Compressed asymptotic analysis of a quantum transport equation

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We perform the asymptotic analysis of the Wigner quantum transport equation in a semiconductor. What we mean as asymptotic analysis:

Let us suppose that the evolution process is characterized by the operator A_ϵ, acting in the Banach space X,

$$\begin{cases} \dfrac{du_\epsilon}{dt} &= A_\epsilon u_\epsilon, \\ u_\epsilon(0) &= u_0 \end{cases}$$

where u_0 is the initial condition. We are interested in finding a new simpler operator, say B_ϵ, such that the new problem

$$\begin{cases} \dfrac{dv_\epsilon}{dt} &= B_\epsilon v_\epsilon, \\ v_\epsilon(0) &= v_0 \end{cases}$$

admits a solution v_ϵ which is sufficiently "close" to the exact solution, u_ϵ, in the

sense

$$\|u_\epsilon - v_\epsilon\|_X = O(\epsilon^2).$$

The main purpose of this paper is to revisit the derivation of drift-diffusion approximation for a linear quantum transport in a semiconductor.

Our aim is to make a rigorous asymptotic analysis using the modified (compressed) Chapman–Enskog method (Mika & Banasiak, 1995a).

This method was applied successfully to more complicated models, yielding a rigorous mathematical theory of the asymptotic expansion. Here we apply the compressed method to the quantum Wigner equation, clarifying the derivation and giving the exact meaning of the hydrodynamic coefficients.

The problem is studied in the L_2 setting, where the norm is the natural one from the physical point of view.

The quantum kinetic model

We consider the linear Wigner equation that describes the evolution in time of the quasi-distribution function $w = w(x, v, t)$ associated to a quantum system with d degrees of freedom, under the effect of a potential $V = V(x)$:

$$\frac{\partial}{\partial t} w + v \cdot \nabla_x w - \Theta[V]w \;=\; 0, \quad t > 0, \quad (x, v) \in \mathbb{R}^{2d}\,.$$

Here the (real-valued) potential V enters through the pseudo-differential operator $\Theta[V]$ defined by

$$(\Theta[V]w)(x, v, t) \;=\; \frac{i}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \delta V(x, \eta) w(x, v', t) e^{i(v-v')\cdot\eta}\, dv'\, d\eta\,,$$

where $\delta V(x, \eta) := \frac{1}{\hbar}\left(V(x + \frac{\hbar\eta}{2m}) - V(x - \frac{\hbar\eta}{2m})\right)$

Let us modify the Wigner equation by adding a collisional term which mimics the dissipative interaction of the quantum system with the environment. The simpler model is a BGK relaxation-time operator, which says that after a time $1/\nu$ the system will relax to a state $w_{\rm eq}$. Accordingly,

$$\frac{\partial}{\partial t}w + v\cdot\nabla_x w - \Theta[V]w \;=\; -\nu(w - w_{\rm eq})\,, \quad t>0, \quad (x,v)\in \mathbb{R}^{2d}.$$

We shall describe the equilibrium state by the $O(\hbar^2)$-quantum corrected thermodynamical equilibrium function calculated by Wigner (1932). Precisely,

$$w_{\rm eq}(x,v) := \left(\frac{m}{2\pi\hbar}\right)^d e^{-\beta H}$$
$$\times\left\{1+\hbar^2\left[\sum_{r=1}^d\left(-\frac{\beta^2}{8m}\frac{\partial^2 V}{\partial x_r^2}+\frac{\beta^3}{24m}\left(\frac{\partial V}{\partial x_r}\right)^2\right)+\frac{\beta^3}{24}\sum_{r,s=1}^d v_r v_s\frac{\partial^2 V}{\partial x_r\partial x_s}\right]+O(\hbar^4)\right\}$$

where $H(x,v) := mv^2/2 + V(x)$ is the Hamiltonian of the system, $\beta\equiv 1/kT$,

with T the (constant) temperature and k the Boltzmann constant. We recall that this follows from an expansion in terms of $\hbar$ of the unconstrained minimizer of the relative (von Neumann) entropy.
Like in Gardner 1994, since the particle density is given by $n(x,t) \equiv n[w](x,t) := \int w(x,v,t)\,dv$, we parametrize the equilibrium function via

$$\int_{\mathbb{R}^d} w_{\mathrm{eq}}(x,v)\,dv = n(x,t)\,.$$

Then we can write the Wigner thermal equilibrium function as

$$w_{\mathrm{eq}}(x,v) = n(x,t)F(v)\left\{1+\hbar^2\left[-\frac{\beta^2}{24m}\sum_{r=1}^{d}\frac{\partial^2 V}{\partial x_r^2}+\frac{\beta^3}{24}\sum_{r,s=1}^{d} v_r v_s \frac{\partial^2 V}{\partial x_r x_s}\right]+O(\hbar^4)\right\},$$

where $F(v)$ is the classical Maxwellian $\left(\frac{\beta m}{2\pi}\right)^{d/2} e^{-\beta m v^2/2}$.

Such function is corrected by a term of order $\hbar^2$.

Formulation of the problem

In the sequel we are interested to study the case in which a "strong" potential is included, accordingly, we introduce the potential characteristic time t_V (i.e. the time needed by a particle of mass m to cover the distance x_0 under the effect of the potential). Moreover we introduce the mean free time t_C between interactions of the system with the background.

The rescaled adimensionalized equation looks as follows

$$\epsilon\frac{\partial}{\partial t}w + \epsilon v\cdot\nabla_x w - \Theta[V]w \;=\; -\nu\left(w - w_{\rm eq}\right), \quad t>0, \quad (x,v)\in \mathbb{R}^{2d}. \qquad (1)$$

where $\epsilon := l/x_0$ with $l := v_0 t_C$ is a characteristic length corresponding to the classical mean free path

We have assumed that the times t_V and t_C are comparable, and $\dfrac{t_V}{t_0} \approx \dfrac{t_C}{t_0} \approx \epsilon\,.$

We are now in position to put (1) in abstract form: let X_k be the space $L^2(\mathbb{R}^{2d}, (1+|v|^{2k})dx\,dv; \mathbb{R})$ with the usual norm

$$\|u\|^2_{X_k} = \int_{\mathbb{R}^{2d}} |u(x,v)|^2 (1+|v|^{2k})\, dxdv\,,$$

and X^v_k be the Hilbert space $L^2(\mathbb{R}^d, (1+|v|^{2k})dv; \mathbb{R})$. We have the following system

$$\begin{cases} \epsilon \dfrac{dw}{dt} = \epsilon\, \mathcal{S}w + \mathcal{A}w + \mathcal{C}w, \\ \lim_{t\to 0^+} \|w(t) - w_0\|_{X_k} = 0 \end{cases} \tag{2}$$

- streaming operator $\qquad \mathcal{S}u := -v\cdot \nabla_x\,,$
- field operator $\qquad \mathcal{A}w := \Theta[V]w,$
- collision operator $\qquad \mathcal{C}w := -(\nu\, w - \Omega\, w)\,,$

where the collision operator Ω is defined by

$$\Omega\, w(x,v,t) := \nu F(v) \left\{ 1 + \hbar^2 \left[-\frac{\beta^2}{24m} \sum_{r=1}^{d} \frac{\partial^2 V}{\partial x_r^2} + \frac{\beta^3}{24} \sum_{r,s=1}^{d} v_r v_s \frac{\partial^2 V}{\partial x_r x_s} \right] \right\} \times \int w(x,v,t)\, dv\,,$$

We shall call $F^{(2)}$ the previous $O(\hbar^2)$-coefficient and

$$\Omega w(x,v) = \nu n[w](x) \left[F(v) + \hbar^2 F^{(2)}(x,v) \right]\,.$$

Remark: A formal integration in the v-variable of the evolution equation gives the continuity equation.

For what the operator $\mathcal{A}+\mathcal{C}$ is concerned, we can state the following preliminary result.

Lemma 1.

If $V \in W^{k,\infty}$ with $2k > d$, then the operator $\mathcal{A}+\mathcal{C}$ is well-defined from X_k into itself and is bounded by

$$\|\mathcal{A}+\mathcal{C}\|_{\mathcal{B}(X_k)} \leq C(d,k)\left[\|V\|_{W^{k,\infty}}(1+\|F\|_{X^v_{k+2}})+\|F\|_{X^v_k}+1\right].$$

Moreover, $\mathcal{A}+\mathcal{C}$ is well-defined (bounded) from X^v_k into itself.

The existence and uniqueness problems for the initial value system (2) for any $\epsilon > 0$ can be investigated by using analogous arguments in Manzini-Barletti(2004), related to semigroup theory in the L^2-setting.

We shall apply the compressed Chapman-Enskog procedure, as proposed by J. Mika (1981) (see also the monograph by Mika & Banasiak (1995)).

Accordingly, it is necessary to study the problem with $\epsilon = 0$, i.e. the equation $(\mathcal{A}+\mathcal{C})f = 0,$ in the space X_k. (in the space X_k^v for any fixed $x \in \mathbb{R}^d$).

Proposition 1.

Under the same assumptions of the previous Lemma, for any fixed x

$$\ker(\mathcal{A}+\mathcal{C}) := \{cM(v), c \in \mathbb{R}\} \subset X_k^v$$

with

$$M(x,v) := \nu\mathcal{F}^{-1}\left\{\frac{\mathcal{F}F(\eta)}{\nu - i\delta V(x,\eta)}\left(1 - \frac{\beta\hbar^2}{24m^2}\sum_{r,s=1}^{d}\eta_r\eta_s\frac{\partial^2 V(x)}{\partial x_s x_s}\right)\right\}(x,v),$$

Moreover, for all $h \in X_k^v$, $(\mathcal{A}+\mathcal{C})u = h$ has a solution if and only if

$$\int_{\mathbb{R}^d} h(v)\,dv = 0\,.$$

Formal expansion

According to Proposition 1., we can decompose the space X_k as follows

$$X_k = (X_k)_M \oplus (X_k)^0$$

where $(X_k)_M$ is the eigenspace spanned by $M := \{\alpha(x)M(v), \alpha \in X_k^x\} \subset X_k$ and

$$(X_k)^0 = \left\{ f \in X_k \,\middle|\, \int f(v)dv = 0 \right\}.$$

We define the corresponding spectral projection $\mathcal{P}$ from (X_k) into $(X_k)_M$

$$\mathcal{P}f = M \int f(v)dv$$

and

$$\mathcal{Q} = \mathcal{I} - \mathcal{P}.$$

We decompose the function $w \in X_k$ as $w = \mathcal{P}w + \mathcal{Q}w$.

From now on we call $\mathcal{P}w$ and $\mathcal{Q}w$ as φ and ψ. We remark that for all $w \in X_k, \mathcal{P}w = Mn[w]$.

φ is called the hydrodynamic part

ψ is called the kinetic part of w.

Operating formally on both sides of the evolution equation (2) for the function w with the projections $\mathcal{P}$ and $\mathcal{Q}$, we obtain the following system of equations

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} &= \mathcal{P}S\mathcal{P}\varphi + \mathcal{P}S\mathcal{Q}\psi \\ \dfrac{\partial \psi}{\partial t} &= \mathcal{Q}S\mathcal{P}\varphi + \mathcal{Q}S\mathcal{Q}\psi + \dfrac{1}{\epsilon}\mathcal{Q}(\mathcal{A}+\mathcal{C})\mathcal{Q}\psi \end{cases} \tag{3}$$

with initial conditions

$$\begin{cases} \varphi(0) &= \varphi_0 = \mathcal{P}f_0 \\ \psi(0) &= \psi_0 = \mathcal{Q}f_0. \end{cases}$$

Applying the compressed asymptotic expansion, we split the solutions φ and ψ into the sums of the "bulk" parts $\bar{\varphi}$ and $\bar{\psi}$ and of the initial layer parts $\tilde{\varphi}$ and $\tilde{\psi}$, which take account of the rapid changes of f for small times

$$\begin{aligned} \varphi(t) &= \bar{\varphi}(t) + \tilde{\varphi}\left(\frac{t}{\epsilon}\right) \\ \psi(t) &= \bar{\psi}(t) + \tilde{\psi}\left(\frac{t}{\epsilon}\right). \end{aligned}$$

The bulk hydrodynamic part $\bar{\varphi}$ is left unexpanded and the other parts are expanded as

$$\begin{aligned} \tilde{\varphi}(\tau) &= \tilde{\varphi}_0(\tau) + \epsilon\tilde{\varphi}_1(\tau) + \epsilon^2\tilde{\varphi}_2(\tau) + \dots \\ \bar{\psi}(t) &= \bar{\psi}_0(t) + \epsilon\bar{\psi}_1(t) + \epsilon^2\bar{\psi}_2(t) + \dots \\ \tilde{\psi}(\tau) &= \tilde{\psi}_0(\tau) + \epsilon\tilde{\psi}_1(\tau) + \epsilon^2\tilde{\psi}_2(\tau) + \dots, \end{aligned}$$

where $\tau = \dfrac{t}{\epsilon}$.

Accordingly, Eqs. (3) for the bulk part terms of the expansion up to the order ϵ^2 become:

$$\left\{\begin{array}{rcl} \dfrac{\partial\bar{\varphi}}{\partial t} & = & \mathcal{P}S\mathcal{P}\bar{\varphi} + \mathcal{P}S\mathcal{Q}\bar{\psi}_0 + \epsilon\mathcal{P}S\mathcal{Q}\bar{\psi}_1 \\ 0 & = & \mathcal{Q}(\mathcal{A}+\mathcal{C})\mathcal{Q}\bar{\psi}_0 \\ 0 & = & \mathcal{Q}S\mathcal{P}\bar{\varphi} + \mathcal{Q}(\mathcal{A}+\mathcal{C})\mathcal{Q}\bar{\psi}_1 \end{array}\right.$$

thus

$$\begin{array}{rcl} \dfrac{\partial\bar{\varphi}}{\partial t} & = & \mathcal{P}S\mathcal{P}\bar{\varphi} - \epsilon\mathcal{P}S\mathcal{Q}(\mathcal{Q}(\mathcal{A}+\mathcal{C})\mathcal{Q})^{-1}\mathcal{Q}S\mathcal{P}\bar{\varphi} \\ \bar{\psi}_0 & \equiv & 0 \\ \bar{\psi}_1 & = & -(\mathcal{Q}(\mathcal{A}+\mathcal{C})\mathcal{Q})^{-1}\mathcal{Q}S\mathcal{P}\bar{\varphi}\,, \end{array} \tag{4}$$

which have to be supplemented by appropriate initial conditions to be determined from the analysis of the initial layer terms and the balance of the initial conditions.

The *unexpanded* function $\bar{\varphi}(x,v,t)$ can be written as the product

$$\bar{\varphi}(x,v,t) = n(x,t)\ M(x,v),$$

since we shall consider the contribution of the initial layer part $\tilde{\varphi}$ via an appropriate initial condition for Eq. (4).

Auxiliary problems. In order to invert the operator $\mathcal{Q}(\mathcal{A}+\mathcal{C})\mathcal{Q}$, we solve in $(X_k)^0$

$$(\mathcal{A}+\mathcal{C})u = \left[-v\cdot\nabla_x M + M\int v\cdot\nabla_x M\,dv\right], \tag{5}$$

and

$$(\mathcal{A}+\mathcal{C})u = \left[M\left(-v+\int vM\,dv\right)\right]. \tag{6}$$

Let us call $D_1(x,v)$ and $D_2(x,v)\equiv(D_2)_i(x,v)$ the respective solutions with $D_1,(D_2)_i\in(X_k)^0$.

Hence the equation (4) takes the form

$$\frac{\partial n}{\partial t} = -\nabla_x \left(n \int v M dv \right) + \epsilon \nabla_x \cdot \left[\left(\int v \otimes D_2 \, dv \right) \cdot \nabla_x n + n \int v D_1 \, dv \right] .$$

Before going further, we derive formally the initial layer conditions. Such initial value can be found by the analysis of the initial layer terms:

$$\tilde{\varphi}_0(\tau), \qquad \tilde{\psi}_0(\tau), \qquad \tilde{\varphi}_1(\tau), \qquad \tilde{\psi}_1(\tau) .$$

We obtain the correct initial value for the bulk hydrodynamic functions

$$\bar{\varphi}(0) = \varphi_0 - \tilde{\varphi}_0(0) - \epsilon \tilde{\varphi}_1(0) = \varphi_0 - \epsilon \mathcal{P} S \mathcal{Q} (\mathcal{Q}(\mathcal{A} + \mathcal{C}) \mathcal{Q})^{-1} \psi_0 .$$

Putting $n_0(x) = \int w_0(x, v) \, dv$

$$n(x, 0) = n_0(x) + \epsilon \int v \cdot \nabla_x D_3(x, v) dv$$

where $D_3 = D_3(x, v)$ is the unique solution of $(\mathcal{A} + \mathcal{C}) D_3(x, v) = \psi_0(x, v)$.

The bulk part

We shall prove that all terms of the bulk part of the expansion are well-defined. Such properties of the expansion terms are relevant into the rigorous asymptotic analysis.

If we neglect the initial layer part (which will be appropriately taken into account via the initial condition), we have

$$J = J^{(0)} + \epsilon J^{(1)} := n \int vM \, dv + \epsilon \int v\bar{\psi}_1 \, dv$$

where

$$\begin{aligned} J^{(0)} &= -\frac{n}{\nu m} \nabla V(x) \quad - \text{Drift term} \\ J^{(1)} &= \int v\bar{\psi}_1 \, dv = -\left[\left(\int v \otimes D_2 \, dv \right) \cdot \nabla_x n + n \int v D_1 \, dv \right] . \end{aligned}$$

Let us call D the matrix given by

$$\mathsf{D}_{ij}(x) := \int v_i (D_2)_j(x,v) dv,$$

Lemma. The first term in $J^{(1)}$ is $\mathsf{D} \cdot \nabla_x \mathsf{n}$ where the tensor D is given by

$$\mathsf{D} = \frac{1}{\nu}\left(\frac{\mathcal{I}}{\beta m} + \frac{1}{\nu^2 m^2}\nabla V \otimes \nabla V + \frac{\beta\hbar^2}{12m^2}(\nabla \otimes \nabla) V\right) + O(\hbar^4)\,.$$

Lemma. The second term in $J^{(1)}$ is given by n times $\int v D_1(x,v)\, dv =$

$$= \frac{1}{\nu}\left(\frac{2}{\nu^2 m^2}(\nabla \otimes \nabla) V \nabla V + \frac{1}{\nu^2 m^2}\Delta V \nabla V + \frac{\beta\hbar^2}{12m^2}\nabla_x \cdot (\nabla \otimes \nabla) V\right) + O(\hbar^4)\,.$$

Accordingly, **the drift-diffusion equation** looks like

$$\begin{aligned}
\frac{\partial n}{\partial t} &= \frac{1}{\nu m}\nabla\cdot(n\nabla V) + \frac{\epsilon}{\nu\beta m}\nabla\cdot\nabla n + \frac{\epsilon}{\nu^3 m^2}\nabla\cdot(\nabla V\otimes\nabla V\nabla n) \\
&+ \frac{\epsilon}{\nu^3 m^2}\nabla\cdot\left[2n\left(\nabla\otimes\nabla\right)V\nabla V + n\Delta V\nabla V\right] \\
&+ \frac{\epsilon\beta\hbar^2}{12\nu m^2}\nabla\cdot\left[\left(\nabla\otimes\nabla\right)V\nabla n + n\nabla\cdot\left(\nabla\otimes\nabla\right)V\right]
\end{aligned}$$

In the same spirit of the approximation

$$\nabla \log n \;=\; -\beta \nabla V + O(\hbar^2)\,,$$

we have the fourth order nonlinear drift-diffusion equation for the electron density n

$$\frac{\partial n}{\partial t} = \frac{1}{\nu m}\nabla\cdot(n\nabla V) + \frac{\epsilon}{\nu\beta m}\nabla\cdot\nabla n + \frac{\epsilon}{\nu^3\beta^2 m^2}\nabla\cdot\left(n\nabla\cdot\frac{(\nabla n\otimes\nabla n)}{n^2} + \frac{(\nabla\otimes\nabla n)\nabla n}{n}\right) - \frac{\epsilon\hbar^2}{12\nu m^2}\left(\nabla^4 n - \nabla\otimes\nabla\frac{(\nabla n\otimes\nabla n)}{n}\right)\,.$$

Rigorous results: bulk and initial layer parts

Bulk and the initial layer solutions, derived formally in Section 3, contain functions whose existence and regularity have to be proved in order to make rigorous the asymptotic analysis.

- The (strongly continuous) semigroup G generated in $(X_k)^0$ by $\mathcal{Q}(\mathcal{A}+\mathcal{C})\mathcal{Q}$ is

$$u(x,v,\tau) = G(\tau)u(x,v,0) \quad = \quad e^{-\tau}\mathcal{F}^{-1}\left(e^{i\,\delta V(x,\eta)\tau}\mathcal{F}u(x,\eta,0)\right) .$$

- Estimates of $\|G(\tau)h\|_{X_k}$ and $\|G(\tau)h\|_{D(S)}$
- All terms of the initial layer expansion are well-defined
- Regularity of the solution of the drift-diffusion equation
- Strong differentiability of the bulk function $\bar{\psi}_1$

Estimate of the error

We prove that the asymptotic expansion up to the first order gives an approximation of order ϵ^2 to the Wigner quantum system (2).

Consider hydrodynamic and kinetic parts, whose errors are given by

$$\begin{aligned} y(t) &= \varphi(t) - [\bar{\varphi}(t) + \tilde{\varphi}_0(\tau) + \epsilon\tilde{\varphi}_1(\tau)] \\ z(t) &= \psi(t) - [\bar{\psi}_0(t) + \epsilon\bar{\psi}_1(t) + \tilde{\psi}_0(\tau) + \epsilon\tilde{\psi}_1(\tau)] \end{aligned}$$

where $\tau = \frac{t}{\epsilon}$.

Taking account of the evolution equations, we can show that the errors y and z satisfy the system

$$\begin{cases} \dfrac{\partial y}{\partial t} &= \mathcal{P}S\mathcal{P}y + \mathcal{P}S\mathcal{Q}z + f \\ \dfrac{\partial z}{\partial t} &= \mathcal{Q}S\mathcal{P}y + \mathcal{Q}S\mathcal{Q}z + \dfrac{1}{\epsilon}\mathcal{Q}(\mathcal{A}+\mathcal{C})\mathcal{Q}z + g \end{cases} \tag{7}$$

with initial condition

$$y(0) = 0\,, \qquad z(0) = 0\,.$$

The inhomogeneous terms f and g are given by

$$\begin{aligned} f(t) &= \epsilon\left[\mathcal{P}S\mathcal{P}\tilde{\varphi}_1(\tau) + \mathcal{P}S\mathcal{Q}\tilde{\psi}_1(\tau)\right] \\ g(t) &= \epsilon\left[-\frac{\partial\bar{\psi}_1}{\partial t} + \mathcal{Q}S\mathcal{Q}\bar{\psi}_1(t) + \mathcal{Q}S\mathcal{P}\tilde{\varphi}_1(\tau) + \mathcal{Q}S\mathcal{Q}\tilde{\psi}_1(\tau)\right]. \end{aligned}$$

It is necessary to separate the evolution of the initial layer part from the bulk part.

Main Theorem

If the initial value w_0 and the potential V are sufficiently smooth, then for any T, $0 < T < \infty$, there is a constant C independent of ϵ but depending on the problem data such that

$$\|y(t) + z(t)\|_X \leq C\epsilon^2$$

uniformly for $0 \leq t \leq T$.

References

[1] A.M. Anile, G. Mascali and V. Romano, Recent development in hydrodynamical modeling of semiconductors, in Mathematical problems in semiconductor physics, edited by Anile A.M., Lecture Notes in Mathematics (Springer) 2003, pp. 1-54

[2] J. Banasiak, Diffusion approximation for the linear Boltzmann equation of semiconductor theory with analysis of the initial layer. *J. Math. Anal. Appl.* **205**, 216-238 (1997).

[3] J. Banasiak, G. Frosali, Modified Chapman-Enskog expansion for an electron transport equation with a constant electric field. Dipartimento di Matematica "V.Volterra", Università di Ancona, Rapporto n.6, 1994.

[4] J. Banasiak, J. Mika, Diffusion limit for the linear Boltzmann equation of the neutron transport theory. *Math. Methods Appl. Sci.* **17**, 1071-1087 (1994)

[5] L. Barletti, L. Demeio, G. Frosali, Multiband Quantum Transport Models for Semiconductor Devices *Proceedings "Kinetic Equations: Direct and Inverse Problems," Mantova, May 15-17, 2005* (to appear)

[6] A. Belleni-Morante, *A concise guide to semigroups and evolution equations.* Singapore: World Scientific. (1994)

[7] C. Manzini, The three dimensional Wigner-Poisson problem with inflow boundary conditions, *J. Math. Anal. Appl.* **313/1**, 184-196 (2006)

[8] G. Frosali, Asymptotic analysis for a particle transport problem in a moving medium, *IMA Journal of Applied Mathematics* **60**, 1-19 (1998)

[9] C. Manzini, L. Barletti, An analysis of the Wigner-Poisson problem with time-dependent, inflow boundary conditions, *Nonlin. Anal.*, **60/1**, 77–100 (2004).

[10] J.R. Mika, New asymptotic expansion algorithm for singularly perturbed evolution equations. *Math. Methods Appl. Sci.* **3**, 172-188 (1981).

[11] J.R. Mika, J. Banasiak, *Singularly perturbed evolution equations with applications to kinetic theory*. Singapore: World Scientific 1995.

[12] J.R. Mika, J. Banasiak, Diffusion limit for a linear kinetic equation, *Transport Theory Stat. Phys.* **24**(1-3), 41-53 (1995).

[13] F. Poupaud, Runaway phenomena and fluid approximation under high fields in semiconductor kinetic theory. *Z. Angew. Math. Mech.* **72**, 359-372 (1992).

[14] G. Alí and G. Frosali, On the quantum hydrodynamic models for the two-band Kane system, *Nuovo Cimento B* **120(12)**, 1279–1298 (2005)

[15] C. Y. Chao and S.L. Chuang, Resonant tunneling of holes in the multiband effective-mass approximation *Phys. Rev. B* **43**, 7027-7039 (1991).

[16] C. Gardner, The Quantum Hydrodynamic Model for Semiconductor Devices *SIAM J.App.Math.* **54(2)**, 409-427 (1994).

[17] I. Gasser and P. Markowich, Quantum hydrodynamics, Wigner transforms and the classical limit, *Asymptotic Analysis* **14**, 97-116 (1997).

[18] A. Jüngel, *Quasi-hydrodynamic Semiconductor Equations*, Birkhäuser, Basel, 2001.

[19] E. O. Kane, Energy band structure in p-type Germanium and Silicon, *J. Phys. Chem. Solids* **1**, 82-89 (1956).

[20] A. Jüngel, D. Matthes, A derivation of the isothermal quantum hydrodynamic equations using entropy minimization, *Z. Angew. Math. Mech.* **85**, 806-814 (2005).

[21] E. Wigner, On the quantum correction for thermodynamic equilibrium, *Phys. Rev.* **40**, 749-759 (1932).