Quantum High-field Corrections To A Drift-Collision Balance Model Of Semiconductor Transport

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We derive a fluid-dynamical system describing electron transport in quantum, high-field regime, starting from Wigner equation. Under the high-field assumption, quantum advection and collisions are comparable and dominant. Consequently diffusive terms shall appear in the system as higher-order corrections.

Key words: Quantum hydro-dynamic models, Wigner transport equation, Chapman-Enskog expansion, High-field asymptotics.

1 Introduction

Fluid-dynamical models are commonly adopted for semiconductor device simulation. The accuracy in predicting device performances depends in particular on the consistency of the model with semiconductor physics and the selection of macroscopic unknown functions. For what concerns the first point, models in literature can be classified as either semi-classical or quantum ones. In the former case, they are derived from a semi-classical description of electron transport (Anile et al. 2003, Krause et al. 2007), and quantum effects are either considered negligible or added a-posteriori (de Falco et al. 2005). In the latter case, instead, models are derived from a quantum picture (Arnold and Jüngel 2006), thus they are capable to describe quantum phenomena in nano-devices. For what concerns the definition of the macroscopic variables, in classical and semi-classical fluid-dynamics they are introduced as moments in the velocity variable of the phase-space distribution-function. Then, fluid-dynamical models are derived by applying the moment method and appropriate closure conditions. Wigner function is the quantum equivalent of a

classical distribution function in the phase-space, $w = w(x, v, t), (x, v) \in \mathbb{R}^{2d}, t > 0$ (Wigner 1932). Accordingly, the macroscopic variables can be formally defined as v-moments of the Wigner function. Wigner equation is the basic kinetic model for quantum transport in nano-devices: it describes time-evolution of the electrons under the effect of a potential $V = V(x)$. Thus, quantum fluid-dynamical models can be obtained analogously by applying the moment method and closure procedures. Quantum transport and quantum hydrodynamical models are analytically challenging (Arnold et al. 2007, Arnold et al. 2007, Dolbeaut et al. 2006) and their numerical implementation requires further investigation (Jüngel and Milišić 2007).
Another crucial ingredient for the accuracy of the model is the suitability to capture the physical regimes at which the devices operate. Nanometric devices nowadays give rise to non-equilibrium hot-electron transport regimes, due to high voltage applied to the contacts, e.g. . In some regimes the electrical field and the interaction with the crystal lattice have comparable strength and are dominant with respect to transport. Accordingly, these regimes are called of "drift-collision balance" (Carrillo et al. 2000). Mathematically speaking, the diffusion is a phenomenon of different order of magnitude than the drift, then it appears as a correction to the drift in terms of some asymptotical parameter. In Carrillo et al. (2000) the corresponding semi-classical model with unknown the electron position density is labelled "augmented drift-diffusion". Many semi-classical models of high-field semiconductor transport are available in literature (Ben Abdallah et al. 1996, Cercignani et al. 2001, Degond and Jüngel 2001, Poupaud 1991, Poupaud 1992).
We are interested instead to describe quantum-driven electron transport in high-field regimes. In Manzini and Frosali (2006) we derive a quantum augmented drift-diffusion model, by using a Chapman-Enskog type procedure. As expected, the equation for the electron position density contains classical and quantum diffusive terms as corrections to the classical drift term, of higher-order in the asymptotic parameter.
In the present paper our aim is deriving a more accurate quantum fluid-dynamical model of semiconductor transport in high-field regimes. Accordingly, we enlarge the set of macroscopic unknown functions: it includes the electron position density n, the fluid velocity $\mathcal{U}$ and the energy density e.
The simpler way to model electron interaction with the lattice in Wigner picture is as the tendency of the electron ensemble to relax to a state of thermal equilibrium with a surrounding phonon bath. Observe that we neglect electron-electron interactions since most devices work at low-density regimes. The corresponding equation is called relaxation-time Wigner equation.
In the present paper, we assume that the dominant mechanisms in the evolution are the relaxation phenomenon and the external-field effect. Therefore, we rescale the relaxation-time Wigner equation with the Knudsen number ϵ as follows

$$\epsilon \frac{\partial w}{\partial t} + \epsilon\, v \cdot \nabla_x w \;=\; \Theta[V]w - \nu(w - w_{\text{eq}}), \quad t > 0\,. \tag{1}$$

The pseudo-differential operator $\Theta[V]$, which will be defined in next section, characterizes quantum description. The scaling above was used for the first time for semiconductor semi-classical modelling by F. Poupaud in (Poupaud 1992). We remark that the different phenomena are described as independent each from the other: the relaxation-time distribution function w_{eq} is the $\mathcal{O}(\hbar^2)$-correct expansion of the state to which the quantum system shall converge due to the presence of a thermal bath at temperature $1/k\beta$ (Wigner 1932), and in absence of some other significant phenomenona.

We shall apply the Chapman-Enskog procedure to Eq. (1) to derive a corrected equation for the unknown n, which shall be identical to the one in Manzini and Frosali (2006). Then, we shall apply the moment method to derive from Eq. (1) equations for $\mathcal{U}$ and e and discuss a closure procedure for these equations. We shall recover a set of equations containing drift terms at zero-th order in the asymptotic parameter ϵ and classical and quantum diffusion terms of the first order in ϵ. We anticipate that both the equation for the fluid velocity $\mathcal{U}$ and for energy density e shall contain $\mathcal{O}(\epsilon)$-corrections that are to be referred to quantum transport, precisely quantum energy-fluxes. Moreover, in the equation for e shall appear quantum–high-field corrections as flux of energy-fluxes. These corrections are due exactly to the use of the pseudo-differential operator instead of the classical Vlasov operator and describe quantum effects induced by the high-field regime.

2 Wigner-BGK equation

The starting point is Wigner equation with unknown the quasi-distribution function $w = w(x, v, t), (x, v) \in \mathbb{R}^{2d}$, $t > 0$, describing, at the kinetic level, the time-evolution of a quantum system with d degrees of freedom, under the effect of an external potential $V = V(x), x \in \mathbb{R}^d$. It reads

$$\frac{\partial w}{\partial t} + v \cdot \nabla_x w - \Theta[V]w = Qw, \quad (x, v) \in \mathbb{R}^{2d}, \quad t > 0 \tag{2}$$

(Wiginer 1932), with the pseudo-differential operator $\Theta[V]$ defined by

$$\begin{aligned}(\Theta[V]w)(x, v, t) &= \frac{i}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \delta V(x, \eta) w(x, v', t) e^{i(v-v')\cdot\eta}\, dv'\, d\eta \\ &= \frac{i}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \delta V(x, \eta) \mathcal{F}w(x, \eta, t) e^{iv\cdot\eta}\, d\eta\,,\end{aligned}$$

where

$$\delta V(x, \eta) := \frac{1}{\hbar}\left[V\left(x + \frac{\hbar\eta}{2m}\right) - V\left(x - \frac{\hbar\eta}{2m}\right)\right]$$

with $\mathcal{F}w(\eta) \equiv [\mathcal{F}_{v\to\eta} w](\eta)$ denotes the Fourier transform of w from v to η, m the electron mass and $\hbar$ the scaled Planck constant. In the Fourier-transformed space $\mathbb{R}^d_x \times \mathbb{R}^d_\eta$ the operator $\Theta[V]$ is the multiplication operator by the function $i\,\delta V$; in symbols,

$$\mathcal{F}\left(\Theta[V]w\right)(x, \eta) = i\,\delta V(x, \eta)\mathcal{F}w(x, \eta)\,. \tag{3}$$

We recall that the expansion of $\Theta[V]$ (for V smooth enough) with respect to $\hbar$ looks as follows (see Gardner 1994):

$$\Theta[V]w = \frac{\nabla_x V}{m}\cdot \nabla_v w - \frac{\hbar^2}{24m^3}\nabla_x\Delta_x V\cdot \nabla_v\Delta_v w + \mathcal{O}(\hbar^4).$$

Accordingly, at the leading order in $\hbar$ the pseudo-differential operator coincides with the operator $\nabla_x V\cdot \nabla_v$ appearing in Vlasov equation and it is easy to prove that, for all multi-index j, $|j| = 0, 1, 2$,

$$\int v^j\, \Theta[V]w\, dv \;=\; \int v^j\, \frac{\nabla_x V}{m}\cdot \nabla_v w\, dv\,, \tag{4}$$

whereas

$$\int v^3\, \Theta[V]w\, dv = \int v^3\, \frac{\nabla_x V}{m}\cdot \nabla_v w\, dv\, + \frac{\hbar^2}{4m^3} n\, \nabla_x\Delta_x V\,. \tag{5}$$

Therefore the quantum correction due to the use of the pseudo-differential operator appears only in the computation of the v-moments of order greater or equal than 3 of the field operator.
The term Qw, added on the right hand side, mimics interactions of the electrons with the lattice. We remark that this point is quite delicate, since the kinetic model has to be consistent with the description in the operatorial (Heisenberg) formulation. In particular, a Wigner equation with a BGK term containing the Wigner transform of the equilibrium operator $\exp(-\beta H)$, H Hamiltonian operator, is the Wigner transform of an equation in Lindblad form, see Lindblad (1976), thus it is quantum-mechanically correct. In Wigner (1932) E. Wigner introduced the $\hbar$-expansion of the Wigner transform of the equilibrium operator $\exp(-\beta H)$. Precisely,

$$\begin{aligned} &w_{\rm W}(x,v) := \left(\frac{m}{2\pi\hbar}\right)^d {\rm e}^{-\beta\mathcal{E}} \\ &\times \left\{ 1 + \hbar^2 \frac{\beta^2}{24}\left[-\frac{3}{m}\Delta_x V(x) + \frac{\beta}{m}|\nabla_x V|^2(x) + \beta \sum_{r,s=1}^{d} v_r v_s \frac{\partial^2 V(x)}{\partial x_r \partial x_s}\right] + O(\hbar^4)\right\} \end{aligned} \tag{6}$$

where $\mathcal{E}(x,v) := mv^2/2 + V(x)$ is the total energy of the system. We call $\widetilde{w_{\rm W}}$ a local (in time and space) modification of $w_{\rm W}$, defined by

$$\widetilde{w_{\rm W}}(x,v,t) \;:=\; C(x,t)\; w_{\rm W}(x,v)\,,$$

with $C = C(x,t)$ to be defined appropriately.
By performing explicit calculations

$$\begin{aligned} \int \widetilde{w_{\rm W}}(x,v,t)\, dv = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{d/2} C(x,t)\, {\rm e}^{-\beta V} \\ \times \left(1 + \hbar^2 \frac{\beta^2}{12m}\left(-\Delta_x V + \frac{\beta}{2}|\nabla_x V|^2\right) + O(\hbar^4)\right)\,, \end{aligned}$$

thus, if we assume that it holds

$$\int \widetilde{w_{\mathrm{W}}}(x,v,t)\,dv \;=\; \int w(x,v,t)\,dv \;=:\; n(x,t)\,,$$

where n is the unknown position density, we get

$$C(x,t) = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{-d/2} n(x,t)\,\mathrm{e}^{\beta V} \times \left\{1 + \hbar^2 \frac{\beta^2}{12m}\left[-\Delta V + \frac{\beta}{2}|\nabla V|^2\right] + \mathcal{O}(\hbar^4)\right\}^{-1}. \quad (7)$$

Eq. (7) defines C in terms of the unknown function n and the assigned function V. By substituting the function C defined by (7) into (6), we can introduce a local version w_{eq} of the equilibrium function w_{W} that looks like

$$w_{\mathrm{eq}}(x,v,t) \;:=\; n(x,t)\left(\frac{\beta m}{2\pi}\right)^{d/2} \mathrm{e}^{-\beta m v^2/2} \times \left\{1 + \hbar^2\frac{\beta^2}{24}\left[-\frac{1}{m}\Delta_x V + \beta\sum_{r,s=1}^{d} v_r v_s \frac{\partial^2 V}{\partial x_r x_s}\right]\right\}. \quad (8)$$

It is convenient to denote for later reference $w_{\mathrm{eq}} = n(F + \hbar^2 F^\hbar)$ with $F = F(v) = (\beta m/(2\pi))^{d/2} \exp\left(-\beta m v^2/2\right)$ and

$$F^\hbar = F^\hbar(x,v) = \left(\frac{\beta m}{2\pi}\right)^{d/2} \mathrm{e}^{-\beta m v^2/2}\left[-\frac{\beta^2}{24m}\Delta_x V + \frac{\beta^3}{24}\sum_{r,s=1}^{d} v_r v_s \frac{\partial^2 V}{\partial x_r x_s}\right].$$

The function w_{eq} is the $\mathcal{O}(\hbar^2)$-accurate approximation of the Wigner function describing the quantum system in the thermodynamical equilibrium state induced by the contact with the phonon bath at temperature $\theta := 1/(k\beta)$. Let us collect here below the first three moments of the equilibrium function w_{eq}:

$$\int w_{\mathrm{eq}}(x,v,t)\,dv \;=\; n(x,t)\,, \quad (9)$$

$$\int v\, w_{\mathrm{eq}}(x,v,t)\,dv \;=\; 0\,, \quad (10)$$

$$\int v\otimes v\, w_{\mathrm{eq}}(x,v,t)\,dv \;=\; n(x,t)\left(\frac{k\theta\mathcal{I}_d}{m} + \frac{\hbar^2}{12m^2k\theta}\nabla_x\otimes\nabla_x V\right), \quad (11)$$

where we use $\mathcal{I}_d$ to denote the $d\times d$ identity matrix. Observe that the contributions coming from the quantum correction $F^\hbar$ appear already in the computation of the second order v-moment of w_{eq}. In the tensor (11) we can indeed distinguish the classical pressure tensor proportional to the phonon temperature θ and the $\mathcal{O}(\hbar^2)$-anisotropic tensor coming from $F^\hbar$ (the $\mathcal{O}(\hbar^2)$-correction to the Maxwellian).

We define the first and second order unknown macroscopic quantities, i.e. the fluid velocity and the energy density

$$\mathcal{U} = \mathcal{U}(x,t) \;:=\; \frac{1}{n}\int v\, w(x,v,t)\, dv\,, \tag{12}$$

$$e = e(x,t) \;:=\; \int \frac{v^2}{2}\, w(x,v,t)\, dv\,. \tag{13}$$

Moreover, we recall that we can split e as

$$e(x,t) \;=\; \int \frac{(v-\mathcal{U})^2}{2}\, w(x,v,t)\, dv + n\,\frac{\mathcal{U}^2}{2} \;=:\; e^{\mathrm{i}}(x,t) + e^{\mathrm{kin}}(x,t) \tag{14}$$

where e^{i} and e^{kin} indicate internal and kinetic energy density, respectively. From the expressions for the moments of w_{eq}, we deduce that in case the system is in the state of thermodynamical equilibrium with the bath individuated by w_{eq}, then the fluid velocity is zero

$$\mathcal{U}_{\mathrm{eq}}(x,t) \equiv 0$$

and the energy density consists only of the internal energy:

$$e_{\mathrm{eq}}(x,t) \;=\; \frac{n(x,t)}{2}\left(\frac{d\,k\theta}{m} + \frac{\hbar^2}{12m^2k\theta}\Delta_x V(x)\right) \;=\; e^{\mathrm{i}}_{\mathrm{eq}}(x,t)\,. \tag{15}$$

Let us consider a more general equilibrium state in which a nonzero fluid velocity field $\mathcal{W} = \mathcal{W}(x)$ is admitted. Such state can be described by $w^{\mathcal{W}}_{\mathrm{eq}} := w_{\mathrm{eq}}(x, v-\mathcal{W}, t)$ and its first three moments are

$$\int w^{\mathcal{W}}_{\mathrm{eq}}(x,v,t)\, dv \;=\; n(x,t)\,, \tag{16}$$

$$\int v\, w^{\mathcal{W}}_{\mathrm{eq}}(x,v,t)\, dv \;=\; n(x,t)\mathcal{W}(x)\,, \tag{17}$$

$$\int v\otimes v\, w^{\mathcal{W}}_{\mathrm{eq}}(x,v,t)\, dv \;=\; n(x,t)\left(\frac{k\theta\mathcal{I}_d}{m} + \frac{\hbar^2}{12m^2k\theta}\nabla_x\otimes\nabla_x V + \mathcal{W}\otimes\mathcal{W}\right)(x)\,. \tag{18}$$

Consequently, the related energy density $e^{\mathcal{W}}_{\mathrm{eq}}$ is

$$e^{\mathcal{W}}_{\mathrm{eq}} \;=\; e^{\mathcal{W},\mathrm{i}}_{\mathrm{eq}} + e^{\mathcal{W},\mathrm{kin}}_{\mathrm{eq}} = \frac{n}{2}\left(\frac{dk\theta}{m} + \frac{\hbar^2}{12m^2k\theta}\Delta_x V\right) + \frac{n}{2}\mathcal{W}^2\,, \tag{19}$$

which equals the equilibrium function internal energy $e^{\mathrm{i}}_{\mathrm{eq}}$ in (15), augmented by the nonzero-kinetic energy.
Both w_{eq} and $w^{\mathcal{W}}_{\mathrm{eq}}$ describe (with $\mathcal{O}(\hbar^2)$-accuracy) equilibrium states that a quantum system attains when in contact with a phonon bath, in case a velocity field $\mathcal{W}$ is not, respectively is, admitted. In the next section we shall describe, instead, a physical regime that is induced by the joint action of an external electrical field and of the phonon bath on the quantum system.

3 The leading order solution of the high-field Wigner-BGK equation

High-field Wigner equation in rescaled form reads

$$\epsilon \frac{\partial w}{\partial t} + \epsilon v \cdot \nabla_x w = \Theta[V]w - \nu(w - w_{\text{eq}})\,, \tag{1}$$

where we have introduced the parameter ϵ for keeping tab on terms of equal order of magnitude (see Cercignani et al. 2001, Manzini and Frosali 2006), for the derivation of the scaling). The leading order term $w^{(0)}$ satisfies the following equation

$$\Theta[V]w^{(0)} - \nu(w^{(0)} - w_{\text{eq}}) = 0\,, \tag{2}$$

thus, it can be expressed as

$$w^{(0)} = (\nu - \Theta[V])^{-1}\,\nu\, w_{\text{eq}}\,. \tag{3}$$

The inverse operator $(\nu - \Theta[V])^{-1}$ is defined in the Fourier space as the multiplication by the factor $(\nu - i\delta V(x,\eta))^{-1}$ (which exists and is bounded for all V since $\nu > 0$), precisely:

$$w^{(0)}(x,v) := \mathcal{F}^{-1}\left(\frac{\nu \mathcal{F} w_{\text{eq}}(x,\eta)}{\nu - i\delta V(x,\eta)}\right)\,. \tag{4}$$

The reader can find an example of functional settings in which the definition above is rigorous in Manzini and Frosali 2006.

Lemma 1 *Let $w^{(0)}$ be defined by Eq. (4). Then the moments of $w^{(0)}$ are*

$$\int w^{(0)} dv = n\,, \tag{5}$$

$$\int v_j\, w^{(0)} dv = -n\frac{\partial_{x_j} V}{\nu m}\,, \tag{6}$$

$$\int v_i\, v_j\, w^{(0)} dv = n\left(\frac{k\theta}{m}\delta_{ij} + 2\frac{\partial_{x_i} V}{\nu m}\frac{\partial_{x_j} V}{\nu m} + \frac{\hbar^2}{12 m^2 k\theta}\partial^2_{x_i x_j} V\right)\,, \tag{7}$$

$$\int \frac{v_j^2}{2} w^{(0)} dv = n\left(d\frac{k\theta}{2m} + \left(\frac{\partial_{x_j} V}{\nu m}\right)^2 + \frac{\hbar^2}{24 m^2 k\theta}\partial^2_{x_j} V\right)\,, \tag{8}$$

$$
\begin{aligned}
\int v_k\, v_i\, v_j\, w^{(0)} dv &= -\frac{\partial_{x_k} V}{\nu m} \int v_i\, v_j\, w^{(0)} dv - \frac{\partial_{x_i} V}{\nu m} \int v_k\, v_j\, w^{(0)} dv \\
&- \frac{\partial_{x_j} V}{\nu m} \int v_k\, v_i\, w^{(0)} dv + n \frac{\hbar^2}{4m^3 \nu} \partial_{x_k} \partial_{x_i} \partial_{x_j} V\,, \qquad (9) \\
\int v_i \frac{v_j^2}{2}\, w^{(0)} dv &= -\frac{\partial_{x_i} V}{\nu m} \int \frac{v_j^2}{2} w^{(0)} dv - \int v_i v_j w^{(0)} dv \frac{\partial_{x_j} V}{\nu m} + n \frac{\hbar^2}{8m^3 \nu} \partial_{x_i} \partial_{x_j}^2 V\,, \\
& \qquad (10) \\
\int v_k\, v_i \frac{v_j^2}{2}\, w^{(0)} dv &= 2 \frac{\partial_{x_k} V}{\nu m} \frac{\partial_{x_i} V}{\nu m} \int \frac{v_j^2}{2} w^{(0)} dv + 2 \frac{\partial_{x_k} V}{\nu m} \int v_i v_j w^{(0)} dv \frac{\partial_{x_j} V}{\nu m} \\
&+ \int v_k v_i w^{(0)} dv \left(\frac{\partial_{x_j} V}{\nu m} \right)^2 + 2 \frac{\partial_{x_i} V}{\nu m} \int v_k v_j w^{(0)} dv \frac{\partial_{x_j} V}{\nu m} \\
&- 2 \frac{\partial_{x_k} V}{\nu m} n \frac{\hbar^2}{8m^3 \nu} \partial_{x_i} \partial_{x_j}^2 V - 2 \frac{\partial_{x_i} V}{\nu m} n \frac{\hbar^2}{8m^3 \nu} \partial_{x_k} \partial_{x_j}^2 V \\
&- n \frac{\hbar^2}{4m^3 \nu} \partial_{x_i} \partial_{x_k} \partial_{x_j} V\, 2 \frac{\partial_{x_j} V}{\nu m} + \int v_k\, v_i \frac{v_j^2}{2}\, w_{\text{eq}} dv\,. \qquad (11)
\end{aligned}
$$

Proof. The moments are computed in the Appendix. The calculations are based on the following well-known identities of Fourier calculus:

$$
\int \mathcal{F}^{-1} \left(\mathcal{F} f \right)(v)\, dv = \mathcal{F} f|_{\eta=0}\,, \qquad \nabla_\eta (\mathcal{F} f)(\eta) = \mathcal{F} \left(-i\, v f(v) \right)\,. \tag{12}
$$

■

Remark 1 The moments of $w^{(0)}$ are computed by taking moments of Eq. (2). For the computation of the moments (5), (6), (7) and (8) we can as well substitute the pseudo-differential operator $\Theta[V]$ with its $\mathcal{O}(1)$-approximation in $\hbar$, $1/m \nabla_x V \cdot \nabla_v$, due to (4). Thus, instead of the expression (4), it can as well be used the "classical" one:

$$
\widetilde{w^{(0)}}(x, v) = \mathcal{F}^{-1} \left(\frac{\nu \mathcal{F} w_{\text{eq}}(x, \eta)}{\nu + i\eta \cdot \nabla_x V} \right)\,.
$$

Accordingly, the only quantum correction appearing in the first four expressions comes from w_{eq}, since $w_{\text{eq}} = n(F + \hbar^2 F^\hbar)$. Precisely, it contributes to the moments (7) and (8). Except for this quantum correction, the first four moments coincide with those computed in Cercignani et al. 2001, relative to Boltzmann equation.
The moments (9), (10) and (11) instead contain as well $\mathcal{O}(\hbar^2)$-corrections due to the pseudo-differential term: more precisely, (9) and (10) contain

$$
\frac{\hbar^2}{4m^3 \nu} \nabla_x \otimes \nabla_x \otimes \nabla_x V\, n\,, \qquad \frac{\hbar^2}{8m^3 \nu} \nabla_x \Delta_x V\, n\,,
$$

respectively (cf. Eq. (5)). Eq. (11) contains additional terms that are due to the co-existence of the quantum and the high-field regimes:

$$
-2 \frac{\partial_{x_k} V}{\nu m} n \frac{\hbar^2}{8m^3 \nu} \partial_{x_i} \partial_{x_j}^2 V - 2 \frac{\partial_{x_i} V}{\nu m} n \frac{\hbar^2}{8m^3 \nu} \partial_{x_k} \partial_{x_j}^2 V - 2 \frac{\partial_{x_j} V}{\nu m} n \frac{\hbar^2}{4 \nu m^3} \partial^3_{x_k x_i x_j} V\,,
$$

being products of the high-field fluid velocity and of the third moments of the pseudo-differential operator.
In conclusion, for an accurate description via a 3-moments fluid-dynamical model of transport in quantum–high-field regimes, it is crucial to use the pseudo-differential operator, i.e. Wigner equation, instead than Vlasov one. ■

We adopt Einstein convenction. We shall use capital greek letters for tensors and calligraphic capital letter for column vectors. Then we can define

$$n = \int w^{(0)} dv \,, \tag{13}$$

$$\mathcal{U}_j^{(0)} := -\frac{\partial_{x_j} V}{\nu m} \,, \tag{14}$$

$$\Pi_{ij}^{(0)} := n\left(\frac{k\theta\delta_{ij}}{m} + 2\mathcal{U}_i^{(0)}\mathcal{U}_j^{(0)} + \frac{\hbar^2}{12m^2 k\theta}\partial^2_{x_i x_j} V\right) \,, \tag{15}$$

$$e^{(0)} := n\left(d\frac{k\theta}{2m} + |\mathcal{U}^{(0)}|^2 + \frac{\hbar^2}{24m^2 k\theta}\Delta_x V\right) \,, \tag{16}$$

$$\mathbf{\Phi}_{kij}^{(0)} := \left(\mathcal{U}_k^{(0)}\Pi_{ij}^{(0)} + \mathcal{U}_i^{(0)}\Pi_{kj}^{(0)} + \mathcal{U}_j^{(0)}\Pi_{ki}^{(0)}\right) - n\frac{\hbar^2}{4m^2}\partial^2_{x_i x_j}\mathcal{U}_k^{(0)} \,, \tag{17}$$

$$\mathcal{J}_i^{(0)} := e^{(0)}\mathcal{U}_i^{(0)} + \Pi_{ij}^{(0)}\mathcal{U}_j^{(0)} - n\frac{\hbar^2}{8m^2}\partial^2_{x_j}\mathcal{U}_i^{(0)} \,, \tag{18}$$

$$\begin{aligned}\Sigma_{ki}^{(0)} :=\ & \Sigma_{ki}^{\text{eq}} + \mathcal{U}_k^{(0)}\mathcal{J}_i^{(0)} + \mathcal{U}_i^{(0)}\mathcal{J}_k^{(0)} + \mathbf{\Phi}_{kij}^{(0)}\mathcal{U}_j^{(0)} \\ & - \mathcal{U}_k^{(0)}\, n\frac{\hbar^2}{8m^2}\partial^2_{x_j}\mathcal{U}_i^{(0)} - n\frac{\hbar^2}{8m^2}\partial^2_{x_j}\mathcal{U}_k^{(0)}\,\mathcal{U}_i^{(0)} - n\frac{\hbar^2}{4m^2}\partial^2_{x_i x_j}\mathcal{U}_k^{(0)}\,\mathcal{U}_j^{(0)} \end{aligned} \tag{19}$$

$$\begin{aligned} =\ & \Sigma_{ki}^{\text{eq}} + e^{(0)}\left(2\mathcal{U}_k^{(0)}\mathcal{U}_i^{(0)}\right) + 2\mathcal{U}_k^{(0)}\left(\Pi_{ij}^{(0)}\mathcal{U}_j^{(0)} - n\frac{\hbar^2}{8m^2}\partial^2_{x_j}\mathcal{U}_i^{(0)}\right) \\ & + 2\left(\Pi_{kj}^{(0)}\mathcal{U}_j^{(0)} - n\frac{\hbar^2}{8m^2}\partial^2_{x_j}\mathcal{U}_k^{(0)}\right)\mathcal{U}_i^{(0)} + \left(\Pi_{ki}^{(0)}\mathcal{U}_j^{(0)} - 2n\frac{\hbar^2}{4m^2}\partial^2_{x_i x_j}\mathcal{U}_k^{(0)}\right)\mathcal{U}_j^{(0)} \,. \end{aligned} \tag{20}$$

where

$$\Sigma_{ki}^{\text{eq}} := \int v_k\, v_i \frac{v_j^2}{2}\, w_{\text{eq}} dv \,.$$

Remark 2 We stress that the term appearing due to the high-field regime is the additional tensor $\mathcal{U}_i^{(0)}\mathcal{U}_j^{(0)}$ in the $\Pi_{ij}^{(0)}$, which consequently appears wherever appears $\Pi_{ij}^{(0)}$, precisely also in $e^{(0)}$ (which is its trace), $\mathbf{\Phi}_{kij}^{(0)}$ and $\mathcal{J}_i^{(0)}$. Moreover, are to be referred to the high-field regime all the terms of the tensor $\Sigma_{ki}^{(0)} - \Sigma_{ki}^{\text{eq}}$. The rewriting (20) of the fourth order moment shall be useful in the last section. ■

From the computation of the v-moments of the leading order state, we can conclude that $w^{(0)}$ describes a nonzero–fluid-velocity state and the velocity field is determined by the applied electrical field, see (14). Observe that the velocity field is constant with respect to time. This means that in the high-field regime the fluid velocity immediately reaches its saturation value. The kinetic energy $e_{\text{kin}}^{(0)}$ is

$$e_{\text{kin}}^{(0)}(x,t) = \frac{n(x,t)}{2}\frac{|\nabla V(x)|^2}{\nu^2 m^2}\,.$$

By using (16), the internal energy density $e_{\text{i}}^{(0)} := e^{(0)} - e_{\text{kin}}^{(0)}$ is

$$e_{\text{i}}^{(0)} = \frac{n}{2}\left(\frac{dk\theta}{m} + \frac{|\nabla V|^2}{\nu^2 m^2} + \frac{\hbar^2}{12m^2 k\theta}\Delta_x V\right)\,. \tag{21}$$

Thus, the high-field action makes the equilibrium internal energy e_{eq} increase of a quantity

$$\frac{n}{2}\frac{|\nabla V|^2}{\nu^2 m^2}.$$

Remark 3 We stress that in general a nonzero-velocity field $\mathcal{W}$ only affects the kinetic energy and does not contribute to increase the internal energy density, see, e.g. (19) relative to the equilibrium regime $w_{\text{eq}}^{\mathcal{W}}$. This means that a field partecipates in modifying the electron internal energy just in case the field has the same order of magnitude as the interaction with the environment, i.e., just in the high-field regime, (Demeio and Frosali 1998, Poupaud 1992). ■

We can now anticipate the aim of next section: we shall derive via a Chapman-Enskog procedure a correction $w^{(1)}$ of $\mathcal{O}(\epsilon)$ for the solution $w^{(0)}$ of Eq. (1).

Remark 4 (Conserved quantities) By formally computing the moments of the high-field Wigner BGK equation (1) we get the continuity equation

$$\frac{\partial n}{\partial t} + \nabla_x \cdot (n\mathcal{U}) = 0\,, \tag{22}$$

by the skew-simmetry of the pseudo-differential operator and the mass-conservation of the BGK operator. For the 1st-order moment, we obtain

$$\epsilon\,\frac{\partial n\mathcal{U}}{\partial t} + \epsilon\,\nabla_x\cdot\int v\otimes vw\,dv + n\frac{\nabla_x V}{m} = -\nu\,n\mathcal{U}\,, \tag{23}$$

and the velocity-momentum is indeed not conserved by the BGK term. In order to resume the conservation of velocity-momentum we can assume that

$$\mathcal{U} \approx -\frac{\nabla_x V}{\nu m}\,,$$

in the $\mathcal{O}(1)$-term. This closure corresponds to assume that $\mathcal{U} \approx \mathcal{U}^{(0)}$ holds, i.e. the system is close to the high-field regime, where the velocity field reaches its asymptotical value. For the 2nd-order moment, instead, we obtain

$$\epsilon\,\frac{\partial e}{\partial t} + \epsilon\,\nabla_x\cdot\int v\frac{v^2}{2}w\,dv + \frac{\nabla_x V}{m}\cdot(n\mathcal{U}) = -\nu(e - e_{\text{eq}})\,. \tag{24}$$

and again for the equation to be in conservative form we can assume that

$$\frac{\nabla_x V}{m} \cdot (n\mathcal{U}) = -\nu(e - e_{\mathrm{eq}}) \,,$$

namely that it holds the following constitutive law

$$e = e_{\mathrm{eq}} - \frac{\nabla_x V}{\nu m} \cdot (n\mathcal{U}) = e_{\mathrm{eq}} + \mathcal{U}^{(0)} \cdot (n\mathcal{U}) \,, \tag{25}$$

in the $\mathcal{O}(1)$-term. Again, this closure means that the regime we are describing is close to the high-field one, and then $e \approx e^{(0)}$, where $e^{(0)} = e_{\mathrm{eq}} + n|\mathcal{U}^{(0)}|^2$, i.e. the difference of energy density, $e - e_{\mathrm{eq}}$, is due to the action of the high field. Observe that we are implicitly assuming that the interactions with the lattice can be considered "elastic" due to the high-field regime: the electrons are so "hot" that the energy loss due to phonon collisions is an unsignificant fraction of their energy (Ben Abdallah 1996). ∎

4 The Chapman-Enskog procedure

The procedure consists of two steps: first, we assume that the microscopic unknown w depends on time only through macroscopic quantities. Among the macroscopic quantities the only one that is conserved by the high-field Wigner-BGK equation is the mass, cf. Remark 4. Then we select the position density n as the sole macroscopic time-dependent unknown and we express

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial n}\frac{\partial n}{\partial t} \,. \tag{26}$$

Secondly, we assume that n is an $\mathcal{O}(1)$ quantity, and we use instead to compute the other v-moments the following expansion

$$w = \sum_{k=1}^{\infty} \epsilon^k w^k \sim w^{(0)} + \epsilon w^{(1)} \,. \tag{27}$$

In order not to contradict the Chapman-Enskog assumptions it must hold $\int w^{(1)}\, dv = 0$, see (31).

We remark that we are applying the Chapman-Enskog procedure to derive a corrected version of the equation with unknown n. Accordingly, we start from the continuity equation for the position density n, (22),

$$\frac{\partial}{\partial t}\int w dv + \nabla_x \cdot \int v w dv = 0 \,,$$

and (26) becomes

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial n}\Big(- \nabla_x \cdot \int v w dv \Big) \,.$$

By substituting this expression in the scaled equation (1), we obtain

$$\epsilon\left(\frac{\partial w}{\partial n}\Big(-\nabla_x\cdot\int vwdv\Big)\right)+\epsilon\, v\cdot\nabla_x w=\Theta[V]w-\nu(w-w_{\rm eq})\,,$$

then, by substituting the truncated asymptotic expansion (27), we get

$$\begin{aligned}&\epsilon\left(\frac{\partial w^{(0)}}{\partial n}\Big(-\nabla_x\cdot\int vw^{(0)}dv\Big)\right)+\epsilon^2\left(\frac{\partial w^{(1)}}{\partial n}\Big(-\nabla_x\cdot\int vw^{(0)}dv\Big)\right)+\\&\epsilon^2\left(\frac{\partial w^{(0)}}{\partial n}\Big(-\nabla_x\cdot\int vw^{(1)}dv\Big)\right)+\epsilon^3\left(\frac{\partial w^{(1)}}{\partial n}\Big(-\nabla_x\cdot\int vw^{(1)}dv\Big)\right)+\\&\epsilon v\cdot\nabla_x w^{(0)}+\epsilon^2 v\cdot\nabla_x w^{(1)}\\=\;&\Theta[V]w^{(0)}+\epsilon\Theta[V]w^{(1)}-\nu(w^{(0)}-w_{\rm eq})-\epsilon\nu\, w^{(1)}\,.\end{aligned}$$

At the 0-th order in ϵ it remains

$$\Theta[V]w^{(0)}-\nu(w^{(0)}-w_{\rm eq})=0\,,$$

whose solution is given formally by (3). By considering $w_{\rm eq}=n(F+\hbar^2F^\hbar)$,

$$\begin{aligned}w^{(0)}&=(\nu-\Theta[V])^{-1}\,\nu\, w_{\rm eq}\\&=n(x,t)\mathcal{F}^{-1}\left(\frac{\nu\mathcal{F}(F+\hbar^2F^\hbar)}{\nu-i\delta V(x,\eta)}\right)(x,v)\,.\end{aligned}$$

Let us define the function $M=M(x,v)$ by

$$M(x,v):=\mathcal{F}^{-1}\left(\frac{\nu\mathcal{F}(F+\hbar^2F^\hbar)}{\nu-i\delta V(x,\eta)}\right)(x,v)\,,\tag{28}$$

see Manzini and Frosali 2006, then it holds $w^{(0)}=n\,M$. At the first order in ϵ

$$\frac{\partial w^{(0)}}{\partial n}\Big(-\nabla_x\cdot\int vw^{(0)}dv\Big)+v\cdot\nabla_x w^{(0)}=(\Theta[V]-\nu)w^{(1)}\,.$$

Taking into account $\dfrac{\partial w^{(0)}}{\partial n}=M$ (cf. (28)) and (13), we obtain

$$M\nabla_x\cdot\left(n\,\frac{\nabla_x V}{\nu m}\right)+v\cdot\nabla_x w^{(0)}=(\Theta[V]-\nu)w^{(1)}\,,$$

Then we have

$$w^{(1)}:=\nabla_x\cdot\left(n\,\frac{\nabla_x V}{\nu m}\right)[\Theta[V]-\nu]^{-1}M+[\Theta[V]-\nu]^{-1}\left(v\cdot\nabla_x(n\,M)\right)\,.\tag{29}$$

Now, we can rewrite the continuity equation with unknown n by using $w=w^{(0)}+\epsilon\,w^{(1)}$ and the expressions for $w^{(0)}$ and $w^{(1)}$. We get a correction of $\mathcal{O}(\epsilon)$ for the continuity equation (22):

$$\frac{\partial n}{\partial t}+\nabla_x\cdot\int vw^{(0)}dv+\epsilon\nabla_x\cdot\int vw^{(1)}dv=0\,.\tag{30}$$

In order to write explicitly the correction term we compute moments of $w^{(1)}$:

Lemma 2 *Let $w^{(1)}$ be defined by (29). Then the moments of $w^{(1)}$ are*

$$\int w^{(1)}dv = 0 \quad (31)$$

$$n\mathcal{U}^{(1)} := \int v\, w^{(1)}dv = -\frac{1}{\nu}\nabla_x\cdot \Pi^{(0)} + \frac{1}{\nu}\nabla_x \cdot \left(n\mathcal{U}^{(0)}\right)\mathcal{U}^{(0)} \quad (32)$$

$$\Pi^{(1)} := \int v\otimes v\, w^{(1)}dv = -\frac{1}{\nu}\nabla_x\cdot \mathbf{\Phi}^{(0)} + \frac{1}{\nu}\nabla_x \cdot \left(n\mathcal{U}^{(0)}\right)\frac{\Pi^{(0)}}{n} + +n\mathcal{U}^{(1)}\otimes\mathcal{U}^{(0)} + \mathcal{U}^{(0)}\otimes n\mathcal{U}^{(1)} \quad (33)$$

$$e^{(1)} := \int \frac{v^2}{2}w^{(1)}dv = -\frac{1}{\nu}\nabla_x\cdot \mathcal{J}^{(0)} + \frac{1}{\nu}\nabla_x \cdot \left(n\mathcal{U}^{(0)}\right)\frac{e^{(0)}}{n} + n\mathcal{U}^{(1)}\cdot\mathcal{U}^{(0)} \quad (34)$$

$$\mathcal{J}^{(1)} = \int v\frac{v^2}{2}w^{(1)}dv = -\frac{1}{\nu}\nabla_x\cdot \Sigma^{(0)} + \frac{1}{\nu}\nabla_x\cdot \left(n\mathcal{U}^{(0)}\right)\frac{\mathcal{J}^{(0)}}{n} + e^{(1)}\mathcal{U}^{(0)} + \Pi^{(1)}\mathcal{U}^{(0)}\,. \quad (35)$$

The proof is in the Appendix. Observe that we use the shortened form $\partial_{x_k}\Pi_{ki} = \nabla_x\cdot \Pi$, as well as $\partial_{x_k}\mathbf{\Phi}_{kij} = \nabla_x\cdot \mathbf{\Phi}$. Then Eq. (30) can be rewritten as

$$\frac{\partial n}{\partial t} - \nabla_x\cdot\left(n\frac{\nabla V}{\nu m}\right) + \frac{\epsilon}{\nu}\nabla_x\cdot\left(\nabla_x\cdot\left(n\frac{\nabla V}{\nu m}\right)\frac{\nabla V}{\nu m}\right) - \frac{\epsilon}{\nu}\nabla_x\cdot\nabla_x\cdot\Pi^{(0)} = 0\,, \quad (36)$$

with $\Pi^{(0)}$ as in (15):

$$\Pi^{(0)} = n\left(\frac{k\theta\mathcal{I}}{m} + 2\frac{\nabla V}{\nu m}\otimes\frac{\nabla V}{\nu m} + \frac{\hbar^2}{12m^2k\theta}\nabla_x\otimes\nabla_x V\right)\,.$$

This is at $\mathcal{O}(1)$ in ϵ a classical drift term with unkwon n with a $\mathcal{O}(\epsilon)$-additional drift due to the high-field regime. By splitting the following term as

$$\frac{\epsilon}{\nu}\nabla_x\cdot\left(\frac{\nabla V}{\nu m}\nabla_x\cdot\left(n\frac{\nabla V}{\nu m}\right)\right) = \frac{\epsilon}{\nu}\nabla_x\cdot\frac{\nabla V}{\nu m}\nabla_x\cdot\left(n\frac{\nabla V}{\nu m}\right) + \frac{\epsilon}{\nu}\frac{\nabla V}{\nu m}\cdot\nabla_x\left(\nabla_x\cdot\left(n\frac{\nabla V}{\nu m}\right)\right)\,,$$

we indeed derive the following high-field corrected version of the (classical) mobility coefficient $\mu_{\mathrm{cl}} = 1/(\nu m)$

$$\mu_{\mathrm{hf}} := \frac{1}{\nu m}\left(1 + \frac{\epsilon}{\nu}\frac{\Delta V}{\nu m}\right)\,.$$

Moreover, there is an $\mathcal{O}(\epsilon)$-diffusive term: to the standard classical and quantum tensors, see Degond et al. (2005), is added an anisotropic tensor, (proportional to) $\nabla V\otimes\nabla V$, that is peculiar of the high-field regime, Cercignani et al. (2001). Eq. (36) is a quantum augmented drift-diffusion model and it coincides with the one derived in Manzini and Frosali (2006) via a different asymptotical procedure. We remark that the quantum correction is due to the $\mathcal{O}(\hbar^2)$-expansion of the Wigner equilibrium function and not to the use of the pseudo-differential operator.
In the next section we shall go further and apply the moment method and then propose a closure procedure suggested by the Chapman-Enskog expansion $w^{(0)} + \epsilon w^{(1)}$.

5 Corrections to moment equations

In the previous section, we have derived a corrected version of the conservation equation for the position density n in a regime close to the high-field one by expressing the position-density flux $n\mathcal{U} := \int v\, w\, dv$ as

$$n\mathcal{U} \approx n\mathcal{U}^{(0)} + \epsilon\, n\mathcal{U}^{(1)} = \int v\,(w^{(0)} + \epsilon\; w^{(1)})\, dv\,,$$

with $w^{(1)}$ obtained by the Chapman-Enskog precedure. In this section we want to obtain corrected equations for the evolution of the fluid velocity $\mathcal{U}^{(0)}$ and the energy density $e^{(0)}$.
Let us start then from the moment equations (23), (24):

$$\begin{aligned}
\epsilon\,\frac{\partial(n\mathcal{U})}{\partial t} + \epsilon\,\nabla_x\cdot\int v\otimes vw\, dv + \frac{\nabla_x V}{m}\int w\, dv &= -\nu\int v\, w\, dv\,,\\
\epsilon\,\frac{\partial e}{\partial t} + \epsilon\,\nabla_x\cdot\int v\frac{v^2}{2}w\, dv + \frac{\nabla_x V}{m}\cdot\int v\, w\, dv &= -\nu\left(\int \frac{v^2}{2}w\, dv - e_{\text{eq}}\right)\,.
\end{aligned}$$

The closure strategy consists in using for the computation of the fluxes $w \approx w^{(0)} + \epsilon\, w^{(1)}$, this means that we are computing with $\mathcal{O}(\epsilon)$-accuracy all the terms that decide the evolution of the macroscopic unknowns. Let us start from Eq. (23), namely,

$$\epsilon\,\frac{\partial(n\mathcal{U})}{\partial t} + \epsilon\,\nabla_x\cdot\int v\otimes vw\, dv + \frac{\nabla_x V}{m}\int w\, dv = -\nu\int v\, w\, dv\,,$$

and substitute $w \approx w^{(0)} + \epsilon\, w^{(1)}$ for the computation of the fluxes. We obtain

$$\frac{\partial(n\mathcal{U})}{\partial t} + \nabla_x\cdot\Pi^{(0)} + \epsilon\,\nabla_x\cdot\Pi^{(1)} + \frac{1}{\epsilon}\frac{n\nabla_x V}{m} = -\frac{1}{\epsilon}\nu\, n(\mathcal{U}^{(0)} + \,\epsilon\mathcal{U}^{(1)})\,.$$

By substiting the expressions for $\mathcal{U}^{(0)}$ and $\mathcal{U}^{(1)}$, precisely,

$$-\nu\,\mathcal{U}^{(0)} := \frac{\nabla_x V}{m}\,, \qquad -\nu\, n\mathcal{U}^{(1)} := \nabla_x\cdot\Pi^{(0)} - \nabla_x\cdot\left(n\mathcal{U}^{(0)}\right)\mathcal{U}^{(0)}\,,$$

it reduces to the following equation with unknown $\mathcal{U} \approx \mathcal{U}^{(0)}$:

$$\frac{\partial(n\mathcal{U}^{(0)})}{\partial t} + \nabla_x\cdot\left(n\mathcal{U}^{(0)}\right)\mathcal{U}^{(0)} + \epsilon\,\nabla_x\cdot\Pi^{(1)} = 0\,.$$

We stress that the standard term $\nabla_x\cdot\Pi^{(0)}$ disappears due to the correction of the BGK moment $-\nu\, n\mathcal{U}^{(1)}$, thus at the leading order there is some drift equation in $\mathcal{U}^{(0)}$. Let us rewrite the definition (33) of $\Pi^{(1)}$ in a more explicit form:

$$\begin{aligned}
\nu\,\nabla_x\cdot\Pi^{(1)} &= \nabla_x\cdot\left[\left(\frac{\Pi^{(0)}}{n} + 2\mathcal{U}^{(0)}\otimes\mathcal{U}^{(0)}\right)\nabla_x\cdot\left(n\mathcal{U}^{(0)}\right)\right]\\
&- \nabla_x\cdot\left(\nabla_x\cdot\Pi^{(0)}\otimes\mathcal{U}^{(0)} + \mathcal{U}^{(0)}\otimes\nabla_x\cdot\Pi^{(0)}\right)\\
&- \nabla_x\cdot\nabla_x\cdot\left(\mathcal{U}^{(0)}\otimes\Pi^{(0)} + \Pi^{(0)}\otimes\mathcal{U}^{(0)} + \Pi^{(0)}_{kj}\mathcal{U}^{(0)}_i\right)\\
&+ \nabla_x\cdot\nabla_x\cdot\left(n\frac{\hbar^2}{4m^2}\nabla\otimes\nabla\otimes\mathcal{U}^{(0)}\right)\,, \qquad (37)
\end{aligned}$$

with the anisotropic tensors $\Pi^{(0)}$ as in (15) and with $\Pi^{(0)}_{kj}\mathcal{U}^{(0)}_i$ componentwise in order to distinguish it from $\mathcal{U}^{(0)}_k\Pi^{(0)}_{ij} = \mathcal{U}^{(0)}\otimes\Pi^{(0)}$ and $\Pi^{(0)}_{ki}\otimes\mathcal{U}^{(0)}_j = \Pi^{(0)}\otimes\mathcal{U}^{(0)}$. Then, at $\mathcal{O}(\epsilon)$ there are diffusive terms in $n\mathcal{U}^{(0)}$ (first line of (37)) and drift terms in $\mathcal{U}^{(0)}$ (second line of (37)). The last two lines are $-\nu\nabla_x\cdot\nabla_x\cdot\mathbf{\Phi}^{(0)}$ as in (17). The third line contains again $\mathcal{O}(\epsilon)$-drift and diffusive terms in $\mathcal{U}^{(0)}$, while the fourth a quantum iper-diffusive term in $\mathcal{U}^{(0)}$.
Starting instead from Eq. (24), namely,

$$\epsilon\,\frac{\partial \mathrm{e}}{\partial t} + \epsilon\,\nabla_x\cdot\int v\frac{v^2}{2}w\,dv + \frac{\nabla_x V}{m}\cdot(n\mathcal{U}) = -\nu(e-e_{\mathrm{eq}})\,,$$

that is the 2nd-order moment of the high-field Wigner-BGK equation, and substituting $w\approx w^{(0)}+\epsilon\,w^{(1)}$ for closing the moments, we get

$$\frac{\partial\,\mathrm{e}^{(0)}}{\partial t} + \nabla_x\cdot\mathcal{J}^{(0)} + \epsilon\,\nabla_x\cdot\mathcal{J}^{(1)} - \nu\mathcal{U}^{(0)}\cdot n\mathcal{U}^{(1)} = -\nu\,e^{(1)}\,. \tag{38}$$

By substituting the explicit expression for $e^{(1)}$, namely

$$e^{(1)} \;=\; n\mathcal{U}^{(1)}\cdot\mathcal{U}^{(0)} + \frac{1}{\nu}\frac{e^{(0)}}{n}\nabla_x\cdot\left(n\mathcal{U}^{(0)}\right) - \frac{1}{\nu}\nabla_x\cdot\mathcal{J}^{(0)}\,,$$

(38) simplifies to

$$\frac{\partial e^{(0)}}{\partial t} + \nabla_x\cdot\left(n\mathcal{U}^{(0)}\right)\frac{e^{(0)}}{n} + \epsilon\,\nabla_x\cdot\mathcal{J}^{(1)} = 0\,,$$

where, once again, at order $\mathcal{O}(1)$ the standard term $\nabla_x\cdot\mathcal{J}^{(0)}$ does not appear, since the regime is of drift-collision balance. Consequently, we expect diffusive terms to appear as order $\mathcal{O}(\epsilon)$ corrections. By writing explicitly all the terms contained in $\nu\mathcal{J}^{(1)}$ (we used (20) for $\Sigma^{(0)}$ and (37) for $\Pi^{(1)}$), we obtain

$$\begin{aligned}
&\nu\mathcal{J}^{(1)} \;=\\
&\nabla_x\cdot\left(n\mathcal{U}^{(0)}\right)\left(2\frac{e^{(0)}}{n}\mathcal{U}^{(0)} + 3|\mathcal{U}^{(0)}|^2\,\mathcal{U}^{(0)} + 2\frac{\Pi^{(0)}}{n}\mathcal{U}^{(0)} - \frac{\hbar^2}{8m^2}\Delta_x\mathcal{U}^{(0)}\right)\\
-\;&\nabla_x\cdot\left[e^{(0)}\mathcal{U}^{(0)} + \Pi^{(0)}\mathcal{U}^{(0)} - n\frac{\hbar^2}{8m^2}\Delta_x\mathcal{U}^{(0)}\right]\mathcal{U}^{(0)} - (\nabla_x\cdot\Pi^{(0)})3|\mathcal{U}^{(0)}|^2\\
-\;&\nabla_x\cdot\left[\Sigma^{\mathrm{eq}} + e^{(0)}\left(2\mathcal{U}^{(0)}\otimes\mathcal{U}^{(0)}\right) + 2\mathcal{U}^{(0)}\otimes\left(\Pi^{(0)}\mathcal{U}^{(0)} - n\frac{\hbar^2}{8m^2}\Delta\mathcal{U}^{(0)}\right)\right]\\
-\;&\nabla_x\cdot\left[2\left(\Pi^{(0)}\mathcal{U}^{(0)} - n\frac{\hbar^2}{8m^2}\Delta\mathcal{U}^{(0)}\right)\mathcal{U}^{(0)} + \left(\Pi^{(0)}\otimes\mathcal{U}^{(0)} - 2n\frac{\hbar^2}{4m^2}\nabla\otimes\nabla\otimes\mathcal{U}^{(0)}\right)\mathcal{U}^{(0)}\right]\\
-\;&\nabla_x\cdot\left[\mathcal{U}^{(0)}\otimes\Pi^{(0)} + \Pi^{(0)}\otimes\mathcal{U}^{(0)} + \Pi^{(0)}_{kj}\mathcal{U}^{(0)}_i\right]\mathcal{U}^{(0)} + \nabla_x\cdot\left[n\frac{\hbar^2}{4m^2}\nabla\otimes\nabla\otimes\mathcal{U}^{(0)}\right]\mathcal{U}^{(0)}\,.
\end{aligned}$$

Among them we isolate the following terms of $\nu\nabla_x\cdot\mathcal{J}^{(1)}$

$$2\nabla_x\cdot\left[\nabla_x\cdot\left(n\mathcal{U}^{(0)}\right)\left(\frac{e^{(0)}}{n}\mathcal{U}^{(0)}\right)\right] - \nabla_x\cdot\left[\mathcal{U}^{(0)}\nabla_x\cdot\left(e^{(0)}\mathcal{U}^{(0)}\right)\right] - 2\nabla_x\cdot\nabla_x\cdot\left[e^{(0)}\mathcal{U}^{(0)}\otimes\mathcal{U}^{(0)}\right]$$

that are $\mathcal{O}(\epsilon)$ drift and diffusive terms in $e^{(0)}$. The remaining terms of $\nu\nabla_x\cdot\mathcal{J}_1$ contain the expected quantum–high-field corrections.

Appendix

Proof of Lemma 1
Since it holds

$$w^{(0)} := \mathcal{F}^{-1}\left\{\frac{\nu\mathcal{F}w_{\rm eq}}{\nu - i\delta V}\right\}$$

and

$$v_j\mathcal{F}^{-1}f = \mathcal{F}^{-1}\left(i\,\partial_{\eta_j}f\right)\,,$$

then

$$v_j\,w^{(0)} \;=\; \mathcal{F}^{-1}\left\{\frac{\nu\mathcal{F}(v_jw_{\rm eq})}{\nu - i\delta V} - \frac{\nu\mathcal{F}w_{\rm eq}\partial_{\eta_j}\delta V}{(\nu - i\delta V)^2}\right\}$$

and

$$\begin{aligned} v_iv_j\,w^{(0)} \;&=\; \mathcal{F}^{-1}\left\{\frac{\nu\mathcal{F}(v_iv_jw_{\rm eq})}{\nu - i\delta V} - \frac{\nu\mathcal{F}(v_jw_{\rm eq})\partial_{\eta_i}\delta V}{(\nu - i\delta V)^2}\right\} \\ &-\; \mathcal{F}^{-1}\left\{\frac{\nu\mathcal{F}(v_iw_{\rm eq})\partial_{\eta_j}\delta V}{(\nu - i\delta V)^2} + \frac{\nu\mathcal{F}w_{\rm eq}\,i\partial^2_{\eta_i\eta_j}\delta V}{(\nu - i\delta V)^2} - \frac{\nu\mathcal{F}w_{\rm eq}\partial_{\eta_j}\delta V\,2\partial_{\eta_i}\delta V}{(\nu - i\delta V)^3}\right\}. \end{aligned}$$

Then, by further differentiation

$$\begin{aligned} v_kv_iv_j\,w^{(0)} \;&=\; \mathcal{F}^{-1}\left\{\frac{\nu\mathcal{F}(v_iv_jv_kw_{\rm eq})}{\nu - i\delta V} - \frac{\nu\mathcal{F}(v_iv_jw_{\rm eq})\partial_{\eta_k}\delta V}{(\nu - i\delta V)^2}\right\} \\ -\; &\mathcal{F}^{-1}\left\{\frac{\nu\mathcal{F}(v_jv_kw_{\rm eq})\partial_{\eta_i}\delta V}{(\nu - i\delta V)^2} + \frac{\nu\mathcal{F}(v_jw_{\rm eq})\,i\partial^2_{\eta_i\eta_k}\delta V}{(\nu - i\delta V)^2} - \frac{\nu\mathcal{F}(v_jw_{\rm eq})\partial_{\eta_i}\delta V\,2\partial_{\eta_k}\delta V}{(\nu - i\delta V)^3}\right\} \\ -\; &\mathcal{F}^{-1}\left\{\frac{\nu\mathcal{F}(v_iv_kw_{\rm eq})\partial_{\eta_j}\delta V}{(\nu - i\delta V)^2} + \frac{\nu\mathcal{F}(v_iw_{\rm eq})\,i\partial^2_{\eta_j\eta_k}\delta V}{(\nu - i\delta V)^2} - \frac{\nu\mathcal{F}(v_iw_{\rm eq})\partial_{\eta_j}\delta V\,2\partial_{\eta_k}\delta V}{(\nu - i\delta V)^3}\right\} \\ -\; &\mathcal{F}^{-1}\left\{\frac{\nu\mathcal{F}(v_kw_{\rm eq})\,i\partial^2_{\eta_i\eta_j}\delta V}{(\nu - i\delta V)^2} + \frac{\nu\mathcal{F}w_{\rm eq}\,i^2\partial^3_{\eta_i\eta_j\eta_k}\delta V}{(\nu - i\delta V)^2} - \frac{\nu\mathcal{F}w_{\rm eq}\,i\partial^2_{\eta_i\eta_j}\delta V\,2\partial_{\eta_k}\delta V}{(\nu - i\delta V)^3}\right\} \\ +\; &\mathcal{F}^{-1}\left\{\frac{\nu\mathcal{F}(v_kw_{\rm eq})\partial_{\eta_j}\delta V\,2\partial_{\eta_i}\delta V}{(\nu - i\delta V)^3} + \frac{\nu\mathcal{F}w_{\rm eq}\,i\partial_{\eta_k}(\partial_{\eta_j}\delta V\,2\partial_{\eta_i}\delta V)}{(\nu - i\delta V)^3}\right\} \\ -\; &\mathcal{F}^{-1}\left\{\frac{6\nu\mathcal{F}w_{\rm eq}\partial_{\eta_j}\delta V\,\partial_{\eta_i}\delta V\,\partial_{\eta_k}\delta V}{(\nu - i\delta V)^4}\right\} \end{aligned}$$

and third order term $v_kv_j^2\,w^{(0)}\,dv$ is computed by taking $i = j$ in the previous calculation. Fourth order terms can be computed analogously.
The corresponding moments are computed by applying the following identity

$$\int\left(\mathcal{F}^{-1}f\right)(v)\,dv = f|_{\eta=0}\,,$$

i.e. by evaluating in $\eta = 0$ the functions between curl parentheses. ■

Proof of Lemma 2

Let us rewrite $w^{(1)}$ as

$$w^{(1)} = \nabla_x \cdot \left(\frac{n \nabla_x V}{\nu m} \right) \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}M}{i\delta V - \nu} \right\} + \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v \cdot \nabla_x w^{(0)})}{i\delta V - \nu} \right\},$$

where we recall $M := w^{(0)}/n$. First we compute the moments of the first addendum.

$$v_j \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}M}{i\delta V - \nu} \right\} = \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_j M)}{i\delta V - \nu} + \frac{\mathcal{F}M \partial_{\eta_j} \delta V}{(i\delta V - \nu)^2} \right\},$$

$$\begin{aligned} v_i v_j \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}M}{i\delta V - \nu} \right\} &= \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_i v_j M)}{i\delta V - \nu} + \frac{\mathcal{F}(v_j M) \partial_{\eta_i} \delta V}{(i\delta V - \nu)^2} \right\} \\ &+ \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_i M) \partial_{\eta_j} \delta V}{(i\delta V - \nu)^2} + \frac{\mathcal{F}M \, i\partial^2_{\eta_i \eta_j} \delta V}{(i\delta V - \nu)^2} + \frac{\mathcal{F}M \partial_{\eta_j} \delta V \, 2\partial_{\eta_i} \delta V}{(i\delta V - \nu)^3} \right\}, \end{aligned}$$

$$\begin{aligned} \frac{v_j^2}{2} \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}M}{i\delta V - \nu} \right\} &= \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_j^2 M/2)}{i\delta V - \nu} + \frac{\mathcal{F}(v_j M) \partial_{\eta_j} \delta V}{(i\delta V - \nu)^2} \right\} \\ &+ \mathcal{F}^{-1} \left\{ \frac{1}{2} \frac{\mathcal{F}M \, i\partial^2_{\eta_j^2} \delta V}{(i\delta V - \nu)^2} + \frac{\mathcal{F}M (\partial_{\eta_j} \delta V)^2}{(i\delta V - \nu)^3} \right\}, \end{aligned}$$

$$\begin{aligned} \frac{v_i v_j^2}{2} \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}M}{i\delta V - \nu} \right\} &= \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_i v_j^2 M/2)}{i\delta V - \nu} + \frac{\mathcal{F}(v_j^2 M/2) \partial_i \delta V}{(i\delta V - \nu)^2} \right\} \\ &+ \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_i v_j M) \partial_{\eta_j} \delta V}{(i\delta V - \nu)^2} + \frac{\mathcal{F}(v_j M) \, i\partial^2_{\eta_i \eta_j} \delta V}{(i\delta V - \nu)^2} \right\} + \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_j M) \partial_{\eta_j} \delta V \, 2\partial_{\eta_i} \delta V}{(i\delta V - \nu)^3} \right\} \\ &+ \frac{1}{2} \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_i M) \, i\partial^2_{\eta_j^2} \delta V}{(i\delta V - \nu)^2} + \frac{\mathcal{F}M \, i^2 \partial^3_{\eta_i \eta_j^2} \delta V}{(i\delta V - \nu)^2} \right\} + \frac{1}{2} \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}M \, i\partial^2_{\eta_j^2} \delta V \, 2\partial_{\eta_i} \delta V}{(i\delta V - \nu)^3} \right\} \\ &+ \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_i M) (\partial_{\eta_j} \delta V)^2}{(i\delta V - \nu)^3} + \frac{\mathcal{F}M \, i\partial_{\eta_i} ((\partial_{\eta_j} \delta V)^2)}{(i\delta V - \nu)^3} \right\} + \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}M (\partial_{\eta_j} \delta V)^2 \, 3\partial_{\eta_i} \delta V}{(i\delta V - \nu)^4} \right\}. \end{aligned}$$

Secondly, we perform analogous calculations with the second addendum

$$v_j \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v \cdot \nabla_x w^{(0)})}{i\delta V - \nu} \right\} = \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}(v_j v \cdot \nabla_x w^{(0)})}{i\delta V - \nu} + \frac{\mathcal{F}(v \cdot \nabla_x w^{(0)}) \partial_{\eta_j} \delta V}{(i\delta V - \nu)^2} \right\},$$

$$
\begin{aligned}
v_i v_j \mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})}{i\delta V-\nu}\right\} &= \mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v_i v_j v\cdot\nabla_x w^{(0)})}{i\delta V-\nu}+\frac{\mathcal{F}(v_j v\cdot\nabla_x w^{(0)})\partial_{\eta_i}\delta V}{(i\delta V-\nu)^2}\right\}\\
&+ \mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v_i v\cdot\nabla_x w^{(0)})\partial_{\eta_j}\delta V}{(i\delta V-\nu)^2}+\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})\, i\partial^2_{\eta_i\eta_j}\delta V}{(i\delta V-\nu)^2}\right\}\\
&+ \mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})\partial_{\eta_j}\delta V\, 2\partial_{\eta_i}\delta V}{(i\delta V-\nu)^3}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{v_j^2}{2}\mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})}{i\delta V-\nu}\right\} &= \mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v_j^2(v\cdot\nabla_x w^{(0)})/2)}{i\delta V-\nu}+\frac{\mathcal{F}(v_j v\cdot\nabla_x w^{(0)})\partial_{\eta_j}\delta V}{(i\delta V-\nu)^2}\right\}\\
&+ \mathcal{F}^{-1}\left\{\frac{1}{2}\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})\, i\partial^2_{\eta_j^2}\delta V}{(i\delta V-\nu)^2}+\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})(\partial_{\eta_j}\delta V)^2}{(i\delta V-\nu)^3}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{v_i v_j^2}{2}\mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})}{i\delta V-\nu}\right\} =\\
&\mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v_i v_j^2(v\cdot\nabla_x w^{(0)})/2)}{i\delta V-\nu}+\frac{\mathcal{F}(v_j^2(v\cdot\nabla_x w^{(0)})/2)\partial_i\delta V}{(i\delta V-\nu)^2}\right\}\\
+\;&\mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v_i v_j v\cdot\nabla_x w^{(0)})\partial_{\eta_j}\delta V}{(i\delta V-\nu)^2}+\frac{\mathcal{F}(v_j v\cdot\nabla_x w^{(0)})\, i\partial^2_{\eta_i\eta_j}\delta V}{(i\delta V-\nu)^2}\right\}\\
+\;&\mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v_j v\cdot\nabla_x w^{(0)})\partial_{\eta_j}\delta V\, 2\partial_{\eta_i}\delta V}{(i\delta V-\nu)^3}\right\}\\
+\;&\frac{1}{2}\mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v_i v\cdot\nabla_x w^{(0)})\, i\partial^2_{\eta_j^2}\delta V}{(i\delta V-\nu)^2}+\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})\, i^2\partial^3_{\eta_i\eta_j^2}\delta V}{(i\delta V-\nu)^2}\right\}\\
+\;&\frac{1}{2}\mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)}\, i\partial^2_{\eta_j^2}\delta V\, 2\partial_{\eta_i}\delta V}{(i\delta V-\nu)^3}\right\}\\
+\;&\mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v_i v\cdot\nabla_x w^{(0)})(\partial_{\eta_j}\delta V)^2}{(i\delta V-\nu)^3}+\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})\, i\partial_{\eta_i}((\partial_{\eta_j}\delta V)^2)}{(i\delta V-\nu)^3}\right\}\\
+\;&\mathcal{F}^{-1}\left\{\frac{\mathcal{F}(v\cdot\nabla_x w^{(0)})(\partial_{\eta_j}\delta V)^2\, 3\partial_{\eta_i}\delta V}{(i\delta V-\nu)^4}\right\}.
\end{aligned}
$$

We now evaluate the expressions in curl parentheses in $\eta = 0$ and recombine them according to the definition of $w^{(1)}$. Observe that we use the writing

$$
\nabla_x\cdot\int v v_j w^{(0)}\, dv = \sum_k \partial_{x_k}\int v_k v_j w^{(0)}\, dv
$$

and analogous ones to shorten the notation. We obtain

$$\begin{aligned}
\nu \int v_j w^{(1)}\, dv &= \nabla_x \cdot \left(n \frac{\nabla_x V}{\nu m} \right) \left\{ - \int v_j M \, dv + \frac{\partial_{x_j} V}{\nu m} \right\} \\
&+ \left\{ -\nabla_x \cdot \int v v_j w^{(0)}\, dv + \nabla_x \cdot \int v w^{(0)}\, dv \frac{\partial_{x_j} V}{\nu m} \right\} \\
&= \nabla_x \cdot \left(n \frac{\nabla_x V}{\nu m} \right) \frac{\partial_{x_j} V}{\nu m} - \nabla_x \cdot \int v v_j w^{(0)}\, dv \,,
\end{aligned}$$

$$\begin{aligned}
\nu \int v_i v_j w^{(1)}\, dv &= \nabla_x \cdot \left(n \frac{\nabla_x V}{\nu m} \right) \left\{ - \int v_i v_j M \, dv - 4 \frac{\partial_{x_j} V}{\nu m} \frac{\partial_{x_i} V}{\nu m} \right\} \\
&+ \left\{ -\nabla_x \cdot \int v v_i v_j w^{(0)}\, dv + 2 \nabla_x \cdot \left(\frac{n \nabla_x V}{\nu m} \right) \frac{\partial_{x_j} V}{\nu m} \frac{\partial_{x_i} V}{\nu m} \right\} \\
&+ \left\{ \nabla_x \cdot \int v v_j w^{(0)}\, dv \frac{\partial_{x_i} V}{\nu m} + \nabla_x \cdot \int v v_i w^{(0)}\, dv \frac{\partial_{x_j} V}{\nu m} \right\} \\
&= -\nabla_x \cdot \left(\frac{n \nabla_x V}{\nu m} \right) \left(\int v_i v_j M \, dv + 2 \frac{\partial_{x_j} V}{\nu m} \frac{\partial_{x_i} V}{\nu m} \right) \\
&+ \nabla_x \cdot \int v v_j w^{(0)}\, dv \frac{\partial_{x_i} V}{\nu m} + \nabla_x \cdot \int v v_i w^{(0)}\, dv \frac{\partial_{x_j} V}{\nu m} \\
&- \nabla_x \cdot \int v v_i v_j w^{(0)}\, dv \,,
\end{aligned}$$

$$\begin{aligned}
\nu \int \frac{v_j^2}{2} w^{(1)}\, dv &= -\nabla_x \cdot \left(n \frac{\nabla_x V}{\nu m} \right) \left(\int \frac{v_j^2}{2} M \, dv + \left(\frac{\partial_{x_j} V}{\nu m} \right)^2 \right) \\
&+ \nabla_x \cdot \int v v_j w^{(0)}\, dv \frac{\partial_{x_j} V}{\nu m} - \nabla_x \cdot \int v \frac{v_j^2}{2} w^{(0)}\, dv \,,
\end{aligned}$$

$$\begin{aligned}
\nu \int \frac{v_i v_j^2}{2} w^{(1)}\, dv &= \\
& \nabla_x \cdot \left(\frac{n \nabla_x V}{\nu m} \right) \left\{ - \int \frac{v_i v_j^2 M}{2} \, dv + \int \frac{v_j^2 M}{2} \, dv \frac{\partial_{x_i} V}{\nu m} + \int v_i v_j M \, dv \frac{\partial_{x_j} V}{\nu m} \right\} \\
+ & \nabla_x \cdot \left(\frac{n \nabla_x V}{\nu m} \right) \left\{ 6 \left(\frac{\partial_{x_j} V}{\nu m} \right)^2 \frac{\partial_{x_i} V}{\nu m} - \frac{\hbar^2 \partial^3_{x_i x_j^2} V}{8 m^3 \nu} \right\} \\
+ & \left\{ -\nabla_x \cdot \int v v_i \frac{v_j^2}{2} w^{(0)}\, dv + \nabla_x \cdot \int v \frac{v_j^2}{2} w^{(0)}\, dv \frac{\partial_{x_i} V}{\nu m} + \nabla_x \cdot \int v v_i v_j w^{(0)}\, dv \frac{\partial_{x_j} V}{\nu m} \right\} \\
+ & \left\{ -2 \nabla_x \cdot \int v v_j w^{(0)}\, dv \frac{\partial_{x_j} V}{\nu m} \frac{\partial_{x_i} V}{\nu m} + \nabla_x \cdot \left(\frac{n \nabla V}{\nu m} \right) \frac{\hbar^2 \partial^3_{x_i x_j^2} V}{8 m^3 \nu} \right\} \\
+ & \left\{ -\nabla_x \cdot \int v v_i w^{(0)}\, dv \left(\frac{\partial_{x_j} V}{\nu m} \right)^2 - 3 \nabla_x \cdot \left(\frac{n \nabla_x V}{\nu m} \right) \left(\frac{\partial_{x_j} V}{\nu m} \right)^2 \frac{\partial_{x_i} V}{\nu m} \right\}
\end{aligned}$$

$$
\begin{aligned}
= & -\nabla_x\cdot\left(\frac{n\nabla_x V}{\nu m}\right)\left(\int\frac{v_iv_j^2M}{2}\,dv-\int\frac{v_j^2M}{2}\,dv\frac{\partial_{x_i}V}{\nu m}-\int v_iv_jM\,dv\frac{\partial_{x_j}V}{\nu m}\right)\\
- & \nabla_x\cdot\left(\frac{n\nabla_x V}{\nu m}\right)\left(-3\left(\frac{\partial_{x_j}V}{\nu m}\right)^2\frac{\partial_{x_i}V}{\nu m}\right)\\
- & \nabla_x\cdot\int vv_i\frac{v_j^2}{2}w^{(0)}\,dv+\nabla_x\cdot\int v\frac{v_j^2}{2}w^{(0)}\,dv\frac{\partial_{x_i}V}{\nu m}+\nabla_x\cdot\int vv_iv_jw^{(0)}\,dv\frac{\partial_{x_j}V}{\nu m}\\
- & 2\nabla_x\cdot\int vv_jw^{(0)}\,dv\frac{\partial_{x_j}V}{\nu m}\frac{\partial_{x_i}V}{\nu m}-\nabla_x\cdot\int vv_iw^{(0)}\,dv\left(\frac{\partial_{x_j}V}{\nu m}\right)^2 .
\end{aligned}
$$

■

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