An Inverse Problem for Two-Frequency Photon Transport in a Slab

Luigi Barletti
Università di Firenze

Federica Dragoni
SNS Pisa

Convegno SIMAI 06 - Ragusa, 22-26 Maggio 2006

Interstellar cloud

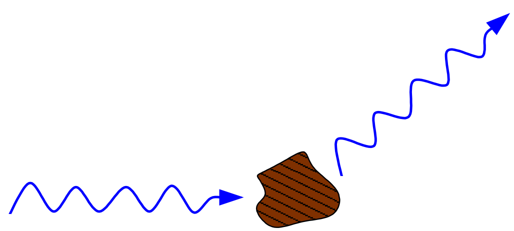
A large, gaseous, gravitational system ($\sim 10\ \mathrm{ly}$) composed by atoms, simple and complex molecules, “dust” grains.

Scattering of light

Light coming from external or internal stars interacts with dust grains and undergoes three main processes:

Scattering of light

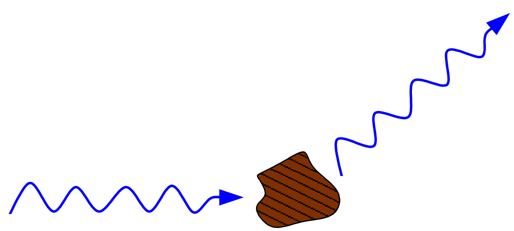
Light coming from external or internal stars interacts with dust grains and undergoes three main processes:



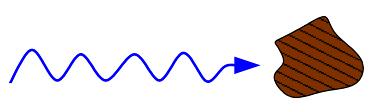
direction scattering

Scattering of light

Light coming from external or internal stars interacts with dust grains and undergoes three main processes:



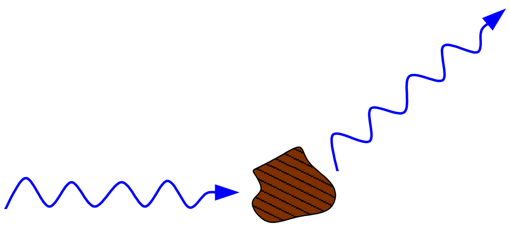
direction scattering



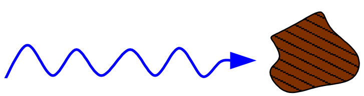
capture

Scattering of light

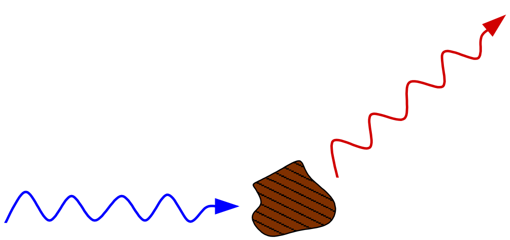
Light coming from external or internal stars interacts with dust grains and undergoes three main processes:



direction scattering

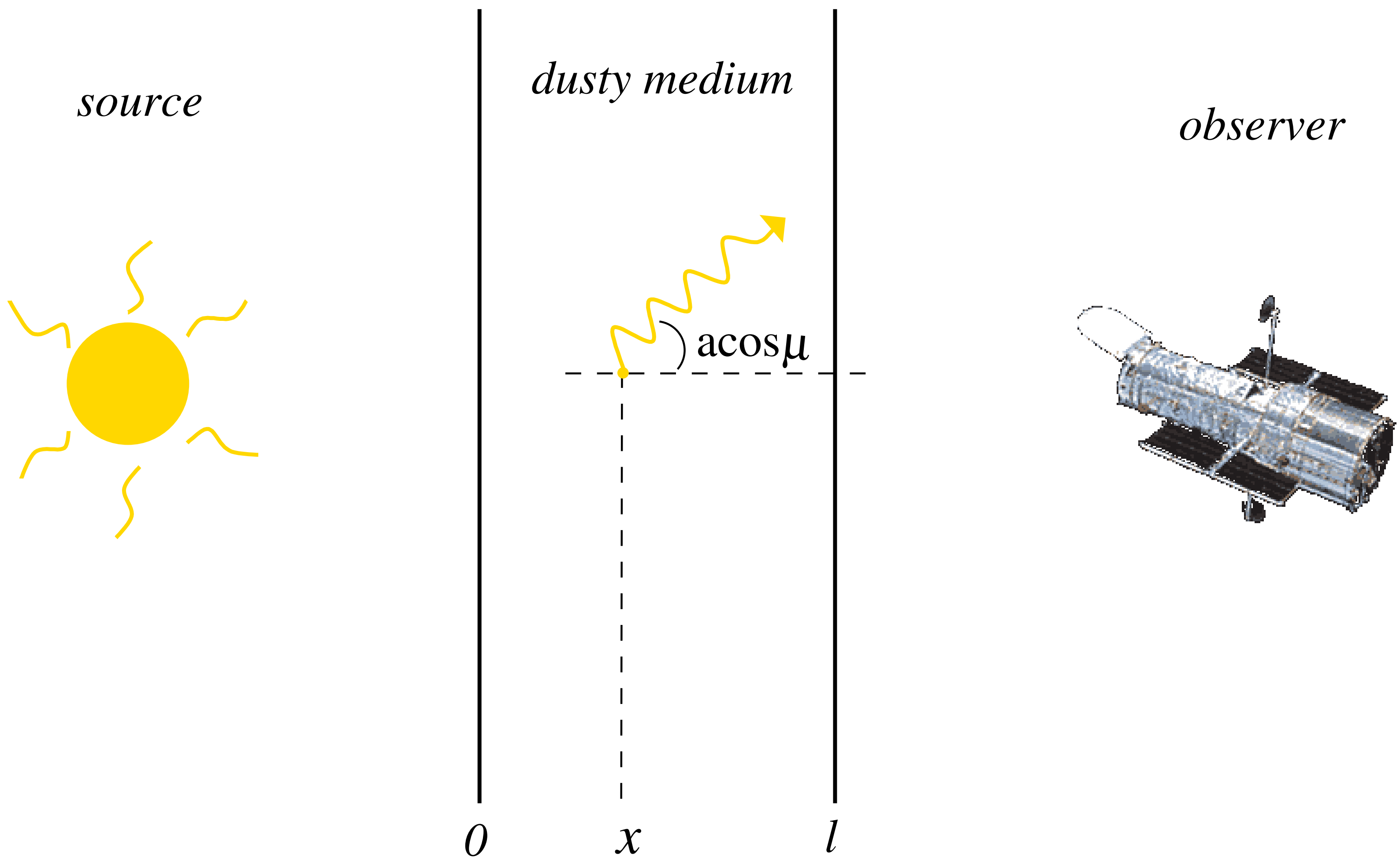


capture



direction/frequency scattering

Slab symmetry



Mathematical model

$f_1(x,\mu)$ = density at x of UV-photons with direction μ

$f_2(x,\mu)$ = density at x of IR-photons with direction μ

Mathematical model

$f_1(x,\mu)$ = density at x of UV-photons with direction μ

$f_2(x,\mu)$ = density at x of IR-photons with direction μ

$\Sigma_{i\to j}$ = macroscopic scattering cross-sections;

$p_{i\to i}(\mu' \to \mu)$ = transition probabilities;

$\Sigma_i := \Sigma_{i,c} + \Sigma_{i\to 1} + \Sigma_{i\to 2}$ = total cross-sections;

$\Sigma_{i,c}$ = capture cross-sections.

Mathematical model

Stationary radiative transfer equations:

$$\mu \frac{\partial f_1}{\partial x}(x,\mu) + \Sigma_1 \, f_1(x,\mu) = \Sigma_{1\to 1} \int_{-1}^{1} p_{1\to 1}(\mu' \to \mu) \, f_1(x,\mu') \, d\mu'$$

$$\mu \frac{\partial f_2}{\partial x}(x,\mu) + \Sigma_2 \, f_2(x,\mu) = \Sigma_{2\to 2} \int_{-1}^{1} p_{2\to 2}(\mu' \to \mu) \, f_2(x,\mu') \, d\mu'$$
$$+ \Sigma_{1\to 2} \int_{-1}^{1} p_{1\to 2}(\mu' \to \mu) \, f_1(x,\mu') \, d\mu$$

Dust densities

We assume that two kinds of dust, with unknown constant, densities ρ_1 and ρ_2 are present in the medium.

Introducing a *dust index* $d = 1, 2$ and denoting by $\sigma^d_{i\to j}$, σ^d_i, $\sigma^d_{i,c}$ the microscopic cross-sections for each kind of dust, we can write:

$$\Sigma_{i\to j} = \rho_1\, \sigma^1_{i\to j} + \rho_2\, \sigma^2_{i\to j}$$

$$\Sigma_i = \rho_1\, \sigma^1_i + \rho_2\, \sigma^2_i$$

$$\Sigma_{i,c} = \rho_1\, \sigma^1_{i,c} + \rho_2\, \sigma^2_{i,c}$$

Inflow conditions

We assume that the *inflows* from the left

$$f_1(0,\mu) = \varphi_1^+(\mu), \quad f_2(0,\mu) = \varphi_2^+(\mu), \quad \text{for } \mu \in (0,1),$$

and from the right

$$f_1(l,\mu) = \varphi_1^-(-\mu), \quad f_2(l,\mu) = \varphi_2^-(-\mu), \quad \text{for } \mu \in (-1,0),$$

are known.

Inflow conditions

We assume that the *inflows* from the left

$$f_1(0,\mu) = \varphi_1^+(\mu), \quad f_2(0,\mu) = \varphi_2^+(\mu), \quad \text{for } \mu \in (0,1),$$

and from the right

$$f_1(l,\mu) = \varphi_1^-(-\mu), \quad f_2(l,\mu) = \varphi_2^-(-\mu), \quad \text{for } \mu \in (-1,0),$$

are known.

The typical direct problem is finding the left and right *outflows*.

The inverse problem

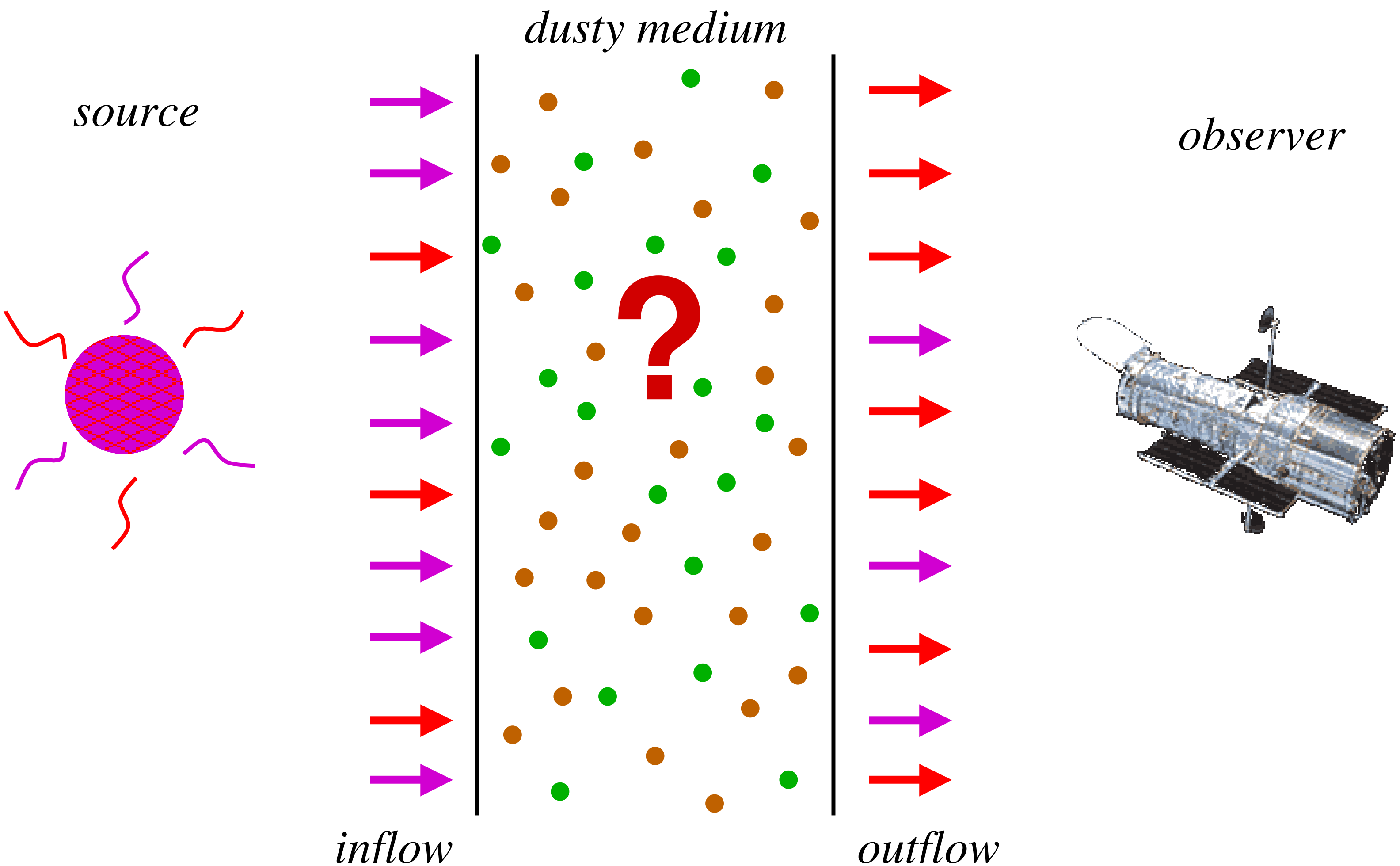
Problem: *find the unknown dust densities* ρ_1 *and* ρ_2*, assuming that the* *integrated right outflows* *at the two frequencies*

$$H_1 := \int_0^1 f_1(l,\mu)\mu\, d\mu,$$

$$H_2 := \int_0^1 f_2(l,\mu)\mu\, d\mu,$$

are known.

The inverse problem



The main result

We make the following assumptions:

A1. *the frequency-scattering vanishes, i.e.* $\Sigma_{1\to 2} \equiv 0$*;*

The main result

We make the following assumptions:

A1. *the frequency-scattering vanishes, i.e.* $\Sigma_{1\to 2} \equiv 0$*;*

A2. *the left-inflow data* φ_1^+ *and* φ_2^+ *are positive on nonzero-measure sets;*

The main result

We make the following assumptions:

A1. *the frequency-scattering vanishes, i.e.* $\Sigma_{1\to 2} \equiv 0$*;*

A2. *the left-inflow data* φ_1^+ *and* φ_2^+ *are positive on nonzero-measure sets;*

A3. $\sigma_{i,c}^j > 0$ *for* $i, j \in \{1, 2\}$ *and*

$$\begin{vmatrix} \sigma_{1,c}^1 & \sigma_{1,c}^2 \\ \sigma_{2,c}^1 & \sigma_{2,c}^2 \end{vmatrix} \neq 0$$

The main result

THEOREM *Under assumptions* **A1**, **A2**, **A3**, *the mapping densities-to-outflows,*

$$(\rho_1, \rho_2) \mapsto (H_1, H_2),$$

is globally invertible and, therefore, the inverse problem is well-posed.

Sketch of the proof

By separating leftward and rightward photons

$$f_i^+(x,\mu) := f_i(x,\mu), \quad f_i^-(x,\mu) := f_i(l-x,-\mu)$$

(for $\mu \in (0,1)$ and $i = 1,2$), the direct problem can be recast into an “evolution” equation (from the inflow to the outflow):

$$\begin{cases} \dfrac{d}{dx}\boldsymbol{f}(x) = A\boldsymbol{f}(x), & x \in [0,l], \\ \boldsymbol{f}(0) = \boldsymbol{\varphi}. \end{cases}$$

Sketch of the proof

where $\boldsymbol{f} := (f_1^+, f_1^-, f_2^+, f_2^-)$, $\boldsymbol{\varphi} := (\varphi_1^+, \varphi_1^-, \varphi_2^+, \varphi_2^-)$
and

$$A = \frac{1}{\mu}\begin{pmatrix} -\Sigma_1 + K_1^{++} & K_1^{-+} & 0 & 0 \\ K_1^{+-} & -\Sigma_1 + K_1^{--} & 0 & 0 \\ 0 & 0 & -\Sigma_2 + K_2^{++} & K_2^{-+} \\ 0 & 0 & K_2^{+-} & -\Sigma_2 + K_2^{--} \end{pmatrix}$$

($K_i^{s_1 s_2}$ are suitable scattering operators).

Sketch of the proof

By using the Lumer-Phillips generation problem we can prove that A generates a contraction semigroup (on the Banch space $L^1([0,1],\mu\,d\mu)^4$), so that the inflow-to-outflow mapping is explicitly given by:

$$\boldsymbol{f}(l) = \mathrm{e}^{lA}\boldsymbol{\varphi}.$$

Sketch of the proof

By using the Lumer-Phillips generation problem we can prove that A generates a contraction semigroup (on the Banch space $L^1([0,1],\mu\, d\mu)^4$), so that the inflow-to-outflow mapping is explicitly given by:

$$\boldsymbol{f}(l) = \mathrm{e}^{lA}\boldsymbol{\varphi}.$$

Now, A is linear with respect to the dust densities:

$$A = \rho_1\, A_1 + \rho_2\, A_2$$

and assumption **A1** implies commutativity:

$$A_1\, A_2 = A_2\, A_1.$$

Sketch of the proof

From commutativity we obtain $\mathrm{e}^{lA} = \mathrm{e}^{l\rho_1 A_1}\,\mathrm{e}^{l\rho_2 A_2}$, which gives

$$\frac{\partial \boldsymbol{f}(l)}{\partial \rho_j} = l\,A_j\,\mathrm{e}^{lA}\,\boldsymbol{\varphi}\,.$$

This allows to prove that the integrated outflows

$$H_i = \int_0^1 f_i^+(l,\mu)\mu\,d\mu = \int_0^1 \left[\mathrm{e}^{lA}\boldsymbol{\varphi}\right]_i^+(\mu)\,\mu\,d\mu$$

are continuously-differentiable with respect to ρ_j.

Sketch of the proof

Moreover, from assumptions **A2** and **A3** we obtain, for all $\rho_1, \rho_2 \geq 0$,

Sketch of the proof

Moreover, from assumptions **A2** and **A3** we obtain, for all $\rho_1, \rho_2 \geq 0$,

$$\begin{vmatrix} \frac{\partial H_1}{\partial \rho_1} & \frac{\partial H_1}{\partial \rho_2} \\ \frac{\partial H_2}{\partial \rho_1} & \frac{\partial H_2}{\partial \rho_2} \end{vmatrix} \neq 0$$

Sketch of the proof

Moreover, from assumptions **A2** and **A3** we obtain, for all $\rho_1, \rho_2 \geq 0$,

$$\begin{vmatrix} \frac{\partial H_1}{\partial \rho_1} & \frac{\partial H_1}{\partial \rho_2} \\ \frac{\partial H_2}{\partial \rho_1} & \frac{\partial H_2}{\partial \rho_2} \end{vmatrix} \neq 0$$

and

$$\frac{\partial H_i}{\partial \rho_j} < 0, \quad i, j \in \{1, 2\}.$$

Sketch of the proof

Moreover, from assumptions **A2** and **A3** we obtain, for all $\rho_1, \rho_2 \geq 0$,

$$\begin{vmatrix} \frac{\partial H_1}{\partial \rho_1} & \frac{\partial H_1}{\partial \rho_2} \\ \frac{\partial H_2}{\partial \rho_1} & \frac{\partial H_2}{\partial \rho_2} \end{vmatrix} \neq 0$$

and

$$\frac{\partial H_i}{\partial \rho_j} < 0, \quad i, j \in \{1, 2\}.$$

These conditions, together, imply the global invertibility of $(\rho_1, \rho_2) \mapsto (H_1, H_2)$. $\square$

An inversion algorithm

A simple bisection-like algorithm can be used to solve the inverse problem:

1. by solving the direct problem, the range of the density-to-outflow mapping is divided into cells;

An inversion algorithm

A simple bisection-like algorithm can be used to solve the inverse problem:

1. by solving the direct problem, the range of the density-to-outflow mapping is divided into cells;
2. the cell containing the measured value (H_1^0, H_2^0) is detected;

An inversion algorithm

A simple bisection-like algorithm can be used to solve the inverse problem:

1. by solving the direct problem, the range of the density-to-outflow mapping is divided into cells;

2. the cell containing the measured value (H_1^0, H_2^0) is detected;

3. a refined grid is produced within that cell.

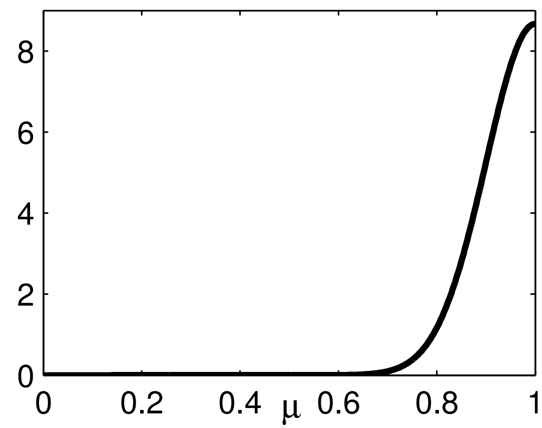
An inversion algorithm

A simple bisection-like algorithm can be used to solve the inverse problem:

1. by solving the direct problem, the range of the density-to-outflow mapping is divided into cells;
2. the cell containing the measured value (H_1^0, H_2^0) is detected;
3. a refined grid is produced within that cell.
4. steps **2** and **3** are repeated up to desired order of accuracy.

Numerical experiments

Inflow datum:



Dust data:

λ (μm)	Dust kind	radius (μm)	σ_c ($\mu\mathrm{m}^2$)	σ_s ($\mu\mathrm{m}^2$)
0.1	graphite	0.25	0.18	0.29
10.0	graphite	0.25	0.93 E-2	0.25 E-3
0.1	silicate	1.00	2.67	4.03
10.0	silicate	1.00	4.22	0.50

Numerical experiments

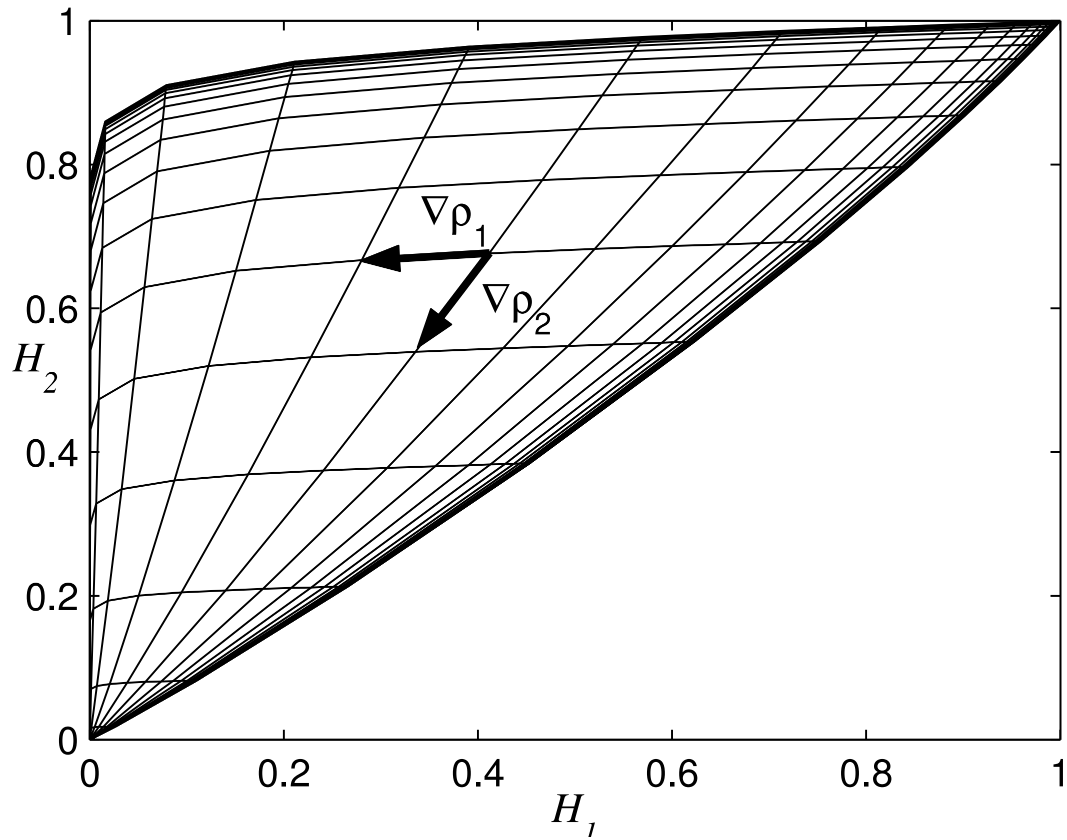
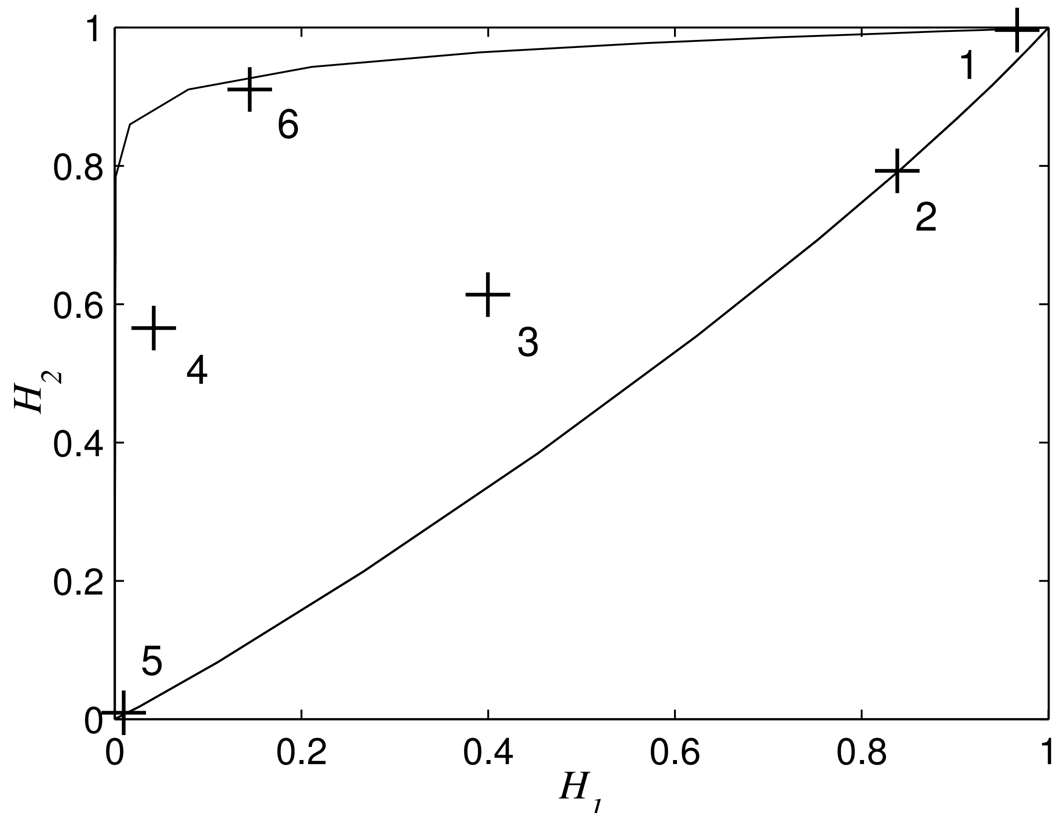


Image of the square $10^{-7}\,\mathrm{m}^{-3} \leq \rho_1, \rho_2 \leq 10^{-1}\,\mathrm{m}^{-3}$ under the mapping $(\rho_1, \rho_1) \mapsto (H_1, H_2)$. The grid is logarithmically spaced.

Numerical experiments

Test points:



Numerical experiments

	H_1^0	H_2^0	ρ_1^0	ρ_2^0	Δ_1	Δ_2	T
1	9.664 E-1	9.962 E-1	1.50 E-4	5.00 E-7	1.5 E-4	1.3 E-4	9.4 E+0
2	8.379 E-1	7.927 E-1	1.00 E-5	5.00 E-5	4.0 E-4	1.9 E-4	2.8 E+1
3	3.996 E-1	6.136 E-1	2.00 E-3	1.00 E-4	4.3 E-4	2.8 E-4	4.4 E+1
4	4.195 E-2	5.652 E-1	1.00 E-2	1.00 E-4	2.0 E-5	7.6 E-4	4.3 E+2
5	9.857 E-3	9.197 E-3	1.25 E-3	1.00 E-3	1.6 E-4	1.9 E-16	4.8 E+2
6	1.446 E-1	9.102 E-1	7.00 E-3	5.00 E-6	1.0 E-4	9.1 E-5	4.8 E+2

(H_1^0, H_2^0) = measured outflows;

(ρ_1^0, ρ_2^0) = computed dust densities;

(Δ_1, Δ_2) = relative difference with the true values of (ρ_1, ρ_2);

T = CPU time.

Conclusions

- we tried to recover the unknown densities of two kinds of absorbing and scattering dust particles from the measurement of outflowing radiation at two different frequencies;

Conclusions

- we tried to recover the unknown densities of two kinds of absorbing and scattering dust particles from the measurement of outflowing radiation at two different frequencies;
- the problem is solvable if the two kinds of dust have different absorption properties on the two measured frequencies, and in absence of frequency-scattering;

Conclusions

- we tried to recover the unknown densities of two kinds of absorbing and scattering dust particles from the measurement of outflowing radiation at two different frequencies;
- the problem is solvable if the two kinds of dust have different absorption properties on the two measured frequencies, and in absence of frequency-scattering;
- the latter assumption seems to be just technical and we hope to get rid of it in future works;

Conclusions

- we tried to recover the unknown densities of two kinds of absorbing and scattering dust particles from the measurement of outflowing radiation at two different frequencies;
- the problem is solvable if the two kinds of dust have different absorption properties on the two measured frequencies, and in absence of frequency-scattering;
- the latter assumption seems to be just technical and we hope to get rid of it in future works;
- a simple bisection algorithm gives good results for realistic values of the physical constants.