$

which is a Kaluza-Klein metric in the spacetime manifold. Another useful choice for the space metric is the optical metric [1], $\mathrm{d}\bar{l} = \mathrm{d}l/(k\cdot k)$. Indeed the geodesics of light, and hence their projections, are invariant under conformal transformations of the spacetime metric g. In the study of light propagation we can reduce the spacetime metric to the form, $\mathrm{d}\bar{s}^2 = \omega^2 - \mathrm{d}\bar{l}^2$ where no conformal factor appears. Now, the projection of a geodesic of a Kaluza-Klein metric with unitary scalar field, $k \cdot k = 1$, satisfies a straightforward generalization of the Lorentz law. In the null case it reads

$$\bar{\nabla}_v v^i = F^i_j v^j\,, \tag{6.20}$$

where $v^i = \mathrm{d}x^i/\mathrm{d}\bar{l}$ is the tangent vector of the projection and $\bar{\nabla}$ is the covariant derivative with respect to the optical Levi-Civita connection. Hence, at least light rays, satisfy a Lorentz like equation in the optical metric background. Here we use the optical metric as a tool for finding the dynamic phase(see figure 6.2).

Let us consider two light beams that leave $\tau^*(0)$ following the path $\tau(\lambda)$, one in the positive direction and the other in the negative direction. Figure 6.2 shows one of the beam and displays some relevant quantities like the dynamic phase Δt and the holonomy δt. As seen from the local inertial frames along τ, with respect to their local Einstein synchronized times, the two light beams have the same speed $c = 1$. Let l be the natural parameter of τ. The dynamic phase is

$$\Delta t = \int_{\tau^*} \mathrm{d}t = \int_{\tau^*} \frac{\mathrm{d}\tau}{\sqrt{k \cdot k}} = \oint_{\tau} \frac{\mathrm{d}l}{c\sqrt{k(l) \cdot k(l)}} = \bar{L}/c\,. \tag{6.21}$$

Where $\bar{L}$ is the total optical length of the path τ. Coming back to the Sagnac effect, the dynamic phase does not depend on the direction followed by the light beam. On the contrary, the holonomy changes sign; as a consequence the difference of the arrival times of the two beams is twice Eq. (6.16), and taking into account the time dilation, that is the difference between Killing time and proper time, the Sagnac effect becomes

$$\delta\tau = -2\sqrt{k \cdot k} \oint_{\tau} A_{\sigma} \, . \tag{6.22}$$

This equation is conveniently written in the coordinates $\{x^i, x^0 = t\}$. Let Σ be a two-dimensional surface such that $\tau = \partial\Sigma$. From $k = \partial_t$ we have $k_\mu = g_{\mu\nu}k^\nu = g_{\mu 0}$ and $A_{\sigma i} = g_{0i}/g_{00}$, hence [9]

$$\delta\tau = -2(g_{00})^{\frac{1}{2}} \oint_{\tau} \frac{g_{0i}}{g_{00}} \, \mathrm{d}x^i = 2(g_{00})^{\frac{1}{2}} \int_{\Sigma} \frac{w_{ij}}{(g_{00})^{\frac{1}{2}}} \, \mathrm{d}x^i \wedge \mathrm{d}x^j \, . \tag{6.23}$$

Notice that the worldline of a point at rest in the stationary frame has equation, $x^i = const.$. This condition, and the independence of $g_{\mu\nu}$ of time, selects those systems of coordinates for which the previous equation holds.

We have completed our exposition of the gauge property of simultaneity in stationary frames. The reader familiar with gauge theories can easily recognize the formal similarity with electrodynamics. Both theories have the mathematical structure of a gauge theory over a one-dimensional group. The only relevant difference is that in the stationary frame the structure group is contractible (elsewhere there would be closed time-like curves), whereas in electrodynamics, $U(1)$, is not. This implies that, in electrodynamics, we can have non trivial principal bundles like monopoles, whereas this is not possible in the present context. Other analogies, like that between the Sagnac effect and the Aharonov-Bohm effect [79], or like that between magnetic forces and Coriolis forces [16, 83], become self evident in light of the gauge interpretation.

6.4 Simultaneity in non-stationary frames

So far we have considered stationary frames. In general, however, the trajectories of the observers which define the frame, will not be generated by a Killing vector field. This does not imply that simultaneity is no more described by a gauge theory, but only that there is no connection preserving structure group. We can still define the space S of trajectories of the observers of the frame. Then, discarding some pathological choices for the trajectories, at least in an open set, the space S will be a differentiable manifold and there will be a differentiable projection $\pi : M \to S$. Hence, again, M can be considered as a fiber bundle over the space S.

It should be noticed that over this bundle one can define the action of the one-parameter group of diffeomorphisms ϕ^u_τ generated by the four-velocity field

Gauge theory	**Simultaneity**
Fiber bundle P	Spacetime manifold M
Structure group G	*Group of time translations* T_1
Base	Space of worldlines S
Fiber	Worldline of an observer at rest in the frame
Vertical space at the point p	Space spanned by the four velocity of an observer at p
Horizontal space	Local Einstein's simultaneous events
Horizontal lift	Pointwise Einstein's synchronization
Connection	Poincaré connection P
Choice of the section	Choice of the simultaneity convention
Gauge transformation	Change of the simultaneity convention
Curvature	Angular velocity of observers at rest in the frame
Bianchi identity	$Dw + a \wedge w = 0$
Holonomy	*Sagnac effect*
Dynamic phase	*Optical round trip time* *in absence of holonomy* $\bar{L}/c$
Kaluza-Klein metric	*Spacetime metric*

Table 6.1: Gauge interpretation of simultaneity. The correspondences in italic hold in stationary frames but do not hold in general.

$u(x)$. With this right action M can be considered, at least locally, as a principal bundle. However, in general, such a structure group do not preserve the simultaneity connection. It is, then, more convenient to treat M as a fiber bundle without structure group since the beginning.

The theory of connections, i.e. gauge theory, was developed by mathematicians in this general context [54] even if, in physics, this generalization has not yet found relevant applications. Here a connection is given by a distribution of planes H_x, that is, by a differentiable assignment, in each point of the bundle, of a plane that we shall call horizontal. The plane, together with the tangent space of the fiber (the observer trajectory in our case) spans the tangent space at x. The relevant difference with the usual gauge theory is the absence of a structure group which sends the fiber to itself and horizontal planes to horizontal planes. Many of the features of the usual gauge theory have a generalization. It is still possible to define the horizontal lift, the curvature and the parallel transport. Even the Bianchi identities have a generalization [54].

In our case, the fiber bundle M has a natural connection. The horizontal planes are given by $H_x = \ker(u_\alpha(x)\mathrm{d}x^\alpha)$ where u^α is the four-velocity of the observer. In this way the connection maintains its original meaning: the horizontal plane still selects, locally, those events which are simultaneous accordantly with the Einstein synchronization convention. Analogously, the horizontal lift τ^* is obtained by a pointwise Einstein synchronization over τ. Notice that the connection $\omega = u_\alpha \mathrm{d}x^\alpha$ is defined only up to a spacetime factor, because its ker does not change after such a multiplication. For this reason, in the generalized gauge theory, one works with the vector valued connection $P^\beta_\alpha = u^\beta u_\alpha$, which is the projector of the tangent space onto the vertical subspace. One then defines the vector valued curvature R (which has nothing to do with the Riemann tensor), and its real counterpart Ω through the formula

$$2R = -[P, P]_{FN} \qquad \text{or} \qquad 2u^\beta \Omega = -[u^\mu \omega, u^\nu \omega]_{FN}, \tag{6.24}$$

where $[,]_{FN}$ is the Frölicher-Nijenhuis bracket. After some algebra the previous equation reads

$$\Omega = D\omega\,, \tag{6.25}$$

that is, the same expression obtained for the usual gauge theory. This expression is invariant under multiplication of the connection and the curvature by the same factor $f(x)$. Previously that factor was fixed by the requirement $\omega(k) = 1$. Here it is fixed by $\omega(u) = 1$. In this normalization the curvature is linked to the vorticity through the relation $\Omega_{\mu\nu} = -2w_{\mu\nu}$.

Let $A = \partial/\partial a$, $B = \partial/\partial b$, be two fields over S of vanishing Lie bracket, and let A^* and B^* be their horizontal lift. Plugging these fields in Eq. (6.25) we find

$$[A^*\Delta a, B^*\Delta b]^\mu u_\mu = 2w_{\alpha\beta}\, A^{*\alpha}\Delta a\, B^{*\beta}\Delta b\,. \tag{6.26}$$

The left hand side expresses the holonomy of a closed path τ of sides $\Delta x^i = A^i \Delta a$ and $\Delta x^i = B^i \Delta b$ in units of proper time. That is, it gives half the Sagnac effect for the path described. The right hand side stands for the product $2\vec{\omega} \cdot \vec{\Delta S}$ where $\vec{\Delta S}$ is the oriented area enclosed by the path τ. Hence, Eq. (6.26) is an infinitesimal, and general relativistic version of the well known formula of the Sagnac effect [85]

$$\delta\tau = \frac{4\vec{\omega} \cdot \vec{\Delta S}}{c^2} \,. \tag{6.27}$$

Notice that this equation holds only for infinitesimal paths. In the case of a stationary frame, there is an integral version given by Eq. (6.23).

In the generalized gauge theory the Bianchi identity reads

$$[P, R]_{FN} = 0 \,. \tag{6.28}$$

After some calculations, and using only the horizontality of the 2-form Ω one arrives at

$$D\Omega + a \wedge \Omega = 0 \,. \tag{6.29}$$

Hence, the Bianchi identity is a non trivial, and quite unexpected, differential relation between the vorticity and the acceleration field. Table 6.1 summarizes the parallelism between simultaneity and gauge theory.

6.5 Back to the usual gauge theory

One can ask what is the condition that the vector field u should satisfy in order for the connection to be principal. In other words, in what circumstance is there a vector field

$$X = \chi(x) u \,, \tag{6.30}$$

whose one parameter group of diffeomorphisms ϕ_t^X with $X = \partial_t$ sends horizontal planes to horizontal planes? If it exists one can choose

$$\omega_\mu = \frac{X_\mu}{X \cdot X} \,, \tag{6.31}$$

and Eqs. (6.3) and (6.4) become both satisfied. The invariance of the horizontal planes under the group of diffeomorphisms ϕ_t^X reads

$$\phi_{t\,*}^X H_x = H_{\phi_t^X(x)} \,. \tag{6.32}$$

Let ω be whatever real valued connection ($\ker \omega(x) = H_x$). The previous equation becomes

$$L_X \omega = f(x) \omega \,, \tag{6.33}$$

for a suitable function $f : M \to \mathbb{R}$. Let us begin with $\omega_\beta = X_\beta$, we find

$$(L_X\, g)_{\beta\nu} X^\nu = f X_\beta \,. \tag{6.34}$$

This equation is equivalent to Eq. (6.32). It is satisfied, for instance, when X is a conformal Killing vector field

$$L_X g = \frac{X^\mu_{;\mu}}{2}\, g \,, \tag{6.35}$$

hence our previous treatment of simultaneity in stationary frames can be easily generalized to conformal stationary frames. In both cases the connection is principal, that is, invariant under the action of a structure group. This should have been expected because the relation of orthogonality follows solely from the causal structure of spacetime, that is, it is invariant under conformal transformations of the metric. A conformal Killing vector field, preserving the causal structure, preserves also the orthogonality relation and therefore sends horizontal planes to horizontal planes. Equation (6.34) does not solve our problem because it is not expressed in terms of the four-velocity field $u(x)$.

Let $\omega_\beta = u_\beta$, plugging this equation in Eq. (6.33) we find after some algebra

$$\nabla_u u_\beta + (\ln \chi)_\beta = u_\beta \nabla_u (\ln \chi) \,. \tag{6.36}$$

The projection of this equation to the vertical space is automatically satisfied. It is convenient to introduce the acceleration 1-form $a_\beta = \nabla_u u_\beta$ and to project the previous equation to the horizontal plane

$$D(\ln \chi) = -a \,. \tag{6.37}$$

The problem is solved if we can establish when this equation has a solution $\ln \chi$ (χ^{-1} is the acceleration potential [25]) over the fiber bundle. The condition we are seeking will be an integrability condition for Eq. (6.37). It should be noticed that Eq. (6.37) gives us the value of $\partial_Y(\ln \chi)$ only for $Y \in H_x$. In order to find $\partial_u(\ln \chi)$ we have to use the non integrability of the distribution of the horizontal planes. When $R = 0$, however, the distribution of horizontal planes becomes integrable and the spacetime admits a foliation of simultaneity slices. In that case there is an infinite number of solutions (if any), each one in correspondence with a different time parameterization of the simultaneity slices. The equivalence relation "$x \sim y$ if there exists a piecewise C^1 horizontal curve which joins x and y", divides the spacetime manifold into sets M_i, $M = \cup_i M_i$. In each set the equation (6.37) has a unique solution $\chi(x)$ up to a constant factor. Indeed if χ' is another solution of Eq. (6.37) then $D(\chi\chi'^{-1}) = 0$ and hence $\chi\chi'^{-1} = c_i$ in each set M_i .

Without losing generality we can restrict ourselves to a region M_i where each pair of events can be joined by a horizontal curve. We assume that M_i is simply connected. Let us consider the 1-form

$$\sigma = f(x)u - a\,, \tag{6.38}$$

where $f(x)$ is a real valued function. With a straightforward calculation it is easy to show that $\mathrm{d}\sigma = 0$ if and only if

$$\begin{aligned} Da &= f\,Du\,, & (6.39)\\ L_u a &= f\,a - Df\,. & (6.40)\end{aligned}$$

In this case, $\sigma = \mathrm{d}\varphi$ and from Eq. (6.38), $f = \partial_u\varphi$ and $D\varphi = -a$. Since φ solves Eq. (6.37) it is the function $\ln\chi$ we are looking for. Conversely if $\ln\chi$, solution of (6.37), exists, it satisfies Eqs. (6.38-6.40) with $f = \partial_u \ln\chi$.

The system of equations (6.39) (6.40) is our necessary and sufficient condition in order to establish whether our connection is principal. The first equation (6.39) states that the two forms Da, Du must be proportional: the proportionality factor gives the derivative $f = \partial_u(\ln\chi)$ that Eq. (6.37) was unable to determine. The second equation (6.40) states a condition that can be checked once we substitute $f(x)$ as given by Eq. (6.39). Hence, we can establish whether or not simultaneity is described by a principal connection looking only at the physical data $u(x)$.

When $a = 0$ the system is solvable with the solution $\chi(x) = 1$. This result says that, when the congruence of time-like curves is given by geodesics, the simultaneity can be treated as a gauge theory over the group of time translations.

Another interesting case, which generalizes the previous one, is that of a perfect fluid of equation of state $p = p(\rho)$. The stress-energy tensor is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p\,g^{\mu\nu}, \tag{6.41}$$

The conservation of 4-momentum $T^{\mu\nu}_{;\nu}$ leads to the local energy conservation and to the Euler equation

$$\begin{aligned} \rho_{,u} &= -(\rho + p)\nabla\cdot u & (6.42)\\ (\rho + p)\,a &= Dp & (6.43)\end{aligned}$$

One can introduce the functions [25]

$$\begin{aligned} \chi &= \exp\left(-\int_{p_0}^{p}\frac{\mathrm{d}p}{\rho + p}\right) & (6.44)\\ R &= \exp\left(-\int_{\rho_0}^{\rho}\frac{\mathrm{d}\rho}{3(\rho + p)}\right) = \chi^{-1/3}\left(\frac{\rho + p}{\rho_0 + p_0}\right)^{-1/3} & (6.45)\end{aligned}$$

which are related to each other and to the pressure. Using the relation between the determinants $g = \chi^2 h$, the energy-momentum conservation becomes

$$\begin{aligned} \left(\ln(R^{-3}|h|^{\frac{1}{2}})\right)_{,u} &= 0 \qquad (6.46) \\ a + D\ln\chi &= 0 \qquad (6.47) \end{aligned}$$

This last equation says that the flow lines of the perfect fluid define a frame where the simultaneity is described by a gauge theory over the group of time translation. The invariance of the connection ω under time translations (see Eq. (6.4) and Eq. (6.13)) implies that the potential A_σ does not depend on time. Making use of Eqs. (6.46) (6.19) (6.13) and introducing $\bar{h}_{ij} = R^{-2}h_{ij}$, the spacetime metric becomes

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \chi(x)^2(\mathrm{d}t + A_i(x^j)\mathrm{d}x^i)^2 + R(x)^2\bar{h}_{ij}(x^k,t)\mathrm{d}x^i\mathrm{d}x^j \qquad (6.48)$$

where the determinant $\bar{h}$ does not depend on time. The function $R(x)$ can be interpreted as the scale factor of the universe since a volume element scales as R^3 with time ($\theta = \nabla \cdot u = 3R_{,u}/R$). This well known result, already obtained by Ehlers, Taub and others [25, 26] arises here as a natural consequence of a careful analysis of simultaneity in the simple case of a perfect fluid with an acceleration potential.

6.6 Simultaneity and gauge theories: other results

I have shown that simultaneity is described by the mathematics of gauge theories. The fiber bundle is the spacetime manifold and the fibers are the trajectories which define the frame. The connection is determined by local simultaneous events accordantly with the Einstein convention, and the horizontal lift follows from a pointwise Einstein synchronization. The curvature is the angular velocity of the observers at rest in the frame and the holonomy is the obstruction against the possibility of applying the Einstein synchronization in the large. A global simultaneity convention allows the introduction of a time coordinate on the manifold. A change in the global simultaneity convention is a gauge transformation. The gauge formalism suggested the calculation of the Bianchi identity which turned out to be a differential relation between the acceleration field and the vorticity field.

A necessary and sufficient condition on the vector field $u(x)$ in order for the connection to principal has been given. In that case the fiber bundle admits a structure group T_1 of time translations which preserves the connection, and the usual gauge theory is recovered. Particular cases are frames of geodesics, frames derived from the flow lines of a perfect fluid and conformal stationary frames. In this last case the Sagnac effect depends straightforwardly on the holonomy of the light path through a generalization of Eq. (6.23). Apart from these simple cases, in order to appreciate the gauge nature of simultaneity, the whole machinery of the Frölicher-Nijenhuis algebra is needed.

The gauge picture of simultaneity is expected to be useful in the search for non trivial solutions of the Einstein equations. Moreover, once generalized to higher dimensions, the mathematics involved can prove to be useful in modern Kaluza-Klein theories, since there, the cylinder condition is dropped.

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