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Quantum hydrodynamical models ari
from the Wigner equation in high-field

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Introduction

- Many devices work at high-field regimes. Huge literature about models "
terms (F. Poupaud, P. Degond, N. Ben Abdallah, I.M. Gamba, A. Ju
many others).

  Common point: semi-classical approach, i.e. Boltzmann equation for se

- Advance in semiconductor technology requires to consider **quantum e**
regimes

  $\Longrightarrow$ **quantum macroscopic models**.

- Quantum Wigner kinetic description.

- Wigner-derived macroscopic models are analytically and numerically cha

- **Goal**: present a rigorous (still in progress) study of the accuracy of a
model derived from the Wigner model via a Chapman-Enskog type
conditions

Wigner-BGK equation

$w = w(x,v,t)$, $(x,v) \in \mathbb{R}^6, t \geq 0$ quasi-distribution function for
$\theta = 1/\kappa\beta$ enviroment temperature, V applied potential (also self-consist

$$\frac{\partial w}{\partial t} + v \cdot \nabla_x w - \Theta[V]w = -\nu(w - w^{\text{eq}}) , \quad t > 0 ,$$

$$\Theta[V]w(x,v) := i\,\mathcal{F}_v^{-1} \quad \frac{1}{\hbar} \; V \quad x + \frac{\hbar\eta}{2m} \quad - V \quad x - \frac{\hbar\eta}{2m}$$

with ν inverse relaxation-time, m effective mass, and $\mathcal{F} = \mathcal{F}_{v\to\eta}$ Fourier
since, by Taylor expansion around x,

$$\frac{i}{\hbar} \; V \quad x + \frac{\hbar\eta}{2m} \quad - V \quad x - \frac{\hbar\eta}{2m} \quad = \frac{i\eta}{m} \cdot \nabla V(x) + \frac{i\eta^2\hbar^2}{24m^3}\,\eta \cdot \nabla$$

then

$$\Theta[V]w(x,v) = -\frac{1}{m}\nabla V(x) \cdot \nabla_v w(x,v) + \frac{\hbar^2}{24m^3}\nabla\Delta V(x) \cdot \nabla_v \Delta$$

Wigner equation: semiclassical limit

From the Taylor expansion of the pseudo-differential term with respect to $\hbar$

$$\Theta[V]w \quad \xrightarrow{\hbar\to 0} \quad -\frac{1}{m}\nabla_x V(x)\cdot\nabla_v w$$ Vlasov ope

$$\Theta[V]w \quad = \quad -\frac{1}{m}\nabla_x V(x)\cdot\nabla_v w + \mathcal{O}(\hbar^2)\,.$$

Fact:

$$\implies \quad \int v^k\,\Theta[V]w(x,v)\,dv \;=\; -(1/m)\int v^k\,\nabla V(x)\cdot\nabla_v w(x,v$$

$\implies$ quantum corrections due to $\Theta[V]$ appear for $k \geq$

the v-moments of $\Theta[V]$ and of $-(1/m)\,\nabla_x V(x)\cdot\nabla_v$ coincide up t
Instead:

$$\int v^3\,\Theta[V]w\,dv = -\frac{1}{m}\int v^3\,\nabla_x V(x)\cdot\nabla_v w\,dv \;+\; \frac{\hbar^2}{4m^3}n$$

The thermal equilibrium state

$$w^{\mathrm{eq}}(x,v,t) = n(x,t)C\,e^{-\beta m v^2/2}\Bigg\{1+\hbar^2\Bigg[-\frac{\beta^2\,\Delta V(x)}{24m}+\frac{\beta^3}{24$$

$w^{\mathrm{eq}}(x,v,t)$, $\mathcal{O}(\hbar^2)$-accurate local thermal equilibrium distribution functio

$$\int w^{\mathrm{eq}}(x,v,t)\,dv \;=\; n(x,t) \;:=\; \int w(x,v,t)\,dv\,, \quad \text{electron p}$$

$$\int w^{\mathrm{eq}}(x,v,t)\,dv \quad = \quad n(x,t) \;:=\; \int w(x,v,t)\,dv\,,$$

$$\int v\,w^{\mathrm{eq}}(x,v,t)\,dv \quad = \quad 0\,,$$

$$\int v^2\,w^{\mathrm{eq}}(x,v,t)\,dv \quad = \quad \frac{3\,\kappa\theta}{m}+\frac{\hbar^2}{12m^2\kappa\theta}\Delta V(x)\Bigg)$$

with $\theta = 1/(\kappa\,\beta)$ enviroment temperature. $\beta, V(x)$ are given!

Quantum corrections due to w^{eq}, which is $\mathcal{O}(\hbar^2)$-accurate, appear already

The Macroscopic Quantities

Let us define the first and second order unknown macroscopic quantities, i.
the energy density

$$u = u(x,t) \quad := \quad \frac{1}{n}\int v\, w(x,v,t)\, dv$$

$$\mathcal{W} = \mathcal{W}(x,t) \quad := \quad \int \frac{v^2}{2}\, w(x,v,t)\, dv$$

Moreover, we recall that we can split $\mathcal{W}$ as

$$\mathcal{W}(x,t) \;=\; \int \frac{(v-u)^2}{2}\, w(x,v,t)\, dv + n\,\frac{u^2}{2} \;=:\; \mathcal{W}_i(x,t$$

- $\mathcal{W}_i$ and $\mathcal{K}$ indicate internal and kinetic velocity, respectively.
- In case of thermodynamical equilibrium with the bath individuated by w
  for the moments of w^{eq} we deduce

$$\Longrightarrow \quad \textbf{the fluid velocity is zero}, \text{ i.e. } u^{\mathrm{eq}}(x,t) \equiv$$

High-field Wigner-BGK equation

$$\epsilon\, w_t + \epsilon\, v \cdot \nabla_x w - \Theta[V]w = -\nu(w - w^{\text{eq}})\,, \quad t > 0\,,$$

is the equation in the high-field scaling, where

$$\epsilon \approx \frac{t_V}{t_0} \approx \frac{t_C}{t_0}$$

with t_V, t_C, t_0 characteristic times.

External potential *and* **interaction with the enviroment** are the **domi**
the evolution and **balance each other.**
At the leading order, $\epsilon = 0$, the solution of $\quad (\nu - \Theta[V])w = \nu w^{\text{eq}} \quad$ is

$$w^{(0)} \;:=\; (\nu\mathcal{I} - \Theta[V])^{-1}\nu w^{\text{eq}} \;=\; \nu\mathcal{F}^{-1} \quad \frac{\mathcal{F}w^{\text{eq}}}{\nu - i\,\delta}$$

The inverse operator $(\nu - \Theta[V])^{-1}$ is defined in the Fourier space as th
factor $\nu(\nu - i\delta V(x,\eta))^{-1}$, which exists and is bounded for all V since ν

Moments at the Leading Order

$$\boxed{\epsilon = 0 \quad \Rightarrow \qquad \Theta[V]w^{(0)} = \nu(w^{(0)} - w^{\mathrm{eq}})}$$

Then the moments of $w^{(0)}$ are

$$\int w^{(0)} dv = n$$

$$\int v\, w^{(0)} dv = -\frac{\nabla_x V}{\nu m} n$$

$$\int v \otimes v\, w^{(0)} dv = \frac{\kappa\theta}{m}\mathcal{I} + 2\frac{\nabla_x V}{\nu m} \otimes \frac{\nabla_x V}{\nu m} + \frac{\hbar^2}{12 m^2 \kappa\theta}\nabla_x \otimes$$

$$\int v \frac{v^2}{2}\, w^{(0)} dv = -\frac{\nabla_x V}{\nu m} \quad \int \frac{v^2}{2}\, w^{(0)} dv + \Pi^{(0)}\Big) + \frac{\hbar^2}{8 m^3 \nu}\nabla$$

We remark that, at the leading order, $u^{(0)} = -\dfrac{\nabla_x V}{\nu m}(x)$ is a nonzero-fl

velocity field is constant: in the high-field regime the fluid velocity reaches

Kinetic Energy at the Leading Order

The kinetic energy is $\mathcal{K}^{(0)} = \frac{n}{2}(x,t)\frac{|\nabla_x V|^2}{(\nu m)^2}(x)$, as a consequence

$$\mathcal{W}^{(0)} \;=\; \mathcal{W}_i^{(0)} + \mathcal{K}^{(0)} = \frac{n}{2} \quad \frac{3\kappa\theta}{m} + \frac{|\nabla_x V|^2}{(\nu m)^2} + \frac{\hbar^2}{12 m^2 \kappa\theta}\Delta_x V \Big)$$

In the internal energy it appears a contribute due the external field, w
high-field assumption:

$$\frac{n}{2}\frac{|\nabla_x V|^2}{(\nu m)^2}\,.$$

Instead, in case of a nonzero-fluid velocity equilibrium function $w^{\mathrm{eq}}(x,$
energy density $\mathcal{W}_{\mathrm{w}}^{\mathrm{eq}}$ is

$$\mathcal{W}_{\mathrm{w}}^{\mathrm{eq}} \;=\; \mathcal{W}_{\mathrm{w},i}^{\mathrm{eq}} + \mathcal{K}_{\mathrm{w}}^{\mathrm{eq}} = \frac{n}{2} \quad \frac{3\kappa\theta}{m} + \frac{\hbar^2}{12 m^2 \kappa\theta}\Delta_x V \Big) +$$

A field contributes in modifying the internal energy $\mathcal{W}_i^{(0)}$ if it ha
magnitude as the interaction with the environment.

From high-field Wigner-BGK equation we can obtain evolution equation
considering $w \simeq w^{(0)}$ and taking v-moments.

Then the energy density $\mathcal{W}^{(0)}$ can change on time only because of *transpo*

$$\frac{\partial n}{\partial t} - \nabla_x \cdot$$

$$\frac{\partial \mathcal{W}^{(0)}}{\partial t} - \nabla_x \cdot \quad \mathcal{W}^{(0)} \frac{\nabla_x V}{\nu m} \quad - \nabla_x \cdot \quad \Pi^{(0)} \frac{\nabla_x V}{\nu m} \quad + \nabla_x \cdot \quad \frac{\hbar^2}{8 m^3 \nu} \nabla$$

Let us repeat that the equations above describe only *transport*, since the e
of **drift-collision balance.**

On the contrary of low-field case, diffusive term and heat-flux term will appe
respectively only as $O(\epsilon)$ corrections via the Chapman-Enskog procedure.

Redistributing the energy density as

$$\mathcal{W}^{(0)} = \int \frac{|v-u^{(0)}|^2}{2} w^{(0)}\,dv + \frac{n|u^{(0)}|^2}{2} = \mathcal{W}_{\mathrm{i}}^{(0)} + \frac{n}{2} \cdots$$

with

$$\mathcal{W}_{\mathrm{i}}^{(0)} = \frac{n(x,t)}{2} \quad \frac{3\kappa\theta}{m} + \frac{|\nabla_x V|^2(x)}{(\nu m)^2} + \frac{\hbar^2}{12 m^2 \kappa\theta}\Delta_x V \cdots$$

and defining

$$\mathbb{P}^{(0)}(x,t) := \int (v-u^{(0)}) \otimes (v-u^{(0)})\, w^{(0)}(x,t)\, d \cdots$$

Consequently, the equations read as follows:

$$\frac{\partial n}{\partial t} - \nabla_x \cdot \cdots$$

$$\frac{\partial \mathcal{W}_{\mathrm{i}}^{(0)}}{\partial t} - \nabla_x \cdot \quad \mathcal{W}_{\mathrm{i}}^{(0)} \frac{\nabla_x V}{\nu m} \quad - \nabla_x \cdot \quad \mathbb{P}^{(0)} \frac{\nabla_x V}{\nu m} \quad + \nabla_x \cdot \quad \frac{\hbar^2}{8 m^3 \nu} \nabla \cdots$$

Next steps: Derive via a Chapman-Enskog procedure corrections of
equations.

The Chapman-Enskog procedure

The procedure consists of two assumptions:

First, we assume that the microscopic unknown w depends on time only th
quantities. Since among the unknown n and $\mathcal{W}^{(0)}$, the only t-dependen
depends on time only through n), we express

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial n}\frac{\partial n}{\partial t}\,.$$

Secondly, we assume that the macroscopic unknown n is an $\mathcal{O}(1)$ quanti
the following expansion

$$w = \sum_{k=1}^{\infty} \epsilon^k w^k \sim w^{(0)} + \epsilon w^{(1)}\,.$$

to compute the other v-moments.
Let us compute the 0th-order moment of the high-field Wigner-BGK equ
continuity equation for n

$$\frac{\partial}{\partial t}\int w dv + \nabla_x \cdot \int v w dv = 0$$

This allows us to express $\dfrac{\partial w}{\partial t}$ in (2) as

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial n} \quad - \nabla_x \cdot \int v w dv \quad .$$

By substituting this expression in the Wigner equation, we obtain

$$\epsilon \quad \frac{\partial w}{\partial n} \quad - \nabla_x \cdot \int v w dv \quad \Bigg) + \epsilon\, v \cdot \nabla_x w = \Theta[V]w - \nu($$

Then we expand the unknown $w \sim w^{(0)} + \epsilon w^{(1)}$.

At the 0**-th order in** ϵ, we obtain

$$\boxed{\Theta[V]w^{(0)} - \nu(w^{(0)} - w^{eq}) = 0\,,}$$

whose solution is given formally by

$$w^{(0)} = (\nu - \Theta[V])^{-1} \nu w^{\text{eq}}$$
$$= \mathcal{F}^{-1} \quad \frac{\nu \mathcal{F} w^{\text{eq}}}{\nu - i\delta V(x,\eta)} \quad (x,v,t) = n(x,t)M$$

where

$$M(x,v) := \mathcal{F}^{-1} \quad \frac{\nu \mathcal{F}(F + \hbar^2 F^{\hbar})}{\nu - i\delta V(x,\eta)}\Bigg) (x,v) \,.$$

At the first order in ϵ

$$\frac{\partial w^{(0)}}{\partial n} \quad - \nabla_x \cdot \int v w^{(0)} dv \quad + v \cdot \nabla_x w^{(0)} = \quad \Theta[V] -$$

Taking into account $\frac{\partial w^{(0)}}{\partial n} = M$, we obtain

$$M \nabla_x \cdot \quad n \frac{\nabla_x V}{\nu m} \quad + v \cdot \nabla_x w^{(0)} = \quad \Theta[V] - \nu$$

Then we have

$$\boxed{w^{(1)} = \nabla_x \cdot \left(n \frac{\nabla_x V}{\nu m} \right) \left(\Theta[V] - \nu\right)^{-1} M + \left(\Theta[V] - \nu\right)^{-1} (}$$

Thus, we can rewrite the continuity equation with unknown n by using t
and $w^{(1)}$. We get a correction of order $\mathcal{O}(\epsilon)$ for the continuity equation.

$$\frac{\partial n}{\partial t} + \nabla_x \cdot \int v w^{(0)} dv + \epsilon \nabla_x \cdot \int v w^{(1)} dv = 0$$

In order to write Eq. (3) in a more explicit form, we need the moments of

$$\begin{aligned}
\int w^{(1)} dv &= 0 \\
\int v\, w^{(1)} dv &= -\frac{1}{\nu} \nabla_x \cdot \Pi_n^{(0)} + \frac{1}{\nu} \nabla_x \cdot \quad n\, u^{(0)} \quad u^{(0)} \\
\int v \otimes v\, w^{(1)} dv &= -\frac{1}{\nu} \nabla_x \cdot \Pi_{nu}^{(0)} + \frac{1}{\nu} \nabla_x \cdot \quad n\, u^{(0)} \quad \frac{\Pi_n^{(0)}}{n} +
\end{aligned}$$

Then Eq. (3) can be rewritten as

$$\boxed{\frac{\partial n}{\partial t} - \nabla_x \cdot \quad n\,\frac{\nabla_x V}{\nu m} \quad + \frac{\epsilon}{\nu}\nabla_x \cdot \quad \nabla_x \cdot \quad n\,\frac{\nabla_x V}{\nu m} \quad \frac{\nabla_x V}{\nu m}}$$

$$\boxed{- \frac{\epsilon}{\nu}\nabla_x \cdot \nabla_x \qquad \frac{\kappa\theta\mathcal{I}}{m} + 2\frac{\nabla_x V}{\nu m} \otimes \frac{\nabla_x V}{\nu m} + \frac{\hbar^2}{12 m^2 \kappa\theta}\nabla_x \otimes}$$

This is a drift equation with an $\mathcal{O}(\epsilon)$-diffusive correction.
Observe that by splitting the following term as

$$\frac{\epsilon}{\nu}\nabla_x \cdot \quad \frac{\nabla_x V}{\nu m}\nabla_x \cdot \quad n\,\frac{\nabla_x V}{\nu m} \qquad = \frac{\epsilon}{\nu}\nabla_x \cdot \frac{\nabla_x V}{\nu m}\nabla_x \cdot \quad n\,\frac{\nabla_x V}{\nu m} \quad + \frac{\epsilon}{\nu}\frac{\nabla_x V}{\nu m}$$

we derive the following high-field corrected version of the (classical) mo
$1/(\nu m)$

$$\mu_{\mathrm{hf}} := \frac{1}{\nu m} \quad 1 + \frac{\epsilon}{\nu^2 m}\Delta_x V \quad .$$

High-field corrections to drift-collision balance s

Starting from

$$\frac{\partial(nu)}{\partial t} + \nabla_x \cdot \int v \otimes v w \, dv + \frac{1}{\epsilon} \frac{\nabla_x V}{m} \int w \, dv = -\frac{\nu}{\epsilon} \int$$

$$\frac{\partial \mathcal{W}}{\partial t} + \nabla_x \cdot \int \frac{v^2}{2} w \, dv + \frac{1}{\epsilon} \frac{\nabla_x V}{m} \cdot (n\, u) = -\frac{\nu}{\epsilon} \int v \frac{v^2}{2} w$$

we investigate two closure strategies to get corrected equations for $u^{(0)}$, $\mathcal{W}$

1. we substitute $w \approx w^{(0)}$ in the $\mathcal{O}(1/\epsilon)$-terms and $w \approx w^{(0)} + \epsilon$ moments (**drift-collision-balance closure**),

2. we substitute $w \approx w^{(0)} + \epsilon w^{(1)}$ everywhere (**CE-corrected closure**).

Remark: With 1. moment conservations hold due to drift-collision balance

High-field corrections to drift-collision balance sy

Drift-collision-balance closure:

$$
\begin{aligned}
\frac{\partial (n\, u^{(0)})}{\partial t} + \nabla_x \cdot \Pi^{(0)} + \frac{\epsilon}{\nu} \nabla_x \cdot \Pi^{(1)} &= 0\,, \\
\frac{\partial \mathcal{W}^{(0)}}{\partial t} + \nabla_x \cdot \mathcal{J_W}^{(0)} + \frac{\epsilon}{\nu} \nabla_x \cdot \mathcal{J_W}^{(1)} &= 0\,.
\end{aligned}
$$

At $\mathcal{O}(1)$ in ϵ drift terms, with $\mathcal{O}(\epsilon)$-diffusive terms in $nu^{(0)}$, $\mathcal{W}^{(0)}$, respe

Chapman-Enskog-corrected closure:

$$
\begin{aligned}
\frac{\partial (n\, u^{(0)})}{\partial t} + \nabla_x \cdot \quad n\, u^{(0)} \quad u^{(0)} + \frac{\epsilon}{\nu} \nabla_x \cdot \Pi^{(1)} &= \\
\frac{\partial \mathcal{W}^{(0)}}{\partial t} + \nabla_x \cdot \quad n u^{(0)} \quad \frac{\mathcal{W}^{(0)}}{n} + \frac{\epsilon}{\nu} \nabla_x \cdot \mathcal{J_W}^{(1)} &=
\end{aligned}
$$

At $\mathcal{O}(1)$ in ϵ *continuity* equation multiplied by $u^{(0)}$, $\mathcal{W}^{(0)}/n$, respective
terms in $n\, u^{(0)}$, $\mathcal{W}^{(0)}$, respectively.

$$\Pi^{(1)} = \underbrace{\Pi^{(0)}/n + 2u^{(0)} \otimes u^{(0)} \quad \nabla_x \cdot \quad nu^{(0)}}_{\text{drift}} - \underbrace{2u^{(0)} \otimes \quad \nabla_x \cdot \Pi}_{\text{drift+diffusion}}$$

$$- \quad \nabla_x \cdot \quad n\frac{\hbar^2}{4m^3}\nabla_x \otimes \nabla_x \otimes \nabla_x V \quad ,$$

$$\mathcal{J}_{\mathcal{W}}^{(1)} = \mathcal{J}_{\mathcal{W}}^{(0)}/n \quad \nabla_x \cdot \quad n\,u^{(0)} \quad + u^{(0)}(\mathcal{W}^{(1)} + \Pi^{(1)}) - \nabla_x \cdot \Pi_{\mathcal{W}}^{(0)}$$

$$\mathcal{W}^{(1)} = \underbrace{\mathcal{W}^{(0)}/n + |u^{(0)}|^2 \quad \nabla_x \cdot \quad n\,u^{(0)}}_{\text{drift}} - \underbrace{u^{(0)} \cdot \quad \nabla_x \cdot \Pi^{(0)}}_{\text{drift+diffusion}} -$$

$$- \quad \nabla_x \cdot \quad n\frac{\hbar^2}{8m^3}\nabla_x \Delta_x V$$

$$\Pi_{\mathcal{W}}^{(0)} = \int v \otimes v \frac{v^2}{2} w_{\text{eq}}\, dv + \Big[\quad 3\Pi^{(0)} \otimes u^{(0)} \quad + \quad n\frac{\hbar^2}{4m^3}\nabla_x \otimes \nabla$$

$$+ \quad \Big[\quad \mathcal{W}^{(0)} + \Pi^{(0)} \quad u^{(0)} + n\frac{\hbar^2}{8m^3}\nabla_x \Delta_x V \Big] \otimes 2u^{(0)} \,.$$

- Standard diffusive terms and heat-flux $\int v \otimes v\, (v^2/2) w_{\text{eq}}$ (**moments u**
- High-field drift, diffusive terms and $\mathcal{O}(\hbar^2)$-high-field terms.

Summary

- We start from an $\mathcal{O}(1)$-in-ϵ drift-collision balance model with unknow
energy densities.

- The equations contain just *drift* terms, and in addition, $\mathcal{O}(1)$-in-ϵ hi
$\mathcal{O}(\hbar^2)$ quantum corrections due to the use of $\Theta[V]$.

- By apply the Chapman-Enskog procedure, we recover a system of equ
energy densities and fluid velocity, that contain standard diffusive terms

- Moreover, we obtain high-field corrections and $\mathcal{O}(\hbar^2)$-high-field correc
the pseudo-differential operator.

In conclusion, we obtain a highly-accurate quantum fluid-dynamical mode
in high-field regime, since it contains moments up to 5^{th}-order and high-fi
peculiar of quantum transport.

Thanks for the attention!

Then the moments of $w^{(0)}$ are

$$
\begin{aligned}
n &= \int w^{(0)} dv \,, \\
\mathcal{J}_n^{(0)} &:= n\, u^{(0)} := \int v\, w^{(0)} dv = -n \frac{\nabla_x V}{\nu m} \,, \\
\Pi_n^{(0)} &:= \int v \otimes v\, w^{(0)} dv = n \quad \frac{k\theta \mathcal{I}}{m} + 2u^{(0)} \otimes u^{(0)} + \frac{\hbar^2}{12m^2k\theta \cdots} \\
\mathcal{W}_n^{(0)} &:= \int \frac{v^2}{2} w^{(0)} dv = n \quad d\frac{k\theta}{m} + |u^{(0)}|^2 + \frac{\hbar^2}{24m^2k\theta} \Delta_x V \Big) \,, \\
\Pi_{nu}^{(0)} &:= \int v \otimes v \otimes v\, w^{(0)} dv = 3u^{(0)} \otimes \Pi^{(0)} + n \frac{\hbar^2}{4m^3\nu} \nabla_x \otimes \cdots \\
\mathcal{J}_{\mathcal{W}}^{(0)} &:= \int v \frac{v^2}{2}\, w^{(0)} dv = \quad \mathcal{W}^{(0)} + \Pi^{(0)} \quad u^{(0)} + n \frac{\hbar^2}{8m^3\nu} \nabla_x \Delta_x \cdots \\
\Pi_{\mathcal{W}}^{(0)} &:= \int v \otimes v \frac{v^2}{2}\, w^{(0)} dv = \int v \otimes v \frac{v^2}{2}\, w_{\text{eq}} dv + 2 \quad \mathcal{W}^{(0)} + \cdots \\
&+ \quad \Pi^{(0)} |u^{(0)}|^2 + 2n \frac{\hbar^2}{4m^3\nu} \nabla_x \Delta_x V \otimes u^{(0)} + 2n \frac{\hbar^2}{4m^3\nu} \nabla_x \cdots
\end{aligned}
$$