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Positive spaces, generalized semi-densities, and quantum interactions

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The basics of quantum particle physics on a curved Lorentzian background are expressed in a formulation which has original aspects and exploits some non-standard mathematical notions. In particular, positive spaces and generalized semi-densities (in a distributional sense) are shown to link, in a natural way, discrete multi-particle spaces to distributional bundles of quantum states. The treatment of spinor and boson fields is partly original also from an algebraic point of view and suggests a non-standard approach to quantum interactions. The case of electroweak interactions provides examples. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3695348]

I. INTRODUCTION

Today, nearly everybody will agree that theoretical physics seems to need some deep rethinking of the foundations. The lack of cogent, unambiguous experimental evidence has opened the way to a plethora of theories and hypotheses, among which one is not yet able to make definite choices.

This paper stems from the somewhat "minimalistic" philosophy that meaningful insight could be gained by careful reconsideration of basic notions which are often taken for granted, and by some viewpoint shifting about them, rather than by generalizing and extending existing formalisms. We focus our attention onto the core topic of quantum particle physics. On one hand, we try and change our point of view about which concepts are to be seen as basic; we propose to replace the "quantization procedure" scheme (a mirror of the historical path) with a more direct approach founded on straightforward definitions of the two basic ingredients: free-particle states and one-point interactions. We also attempt a clarification, in precise mathematical terms, of the relation between the "discrete" and "continuous" descriptions and look for a consistent formalism in curved spacetime. Furthermore, we use some non-standard mathematical ideas, though firmly established in the literature for the most part: positive spaces, the geometry of distributional bundles of generalized semi-densities, and a partly original approach to two-spinors and gauge fields which was discussed in previous papers.[1–4] We now briefly discuss these points.

A covariant formulation on curved spacetime, with assigned background metric, has not only the purpose of examining the possible effects of gravitation on particle physics; even more importantly, it constraints our langauge to be meaningful at a higher level. On the other hand, experimentally well-grounded quantum theory is heavily dependent on the flat spacetime structure. Moreover, the standard theory also uses a distinguished time; hence, it depends on the choice of an observer (though, eventually, a kind of invariance can be recovered). In curved spacetime one has to give up Fourier transforms and the induced relation between the "position space representation" and the "momentum space representation" of a quantum state. In this paper's approach, quantum states are described as elements of distributional *quantum bundles*, defined as bundles of generalized sections (in a distributional sense) of bundles over the bundle of momenta over spacetime; hence the underlying finite-dimensional structure is that of a two-fibered bundle, so that, in a sense, we start with "quantum states" which can be seen as labeled by momentum *and* spacetime point. However,

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elaborating on ideas presented in a previous paper,[5] we show how parallel transport in quantum bundles, along a world-line, determines a full momentum-space formalism "carried" by the *detector* (i.e., a point observer) represented by the world-line. The local spacetime splitting associated with the detector also determines, via exponentiation, a sort of position space representation (see the remark at the end of Sec. IV D); in the flat case one essentially recovers the standard setting.

The debate about the discrete/continuous fundamental nature of the universe still continues today (despite Niels Bohr's belief he had settled the question once for all). It may be also worthwhile to note that the way quantum field theory (QFT) is usually introduced, following the historical path, draws on analogy with finite-dimensional linear algebra, despite the fact that some entities turn out to be ill defined. Moreover, while one usually speaks of a "Hilbert space of states," free-particle states are actually "generalized states," i.e., elements of a larger distributional space. These considerations suggested an approach, offered as a provisional step towards further clarification of these matters, in which the "abstract" space of free-particle states, each one constituted of a finite but arbitrary large number of particles, can be identified with a space of finite linear combinations of delta-type semi-densities; the identification needs the spacetime volume form. This space is dense in a larger "rigged Hilbert space" which is needed in order to do calculations.

Quantum interactions, too, can be introduced in terms of generalized semi-densities. We bypass the usual quantization procedures and give a direct definition of the interaction among quantum states, which is essentially determined by the underlying "classical" (i.e., finite-dimensional) geometric structures, without having to deal with the classical fields and Lagrangians. This approach, of course, also bears on the way we look at gauge fields. A classical gauge field is a connection, and its quantization requires the choice of a gauge. Furthermore, virtual gauge particle states have more "degrees of freedom" than free states. Accordingly, we propose to look at quantum gauge particles and classical gauge fields as separated entities to begin with; their precise relation must be the object of further study, but the general idea is that it should fall into the "matter defines the geometry" philosophy. If we accept this approach then all gauge field states can be shown to arise from a rather simple recipe.

We now sketch the roles of the non-standard mathematical notions employed.

- Positive spaces play a role in various places. In addition to allowing a mathematically precise treatment of physical scales, they constitute a necessary ingredient in discussing the link between the formalism of finite-dimensional linear algebra and the needed extension to functional multi-particle spaces, and in understanding the detailed structure of the quantum interaction. Furthermore, a distinguished positive space is shown to arise in the context of spinor geometry (and is naturally interpreted as the space of lengths).
- The basics of the geometry of *distributional bundles* were studied in previous papers.[6] In particular, the notion of a bundle of generalized semi-densities turns out to be a natural extension of the notion of *rigged Hilbert space* found in the literature.[7] We introduce *quantum bundles* (used in the description of quantum states and quantum interactions) as distributional bundles constructed from an underlying two-fibered classical structure.
- Various aspects of two-spinor geometry bear on the way fermion and boson fields are treated, both from an algebraic point of view ("internal" particle structure) and with regard to the construction of the appropriate quantum bundles. A discussion of electroweak geometry in this context, improving and extending previous work,[4] is used in the examples of Sec. V. All boson fields, including the Higgs field, derive from a unique general construction.

Some readers will notice a further difference in language between this presentation and more usual ones: we hardly make any reference to symmetry groups. Of course these are implicit in the underlying classical geometric structures, but explicitly using them is not essential at this level.

The attitude of looking directly to free-particle states and one-point interactions is consistent with a view of particle physics which can be roughly sketched as follows (e.g., see Veltman[8]). Let $\boldsymbol{\mathcal{Q}}$ be a complex vector space, whose elements represent the states of quantum systems containing certain particle types. A time-dependent vector $\psi : \boldsymbol{T} \to \boldsymbol{\mathcal{Q}}$ is called a *quantum history* if its evolution is governed by the law

$$\psi(t) = \mathcal{U}_{t_0}(t)\,\psi(t_0),$$

where $t_0 \in \boldsymbol{T}$ is an arbitrarily chosen "initial" time and

$$\mathcal{U}_{t_0} : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{Q}})$$

is a time-dependent endomorphism; on turn, $\mathcal{U}_{t_0}$ is determined by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}_{t_0}(t) = -\mathrm{i}\,\mathfrak{H}(t)\circ\mathcal{U}_{t_0}(t), \qquad \mathfrak{H} : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{Q}}), \quad \mathcal{U}_{t_0}(t_0) = \mathbb{1}_{\boldsymbol{\mathcal{Q}}}.$$

Here, $\mathfrak{H}$ is a time-dependent endomorphism which is dictated, in an essentially elementary way, by the underlying "classical structure," namely, by the geometry of certain finite-dimensional vector spaces and manifolds. The problem of determining $\mathcal{U}$ is on a different footing: the above differential equation has the formal solution

$$\mathcal{U}_{t_0}(t) := \mathbb{1} + \sum_{N=1}^{\infty}\frac{(-\mathrm{i})^N}{N!}\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t}\mathrm{d}t_2\ldots\int_{t_0}^{t}\mathrm{d}t_N\,\Uparrow\mathfrak{H}(t_1)\,\mathfrak{H}(t_2)\,\ldots\,\mathfrak{H}(t_N)\Uparrow,$$

where $\Uparrow\mathfrak{H}(t_1)\,\mathfrak{H}(t_2)\,\ldots\,\mathfrak{H}(t_N)\Uparrow$ stands for the *time-ordered* product of the N endomorphisms $\mathfrak{H}(t_1), \ldots, \mathfrak{H}(t_N)$. The basic object of research is rather the *scattering operator*

$$\mathcal{S} := \lim_{\substack{t\to+\infty \\ t_0\to-\infty}} \mathcal{U}_{t_0}(t) = \sum_{N=0}^{\infty}\mathcal{S}_N,$$

which, intuitively, relates asymptotical states of "incoming" and "outgoing" free particles interacting in a small spacetime region (if $\boldsymbol{\mathcal{Q}}$ is endowed with a Hermitian scalar product, and $\psi_{\mathrm{in}}, \psi_{\mathrm{out}} \in \boldsymbol{\mathcal{Q}}$ represent states of incoming and outgoing particles, then the "scattering matrix element" $\langle\psi_{\mathrm{out}}, \mathcal{S}\psi_{\mathrm{in}}\rangle$ is directly related to measurable physical quantities).

However, a formalism based on these ideas may not yield finite results even if $\boldsymbol{\mathcal{Q}}$ is finite-dimensional (integrals may diverge). In the infinite-dimensional case there is a further complication: the composition $\mathfrak{H}(t_1)\,\mathfrak{H}(t_2)\ldots$ might be not defined at all. Thus one has to conclude that the theory is ill defined, though suitable procedures for extracting its physical content can be eventually devised. The role of time, which must be associated with a chosen observer of some kind, is also somewhat unsatisfactory from the point of view of the general relativist. Up to now, no definitive answer to these fundamental issues has been found. Accepting that, we just attempt a clarification of some of the fundamental notions as a provisional step.

II. POSITIVE SPACES AND PHYSICAL SCALES

An algebraically precise treatment of physical scales was introduced around 1995 after an idea of M. Modugno, and has been used, since then, in papers of various authors.[9–15] In particular, this approach has been systematically exploited in the context of the "covariant quantum mechanics" program (somewhat affine to geometric quantization but with clear-cut differences). Though physical scales (or "dimensions") are usually dealt with in an "informal" way, without a precise mathematical setting, a more formal approach has various advantages in clarifying the geometric background of a theory, and has even found to have heuristic value (see Sec. 1 of Ref. 16 for a more detailed discussion).

The basic notion is that of a *positive space* (or *scale space*, or *unit space*), namely, a one-dimensional "semi-vector space" without the zero element. The notion of a scale space arises quite naturally from simple arguments. The distance of two points in Euclidean space, for example, can be expressed as a real number *only if a length unit has been fixed*. On the other hand, the product of a non-zero distance by a positive number is again a well-defined non-zero distance, namely, the set $\mathbb{L}$ of lengths is naturally endowed with a left action $\mathbb{R}^+ \times \mathbb{L} \to \mathbb{L}$; this turns out to be free and transitive, and so it determines an algebraic structure of semi-vector space over $\mathbb{R}^+$ (note that $\mathbb{L}$ has no naturally distinguished element).

A rigorous study of the tensor algebra of positive spaces turns out to be more delicate than one expects at first sight (for a complete treatment see the recent paper by Janiška, Modugno and Vitolo[16]). Here, we will just sketch the basic notions, needed for "everyday use."

A. Positive spaces

A *semi-vector space* is defined to be a set $\boldsymbol{A}$ equipped with an addition map $\boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A}$ and a multiplication map $\mathbb{R}^+ \times \boldsymbol{A} \to \boldsymbol{A}$, fulfilling the usual axioms of vector spaces except those properties which involve opposites and the zero element. Then, in particular, any vector space is a semi-vector space, and the set of linear combinations over $\mathbb{R}^+$ of n independent vectors in a vector space is a semi-vector space.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are semi-vector spaces, the notion of a *semi-linear* map $f : \boldsymbol{A} \to \boldsymbol{B}$ is defined in an obvious way; then we have the new semi-vector space $\mathrm{sLin}(\boldsymbol{A}, \boldsymbol{B})$ of all semi-linear maps $\boldsymbol{A} \to \boldsymbol{B}$. In particular, the *semi-dual space* (or simply the "dual space") of a semi-vector space $\boldsymbol{A}$ is defined to be the semi-vector space $\boldsymbol{A}^* := \mathrm{sLin}(\boldsymbol{A}, \mathbb{R}^+)$.

A semi-vector space $\mathbb{U}$ is called a *positive space* if the multiplication $\mathbb{R}^+ \times \mathbb{U} \to \mathbb{U}$ is a transitive left action of the group $(\mathbb{R}^+, \cdot\,)$ on $\mathbb{U}$ (then a positive space cannot have a zero element). If $b \in \mathbb{U}$, then any other element $u \in \mathbb{U}$ can be written as $u^0\, b$ with $u^0 \in \mathbb{R}^+$. Quite naturally we can write $u^0 \equiv u/b$, that is $u = (u/b)\, b$, $(u/b) \in \mathbb{R}^+$. So we might also say that a positive space is a "one-dimensional" semi-vector space. Actually, the map $u \mapsto u/b$ turns out to be a semi-linear isomorphism. This fact and the cancellation law for $\mathbb{R}^+$ imply the cancellation law for $\mathbb{U}$, i.e., $u + w = v + w \Rightarrow u = v$ $(u, v, w \in \mathbb{U})$.

If $\mathbb{U}$ and $\mathbb{V}$ are positive spaces, then the semi-vector space $\mathrm{sLin}(\mathbb{U}, \mathbb{V})$ turns out to be a positive space. In particular, we have the positive spaces $\mathbb{U}^* := \mathrm{sLin}(\mathbb{U}, \mathbb{R}^+)$ and $\mathrm{sLin}(\mathbb{U}, \mathbb{U})$. The latter is naturally isomorphic to $\mathbb{R}^+$, since any semi-linear map $f : \mathbb{U} \to \mathbb{U}$ is of the type $f\colon u \mapsto r\,u$ with $r \in \mathbb{R}^+$.

B. Tensor products of positive spaces

With regard to basic notions, the algebra of positive spaces is nearly a straightforward rephrasing of vector space algebra. Tensor products require some care.

We denote by $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{W}$ arbitrary positive spaces, and by $\boldsymbol{V}$ and $\boldsymbol{W}$ arbitrary real vector spaces of finite dimension. A map $\mathbb{U} \times \boldsymbol{V} \to \boldsymbol{W}$ which is semi-linear with respect to the first factor and linear with respect to the second factor is called *sesqui-linear*.

A *(left) sesqui-tensor product* of a positive space $\mathbb{U}$ and a vector space $\boldsymbol{V}$ is defined to be a vector space $\mathbb{U} \otimes \boldsymbol{V}$ along with a sesqui-linear map $\mathbb{U} \times \boldsymbol{V} \to \mathbb{U} \otimes \boldsymbol{V}$ fulfilling the following universal property: if $f : \mathbb{U} \times \boldsymbol{V} \to \boldsymbol{W}$ is a sesqui-linear map, then there is a unique linear map $\tilde{f} : \mathbb{U} \otimes \boldsymbol{V} \to \boldsymbol{W}$ such that $f = \tilde{f} \circ \otimes$. It can be proved[16] that the sesqui-tensor product indeed exists, is unique up to a distinguished linear isomorphism and is linearly generated by the image of the map $\otimes : \mathbb{U} \times \boldsymbol{V} \to \mathbb{U} \otimes \boldsymbol{V}$.

If $\{\mathsf{b}_i\} \subset \boldsymbol{V}$, $i = 1, \ldots, \dim \boldsymbol{V}$ is a basis of $\boldsymbol{V}$, and if $\mathsf{b} \in \mathbb{U}$, then it is not difficult to prove that $\{\mathsf{b} \otimes \mathsf{b}_i\}$ is a basis of $\mathbb{U} \otimes \boldsymbol{V}$. Thus $\dim(\mathbb{U} \otimes \boldsymbol{V}) = \dim \boldsymbol{V}$. Moreover, the right semi-tensor product $\boldsymbol{V} \otimes \mathbb{U}$ can be defined similarly, and turns out to be naturally isomorphic to $\mathbb{U} \otimes \boldsymbol{V}$; thus, we identify $v \otimes u \in \boldsymbol{V} \otimes \mathbb{U}$ with $u \otimes v$, getting the "numberlike" behavior of elements in positive spaces.

A particular case is that of the *universal vector extension* $\mathbb{R} \otimes \mathbb{U}$ of $\mathbb{U}$, which turns out to be the disjoint union $\mathbb{R} \otimes \mathbb{U} = \mathbb{U}_- \cup \{0\} \cup \mathbb{U}_+$, where $\mathbb{U}_+ := \{1 \otimes u : u \in \mathbb{U}\}$ and $\mathbb{U}_- := \{(-1) \otimes u : u \in \mathbb{U}\}$ are positive spaces. By virtue of the universal property one also proves the following:

- Every semi-linear map $\mathbb{U} \to \boldsymbol{W}$ can be uniquely extended to a linear map $\mathbb{R} \otimes \mathbb{U} \to \boldsymbol{W}$;
- there is a natural isomorphism $\mathbb{R} \otimes (\mathbb{U}^*) \leftrightarrow (\mathbb{R} \otimes \mathbb{U})^*$;
- there is a natural isomorphism $\mathbb{U} \otimes \boldsymbol{V} \leftrightarrow (\mathbb{R} \otimes \mathbb{U}) \otimes \boldsymbol{V}$.

A *semi-tensor product* of positive spaces $\mathbb{U}$ and $\mathbb{V}$ is a positive space $\mathbb{U} \otimes \mathbb{V}$ along with a semi-bilinear map $\otimes : \mathbb{U} \times \mathbb{V} \to \mathbb{U} \otimes \mathbb{V}$ fulfilling the following universal property: if $\mathbb{W}$ is a positive space and $f : \mathbb{U} \times \mathbb{V} \to \mathbb{U} \otimes \mathbb{V}$ is a semi-bilinear map, then there exists a unique semi-linear map $\tilde{f} : \mathbb{U} \otimes \mathbb{V} \to \mathbb{W}$ such that $f = \tilde{f} \circ \otimes$.

While the uniqueness of the semi-tensor product is easily established by a standard procedure, proving its existence is a more intricate task,[16] which involves the universal vector extensions of $\mathbb{U}$ and $\mathbb{V}$. Eventually everything works fine, and one has natural semi-linear isomorphisms

$$\mathbb{R}^+ \otimes \mathbb{U} \cong \mathbb{U} \otimes \mathbb{R}^+ \cong \mathbb{U}, \qquad \mathbb{R} \otimes (\mathbb{U} \otimes \mathbb{V}) \cong (\mathbb{R} \otimes \mathbb{U}) \otimes (\mathbb{R} \otimes \mathbb{V}),$$

$$\mathbb{V} \otimes \mathbb{U}^* \cong \mathrm{sLin}(\mathbb{U}, \mathbb{V}), \qquad \mathrm{Tr} : \mathbb{U} \otimes \mathbb{U}^* \to \mathbb{R}^+.$$

The semi-tensor product can be easily generalized to any number of factors and turns out to be associative. In particular, for any $n \in \mathbb{N}$ we consider the nth semi-tensor power

$$\otimes^n \mathbb{U} := \underbrace{\mathbb{U} \otimes \ldots \otimes \mathbb{U}}_{n \text{ times}}, \quad \otimes^0 \mathbb{U} := \mathbb{R}^+, \quad \otimes^1 \mathbb{U} := \mathbb{U},$$

so that we obtain a "semi-tensor algebra" $\oplus_{n \in \mathbb{N}} \otimes^n \mathbb{U}$. Usually, it will be convenient to adopt a "numberlike" notation: we write $uv \equiv u \otimes v$ if either u or v (or both) is an element in a positive space. If $u \in \mathbb{U}$, then the unique $u^{-1} \in \mathbb{U}^*$ such that $\langle u^{-1}, u \rangle = 1$, namely, the *dual* element of u, is also called the *inverse* of u.

C. Rational powers of positive spaces

We say that a function $f : \mathbb{U} \to \mathbb{R}^+$ is of *degree* $\alpha \in \mathbb{R}$ if

$$f(r\, u) = r^\alpha\, f(u)\ \forall\, r \in \mathbb{R}^+, \quad u \in \mathbb{U}.$$

The set $\mathrm{F}^\alpha(\mathbb{U})$ of all such functions turns out to be a positive space. Note that each element in $\mathrm{F}^\alpha(\mathbb{U})$ is determined by the value it takes on any fixed element in $\mathbb{U}$. Conversely, each $u \in \mathbb{U}$ determines a distinguished element $f_u \in \mathrm{F}^\alpha(\mathbb{U})$ by the rule $f_u(u) = 1$.

In particular, $\mathrm{F}^0(\mathbb{U}) \cong \mathbb{R}^+$ and $\mathrm{F}^1(\mathbb{U}) \cong \mathbb{U}^*$. If $n \in \mathbb{N}$, then $\mathrm{F}^n(\mathbb{U}) \cong \otimes^n \mathbb{U}^*$. A natural semi-linear isomorphism $\mathrm{F}^{-1}(\mathbb{U}) \cong \mathbb{U}^{**} \cong \mathbb{U}$ is determined by the identification of $f \in \mathrm{F}^{-1}(\mathbb{U})$ with u^{-1}, where $u \in \mathbb{U}$ is characterized by $f(u) = 1$. More generally, $\mathrm{F}^{-\alpha}(\mathbb{U}) \cong \mathrm{F}^\alpha(\mathbb{U}^*)$. We will use the convenient shorthand

$$\mathbb{U}^\alpha \equiv \mathrm{F}^{-\alpha}(\mathbb{U}).$$

If $u \in \mathbb{U}$, then u^α is defined to be the unique element in $\mathbb{U}^\alpha$ such that $u^\alpha(u^{-1}) = 1$.

Essentially, we are interested in the case when α is rational. The reason is that if $n \in \mathbb{N}$, then we have a natural semi-linear isomorphism

$$(\mathbb{U}^{1/n})^n \equiv \underbrace{\mathbb{U}^{1/n} \otimes \ldots \otimes \mathbb{U}^{1/n}}_{n \text{ factors}} \leftrightarrow \mathbb{U} : u^{1/n} \otimes \ldots \otimes u^{1/n} \leftrightarrow u.$$

Then we are led to regard the positive space $\mathbb{U}^{1/n} \equiv \mathrm{F}^{1/n}(\mathbb{U}^*)$ as the *n-root* of $\mathbb{U}$. We find that *rational powers* of positive spaces behave quite naturally, since for any $p, q \in \mathbb{Q}$, one has

$$(\mathbb{U}^q)^p \cong \mathbb{U}^{pq}, \quad \mathbb{U}^p \otimes \mathbb{U}^q \cong \mathbb{U}^{p+q}.$$

In particular, $(\mathbb{U}^q)^* \cong (\mathbb{U}^*)^q$.

D. Physical scales

In many physical theories it is convenient to take the following positive spaces as *basic spaces of scales*: the space $\mathbb{T}$ of *time scales*, the space $\mathbb{L}$ of *length scales*, the space $\mathbb{M}$ of *mass scales*. An arbitrary *scale space* is then a positive space of the type

$$\mathbb{S} \equiv \mathbb{S}[d_1, d_2, d_3] := \mathbb{T}^{d_1} \otimes \mathbb{L}^{d_2} \otimes \mathbb{M}^{d_3}, \quad d_i \in \mathbb{Q}.$$

An element $\mathsf{s} \in \mathbb{S}$ is also called a *scale* (or possibly a *unit of measurement*). A *variable scale* is an $\mathbb{S}$-valued map.

A "scaled" version of a vector bundle $\boldsymbol{E} \rightarrowtail \boldsymbol{M}$ is a fibered tensor product $\mathbb{S} \otimes \boldsymbol{E} \rightarrowtail \boldsymbol{M}$. Fibered operations on $\boldsymbol{E}$ translate to scaled operations so, for example, if $\boldsymbol{F} \rightarrowtail \boldsymbol{M}$ is another bundle over the same base manifold and $\mathcal{D} : \mathrm{sec}(\boldsymbol{M}, \boldsymbol{E}) \to \mathrm{sec}(\boldsymbol{M}, \boldsymbol{F})$ is a linear differential operator, then one has the linear differential operator (denoted by the same symbol)

$$\mathcal{D} : \mathrm{sec}(\boldsymbol{M}, \mathbb{S} \otimes \boldsymbol{E}) \to \mathrm{sec}(\boldsymbol{M}, \mathbb{S} \otimes \boldsymbol{F}) : \sigma \mapsto \mathcal{D}\sigma := \mathsf{s} \otimes \mathcal{D}(\mathsf{s}^{-1}|\sigma),$$

which is clearly independent of the choice of $\mathsf{s} \in \mathbb{S}$. If $\alpha : \boldsymbol{M} \to \mathbb{S} \otimes \wedge^p \mathrm{T}^*\boldsymbol{M}$ is a p-form ($p \in \mathbb{N}$) on a manifold $\boldsymbol{M}$, then one gets the "scaled exterior differential" $\mathrm{d}\alpha : \boldsymbol{M} \to \mathbb{S} \otimes \wedge^{p+1}\mathrm{T}^*\boldsymbol{M}$; if v is a vector field on $\boldsymbol{M}$, then one gets the "scaled Lie derivative" $\mathcal{L}[v]\alpha : \boldsymbol{M} \to \mathbb{S} \otimes \wedge^p \mathrm{T}^*\boldsymbol{M}$. A linear connection Γ of $\boldsymbol{E} \rightarrowtail \boldsymbol{M}$ determines a linear connection of $\mathbb{S} \otimes \boldsymbol{E} \rightarrowtail \boldsymbol{M}$: the covariant derivative of a section $\sigma : \boldsymbol{M} \to \mathbb{S} \otimes \boldsymbol{E}$ is a section $\nabla\sigma : \boldsymbol{M} \to \mathbb{S} \otimes \mathrm{T}^*\boldsymbol{M} \otimes \boldsymbol{E}$; note that the coefficients of Γ in a frame (θ_i) of $\boldsymbol{E}$ coincide with the coefficients of the induced connection in the frame $(\mathsf{s} \otimes \theta_i)$, where $\mathsf{s} \in \mathbb{S}$ is any constant scale. Two sections $\sigma : \boldsymbol{M} \to \boldsymbol{E}$ and $\sigma' : \boldsymbol{M} \to \mathbb{S} \otimes \boldsymbol{E}$ of differently scaled vector bundles can be compared if we avail of a scale factor $\mathsf{s} : \boldsymbol{M} \to \mathbb{S}$, called a *coupling scale*, or possibly a *coupling constant*.

The commonest coupling constants are the speed of light $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$; Planck's constant $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$; Newton's gravitational constant $\mathbb{G} \in \mathbb{T}^{-2} \otimes \mathbb{L}^3 \otimes \mathbb{M}^{-1}$; the positron charge $e \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$; a particle's mass $m \in \mathbb{M}$. Note that c determines an isomorphism $\mathbb{T} \cong \mathbb{L}$, and together with $\hbar$ also an isomorphism $\mathbb{M} \cong \mathbb{L}^{-1}$. Actually in QFT, one usually expresses all physical dimensions as powers of just one scale (namely, the setting $c = \hbar = 1$).

The metric, either Euclidean or Lorentz-type, is appropriately described as a scaled tensor field. Focusing our attention on the spacetime $(\boldsymbol{M}, g)$ of general relativity, the metric is a section

$$g : \boldsymbol{M} \to \mathbb{L}^2 \otimes \mathrm{T}^*\boldsymbol{M} \otimes \mathrm{T}^*\boldsymbol{M},$$

so that the scalar product of vectors is valued into $\mathbb{R} \otimes \mathbb{L}^2 \equiv \mathbb{R} \otimes \mathbb{L} \otimes \mathbb{L}$. On the other hand, g can be seen as an $\mathbb{R}$-valued Lorentz metric on the fibers of the bundle

$$\boldsymbol{H} \equiv \mathbb{L}^{-1} \otimes \mathrm{T}\boldsymbol{M} \equiv \mathbb{L}^* \otimes \mathrm{T}\boldsymbol{M} \longrightarrow \boldsymbol{M}.$$

Conversely, we write $\mathrm{T}\boldsymbol{M} \equiv \mathbb{L} \otimes \boldsymbol{H}$. We also get

$$\wedge^4\mathrm{T}\boldsymbol{M} \cong \mathbb{L}^4 \otimes \wedge^4\boldsymbol{H}, \quad \wedge^4\mathrm{T}^*\boldsymbol{M} \cong \mathbb{L}^{-4} \otimes \wedge^4\boldsymbol{H}^*.$$

If $\boldsymbol{X}$ is an oriented n-dimensional manifold, then we write $\mathbb{V}\boldsymbol{X} \equiv (\wedge^n\boldsymbol{X})^+$, so that $\mathbb{V}^{-1}\boldsymbol{X} \rightarrowtail \boldsymbol{X}$ is the bundle of all positive volume forms on $\boldsymbol{X}$. In the case of spacetime, the metric determines a unique positive scaled normalized volume form

$$\eta : \boldsymbol{M} \to \wedge^4\boldsymbol{H}^* \equiv \mathbb{L}^4 \otimes \mathbb{V}^{-1}\boldsymbol{M}.$$

Remark: In Einstein spacetime, let $\boldsymbol{T} \subset \boldsymbol{M}$ be the world-line of a classical particle. Then the particle's velocity is the tangent map $\mathrm{T}\boldsymbol{T} \to \mathrm{T}\boldsymbol{M}$ of the inclusion. The *good clock* hypothesis implies $\mathrm{T}\boldsymbol{T} \cong \boldsymbol{T} \times \mathbb{T}$, a trivial bundle, so that the velocity can be seen as a map $\boldsymbol{T} \to \mathbb{T}^{-1} \otimes \mathrm{T}\boldsymbol{M}$. Multiplying by the particle's mass we obtain 4-momentum $\boldsymbol{T} \to \mathbb{M} \otimes \mathbb{T}^{-1} \otimes \mathrm{T}\boldsymbol{M}$. Since the Lorentz metric is $\mathbb{L}^2$-scaled, the covariant form of 4-momentum is valued in

$$\boldsymbol{P} \cong \mathbb{M} \otimes \mathbb{L}^2 \otimes \mathbb{T}^{-1} \otimes \mathrm{T}^*\boldsymbol{M}.$$

The speed of light and the Planck constant jointly yield semi-linear isomorphisms $\mathbb{T} \cong \mathbb{L}$ and $\mathbb{M} \cong \mathbb{L}^{-1}$, hence we write $\boldsymbol{P} \cong \mathrm{T}^*\boldsymbol{M}$.

III. CLASSICAL GEOMETRY

By a "classical" manifold I mean a finite-dimensional Hausdorff manifold. More generally, the "classical geometry" underlying a quantum theory is constituted by the finite-dimensional geometric structures (manifolds, bundles) needed for the corresponding classical field theory.

A. Two-spinors basics

In previous papers,[1–3] I discussed a partly original approach to 2-spinors, which turns out to be convenient for an integrated approach to Einstein-Cartan-Maxwell-Dirac fields starting from minimal geometric assumptions. More recently,[4] I also discussed the extension of that setting to more general gauge field theories. This section contains a summary of some results to be used later. We first state some purely algebraic results about certain complex vector spaces. Then we will deal with vector bundles having fiber structures of the considered types; by abuse of language, these bundles will be denoted with the same symbol as the corresponding spaces.

First, we recall that if $\boldsymbol{V}$ is a complex vector space, then Hermitian transposition is a natural anti-linear involution of $\boldsymbol{V}\otimes\overline{\boldsymbol{V}}$ (where $\overline{\boldsymbol{V}}$ denotes the conjugate space). Thus one has the decomposition into the direct sum of the *real* eigenspaces of the involution corresponding to eigenvalues ± 1, namely,

$$\boldsymbol{V}\otimes\overline{\boldsymbol{V}} = \mathrm{H}(\boldsymbol{V}\otimes\overline{\boldsymbol{V}}) \oplus \mathrm{i}\,\mathrm{H}(\boldsymbol{V}\otimes\overline{\boldsymbol{V}}).$$

Starting from a two-dimensional complex vector space $\boldsymbol{S}$, with no further assumption, the above basic construction gives rise to a rich algebraic structure:

- The Hermitian subspace of $\wedge^2\boldsymbol{S}\otimes\wedge^2\overline{\boldsymbol{S}}$ is a real one-dimensional vector space with a distinguished orientation; its positively oriented semispace $\mathbb{L}^2$ (whose elements are of the type $w\otimes\bar{w}$, $w\in\wedge^2\boldsymbol{S}$) has the square root semispace $\mathbb{L}$, which can be identified with the space of *length units*.
- The 2-spinor space is defined to be $\boldsymbol{U} := \mathbb{L}^{-1/2}\otimes\boldsymbol{S}$. The space $\wedge^2\boldsymbol{U}$ is naturally endowed with a Hermitian metric, namely, the identity element in
$$\mathrm{H}[(\wedge^2\overline{\boldsymbol{U}}^\star)\otimes(\wedge^2\boldsymbol{U}^\star)] \cong \mathbb{L}^2\otimes\mathrm{H}[(\wedge^2\boldsymbol{S}^\star)\otimes(\wedge^2\boldsymbol{S}^\star)],$$
so that normalized "symplectic forms" $\varepsilon\in\wedge^2\boldsymbol{U}^\star$ constitute a U(1)-space (any two of them are related by a phase factor). Each ε yields the isomorphism $\varepsilon^\flat : \boldsymbol{U}\to\boldsymbol{U}^\star : u\mapsto u^\flat := \varepsilon(u, \_)$.
- The identity element in $\mathrm{H}[(\wedge^2\overline{\boldsymbol{U}}^\star)\otimes(\wedge^2\boldsymbol{U}^\star)]$ can be written as $\varepsilon\otimes\bar{\varepsilon}$ where $\varepsilon\in\wedge^2\boldsymbol{U}^\star$ is any normalized element. This natural object can also be seen as a bilinear form g on $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$, via the rule $g(u\otimes\bar{v}, r\otimes\bar{s}) = \varepsilon(u, r)\,\bar{\varepsilon}(\bar{v}, \bar{s})$ extended by linearity. Its restriction to the Hermitian subspace $\boldsymbol{H}\equiv\mathrm{H}(\boldsymbol{U}\otimes\overline{\boldsymbol{U}})$ turns out to be a Lorentz metric. Null elements in $\boldsymbol{H}$ are of the form $\pm u\otimes\bar{u}$ with $u\in\boldsymbol{U}$ (thus there is a distinguished time-orientation in $\boldsymbol{H}$).
- Let $\boldsymbol{W}\equiv\boldsymbol{U}\oplus\overline{\boldsymbol{U}}^\star$. The linear map $\gamma : \boldsymbol{U}\otimes\overline{\boldsymbol{U}}\to\mathrm{End}(\boldsymbol{W}) : y\mapsto\gamma(y)$ acting as
$$\gamma(r\otimes\bar{s})(u,\chi) = \sqrt{2}\big(\langle\chi,\bar{s}\rangle\,p,\ \langle r^\flat, u\rangle\,\bar{s}^\flat\big)$$
is well-defined independently of the choice of the normalized $\varepsilon\in\wedge^2\boldsymbol{U}^\star$ yielding the isomorphism $\varepsilon^\flat$. Its restriction to $\boldsymbol{H}$ turns out to be a Clifford map. Thus one is led to regard $\boldsymbol{W}\equiv\boldsymbol{U}\oplus\overline{\boldsymbol{U}}^\star$ as the space of Dirac spinors, decomposed into its Weyl subspaces. The anti-isomorphism $\boldsymbol{W}\to\boldsymbol{W}^\star : (u,\chi)\mapsto(\bar{\chi},\bar{u})$ is the usual *Dirac adjunction* ($\psi\mapsto\bar{\psi}$ in traditional notation), associated with a Hermitian product k having the signature (+ , + , − , −).

An arbitrary basis (ξ_A) of $\boldsymbol{S}$, $A{=}1{,}2$, determines bases of the various associated spaces, in particular the bases $l\in\mathbb{L}$ (a length unit), $\big(\zeta_A\big)\equiv\big(l^{-1/2}\,\xi_A\big)\subset\boldsymbol{U}$, $\varepsilon\in\wedge^2\boldsymbol{U}^\star$. We have $\varepsilon=\varepsilon_{AB}\,\zeta^A\wedge\zeta^B$, where $\big(\zeta^A\big)\subset\boldsymbol{U}^\star$ is the dual basis of $\big(\zeta_A\big)$ and $\big(\varepsilon_{AB}\big)$ denotes the antisymmetric Ricci matrix. As for the basis of $\boldsymbol{H}\equiv\mathrm{H}(\boldsymbol{U}\otimes\overline{\boldsymbol{U}})$ associated with (ζ_A) one usually considers the *Pauli basis* $\big(\tau_\lambda\big)$, given by $\tau_\lambda\equiv\frac{1}{\sqrt{2}}\,\sigma_\lambda{}^{AA'}\,\zeta_A\otimes\bar{\zeta}_{A'}$ where $(\sigma_\lambda{}^{AA'})$, $\lambda=0,1,2,3$, denotes the λth Pauli matrix (dotted indices refer to components in conjugate spaces). This basis is readily seen to be g-orthonormal. The associated *Weyl basis* of $\boldsymbol{W}$ is defined to be the basis (ζ_α), $\alpha=1,2,3,4$, given by

$$(\zeta_1,\zeta_2,\zeta_3,\zeta_4) := (\zeta_1,\zeta_2,-\bar{\zeta}^1,-\bar{\zeta}^2),$$

where ζ_1 is a simplified notation for $(\zeta_1, 0)$, and the like.

Remark: In contrast to the usual 2-spinor formalism, no symplectic form is fixed. The 2-form ε is unique up to a phase factor which depends on the chosen 2-spinor basis, and determines isomorphisms $\varepsilon^\flat : \boldsymbol{U} \to \boldsymbol{U}^\star$ and $\varepsilon^\# : \boldsymbol{U}^\star \to \boldsymbol{U}$. Also note that no Hermitian form on $\boldsymbol{S}$ or $\boldsymbol{U}$ is assigned; actually, because of the Lorentz structure of $\boldsymbol{H}$, the choice of such an object turns out to be equivalent to the choice of an "observer".

We now consider a complex vector bundle $\boldsymbol{S} \rightarrowtail \boldsymbol{M}$ with two-dimensional fibers. By performing the above sketched constructions fiberwise we obtain various vector bundles, which are denoted, for simplicity, by the corresponding symbols. We observe that some appropriate topological restrictions are implicit in what follows; we will assume the needed hypotheses to hold without further comment.

A linear connection Γ on $\boldsymbol{S}$ determines linear connections on the associated bundles, and, in particular, connections G of $\mathbb{L}$, Y of $\wedge^2\boldsymbol{U}$ and $\tilde{\Gamma}$ of $\boldsymbol{H}$; on turn, it can be expressed in terms of these as

$$\Gamma_{a}{}^{A}{}_{B} = (G_a + \mathrm{i}\, Y_a)\delta^{A}{}_{B} + \tfrac{1}{2}\,\tilde{\Gamma}_{a}{}^{AA^{\cdot}}{}_{BA^{\cdot}}\,.$$

If $\boldsymbol{M}$ is four-dimensional, then a *tetrad* is defined to be a linear morphism $\Theta : \mathrm{T}\boldsymbol{M} \to \mathbb{L}\otimes\boldsymbol{H}$. An invertible tetrad determines, by pull-back, a Lorentz metric on $\boldsymbol{M}$ and a metric connection of $\mathrm{T}\boldsymbol{M} \rightarrowtail \boldsymbol{M}$, as well as a Dirac morphism $\mathrm{T}\boldsymbol{M} \to \mathbb{L}\otimes\mathrm{End}\,\boldsymbol{W}$.

A non-singular field theory in the above geometric environment can be naturally formulated[1] even if Θ is not required to be invertible everywhere. If the invertibility requirement is satisfied, then one gets essentially the standard Einstein-Cartan-Maxwell-Dirac theory, but with some redefinition of the fundamental fields: these are now the 2-spinor connection Γ, the tetrad Θ, the Maxwell field F, and the Dirac field $\psi : \boldsymbol{M} \to \mathbb{L}^{-3/2}\otimes\boldsymbol{W}$. Gravitation is represented by Θ and $\tilde{\Gamma}$ together. G is assumed to have vanishing curvature, $\mathrm{d}G = 0$, so that we can find local charts such that $G_a = 0$; this amounts to "gauging away" the conformal "dilaton" symmetry. Coupling constants arise as covariantly constants sections of $\mathbb{L}^r$ (r rational). One then writes a natural Lagrangian which yields all the field equations: the Einstein equation and the equation for torsion; the equation $F = 2\,\mathrm{d}Y$ (thus Y is essentially the electromagnetic potential) and the other Maxwell equation; the Dirac equation.[2]

B. Fermi transport of spinors

The usual Fermi transport of spacetime tensors along a timelike one-dimensional submanifold $\boldsymbol{T} \subset \boldsymbol{M}$ can be naturally (though somewhat not uniquely) extended to spinors.[17] We sketch the basic result, because it enters our definition of free-particle states.

Let $\boldsymbol{H}_{\boldsymbol{T}} \rightarrowtail \boldsymbol{T}$ denote the restriction of $\boldsymbol{H}$ to the base $\boldsymbol{T}$. We obtain a distinguished section $\Phi : \boldsymbol{T} \to \mathrm{T}^*\boldsymbol{T}\otimes\wedge^2\boldsymbol{H}_{\boldsymbol{T}}$ by the rule

$$v\rfloor\Phi = 2\,(\nabla_v\tau)\wedge\tau, \qquad v \in \mathrm{T}\,\boldsymbol{T},$$

where $\tau : \boldsymbol{T} \to \mathbb{L}^{-1}\otimes\mathrm{T}\boldsymbol{T} \subset \mathbb{L}^{-1}\otimes\mathrm{T}\boldsymbol{M}$ is the unit future-oriented (scaled) tangent vector field of $\boldsymbol{T}$. By lowering the second index in Φ through the Lorentz metric we then get a section $\Phi^\flat : \boldsymbol{T} \to \mathrm{T}^*\boldsymbol{T}\otimes\boldsymbol{H}_{\boldsymbol{T}}\otimes\boldsymbol{H}^*_{\boldsymbol{T}}$. Covariant derivation along $\boldsymbol{T}$ determines a connection $\Gamma_{\boldsymbol{T}}$ of $\boldsymbol{H}_{\boldsymbol{T}} \rightarrowtail \boldsymbol{T}$, which can be modified as $\Gamma_{\mathrm{F}} := \Gamma_{\boldsymbol{T}} + \Phi^\flat$; this provides the standard Fermi transport. By taking half the trace of $\Phi^\flat$ with respect to conjugate 2-spinor indices we obtain a section $\phi : \boldsymbol{T} \to \mathrm{T}^*\boldsymbol{T}\otimes\boldsymbol{U}_{\boldsymbol{T}}\otimes\boldsymbol{U}^\star_{\boldsymbol{T}}$, where $\boldsymbol{U}_{\boldsymbol{T}} \rightarrowtail \boldsymbol{T}$ denotes the base $\boldsymbol{T}$ restricted bundle $\boldsymbol{U}$.

Using base coordinates $(\mathsf{x}^a) \equiv (\mathsf{x}^1, \mathsf{x}^2, \mathsf{x}^3, \mathsf{x}^4)$ such that $\partial\mathsf{x}_4 \equiv \partial/\partial\mathsf{x}^4$ is tangent to $\boldsymbol{T}$, and a Pauli frame (τ_λ) of $H_{\boldsymbol{T}}$ such that $\tau_0 \equiv \tau$, we have the coordinate expressions

$$\Phi_{4}{}^{\lambda}{}_{\mu} \equiv (\Phi^\flat)_{4}{}^{\lambda}{}_{\mu} = -\tilde{\Gamma}_{4}{}^{\lambda}{}_{\mu}, \qquad \phi_{4}{}^{A}{}_{B} = \tfrac{1}{2}\,\Phi_{4}{}^{AA^{\cdot}}{}_{BA^{\cdot}} = \tfrac{1}{2}\,\tilde{\Gamma}_{4}{}^{0j}\,\sigma_{j}{}^{A}{}_{B}$$

(hence $\Phi_{4}{}^{0}{}_{0} = \Phi_{4}{}^{j}{}_{j} = \Phi_{4}{}^{\lambda}{}_{\lambda} = 0$ and $\phi_{4}{}^{A}{}_{A} = 0$). Let now $\Gamma_{\boldsymbol{T}}$ be the connection of $\boldsymbol{U}_{\boldsymbol{T}} \rightarrowtail \boldsymbol{T}$ determined by Γ; the sum

$$\Gamma_{\mathrm{F}} := \Gamma_{\boldsymbol{T}} + \phi$$

is a new connection on the same bundle, which we call the *spinor Fermi connection*. It turns out that the induced connection $\Gamma_{\mathrm{F}}\otimes\bar{\Gamma}_{\mathrm{F}}$ of $H_{\boldsymbol{T}} \rightarrowtail \boldsymbol{T}$ coincides with the Fermi connection Γ_{F}; moreover, any other linear connection Γ'_{F} of $\boldsymbol{U}_{\boldsymbol{T}} \rightarrowtail \boldsymbol{T}$ yielding Γ_{F} differs from Γ_{F} by a term of the type $\mathrm{i}\,\alpha\otimes\mathbb{1}$ with

$\alpha : \boldsymbol{T} \to \mathrm{T}^*\boldsymbol{T}$, namely,

$$(\Gamma'_{\mathrm{F}})^{A}_{4\,B} = (\Gamma_{\mathrm{F}})^{A}_{4\,B} + \mathrm{i}\,\alpha_4\,\delta^{A}{}_{B}.$$

So we get a family of connections of the restricted bundle $\boldsymbol{U}_{\boldsymbol{T}} \rightarrowtail \boldsymbol{T}$. Each element of the family yields the standard Fermi transport of vectors, and is characterized by the arbitrary choice of an imaginary function on $\boldsymbol{T}$. Since Γ_{F} $(\alpha = 0)$ is a distinguished element of the family, we see it as the natural generalization of Fermi transport to 2-spinors.

C. Electromagnetic interaction at finite-dimensional algebraic level

In order to deal with quantum particles we will see the gravitational field as a fixed background structure; this means that the tetrad Θ and the gravitational part of the spin connection are fixed (rather than "field variables"). If no confusion arises, by using Θ we make the identification $\mathrm{T}\boldsymbol{M} \cong \mathbb{L} \otimes \boldsymbol{H}$, and view 1-forms of $\boldsymbol{M}$ as *scaled* sections $\boldsymbol{M} \to \mathbb{L}^{-1} \otimes \boldsymbol{H}^*$.

The classical interaction between the Dirac field and the e.m. potential can be deduced (following the usual procedure) from the Dirac Lagrangian (a density constructed through certain contractions of $\bar{\psi} \otimes \nabla\psi$) by extracting the relevant term after writing $Y = eA$, where A is a true 1-form, via the choice of an e.m. gauge. Somewhat differently we can see the classical interaction as directly deriving from the underlying geometric structure, namely, as the natural contraction

$$\ell_{\mathrm{int}} : \overline{\boldsymbol{W}} \otimes \boldsymbol{H} \otimes \boldsymbol{W} \to \mathbb{C} : (\bar{\phi}, A, \psi) \mapsto -e\,\langle \bar{\phi}, \gamma[A]\psi \rangle.$$

We can also see ℓ_{int} as a tensor field $\boldsymbol{M} \to \overline{\boldsymbol{W}}^{\star} \otimes \boldsymbol{H}^* \otimes \boldsymbol{W}^{\star}$. From this, using the algebraic structures of the fibers of the involved bundles, we can obtain eight tensor fields of different index types; these correspond to different combinations of particle absorption and creation, respectively, represented by covariant and contravariant indices.

In order to see how all this works precisely in the context of Dirac electrodynamics, one has to introduce a few further notions. Let $\boldsymbol{P}_m \subset \mathrm{T}^*\boldsymbol{M}$ the subbundle over $\boldsymbol{M}$ whose fibers are the future hyperboloids ("mass-shells") corresponding to mass $m \in \{0\} \cup \mathbb{L}^{-1}$. If $p \in (\boldsymbol{P}_m)_x$, $x \in \boldsymbol{M}$, then we have the *Dirac splitting*

$$\boldsymbol{W}_x = \boldsymbol{W}_p^+ \oplus \boldsymbol{W}_p^-, \quad \boldsymbol{W}_p^{\pm} := \mathrm{Ker}(\gamma[p^{\#}] \mp m),$$

where $p^{\#} \equiv g^{\#}(p) \in \mathbb{L}^{-2} \otimes \mathrm{T}\boldsymbol{M}$ is the contravariant form of p. Thus we obtain two-fibered bundles $\boldsymbol{W}_m^{\pm} \rightarrowtail \boldsymbol{P}_m \rightarrowtail \boldsymbol{M}$, where

$$\boldsymbol{W}_m^{\pm} := \bigsqcup_{p \in \boldsymbol{P}_m} \boldsymbol{W}_p^{\pm} \subset \boldsymbol{P}_m \underset{\boldsymbol{M}}{\times} \boldsymbol{W}.$$

We call $\boldsymbol{W}_m^+$ and $\overline{\boldsymbol{W}}_m^-$ the *electron bundle* and the *positron bundle*, respectively. If $\big(\zeta_A(p)\big)$ is a 2-spinor frame such that $p^{\#} \propto \tau_0$ in the associated Pauli frame, then the *Dirac frame*$\big(\mathsf{u}_A(p), \mathsf{v}_B(p)\big)$ is k-orthonormal and adapted to the Dirac splitting; it is defined to be

$$\mathsf{u}_1 \equiv \tfrac{1}{\sqrt{2}}(\zeta_1, \bar{\zeta}^1), \quad \mathsf{u}_2 \equiv \tfrac{1}{\sqrt{2}}(\zeta_2, \bar{\zeta}^2), \quad \mathsf{v}_1 \equiv \tfrac{1}{\sqrt{2}}(\zeta_1, -\bar{\zeta}^1), \quad \mathsf{v}_2 \equiv \tfrac{1}{\sqrt{2}}(\zeta_2, -\bar{\zeta}^2).$$

The special algebraic structure of $\boldsymbol{W}$ yields a natural restriction of ℓ_{int}, namely,

$$\ell'_{\mathrm{int}} : \boldsymbol{M} \to (\overline{\boldsymbol{W}}_m^- \otimes \boldsymbol{H} \otimes \boldsymbol{W}_m^+)^{\star}.$$

Since $(\boldsymbol{W}_m^+)^{\star} \cong \overline{\boldsymbol{W}}_m^+$ and $(\overline{\boldsymbol{W}}_m^-)^{\star} \cong \boldsymbol{W}_m^-$ by virtue of Dirac adjunction, other fermion factors in $\overline{\boldsymbol{W}}^{\star} \otimes \boldsymbol{H}^* \otimes \boldsymbol{W}^{\star}$ are seen as contravariant, and as such correspond to particle creation. The anticipated eight tensor fields obtained from ℓ_{int} then derive from the expansion of

$$\ell_{\mathrm{int}} : \boldsymbol{M} \to \Big(\boldsymbol{W}_m^+ \oplus (\overline{\boldsymbol{W}}_m^-)^{\star}\Big) \otimes \Big(\boldsymbol{H} \oplus \boldsymbol{H}^*\Big) \otimes \Big((\boldsymbol{W}_m^+)^{\star} \oplus \overline{\boldsymbol{W}}_m^-\Big) \cong \overline{\boldsymbol{W}} \otimes \Big(\boldsymbol{H} \oplus \boldsymbol{H}^*\Big) \otimes \boldsymbol{W},$$

where direct sums and tensor products are fibered over the appropriate momentum bundles. If $\alpha \in \boldsymbol{H}^*$, then $\ell_{\mathrm{int}}(\bar{\phi}, \alpha, \psi) = \ell_{\mathrm{int}}(\bar{\phi}, \alpha^{\#}, \psi)$, with $\alpha^{\#} \equiv g^{\#}(\alpha) \in \boldsymbol{H}$. The extension of ℓ_{int} to complexified e.m. vector fields $\alpha : \boldsymbol{M} \to \mathbb{C} \otimes \boldsymbol{H}$ is straightforward. The ordering is inessential, so one recovers the possible point interactions, represented in a Feynman diagram as shown in Fig. 1. Wavy lines,

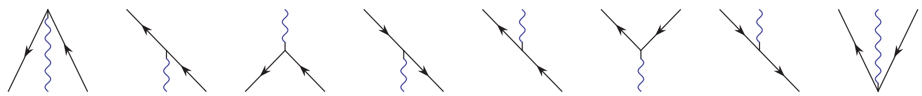

FIG. 1. Point interactions of electrodynamics.

upward arrows, and downward arrows, respectively, represent photons, electrons, and positrons. Time is flowing upwards, ingoing lines correspond to covariant factors (starred spaces) that is to particle absorption, outgoing lines correspond to contravariant factors that is to particle creation. As for the expression of ℓ_{int} in a Dirac frame, a dotted index always corresponds to a positron and a plain index always corresponds to an electron (low and high indices correspond to ingoing and outgoing particles, respectively).

Remark: We do *not* consider different interactions generated by moving indices through some *positive* Hermitian metric h in the fibers of $\boldsymbol{W}$ or $\boldsymbol{U}$. There is actually no need to include any such structure in the fundamental assumptions; though this is actually considered in standard presentations (and usually indicated as $\psi \mapsto \psi^\dagger$), it can be shown[18,1,3] that its assignment is equivalent to the assignment of an observer.

In order to look at e.m. interaction from the point of view of 2-spinors, let

$$\beta \in \mathbb{C}\otimes \boldsymbol{H} \equiv \boldsymbol{U}\otimes\overline{\boldsymbol{U}}, \quad \phi, \psi \in \boldsymbol{U}\oplus\overline{\boldsymbol{U}}^{\star} \equiv \boldsymbol{W},$$

with $\phi \equiv (s, \sigma)$ and $\psi \equiv (u, \chi)$. Then

$$\ell_{\text{int}}(\bar\phi, \beta, \psi) \equiv -\sqrt{2}\,e\,\big(\langle\beta, \bar\sigma\otimes\chi\rangle + \langle\beta^\flat, u\otimes\bar s\rangle\big) = -\sqrt{2}\,e\,\big(\beta^{AA^{\cdot}}\,\bar\sigma_A\,\chi_{A^{\cdot}} + \beta_{AA^{\cdot}}\,u^A\,\bar s^{A^{\cdot}}\big),$$

which is the sum of two interactions: one interaction is the absorption of a photon and two left-handed fermions, the other is the absorption of a photon and two right-handed fermions. Note that, since left-handed fermions are valued either in $\boldsymbol{U}^\star$ or in $\overline{\boldsymbol{U}}^\star$, index positions in ℓ_{int} relatively to these particles are actually *inverted* with respect to the basic prescriptions of the general formalism. Similarly, the absorption of a photon and of an electron and the creation of an electron is represented by

$$\ell_{\text{int}}\rfloor(\beta\otimes\psi) = -\sqrt{2}\,e\,\big(\beta\rfloor\chi,\; u\rfloor\beta^\flat\big) = -\sqrt{2}\,e\,\big(\beta^{AA^{\cdot}}\,\chi_{A^{\cdot}}\,\zeta_A,\; \beta_{AA^{\cdot}}\,u^A\,\bar\zeta^{A^{\cdot}}\big),$$

which again can be seen as the sum of two interactions: the absorption of a photon and a left-handed fermion with the creation of a right-handed fermion, and the absorption of a photon and a right-handed fermion with the creation of a left-handed fermion.

D. Boson and gauge fields

We deal with linear connections of a vector bundle $\boldsymbol{E} \rightarrowtail \boldsymbol{M}$. A *gauge* is essentially a local connection γ_0 with vanishing curvature tensor (a "flat" connection). Then an arbitrary connection γ is characterized locally by the difference $\alpha \equiv \gamma - \gamma_0 : \boldsymbol{M} \to \mathrm{T}^*\boldsymbol{M}\otimes\boldsymbol{E}\otimes\boldsymbol{E}^*$, a true tensor field. Conversely, the field α together with the choice of a gauge γ_0 determines the connection $\gamma = \gamma_0 + \alpha$. The need for a gauge choice derives from the fact that certain classical fields are described as connections, while the description of the corresponding quantum particles requires sections of vector bundles.

Let us focus on a "pre-quantum" treatment of gauge fields as tensor fields, based on the assumption that fermion fields are described as sections of a bundle $\boldsymbol{Y} \rightarrowtail \boldsymbol{M}$, where

$$\boldsymbol{Y} \equiv \boldsymbol{Y}_{\text{R}}\oplus\boldsymbol{Y}_{\text{L}} \equiv (\boldsymbol{F}_{\text{R}}\otimes\boldsymbol{U})\oplus(\boldsymbol{F}_{\text{L}}\otimes\overline{\boldsymbol{U}}^\star),$$

and where $\boldsymbol{F}_{\text{R}} \rightarrowtail \boldsymbol{M}$ and $\boldsymbol{F}_{\text{L}} \rightarrowtail \boldsymbol{M}$ are complex vector bundles (describing the internal fermion structure besides spin) endowed with fibered Hermitian structures. In a previous paper,[4] I proposed to describe the fundamental boson fields in terms of sections of the vector bundles arising from expanding $\overline{Y}\otimes Y$, namely,

$$\overline{Y}\otimes Y \cong (\overline{Y}_{\mathrm{R}}\otimes Y_{\mathrm{R}})\oplus(\overline{Y}_{\mathrm{L}}\otimes Y_{\mathrm{L}})\oplus(\overline{Y}_{\mathrm{R}}\otimes Y_{\mathrm{L}})\oplus(\overline{Y}_{\mathrm{L}}\otimes Y_{\mathrm{R}})\cong$$

$$\cong(\overline{F}_{\mathrm{R}}\otimes F_{\mathrm{R}}\otimes\overline{U}\otimes U)\oplus(\overline{F}_{\mathrm{L}}\otimes F_{\mathrm{L}}\otimes U^{\star}\otimes\overline{U}^{\star})\oplus$$

$$\oplus(\overline{F}_{\mathrm{R}}\otimes F_{\mathrm{L}}\otimes\overline{U}\otimes\overline{U}^{\star})\oplus(\overline{F}_{\mathrm{L}}\otimes F_{\mathrm{R}}\otimes U^{\star}\otimes U).$$

Because of the algebraic structure of the fibers, one gets various contractions among fermion and bosons. We suppose that these contractions are related to the possible particle interactions (roughly analogue to "chemical bonds"). In Sec. V, in particular, we will recover the interactions of the electroweak theory.

Next, we observe that the Hermitian structures of F_{R} and F_{L} determine fibered isomorphisms $\overline{F}_{\mathrm{R}}\cong F_{\mathrm{R}}^{\star}$ and $\overline{F}_{\mathrm{L}}\cong F_{\mathrm{L}}^{\star}$; one also has (Sec. III A) $U\otimes\overline{U}\cong\mathbb{C}\otimes H$ and $U^{\star}\otimes\overline{U}^{\star}\cong\mathbb{C}\otimes H^{*}$. Furthermore, the Lorentz metric yields the isomorphism $H\leftrightarrow H^{*}$, and the tetrad Θ yields the scaled isomorhism $H^{*}\leftrightarrow\mathbb{L}\otimes\mathrm{T}^{*}M$. Hence, after rearranging the order of tensor factors, sections $M\to\mathbb{L}^{-1}\otimes\overline{Y}_{\mathrm{R}}\otimes Y_{\mathrm{R}}$ and $M\to\mathbb{L}^{-1}\otimes\overline{Y}_{\mathrm{L}}\otimes Y_{\mathrm{L}}$ can be seen as fields $M\to\mathrm{T}^{*}M\otimes F_{\mathrm{R}}\otimes F_{\mathrm{R}}^{\star}$ and $M\to\mathrm{T}^{*}M\otimes F_{\mathrm{L}}\otimes F_{\mathrm{L}}^{\star}$, respectively, and are obvious candidates for the role of gauge fields.

The case of sections $M\to\overline{Y}_{\mathrm{R}}\otimes Y_{\mathrm{L}}$ and $M\to\overline{Y}_{\mathrm{L}}\otimes Y_{\mathrm{R}}$ is somewhat different. Among all these one can consider, in particular, those sections which are proportional to the identity of $\overline{U}$ or U; the Higgs field of the electroweak theory can be seen to arise exactly in this way (Sec. V A). On the other hand, one could be lead to consider a larger class of fields in this sector, not obeying the condition of proportionality to the identity.

The question of which kinds of particles and interactions one ought eventually consider is rather complex, and interwoven with the issue of constraints and the degrees of freedom of the gauge fields. These will not be discussed here. We note that eventually, in general, one has to deal with many more particle types than would be implied by a naïf application of "quantization rules" based on the underlying classical-field structure (in which the gauge fields are derived from a connection).

Hence the presented approach must be completed by further study concerning symmetries and constraints. We hope that this may shed some light on the relation between connections (i.e., classical gauge fields) and quantum gauge particles.

IV. MULTI-PARTICLE SPACES, QUANTUM BUNDLES, AND INTERACTIONS

A. Discrete multi-particle spaces

Let X be any set, and consider the complex vector space *freely generated* by X, namely, the space $\mathcal{F}(X)$ of all *finite* formal linear combinations of elements in X. We write

$$\mathcal{F}(X)\equiv\Big\{\phi=\sum_{x\in X_{\phi}}\phi(x)\,|x\rangle\Big\},\quad \phi(x)\in\mathbb{C},\quad X\supset X_{\phi}\text{ finite},$$

namely, $x\in X$ is written as $|x\rangle$ when seen as an element in $\mathcal{F}(X)$.

The dual space $\mathcal{F}^{\backslash}(X)\equiv[\mathcal{F}(X)]^{\backslash}$ is the complex vector space $\mathcal{F}(X)$ of *all* functions $X\to\mathbb{C}$, which we formally write as infinite sums

$$\mathcal{F}^{\backslash}(X)=\Big\{\theta=\sum_{x\in X}\theta(x)\,\langle x|\Big\},\quad \theta(x)\in\mathbb{C},\quad \langle x|x'\rangle=\delta(x,x'),$$

where $\delta(x,x')$ is the ordinary *Kronecker delta*. When $\theta\in\mathcal{F}^{\backslash}(X)$ is applied to $\phi\in\mathcal{F}(X)$, one gets the finite sum

$$\langle\theta,\phi\rangle=\sum_{\substack{x\in X\\ x'\in X_{\phi}}}\theta(x)\,\phi(x')\,\langle x|x'\rangle=\sum_{x\in X_{\phi}}\theta(x)\,\phi(x).$$

Also note that there is a natural antilinear inclusion $\boldsymbol{\mathcal{F}}(\boldsymbol{X}) \hookrightarrow \boldsymbol{\mathcal{F}}^{\backprime}(\boldsymbol{X})$, given by

$$\phi = \sum_{x \in \boldsymbol{X}_\phi} \phi(x)\,|x\rangle \longmapsto \phi^\dagger = \sum_{x \in \boldsymbol{X}_\phi} \bar{\phi}(x)\,\langle x|;$$

this is obviously associated with a Hermitian scalar product in $\boldsymbol{\mathcal{F}}(\boldsymbol{X})$.

Let now $\boldsymbol{Z}$ be a finite-dimensional complex vector space and set

$$\boldsymbol{\mathcal{Z}}^1 := \boldsymbol{\mathcal{F}}(\boldsymbol{X}) \otimes \boldsymbol{Z}, \qquad \boldsymbol{\mathcal{Z}}^{\backprime 1} := \boldsymbol{\mathcal{F}}^{\backprime}(\boldsymbol{X}) \otimes \boldsymbol{Z},$$

namely, $\boldsymbol{\mathcal{Z}}^{\backprime 1}$ is the vector space of all maps $\boldsymbol{X} \to \boldsymbol{Z}$, while $\boldsymbol{\mathcal{Z}}^1$ is its subspace of all such maps which vanish outside a set of finite cardinality. Next, for $n \in \mathbb{N}$ we consider the *n-particles space*

$$\boldsymbol{\mathcal{Z}}^n := \between^n \boldsymbol{\mathcal{Z}}^1 \equiv \begin{cases} \vee^n \boldsymbol{\mathcal{Z}}^1 & \text{(bosons)}, \\ \wedge^n \boldsymbol{\mathcal{Z}}^1 & \text{(fermions)}, \end{cases}$$

where the shorthand $\between$ is used, whenever the involved expressions are formally identical, to denote either the symmetrised or the antisymmetrized tensor product. Moreover, we consider its extension

$$\boldsymbol{\mathcal{Z}}^{\backprime n} \subset \boldsymbol{\mathcal{F}}(\boldsymbol{X}^n) \otimes \between^n \boldsymbol{Z}$$

defined to be the vector space of all maps $\boldsymbol{X} \times \boldsymbol{X} \times \cdots \times \boldsymbol{X} \equiv \boldsymbol{X}^n \to \between^n \boldsymbol{Z}$ which are either symmetric or anti-symmetric in their arguments. Finally, we introduce the *multi-particle spaces*

$$\boldsymbol{\mathcal{Z}} := \bigoplus_{n=0}^{\infty} \boldsymbol{\mathcal{Z}}^n, \qquad \boldsymbol{\mathcal{Z}}^{\backprime} := \bigoplus_{n=0}^{\infty} \boldsymbol{\mathcal{Z}}^{\backprime n}.$$

These are defined by analogy with the usual *Fock spaces* (we take no completion, namely, we consider states constituted by a finite, arbitrarily large number of particles).

All the above constructions can be repeated with the same "source" set $\boldsymbol{X}$ and with the "target" space $\boldsymbol{Z}$ replaced by either the dual space $\boldsymbol{Z}^\star$, or the *anti-dual* space $\overline{\boldsymbol{Z}}^\star \cong \boldsymbol{Z}^{\overline{\star}}$ or the *conjugate space* $\overline{\boldsymbol{Z}} \cong \boldsymbol{Z}^{\star\overline{\star}}$. We obtain the multi-particle spaces indicated as

$$\boldsymbol{\mathcal{Z}}^\star := \bigoplus_{n=0}^{\infty} \boldsymbol{\mathcal{Z}}^{\star n}, \qquad \boldsymbol{\mathcal{Z}}^{\overline{\star}} := \bigoplus_{n=0}^{\infty} \boldsymbol{\mathcal{Z}}^{\overline{\star} n}, \qquad \boldsymbol{\mathcal{Z}}^{\bullet} := \bigoplus_{n=0}^{\infty} \boldsymbol{\mathcal{Z}}^{\bullet n},$$

as well as the respective extended (primed) spaces. Note that we get couples of mutually dual spaces; so, $\boldsymbol{\mathcal{Z}}^n$ and $\boldsymbol{\mathcal{Z}}^{\backprime\star n}$ are mutually dual, and the like. For example, the contraction between $z = \sum_{x \in \boldsymbol{X}_z} |x\rangle \otimes z(x) \in \boldsymbol{\mathcal{Z}}^1$ and $\zeta = \sum_{x \in \boldsymbol{X}} \langle x| \otimes \zeta(x) \in \boldsymbol{\mathcal{Z}}^{\backprime\star 1}$ is given by

$$\langle \zeta, z\rangle = \sum_{x \in \boldsymbol{X}_z} \langle x'|x\rangle\,\langle \zeta(x'), z(x)\rangle = \sum_{x \in \boldsymbol{X}_z} \langle \zeta(x), z(x)\rangle.$$

Remark: More generally, the above constructions and notations apply if $\boldsymbol{Z} \rightarrowtail \boldsymbol{X}$ is a vector bundle: then $\boldsymbol{\mathcal{Z}}^1$ is defined to be the space of all sections $\boldsymbol{X} \to \boldsymbol{Z}$ which vanish outside a set of finite cardinality, while $\boldsymbol{\mathcal{Z}}^{\backprime 1}$ is defined to be the space of all sections $\boldsymbol{X} \to \boldsymbol{Z}$ (here we are using the term "section" in the extended meaning of fibered sets, namely, to denote an arbitrary map $\boldsymbol{X} \to \boldsymbol{Z}$ whose composition with the projection $\boldsymbol{X} \rightarrowtail \boldsymbol{X}$ is the identity: a section in this sense needs not be smooth).

B. Generalized semi-densities and generalized frames

Let $\boldsymbol{Z} \rightarrowtail \boldsymbol{X}$ be a classical complex vector bundle over the real m-dimensional manifold $\boldsymbol{X}$. Denote as $\boldsymbol{\mathcal{D}}_\circ(\boldsymbol{X}, \wedge^m \mathrm{T}^*\boldsymbol{X} \otimes \boldsymbol{Z}^\star)$ the vector space of all smooth sections $\boldsymbol{X} \to \wedge^m \mathrm{T}^*\boldsymbol{X} \otimes \boldsymbol{Z}^\star$ (i.e., $\boldsymbol{Z}^\star$-valued densities on $\boldsymbol{X}$) which have compact support. This space is endowed with a standard topology;[19] its dual space $\boldsymbol{\mathcal{D}}(\boldsymbol{X}, \boldsymbol{Z})$ is called the space of *generalized sections* of $\boldsymbol{Z} \rightarrowtail \boldsymbol{X}$. In particular, any sufficiently regular ordinary section $\theta : \boldsymbol{X} \to \boldsymbol{Z}$ can be seen as an element of $\boldsymbol{\mathcal{D}}(\boldsymbol{X}, \boldsymbol{Z})$

via the rule

$$\langle\theta,\sigma\rangle := \int_X \theta\rfloor\sigma, \quad \sigma\in\mathcal{D}_\circ(\boldsymbol{X},\wedge^m\mathrm{T}^*\boldsymbol{X}\otimes\boldsymbol{Z}^\star),$$

the fiberwise contraction $\theta\rfloor\sigma$ being an ordinary $\mathbb{C}$-valued m-form on $\boldsymbol{X}$. For an arbitrary generalized section we write $\theta:\boldsymbol{X}\rightsquigarrow\boldsymbol{Z}$.

Suppose now that $\boldsymbol{X}$ is *orientable* and choose a positive semi-vector bundle $\mathbb{V}\equiv\mathbb{V}\boldsymbol{X}\equiv(\wedge^n\mathrm{T}\boldsymbol{X})^+$. Its square-root bundle $\mathbb{V}^{1/2}\rightarrowtail\boldsymbol{X}$ and the dual bundle $\mathbb{V}^{-1/2}\equiv(\mathbb{V}^{1/2})^*\rightarrowtail\boldsymbol{X}$ are then well defined. *Generalized* $\boldsymbol{Z}$*-valued semi-densities* (or *half-densities*) on $\boldsymbol{X}$ are now defined to be generalized sections $\boldsymbol{X}\rightsquigarrow\mathbb{V}^{-1/2}\otimes\boldsymbol{Z}$, namely, elements of the vector space

$$\not{\mathcal{D}}(\boldsymbol{X},\boldsymbol{Z})\equiv\mathcal{D}(\boldsymbol{X},\mathbb{V}^{-1/2}\otimes\boldsymbol{Z})$$

dual to

$$\not{\mathcal{D}}_\circ(\boldsymbol{X},\boldsymbol{Z}^\star)\equiv\mathcal{D}_\circ\big(\boldsymbol{X},\mathbb{V}^{-1/2}\otimes\boldsymbol{Z}^\star\big)\cong\mathcal{D}_\circ\big(\boldsymbol{X},\mathbb{V}^{-1}\otimes(\mathbb{V}^{1/2}\otimes\boldsymbol{Z}^\star)\big).$$

By varying the "target bundle" one also gets other kinds of generalized sections; in particular, one gets generalized densities and generalized currents.[19,20] Semi-densities have a special status because of the natural inclusion $\not{\mathcal{D}}_\circ(\boldsymbol{X},\boldsymbol{Z})\subset\not{\mathcal{D}}(\boldsymbol{X},\boldsymbol{Z})$, but even more so if the fibers of the $\boldsymbol{Z}\rightarrowtail\boldsymbol{X}$ are endowed with a Hermitian metric. In fact, in this case one has the space $\mathcal{L}^2(\boldsymbol{X},\boldsymbol{Z})$ of all ordinary semi-densities $\theta:\boldsymbol{X}\to\mathbb{V}^{-1/2}\otimes\boldsymbol{Z}$ such that

$$\int_X\bar\theta\rfloor\theta<\infty.$$

The quotient $\mathcal{H}(\boldsymbol{X},\boldsymbol{Z})=\mathcal{L}^2(\boldsymbol{X},\boldsymbol{Z})/\boldsymbol{0}$ is then a Hilbert space, where $\boldsymbol{0}\subset\mathcal{L}^2(\boldsymbol{X},\boldsymbol{Z})$ denotes the subspace of all almost-everywhere vanishing sections, and we get a so-called *rigged Hilbert space*[7]

$$\not{\mathcal{D}}_\circ(\boldsymbol{X},\boldsymbol{Z})\subset\mathcal{H}(\boldsymbol{X},\boldsymbol{Z})\subset\not{\mathcal{D}}(\boldsymbol{X},\boldsymbol{Z}).$$

Elements in $\not{\mathcal{D}}(\boldsymbol{X},\boldsymbol{Z})\setminus\mathcal{H}(\boldsymbol{X},\boldsymbol{Z})$ can then be identified with the "generalized states" (or "non-normalizable" states) of the common terminology.

Let $\delta[x]$ be the *Dirac density* on $\boldsymbol{X}$ with support $\{x\}$, $x\in\boldsymbol{X}$; namely, $\delta[x]$ is the $\mathbb{C}$-valued generalized density acting as $\langle\delta[x],f\rangle=f(x)$ for all functions $f:\boldsymbol{X}\to\mathbb{C}$. If $s:\boldsymbol{X}\to\boldsymbol{Z}$ is any ordinary section, then we say that $\delta[x]\otimes s$ is a $\boldsymbol{Z}$-valued generalized density *of Dirac-type*. In particular, a generalized semi-density of Dirac-type is of the form $\delta[x]\otimes v\in\not{\mathcal{D}}(\boldsymbol{X},\boldsymbol{Z})$ with $v:\boldsymbol{X}\to\mathbb{V}^{1/2}\otimes\boldsymbol{Z}$, and a generalized section of Dirac-type is of the form $\delta[x]\otimes\alpha\in\mathcal{D}(\boldsymbol{X},\boldsymbol{Z})$ with $\alpha:\boldsymbol{X}\to\mathbb{V}\otimes\boldsymbol{Z}$.

Consider now the complex vector spaces

$$\underline{\mathcal{D}}(\boldsymbol{X},\boldsymbol{Z}),\quad \underline{\not{\mathcal{D}}}(\boldsymbol{X},\boldsymbol{Z}),\quad \underline{\mathcal{D}}(\boldsymbol{X},\mathbb{V}^{-1}\otimes\boldsymbol{Z}),$$

defined to be the spaces of all *finite linear combinations* of Dirac-type generalized sections, semi-densities, and densities, respectively. If a volume form $\eta:\boldsymbol{X}\to\mathbb{V}^{-1}$ *is assigned*, then one gets distinguished isomorphisms among the above spaces, determined by

$$\delta[x]\otimes\eta^{-1}\longleftrightarrow\delta[x]\otimes\eta^{-1/2}\longleftrightarrow\delta[x].$$

Remark: An important result in the theory of distributions (see Schwartz,[19] Chap. III, p. 75) implies that these $\underline{\mathcal{D}}$ spaces are dense in the respective "non-underlined" spaces; so, for example, any element in $\mathcal{D}(\boldsymbol{X},\boldsymbol{Z})$ can be approximated with arbitrary precision (in the sense of the topology of distributional spaces) by a finite linear combination of Dirac-type densities.

We can relate the above spaces to the space $\mathcal{Z}^1$ introduced in Sec. IV A (see the concluding remark there); in fact there is a natural identification $\mathcal{Z}^1\cong\underline{\mathcal{D}}(\boldsymbol{X},\mathbb{V}^{-1}\otimes\boldsymbol{Z})$, given by

$$|x\rangle\otimes z\longleftrightarrow\delta[x]\otimes\tilde z,\quad x\in\boldsymbol{X},\,z\in\boldsymbol{Z}_x,$$

where $\tilde z:\boldsymbol{X}\to\boldsymbol{Z}$ in any section such that $\tilde z(x)=z$. If a volume form η is assigned, then one also gets distinguished identifications $\mathcal{Z}^1\leftrightarrow\underline{\mathcal{D}}(\boldsymbol{X},\boldsymbol{Z})\leftrightarrow\underline{\not{\mathcal{D}}}(\boldsymbol{X},\boldsymbol{Z})$. Consider, in particular, the latter space and the correspondence

$$|x\rangle \otimes \mathsf{b}_\alpha(x) \leftrightarrow \mathsf{B}_{x,\alpha} \equiv \delta[x] \otimes \eta^{-1/2} \otimes \mathsf{b}_\alpha(x),$$

where (b_α) is a frame of $\boldsymbol{Z} \rightarrowtail \boldsymbol{X}$ (for simplicity we assume here that the domain of the "classical" frame (b_α) is the whole $\boldsymbol{X}$; this assumption is actually true in the main cases of interest and could be dropped at the price of a lengthier discussion).

We say that $(\mathsf{B}_{x,\alpha})$ is a *generalized frame* of the space $\boldsymbol{\mathcal{D}}(\boldsymbol{X}, \boldsymbol{Z})$ of all generalized semi-densities. This is natural in view of the theorem cited in the above remark, and a "generalized index" notation turns out to be handy: we write $\mathsf{B}^{x,\alpha} \equiv \delta[x] \otimes \eta^{-1/2} \otimes \mathsf{b}^\alpha(x)$, where (b^α) is the dual classical frame. Though a contraction of two distributions is not defined in general, a straightforward extension of the discrete-space operation yields

$$\langle \mathsf{B}^{x',\alpha'}, \mathsf{B}_{x,\alpha}\rangle = \delta^{x',\alpha'}_{x,\alpha}\, \eta(x).$$

This is consistent with "index summation" in a generalized sense. Actually, let $f \in \boldsymbol{\mathcal{D}}_\circ(\boldsymbol{X}, \boldsymbol{Z})$ and $\lambda \in \boldsymbol{\mathcal{D}}_\circ(\boldsymbol{X}, \boldsymbol{Z}^\star)$ be test semi-densities and write

$$f^{x,\alpha} \equiv f^\alpha(x) \equiv \langle \mathsf{B}^{x,\alpha}, f\rangle, \quad \lambda_{x,\alpha} \equiv \lambda_\alpha(x) \equiv \langle \lambda, \mathsf{B}_{x,\alpha}\rangle.$$

Then also

$$\langle \lambda, f\rangle \equiv \lambda_{x',\alpha'}\, f^{x,\alpha}\, \langle \mathsf{B}^{x',\alpha'}, \mathsf{B}_{x,\alpha}\rangle \equiv \int_{\boldsymbol{X}} \lambda_\alpha(x)\, f^\alpha(x)\, \eta(x),$$

namely, we interpret index summation with respect to the continuous variable x as integration (provided by the chosen volume form). Possibly, this formalism can be extended to the contraction of two generalized semi-densities whenever it makes sense.

The identification $\mathcal{Z}^1 \leftrightarrow \underline{\boldsymbol{\mathcal{D}}}(\boldsymbol{X}, \boldsymbol{Z})$ naturally extends to $\mathcal{Z}^n \leftrightarrow \bigvee^n \underline{\boldsymbol{\mathcal{D}}}(\boldsymbol{X}, \boldsymbol{Z})$, and the latter space turns out to be dense in $\boldsymbol{\mathcal{D}}(\boldsymbol{X}^n, \bigvee^n \boldsymbol{Z})$, where $\boldsymbol{X}^n \equiv \boldsymbol{X} \times \cdots \times \boldsymbol{X}$. Then one straightforwardly introduces frames of multi-particle spaces of generalized semi-densities.

C. Distributional quantum bundles

The notion of smoothness introduced by Frölicher,[21] or *F-smoothness*, provides a general setting for calculus in functional spaces[22,23] and differential geometry in functional bundles.[10,23–25] An important aspect of that approach is that the essential results can be formulated in terms of finite-dimensional spaces and maps, without heavy involvement in infinite-dimensional topology and other intricated questions. In particular, the notion of a smooth connection on a functional bundle has been applied in the context of the "covariant quantization" approach to quantum mechanics,[9–11] The F-smooth geometry of distributional bundles and quantum connections has been studied in a previous paper.[6]

Let $(\boldsymbol{M}, g)$ be Einstein's spacetime. Taking the speed of light and the Planck constant into account (see the remark concluding Sec. II D), the covariant form of a particle's 4-momentum turns out to be valued into $\boldsymbol{P}_m \subset \boldsymbol{P} \cong \mathrm{T}^*\boldsymbol{M}$, where

$$\boldsymbol{P}_m := \{p \in \boldsymbol{P} : g^\#(p, p) = -m^2,\ p \text{ future pointing}\} \rightarrowtail \boldsymbol{M}$$

is the bundle of "mass -shells" corresponding to mass $m \in \{0\} \cup \mathbb{L}^{-1}$.

Let now $\boldsymbol{Z} \to \boldsymbol{P}_m$ be a vector bundle (representing the "internal degrees of freedom" of the considered particle type). At each $x \in \boldsymbol{M}$ we can perform the constructions of Sec. IV A, with the generic manifold $\boldsymbol{X}$ replaced now by $(\boldsymbol{P}_m)_x$. In particular, we have spaces

$$\mathcal{Z}^1_x := \mathcal{F}((\boldsymbol{P}_m)_x) \otimes \boldsymbol{Z}_x, \qquad \mathcal{Z}^{\backslash 1}_x := \mathcal{F}^{\backslash}((\boldsymbol{P}_m)_x) \otimes \boldsymbol{Z}_x,$$

and the like. The fibered sets

$$\mathcal{Z}^1 := \bigsqcup_{x\in\boldsymbol{M}} \mathcal{Z}^1_x, \quad \mathcal{Z}^{\backslash 1} := \bigsqcup_{x\in\boldsymbol{M}} \mathcal{Z}^{\backslash 1}_x$$

have natural *F-smooth* structures of vector bundles over $\boldsymbol{M}$. Furthermore, the multi-particle bundles $\mathcal{Z} := \bigoplus_n \mathcal{Z}^n \rightarrowtail \boldsymbol{M}$ and $\mathcal{Z}^{\backslash} := \bigoplus_n \mathcal{Z}^{\backslash n} \rightarrowtail \boldsymbol{M}$, $n \in \{0\} \cup \mathbb{N}$, can be straightforwardly constructed by an obvious extension of the procedure of Sec. IV A. Their role in the representation of asymptotic quantum states will be made precise below.

Analogous constructions yield distributional bundles over $\boldsymbol{M}$. In particular, we are interested in the bundle of generalized semi-densities: for each $x \in \boldsymbol{M}$ we consider the vector space $\boldsymbol{\mathcal{D}}((\boldsymbol{P}_m)_x, \boldsymbol{Z}_x)$ and get the F-smooth vector bundle

$$\boldsymbol{\mathcal{D}}(\boldsymbol{P}_m, \boldsymbol{Z}) := \bigsqcup_{x \in \boldsymbol{M}} \boldsymbol{\mathcal{D}}((\boldsymbol{P}_m)_x, \boldsymbol{Z}_x) \rightarrowtail \boldsymbol{M},$$

and its sub-bundle $\underline{\boldsymbol{\mathcal{D}}}(\boldsymbol{P}_m, \boldsymbol{Z})$ whose fibers are constituted of finite linear combinations of Dirac-type semi-densities. In order to describe quantum states as sections of bundles of generalized semi-densities, we need an isomorphism $\boldsymbol{\mathcal{Z}}^1 \leftrightarrow \underline{\boldsymbol{\mathcal{D}}}(\boldsymbol{P}_m, \boldsymbol{Z})$, namely, we need a volume form on the fibers of $\boldsymbol{P}_m \to \boldsymbol{M}$. There is actually a distinguished *scaled* such form, namely, the *Leray form* $\omega_m : \boldsymbol{P}_m \to \mathbb{L}^{-2} \otimes \wedge^3 \mathrm{T}^* \boldsymbol{P}_m$ of the mass-shells;[26] this is usually indicated as $\delta(p^2 - m^2)$ where $p^2 \equiv g^\#(p, p)$. In practice, however, one uses a related but somewhat different form, determined by the choice of an observer (Sec. IV D).

Remark: Summarizing, a finite-dimensional two-fibered bundle $\boldsymbol{Z} \rightarrowtail \boldsymbol{P}_m \rightarrowtail \boldsymbol{M}$ is the basic classical geometric datum underlying the description of a particle type. Now the spacetime connection Γ determines a connection Γ_m of $\boldsymbol{P}_m \to \boldsymbol{M}$; this, together with a linear connection of $\boldsymbol{Z} \rightarrowtail \boldsymbol{M}$ projectable over Γ, determines a linear connection of the distributional bundle $\boldsymbol{\mathcal{D}}(\boldsymbol{P}_m, \boldsymbol{Z}) \rightarrowtail \boldsymbol{M}$ which can be characterized in the simplest way as follows. Let $\boldsymbol{N} \subset \boldsymbol{M}$ be any one-dimensional submanifold; let $p : \boldsymbol{N} \to \boldsymbol{P}_m$ and $z(p) : \boldsymbol{N} \to \boldsymbol{Z}$ be parallely transported sections of the restricted bundles; then

$$\delta[p] \otimes \omega_m^{-1/2} \otimes z(p) : \boldsymbol{N} \to \mathbb{L} \otimes \boldsymbol{\mathcal{D}}(\boldsymbol{P}_m, \boldsymbol{Z})$$

is parallely transported. Actually, $z(p)$ projects over p since the connection of $\boldsymbol{Z} \rightarrowtail \boldsymbol{M}$ is projectable over Γ_m (which is true for most relevant physical theories). We could use this parallel transport, along a detector's world-line (Sec. IV D), in order to define the *free-particle states*. However, we will actually consider a variation of this construction.

D. Detectors and free-particle states

In particle physics, one needs some kind of an observer. We will now see how considering a *detector*, represented by a given (timelike) world-line $\boldsymbol{T} \subset \boldsymbol{M}$, suffices for reproducing, in terms of generalized semi-densities, essentially the standard momentum space formalism. This can be seen as a sort of a complicated "clock" carried by the detector.

If $\boldsymbol{P}_{\boldsymbol{T}} \rightarrowtail \boldsymbol{T}$ is the restriction of $\boldsymbol{P} \equiv \mathrm{T}^*\boldsymbol{M} \rightarrowtail \boldsymbol{M}$, then one has the g-orthogonal splitting

$$\boldsymbol{P}_{\boldsymbol{T}} = \boldsymbol{P}_\parallel \oplus_{\boldsymbol{T}} \boldsymbol{P}_\perp.$$

The restriction of $g^\#$ to the fibers of $\boldsymbol{P}_\perp \rightarrowtail \boldsymbol{M}$ is an $\mathbb{L}^{-2}$-scaled Euclidean metric, yielding an $\mathbb{L}^{-3}$-scaled volume form

$$\eta_\perp : \boldsymbol{P}_\perp \to \mathbb{L}^{-3} \otimes \wedge^3 (\boldsymbol{P}_\perp)^*.$$

If $\tau_0 : \boldsymbol{T} \to \mathbb{L}^{-1} \otimes \mathrm{T}\boldsymbol{M}$ is the normalized scaled vector field tangent to $\boldsymbol{T}$, then $\tau_0 \wedge \eta_\perp : \boldsymbol{T} \to \mathbb{L}^{-4} \otimes \wedge^4 \mathrm{T}\boldsymbol{M}$ is the spacetime contravariant scaled volume form determined by g.

The restriction of the orthogonal projection $\boldsymbol{P} \to \boldsymbol{P}_\perp$ over $\boldsymbol{T}$ yields distinguished diffeomorphisms $\boldsymbol{P}_m \leftrightarrow \boldsymbol{P}_\perp$ for all $m \in \{0\} \cup \mathbb{L}^{-1}$. The pull-back of $\eta_\perp$ is then an $\mathbb{L}^{-3}$-scaled volume form on $\boldsymbol{P}_m$, which is denoted, by abuse of language, by the same symbol. Then we have

$$\omega_m(p) = \frac{1}{2\, p_0}\, \eta_\perp = \frac{1}{2\, \mathrm{E}_m(p_\perp)}\, \eta_\perp, \qquad p \in \boldsymbol{P}_m,$$

where $p = p_\parallel + p_\perp$ with $p_\parallel \in \boldsymbol{P}_\parallel$ and $p_\perp \in \boldsymbol{P}_\perp$ and

$$p_0 \equiv \mathrm{E}_m(p_\perp) \equiv \sqrt{{p_\parallel}^2} = \sqrt{m^2 + p_\perp^2} \in \mathbb{L}^{-1}.$$

Since ω_m and $\eta_\perp$ are scaled volume forms, they yield isomorphisms

$$\mathcal{Z}^1 \leftrightarrow \mathbb{L}\otimes\underline{\mathcal{D}}(\boldsymbol{P}_m,\boldsymbol{Z}), \qquad \mathcal{Z}^1 \leftrightarrow \mathbb{L}^{3/2}\otimes\underline{\mathcal{D}}(\boldsymbol{P}_m,\boldsymbol{Z}),$$

determined respectively by the correspondences

$$|p\rangle \leftrightarrow \delta[p]\otimes\omega_m^{-1/2}, \qquad |p\rangle \leftrightarrow \delta[p]\otimes\eta_\perp^{-1/2},$$

where $\delta[p]\in\mathcal{D}(\boldsymbol{P}_m,\mathbb{V}^{-1}\boldsymbol{P}_m)$ is the Dirac density with support $\{p\}$, $p\in\boldsymbol{P}_m$. We need a fixed length unit $l\in\mathbb{L}$ in order to obtain unscaled correspondences; in particular, we write

$$\mathcal{Z}^1 \leftrightarrow \underline{\mathcal{D}}(\boldsymbol{P}_m,\boldsymbol{Z}) \;:\; |p\rangle \leftrightarrow \mathsf{X}_p \equiv l^{-3/2}\,\delta[p]\otimes\eta_\perp^{-1/2}.$$

Remark: Unscaled reference states will be needed in order to recover a consistent theory; the need for the choice of a unit of length $l\in\mathbb{L}$ comes at this point (compare with the usual "box quantization" argument).

We will use orthonormal $\mathbb{L}^{-1}$-scaled coordinates $(\mathsf{p}_\lambda)\equiv(\mathsf{p}_0,\mathsf{p}_1,\mathsf{p}_2,\mathsf{p}_3)$ in the fibers of $\boldsymbol{P}_m$, and write

$$\eta_\perp = \mathsf{p}_1\wedge\mathsf{p}_2\wedge\mathsf{p}_3 \equiv \mathrm{d}^3\mathsf{p} \quad\Rightarrow\quad \omega_m = \frac{1}{2\,\mathsf{p}_0}\,\mathrm{d}^3\mathsf{p}.$$

We will also use the shorthand $\mathsf{p}_\perp\equiv(\mathsf{p}_1,\mathsf{p}_2,\mathsf{p}_3)$.

We obtain a *generalized frame of free quantum states* along $\boldsymbol{T}$ as follows. First, at some event $t_0\in\boldsymbol{T}\subset\boldsymbol{M}$ we choose a frame of $\boldsymbol{Z}\to(\boldsymbol{P}_m)_{t_0}$, namely, we smoothly choose a basis $(\mathsf{b}_\alpha(p))$ of $\boldsymbol{Z}_p$ for each $p\in(\boldsymbol{P}_m)_{t_0}$; hence the family of generalized semi-densities

$$\mathsf{A}_{p,\alpha}\equiv\mathsf{X}_p\otimes\mathsf{b}_\alpha(p)\in\mathcal{D}(\boldsymbol{P}_m,\boldsymbol{Z})_{t_0}$$

is a generalized frame at t_0. Next, $\mathsf{A}_{p,\alpha}$ has to be to transported along $\boldsymbol{T}$, via some transport of p and $z(p)$. As observed at the end of Sec. IV C, one could use the parallel transport determined by the spacetime connection and by a connection of $\boldsymbol{Z}\rightarrowtail\boldsymbol{M}$ projectable onto the first. We note, however, that $\eta_\perp$ (appearing in the definition of X_p) is not parallely transported along $\boldsymbol{T}$ unless $\boldsymbol{T}$ is geodesic; thus using Fermi transport for p seems natural, since $\eta_\perp$ is actually Fermi-transported. As for $z(p)$, it will typically have spinor and non-spinor factors; the former can be Fermi transported (Sec. III B), while the latter is parallely transported along p or just $\boldsymbol{T}$ (the non-spinor part of the internal structure is often described by a vector bundle over $\boldsymbol{M}$).

Remark: We then realize that different types of parallel transport can be considered for the definition of free states. A detailed comparison among these may allow insight into gravitational effects in quantum observations.

We use, along $\boldsymbol{T}$, the linear correspondence

$$\mathcal{Z}^1_{\boldsymbol{T}}\equiv\underline{\mathcal{F}}(\boldsymbol{P}_m,\boldsymbol{Z})_{\boldsymbol{T}} \longleftrightarrow \underline{\mathcal{D}}(\boldsymbol{P}_m,\boldsymbol{Z})_{\boldsymbol{T}} \hookrightarrow \mathcal{D}(\boldsymbol{P}_m,\boldsymbol{Z})_{\boldsymbol{T}}$$

characterized by

$$|p,{\scriptstyle A}\rangle\equiv|p,\mathsf{b}_\alpha(p)\rangle \longleftrightarrow \mathsf{A}_{p,\alpha}\equiv l^{-3/2}\,\delta[p]\otimes\eta_\perp^{-1/2}\otimes\mathsf{b}_\alpha(p),$$

and determined by the spacelike volume form associated with the detector and the chosen length unit. The free-particle states determine trivializations

$$\mathcal{D}(\boldsymbol{P}_m,\boldsymbol{Z})_{\boldsymbol{T}}\cong\boldsymbol{T}\times\mathcal{D}(\boldsymbol{P}_m,\boldsymbol{Z})_{t_0}$$

and the like, where $t_0\in\boldsymbol{T}$ is some arbitrarily chosen event. In particular, $(\mathcal{Z}^1)_{t_0}$ can be identified with the space of free one-particle states; the multi-particle space

$$\mathcal{Z}_{t_0}\equiv\bigoplus_{n=0}^{\infty}(\mathcal{Z}^n)_{t_0}, \qquad (\mathcal{Z}^n)_{t_0}\equiv\mathbb{X}^n(\mathcal{Z}^1)_{t_0}$$

(see Sec. IV A) is the space of all quantum states for a given particle type.

A realistic physical theory must have more than one particle types, of masses m', m'', ... and internal structure bundles $\boldsymbol{Z}' \to \boldsymbol{P}_{m'}$, $\boldsymbol{Z}'' \to \boldsymbol{P}_{m''}$, ... We get multi-particle bundles $\boldsymbol{\mathcal{Z}}'$, $\boldsymbol{\mathcal{Z}}''$ and the like, and the total quantum bundle

$$\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{Z}}' \underset{\boldsymbol{M}}{\otimes} \boldsymbol{\mathcal{Z}}'' \underset{\boldsymbol{M}}{\otimes} \cdots$$

of the considered theory (a tensor product of multi-particle bundles, one for each particle type). Trivialization along a given detector naturally extends to $\boldsymbol{\mathcal{V}}$, namely,

$$\boldsymbol{\mathcal{V}}_{\boldsymbol{T}} \cong \boldsymbol{T} \times \boldsymbol{\mathcal{V}}_{t_0},$$

so that $\boldsymbol{\mathcal{V}}_{t_0}$ can be identified with the space $\boldsymbol{\mathcal{Q}}$ of all quantum states considered in the Introduction. By construction, the free-particle trivialization preserves particle type and number; the interaction changes them by mixing the various subspaces of $\boldsymbol{\mathcal{Q}}$.

Remark: Let $(\mathrm{T}\boldsymbol{M})_{\boldsymbol{T}} \rightarrowtail \boldsymbol{T}$ be the restriction of the tangent bundle $\mathrm{T}\boldsymbol{M} \rightarrowtail \boldsymbol{M}$. Similar to the g-orthogonal splitting of $\boldsymbol{P}_{\boldsymbol{T}}$ one has the splitting $(\mathrm{T}\boldsymbol{M})_{\boldsymbol{T}} = (\mathrm{T}\boldsymbol{M})^{\parallel}_{\boldsymbol{T}} \oplus_{\boldsymbol{T}} (\mathrm{T}\boldsymbol{M})^{\perp}_{\boldsymbol{T}}$. Exponentiation determines, for each $t \in \boldsymbol{T}$, a diffeomorphism from a neighbourhood of 0 in $(\mathrm{T}\boldsymbol{M})^{\perp}_t$ to a spacelike submanifold $\boldsymbol{M}_t \subset \boldsymbol{M}$, and so a three-dimensional foliation of a neighbourhood of $\boldsymbol{T}$. If $\psi \in \boldsymbol{\mathcal{D}}(\boldsymbol{P}_m, \boldsymbol{Z})_t$ is a *tempered* generalized semi-density, then via Fourier transform we obtain a generalized semi-density $\check{\psi} \in \boldsymbol{\mathcal{D}}((\mathrm{T}\boldsymbol{M})^{\perp}_t, \boldsymbol{Z})$. A suitable restriction of $\check{\psi}$ then yields, via exponentiation, a semi-density on $\boldsymbol{M}_t$. In flat spacetime, the correspondence $\psi \leftrightarrow \check{\psi}$ essentially amounts to the usual correspondence between momentum-space and position-space representation.

E. Free-particle states in QED and gauge theories

In order to define appropriate generalized frames for free electron and positron states one needs, for each $p \in (\boldsymbol{P}_m)_{\boldsymbol{T}}$, a "classical" frame of $\boldsymbol{W}_p$ adapted to the splitting $\boldsymbol{W}_p = \boldsymbol{W}^+_p \oplus \boldsymbol{W}^-_p$. We use a Dirac frame (Sec. III C)

$$\big(\mathsf{u}_{A}(p),\, \mathsf{v}_{A}(p)\big), \quad A = 1, 2,$$

required to be Fermi transported along $\boldsymbol{T}$ as well as p (Sec. III B), and get the generalized frames

$$\mathsf{A}_{pA} := \mathsf{X}_p \otimes \mathsf{u}_{A}(p) : \boldsymbol{T} \to \boldsymbol{\mathcal{D}}(\boldsymbol{P}_m, \boldsymbol{W}^+_m),$$

$$\mathsf{C}_{pA\cdot} := \mathsf{X}_p \otimes \bar{\mathsf{v}}_{A\cdot}(p) : \boldsymbol{T} \to \boldsymbol{\mathcal{D}}(\boldsymbol{P}_m, \overline{\boldsymbol{W}}{}^-_m),$$

respectively, for electrons and positrons.

Next, we consider the zero-mass sub-bundle $\boldsymbol{P}_0 \subset \mathrm{T}^*\boldsymbol{M}$ of future null half-cones. We will use the identifications $\boldsymbol{H} \cong \mathbb{L}^{-1} \otimes \mathrm{T}\boldsymbol{M}$ and $\boldsymbol{H}^* \cong \mathbb{L} \otimes \mathrm{T}^*\boldsymbol{M}$ determined by the fixed tetrad (Sec. III C). Let $\boldsymbol{H}' \rightarrowtail \boldsymbol{P}_0 \rightarrowtail \boldsymbol{M}$ be the two-fibered bundle whose fiber over any $k \in (\boldsymbol{P}_0)_x$, $x \in \boldsymbol{M}$, is the three-dimensional real vector space $\boldsymbol{H}'_k := \{y \in \boldsymbol{H} : \langle k, y\rangle = 0\}$. Since $k^\# \equiv g^\#(k) \in \mathbb{L}^{-1} \otimes \boldsymbol{H}'$, we have the real vector bundle $\boldsymbol{B}_{\mathbb{R}} \rightarrowtail \boldsymbol{P}_0$ whose fiber over any $k \in \boldsymbol{P}_0$ is the two-dimensional quotient space $\boldsymbol{H}'_k/\langle k\rangle$ (here $\langle k\rangle$ denotes the vector space generated by k).

It turns out that the spacetime metric "passes to the quotient," so it naturally determines a negative metric $g_{\boldsymbol{B}}$ in the fibers of $\boldsymbol{B}_{\mathbb{R}} \rightarrowtail \boldsymbol{P}_0$, as well as a "Hodge" isomorphism $*_{\boldsymbol{B}}$ which can be characterized through the rule $*(k \wedge \beta) = -k \wedge (*_{\boldsymbol{B}}\beta)$.

The complexified two-fibered bundle $\boldsymbol{B} := \mathbb{C} \otimes \boldsymbol{B}_{\mathbb{R}} \rightarrowtail \boldsymbol{P}_0 \rightarrowtail \boldsymbol{M}$ (which we may call the *optical bundle*, though this term has no unique meaning in the literature[27]) has the canonical splitting

$$\boldsymbol{B} = \boldsymbol{B}^+ \underset{\boldsymbol{P}_0}{\oplus} \boldsymbol{B}^-,$$

where the fibers of $\boldsymbol{B}^{\pm} \to \boldsymbol{P}_0$ are complex one-dimensional $g_{\boldsymbol{B}}$-null subspaces defined to be the eigenspaces of $-\mathrm{i}\,*_{\boldsymbol{B}}$ with eigenvalues ± 1 (*self-dual* and *anti-self-dual* subspaces).

If we restrict the above bundles to the detector's world line $\boldsymbol{T} \subset \boldsymbol{M}$, then we can identify $\boldsymbol{B}_{\mathbb{R}} \rightarrowtail \boldsymbol{P}_0 \rightarrowtail \boldsymbol{T}$ with $\boldsymbol{H}' \cap \boldsymbol{H}^{\perp} \rightarrowtail \boldsymbol{P}_0 \rightarrowtail \boldsymbol{T}$ ("radiation gauge"). For any $k \in (\boldsymbol{P}_0)_x$, $x \in \boldsymbol{M}$, let (τ_λ)

be a Pauli basis of $\boldsymbol{H}$ at x such that τ_0 is tangent to $\boldsymbol{T}$ and $k^{\#} \propto \tau_0 + \tau_3$; setting

$$(\mathsf{b}_1, \mathsf{b}_2) := \Big(\tfrac{1}{\sqrt{2}}\,(\tau_1 + \mathrm{i}\,\tau_2),\ \tfrac{1}{\sqrt{2}}\,(\tau_1 - \mathrm{i}\,\tau_2)\Big) \subset \mathbb{C} \otimes \boldsymbol{H}' \cap \boldsymbol{H}^{\perp},$$

$$\mathsf{B}_{\kappa\varrho} := \mathsf{X}_k \otimes \mathsf{b}_{\varrho}(k), \quad k \in \boldsymbol{P}_0,\ \varrho = 1, 2,$$

one gets, by Fermi transport, a generalized frame $\{\mathsf{B}_{\kappa\varrho}\}$ of the quantum bundle $\boldsymbol{\mathcal{D}}_M(\boldsymbol{P}_0, \boldsymbol{B}) \rightarrowtail \boldsymbol{M}$. This frame is adapted to the self-dual/anti-self-dual splitting.

Virtual photons, on the other hand, span a larger bundle: they are described as Fermi-transported sections $\boldsymbol{T} \to \boldsymbol{\mathcal{D}}(\boldsymbol{P}, \mathbb{C} \otimes \boldsymbol{H})$, where one uses the generalized frame $\{\mathsf{B}_{k\lambda}\} := \{\mathsf{X}_k \otimes \tau_\lambda\}$, $\lambda = 0, 1, 2, 3$.

Generalized frames of free-particle states for more general gauge theories, of the kind sketched in Sec. III D, can be introduced by a natural extension of the above constructions, by inserting appropriate frames of the bundles $\boldsymbol{F}_{\mathrm{R}}$ and $\boldsymbol{F}_{\mathrm{L}}$. Basically, we need an extended transport mechanism along $\boldsymbol{T}$. Fermi transport is not applicable to internal degrees of freedom which are not soldered to spinors or spacetime vectors, so we will use covariant constancy relatively to background connections of $\boldsymbol{F}_{\mathrm{R}}$ and $\boldsymbol{F}_{\mathrm{L}}$ which have to be assumed.

F. Quantum interactions

The main purpose of this paper is to discuss criteria by which a sensible interaction can be constructed. While one could think of considering an interaction acting directly on the discrete multi-particle spaces, a few attempts show that, in general, an approach of this kind can only work at the lowest order. The full quantum formalism can be recovered if free-particle states are represented as semi-densities, so that their interactions are described by a suitable endomorphism in the state space of the theory. More precisely, interactions are described by a section

$$-\mathrm{i}\,\mathrm{d}\mathsf{t} \otimes \mathfrak{H} : \boldsymbol{T} \to \mathrm{T}^*\boldsymbol{T} \otimes \mathrm{End}(\boldsymbol{\mathcal{V}}),$$

where the function $\mathsf{t} : \boldsymbol{T} \to \mathbb{R} \otimes \mathbb{T} \cong \mathbb{R} \otimes \mathbb{L}$ (taking the speed of light $c \in \mathbb{L} \otimes \mathbb{T}^{-1}$ into account) is the detector's *proper time*, unique up to the choice of an "initial" time; thus $\mathrm{d}\mathsf{t} : \boldsymbol{T} \to \mathbb{L} \otimes \mathrm{T}^*\boldsymbol{T}$. This means that $\mathfrak{H}$ itself has to be an $\mathbb{L}^{-1}$-scaled morphism $T \to \mathbb{L}^{-1} \otimes \mathrm{End}(\boldsymbol{\mathcal{V}})$.

Remark: We can see the free-particle trivialization as determined by a *free-particle connection* of the functional bundle $\boldsymbol{\mathcal{V}} \rightarrowtail \boldsymbol{T}$. Thus the interaction is a tensor field which modifies that connection; using the trivialization, the interaction can be seen as a 1-form on $\boldsymbol{T}$ valued into the endomorphisms of a fixed space $\boldsymbol{\mathcal{Q}}$ (say $\boldsymbol{\mathcal{V}}_{t_0}$) as sketched in the Introduction.

Essentially, $\mathfrak{H}$ arises as the tensor product of the classical interaction and a certain semi-density $\underline{\Lambda}$ on the fibers of particle momenta (the "quantum ingredient" of the interaction). In order to make this idea more precise we have to introduce some further notations. Consider r masses $m', m'', \dots, m^{(r)}$, and let the shorthand

$$\boldsymbol{P}_{\times} := \boldsymbol{P}_{m'} \underset{\boldsymbol{M}}{\times} \boldsymbol{P}_{m''} \underset{\boldsymbol{M}}{\times} \cdots \underset{\boldsymbol{M}}{\times} \boldsymbol{P}_{m^{(r)}} \rightarrowtail \boldsymbol{M}$$

denote the bundle of r particle momenta corresponding to these masses. We can exhibit a distinguished *scaled* generalized density on the fibers of the restricted bundle $\boldsymbol{P}_{\times} \to \boldsymbol{T}$; this is characterized by the requirement that it acts on a test element $f \in \boldsymbol{\mathcal{D}}_{\circ}(\boldsymbol{P}_{\times}, \mathbb{C})$ as

$$f \mapsto \iint f\big(\mathsf{p}'_{\perp}, \mathsf{p}''_{\perp}, \dots, \mathsf{p}^{(r-1)}_{\perp}, -\textstyle\sum_{i=1}^{r-1} \mathsf{p}^{(i)}_{\perp}\big)\, \mathrm{d}^3\mathsf{p}'\, \mathrm{d}^3\mathsf{p}'' \dots \mathrm{d}^3\mathsf{p}^{(r-1)}.$$

If we write this generalized density as

$$\delta^{(r)}\, \eta'_{\perp} \otimes \eta''_{\perp} \otimes \cdots \otimes \eta^{(r)}_{\perp} \equiv \delta(\mathsf{p}'_{\perp} + \mathsf{p}''_{\perp} + \dots + \mathsf{p}^{(r)}_{\perp})\, \mathrm{d}^3\mathsf{p}' \otimes \mathrm{d}^3\mathsf{p}'' \otimes \dots \otimes \mathrm{d}^3\mathsf{p}^{(r)},$$

then $\delta^{(r)}$ turns out to be an $\mathbb{L}^3$-scaled generalized function (this follows from the fact that $\eta_{\perp} = \mathrm{d}^3\mathsf{p}$ is $\mathbb{L}^{-3}$-scaled). Mimicrying traditional notation we also write $\delta^{(r)} \equiv \delta(\mathsf{p}'_{\perp} + \mathsf{p}''_{\perp} + \dots + \mathsf{p}^{(r)}_{\perp})$.

Next, we introduce the scaled generalized half-density

$$\underline{\Lambda}^{(r)} := \delta^{(r)}\sqrt{\omega_{m'}}\otimes\sqrt{\omega_{m''}}\otimes\cdots\otimes\sqrt{\omega_{m^{(r)}}} : \boldsymbol{T}\to\mathbb{L}^{3-r}\otimes\boldsymbol{\mathcal{D}}(\boldsymbol{P}_{\times},\mathbb{C}),$$

which has the coordinate expression

$$\underline{\Lambda}^{(r)} = \frac{\delta(\mathsf{p}'_\perp+\mathsf{p}''_\perp+\cdots+\mathsf{p}^{(r)}_\perp)}{\sqrt{2^r\,\mathsf{p}'_0\,\mathsf{p}''_0\ldots\mathsf{p}^{(r)}_0}}\sqrt{\mathrm{d}^3\mathsf{p}'}\otimes\sqrt{\mathrm{d}^3\mathsf{p}''}\otimes\ldots\otimes\sqrt{\mathrm{d}^3\mathsf{p}^{(r)}}.$$

If $\ell^{(r)}$ is a tensor field describing the classical interaction of r particles, then

$$\ell^{(r)}\otimes\underline{\Lambda}^{(r)} : \boldsymbol{T}\to\mathbb{L}^{3-r}\otimes\boldsymbol{\mathcal{V}}^r,$$

where $\boldsymbol{\mathcal{V}}^r\subset\boldsymbol{\mathcal{V}}$ denotes the sub-bundle of all tensors of rank r; by this tensor we construct the quantum interaction through an analogue of the classical mechanism: contraction with s free-particle states, $s\le r$, describes the absorption of s particles and the creation of $r - s$. The quantum part of this operation, that is the contraction of semi-density distributions, is defined in a generalized sense as discussed in Sec. IV B.

We can restate the above considerations by saying, by analogy with the finite-dimensional situation, that the above tensor comes in several different "index types": absorption corresponds to a "covariant" index, creation to a "contravariant" index.

Now the classical part $\ell^{(r)}$ already comes in various types, related by the underlying algebraic structure. One must then allow analogous corresponding types for the quantum part $\underline{\Lambda}^{(r)}$: these are distinguished by the signs of the spatial momenta, a minus sign for the momentum of an incoming particle. The complete prescription also requires a phase factor and a coupling constant $\lambda\in\mathbb{L}^{r-4}$, which will be expressed in terms of particle masses. Hence, eventually, one considers

$$\Lambda^{(r)} := \lambda\,\mathrm{e}^{-\mathrm{i}\,(\pm\mathsf{p}'_0\pm\mathsf{p}''_0\pm\cdots\pm\mathsf{p}^{(r)}_0)\,\mathsf{t}}\,\underline{\Lambda}^{(r)}\otimes\ell^{(r)} : \boldsymbol{T}\to\mathbb{L}^{-1}\otimes\boldsymbol{\mathcal{V}}^r,$$

where now

$$\underline{\Lambda}^{(r)} = \frac{\delta(\pm\mathsf{p}'_\perp\pm\mathsf{p}''_\perp\pm\cdots\pm\mathsf{p}^{(r)}_\perp)}{\sqrt{2^r\,\mathsf{p}'_0\,\mathsf{p}''_0\ldots\mathsf{p}^{(r)}_0}}\sqrt{\mathrm{d}^3\mathsf{p}'}\otimes\sqrt{\mathrm{d}^3\mathsf{p}''}\otimes\ldots\otimes\sqrt{\mathrm{d}^3\mathsf{p}^{(r)}}$$

(we could distinguish the various types of $\Lambda^{(r)}$ by different symbols,[5] but the notation becomes rather clumsy in general).

Summarizing, each one of these 2^r types of $\Lambda^{(r)}$ can be viewed as a section $\boldsymbol{T}\to\mathrm{End}(\boldsymbol{\mathcal{V}})$, and $\mathfrak{H}$ is the sum of all these endomorphisms for all suitable values of r. Usually, one only deals with three or four legs point interactions, namely, $r = 3, 4$. We will consider a few typical cases in the context of electroweak theory.

In a previous paper,[5] I showed how one recovers the electron and photon propagators from the above ideas, and how one sets to calculate matrix elements. It is not difficult to convince oneself that essentially the same arguments also work in a more general gauge theory situation such as that sketched in Sec. III D.

Remark: A general discussion of quantum fields in the context of quantum connections on distributional bundles was presented in a previous paper.[28] Those notions can be used to explicitly relate the setting of this paper to standard presentations. Here, we just note that a *creation operator* in the fibers of $\boldsymbol{\mathcal{V}}\rightarrowtail\boldsymbol{T}$ can be naturally defined as the tensor product (either symmetric or antisymmetric) by some given state vector, while an *absorption operator* is defined as the contraction by a given vector. A quantum field is then defined to be a certain combination of creation and absorption operators. The various terms which constitute $\mathfrak{H}$ can be seen as compositions of creation and absorption operators.

V. EXAMPLES: ELECTROWEAK INTERACTIONS

A. Electroweak geometry and fields

Electroweak geometry can be viewed as an extension of 2-spinor geometry. The main further ingredient is a new complex vector bundle $\boldsymbol{I} \to \boldsymbol{M}$, called the *isospin* bundle, whose two-dimensional fibers are endowed with a positive Hermitian metric $h : \boldsymbol{M} \to \overline{\boldsymbol{I}}^\star \otimes_{\boldsymbol{M}} \boldsymbol{I}$. In a previous paper,[4] I discussed a setting in which the fermions of electroweak theory are described as sections of the vector bundle

$$\boldsymbol{Y} \equiv \boldsymbol{Y}_{\rm R} \oplus \boldsymbol{Y}_{\rm L} \equiv \big(\wedge^2 \boldsymbol{I} \otimes \boldsymbol{U}\big) \oplus \big(\boldsymbol{I} \otimes \overline{\boldsymbol{U}}^\star\big),$$

namely, the setting sketched in Sec. III D with $\boldsymbol{F}_{\rm R} \equiv \wedge^2 \boldsymbol{I}$ and $\boldsymbol{F}_{\rm L} \equiv \boldsymbol{I}$. Here, $\boldsymbol{U} \rightarrowtail \boldsymbol{M}$ is the same 2-spinor bundle of electrodynamics, with the further assumption that the 2-spinor connection Γ determines a curvature-free connection of $\wedge^2 \boldsymbol{U}$. Then we can choose a 2-spinor frame $\big(\zeta_{\scriptscriptstyle A}\big)$ such that $\nabla\epsilon \equiv \nabla(\zeta^1 \wedge \zeta^2) = 0$, i.e., $\Gamma_{a\,A}^{\;\;A} = 0$.

An h-orthonormal local frame of $\boldsymbol{I} \rightarrowtail \boldsymbol{M}$ (*isospin frame*) will be denoted by $\big(\xi_\alpha\big)$, $\alpha = 1, 2$. We write the coefficients of a linear connection X of $\boldsymbol{I} \rightarrowtail \boldsymbol{M}$ as

$$X_{a\;\beta}^{\;\alpha} = X_a^\lambda\, \sigma_{a\;\beta}^{\;\alpha}, \quad X_a^\lambda : \boldsymbol{M} \to \mathbb{C}, \quad a = 1, 2, 3, 4, \quad \lambda = 0, 1, 2, 3.$$

Though Γ is traceless, the connection of $\boldsymbol{Y} \rightarrowtail \boldsymbol{M}$ induced by Γ and X has a non-vanishing trace part coming from X. The coefficients X_a^λ are imaginary if and only if X is Hermitian, i.e., fulfills the condition $\nabla[X]h = 0$.

By examining the conformal invariance requirements of the classical theory, one easily sees that the fields must be scaled: the Fermion field has to be $\mathbb{L}^{-3/2}$-scaled, while the gauge fields and the Higgs field have to be $\mathbb{L}^{-1}$-scaled. The coordinate expression of a field $\Psi \equiv \Psi_{\rm R} + \Psi_{\rm L} : \boldsymbol{M} \to \mathbb{L}^{-3/2} \otimes (\boldsymbol{Y}_{\rm R} \oplus \boldsymbol{Y}_{\rm L})$ will be written as

$$\Psi = \Psi^{\scriptscriptstyle A}\, \hat{\xi} \otimes \zeta_{\scriptscriptstyle A} + \Psi^\alpha_{\scriptscriptstyle A^\cdot}\, \xi_\alpha \otimes \bar{\zeta}^{\scriptscriptstyle A^\cdot},$$

where $\hat{\xi} \equiv \xi_1 \wedge \xi_2 : \boldsymbol{M} \to \wedge^2 \boldsymbol{I}$ (the scaling is carried by the field's components).

According to the ideas sketched in Sec. III D boson fields are viewed as sections of bundles obtained from expanding $\overline{\boldsymbol{Y}} \otimes \boldsymbol{Y}$, namely,

$$\overline{\boldsymbol{Y}} \otimes \boldsymbol{Y} \cong (\wedge^2 \overline{\boldsymbol{I}} \otimes \wedge^2 \boldsymbol{I} \otimes \boldsymbol{H}) \oplus (\overline{\boldsymbol{I}} \otimes \boldsymbol{I} \otimes \boldsymbol{H}^*) \oplus$$

$$\oplus\, (\wedge^2 \overline{\boldsymbol{I}} \otimes \boldsymbol{I} \otimes \mathrm{End}\overline{\boldsymbol{U}}) \oplus (\wedge^2 \boldsymbol{I} \otimes \overline{\boldsymbol{I}} \otimes \mathrm{End}\boldsymbol{U}).$$

The gauge fields are represented by a section

$$W : \boldsymbol{M} \to \mathbb{L}^{-1} \otimes \overline{\boldsymbol{I}} \otimes \boldsymbol{I} \otimes \boldsymbol{H}^* \cong \mathbb{L}^{-1} \otimes (\overline{\boldsymbol{Y}}_{\rm L} \otimes \boldsymbol{Y}_{\rm L}),$$

which by natural operations also determines a section $\widehat{W} : \boldsymbol{M} \to \mathbb{L}^{-1} \otimes \wedge^2 \overline{\boldsymbol{I}} \otimes \wedge^2 \boldsymbol{I} \otimes \boldsymbol{H}$.

Next, we note that the two last bundles, in the above decomposition of $\overline{\boldsymbol{Y}} \otimes \boldsymbol{Y}$, are mutually conjugate, and that we can consider sections of the special form

$$\phi \otimes 1\!\!1_{\overline{\boldsymbol{U}}} : \boldsymbol{M} \to \mathbb{L}^{-1} \otimes \wedge^2 \overline{\boldsymbol{I}} \otimes \boldsymbol{I} \otimes \mathrm{End}\overline{\boldsymbol{U}}, \quad \phi : \boldsymbol{M} \to \mathbb{L}^{-1} \otimes \wedge^2 \overline{\boldsymbol{I}} \otimes \boldsymbol{I},$$

where $1\!\!1_{\overline{\boldsymbol{U}}} : \boldsymbol{M} \to \overline{\boldsymbol{U}} \otimes \overline{\boldsymbol{U}}^\star \cong \mathrm{End}\overline{\boldsymbol{U}}$ denotes identity of $\overline{\boldsymbol{U}} \rightarrowtail \boldsymbol{M}$. The section ϕ will be called a *Higgs field*. Then also $\bar{\phi} \otimes 1\!\!1_{\boldsymbol{U}} : \boldsymbol{M} \to \mathbb{L}^{-1} \otimes \wedge^2 \boldsymbol{I} \otimes \overline{\boldsymbol{I}} \otimes \mathrm{End}\boldsymbol{U}$.

Using the frame $\big(\iota_\lambda\big) \equiv \big(\sigma_\lambda^{\alpha\alpha^\cdot}\, \bar{\xi}_{\alpha^\cdot} \otimes \xi_\alpha\big)$ of $\overline{\boldsymbol{I}} \otimes \boldsymbol{I}$ ($\lambda = 0, 1, 2, 3$) we write $W = \frac{\mathrm{i}}{2}\, W_\lambda^\mu\, \iota_\mu \otimes \tau^\lambda$. If the field W is known, then the choice of an isospin gauge determines a isospin connection with components $X_{a\;\beta}^{\;\alpha} = q\, \Theta_a^\lambda\, h_{\alpha^\cdot\beta}\, W_\lambda^{\;\alpha^\cdot\alpha} = \frac{\mathrm{i}}{2}\, q\, \Theta_a^\lambda\, W_\lambda^\mu\, \sigma_{\mu\;\beta}^{\;\alpha}$, $q \in \mathbb{R}^+$ (these are *unscaled* because of the scaling of the tetrad Θ, see Sec. III A).

B. Symmetry breaking

In standard electroweak theory, one assumes that there is one special section $\phi_0 : \boldsymbol{M} \to \mathbb{L}^{-1} \otimes \wedge^2 \overline{\boldsymbol{I}} \otimes \boldsymbol{I}$, the "vacuum expectation value" of ϕ, supposedly arising as a minimum of the

"Higgs potential" $V[\phi] := \lambda\,(2\,m^2\,\langle\bar\phi,\phi\rangle - \langle\bar\phi,\phi\rangle^2)$, with $m\in\mathbb{L}^{-1}$ and $\lambda\in\mathbb{R}^+$. This determines an h-orthogonal decomposition $\boldsymbol{I}=\boldsymbol{I}_1\oplus\boldsymbol{I}_2$ characterized by $\phi_0:\boldsymbol{M}\to\mathbb{L}^{-1}\otimes\wedge^2\overline{\boldsymbol{I}}\otimes\boldsymbol{I}_1$. We can choose the h-orthonormal isospin frame (ξ_α) in such a way that $\phi_0 = m\,\hat{\bar\xi}\otimes\xi_1$.

Next, one gets a decomposition of $\boldsymbol{E}\equiv\overline{\boldsymbol{I}}\otimes\boldsymbol{I}$, yielding the targets of the gauge fields. We write $\boldsymbol{E}=\boldsymbol{E}_A\oplus\boldsymbol{E}_Z\oplus\boldsymbol{E}_+\oplus\boldsymbol{E}_-$, where

$$\boldsymbol{E}_A\equiv\overline{\boldsymbol{I}}_1\otimes\boldsymbol{I}_1,\quad \boldsymbol{E}_+\equiv\overline{\boldsymbol{I}}_1\otimes\boldsymbol{I}_2,\quad \boldsymbol{E}_-\equiv\overline{\boldsymbol{I}}_2\otimes\boldsymbol{I}_1,$$

and $\boldsymbol{E}_Z$ is generated by

$$\mathsf{e}_Z\equiv-\sin\theta\,\tan\theta\,\iota_0+\cos\theta\,\iota_3=\sec\theta\left[\cos(2\theta)\,\bar\xi_1\otimes\xi_1-\bar\xi_2\otimes\xi_2\right].$$

The parameter $\theta\in(0,\pi/2)$, a necessary ingredient of the theory, is called the *Weinberg angle*.

The gauge field is then written as the sum $W=\frac{\mathrm{i}}{2}\,(A+Z+W^++W^-)$, in which the four terms are, respectively, valued into

$$\mathbb{L}^{-1}\otimes\boldsymbol{H}^*\otimes\boldsymbol{E}_A,\quad \mathbb{L}^{-1}\otimes\boldsymbol{H}^*\otimes\boldsymbol{E}_Z,\quad \mathbb{L}^{-1}\otimes\boldsymbol{H}^*\otimes\boldsymbol{E}_+,\quad \mathbb{L}^{-1}\otimes\boldsymbol{H}^*\otimes\boldsymbol{E}_-,$$

and their coordinate expressions are written as

$$A=A_\lambda\,\tau^\lambda\otimes\mathsf{e}_A\equiv-2\,\sin\theta\,A_\lambda\,\tau^\lambda\otimes\bar\xi_1\otimes\xi_1,$$

$$Z=Z_\lambda\,\tau^\lambda\otimes\mathsf{e}_Z\equiv\sec\theta\,Z_\lambda\,\tau^\lambda\otimes\left[\cos(2\theta)\,\bar\xi_1\otimes\xi_1-\bar\xi_2\otimes\xi_2\right],$$

$$W^+=W^+_\lambda\,\tau^\lambda\otimes\mathsf{e}_+\equiv\sqrt{2}\,W^+_\lambda\,\tau^\lambda\otimes\bar\xi_1\otimes\xi_2,$$

$$W^-=W^-_\lambda\,\tau^\lambda\otimes\mathsf{e}_-\equiv\sqrt{2}\,W^-_\lambda\,\tau^\lambda\otimes\bar\xi_2\otimes\xi_1$$

(in order to match the main usual formulas as closely as possible, some conventions have been changed relatively to a previous paper[4]).

With the choice of a gauge, the above fields are supposed to determine a Hermitian connection of $\boldsymbol{I}\rightarrowtail\boldsymbol{M}$. This implies that the coefficients A_λ and Z_λ are real, while W^+_λ and W^-_λ are mutually complex conjugate.

As for a Fermion field, after symmetry breaking it splits as $\Psi\equiv u\,+\,\chi\,+\,\nu\equiv\psi\,+\,\nu$, with

$$u\equiv\Psi_{\mathrm{R}}=u^A\,\hat{\bar\xi}\otimes\zeta_A,\quad \chi=\Psi^1_{A^\cdot}\,\xi_1\otimes\bar\zeta^{A^\cdot}\equiv\chi_{A^\cdot}\,\xi_1\otimes\bar\zeta^{A^\cdot},\quad \nu=\Psi^2_{A^\cdot}\,\xi_2\otimes\bar\zeta^{A^\cdot}\equiv\nu_{A^\cdot}\,\xi_2\otimes\bar\zeta^{A^\cdot}.$$

Now $\psi\equiv(u,\chi)\equiv\Psi_{\mathrm{R}}+\chi\equiv\psi_{\mathrm{R}}+\psi_{\mathrm{L}}$ is viewed as the electron field, and ν is the neutrino.

C. Interactions among gauge particles

After symmetry breaking we identify the electron bundle as $\boldsymbol{W}\equiv(\wedge^2\boldsymbol{I}\otimes\boldsymbol{U})\oplus(\boldsymbol{I}_1\otimes\overline{\boldsymbol{U}}^\star)$, and construct electron and positron generalized frames essentially as in QED. From the neutrino bundle we obtain the generalized frame $\{\mathsf{N}_{kA^\cdot}\}\equiv\{\mathsf{X}_k\otimes\xi_2\otimes\bar\zeta_{A^\cdot}\}$. Similarly, from classical frames for gauge and Higgs fields one constructs the corresponding quantum frames via tensor product by the generalized semi-densities X_k.

In standard approaches one finds the allowed interactions as terms of the total Lagrangian, which in coordinates can be written as $\mathcal{L}=(\ell_\Psi+\ell_\phi+\ell_X+\ell_{\mathrm{int}})\,\mathrm{d}^4\mathsf{x}$ with

$$\ell_\Psi=\tfrac{\mathrm{i}}{\sqrt{2}}\,(\Theta^{-1})^a_{AA^\cdot}\left(\nabla_a\Psi^A\,\bar\Psi^{A^\cdot}-\Psi^A\,\nabla_a\bar\Psi^{A^\cdot}+\varepsilon^{AB}\bar\varepsilon^{A^\cdot B^\cdot}\,(\bar\Psi_{B\alpha}\,\nabla_a\Psi^\alpha_{B^\cdot}-\nabla_a\bar\Psi_{B\alpha}\,\Psi^\alpha_{B^\cdot})\right)\det\Theta,$$

$$\ell_\phi=\left(g^{ab}\,\nabla_a\bar\phi_\alpha\,\nabla_b\phi^\alpha+2\,\lambda\,m^2\,(\bar\phi_\alpha\,\phi^\alpha)-\lambda\,(\bar\phi_\alpha\,\phi^\alpha)^2\right)\det\Theta,$$

$$\ell_X=-\frac{1}{2\,q^2}\,g^{ac}\,g^{bd}\,h_{\alpha^\cdot\alpha}\,h^{\beta^\cdot\beta}\,\bar R_{ab}{}^{\alpha^\cdot}{}_{\beta^\cdot}\,R_{cd}{}^{\alpha}{}_{\beta}\,\det\Theta,\quad \ell_{\phi\Psi}=-\left(\bar\Psi_{A\alpha}\,\phi^\alpha\,\Psi^A+\bar\Psi^{A^\cdot}\,\bar\phi_\alpha\,\Psi^\alpha_{A^\cdot}\right)\det\Theta,$$

where $\bar{\Psi}_{B\alpha} \equiv h_{\alpha\cdot\alpha}\,\bar{\Psi}^{\alpha\cdot}_{B}$, $\bar{\phi}_\alpha \equiv h_{\alpha\cdot\alpha}\,\bar{\phi}^{\alpha\cdot}$, X is the isospin connection and $R_{ab}{}^{\alpha}{}_{\beta}$ denotes the components of its curvature tensor.

Here we do not aim at a thorough examination of all possible interactions (by the way, a discussion of QED interaction in terms of generalized semi-densities can be found in a previous paper[5]). Instead we will recover, as an example, the interactions among the gauge fields, and we will do that by a somewhat different procedure than usual.

The starting point of our discussion is the observation that a radiative e.m. field in flat spacetime can be written in the form $F = k\wedge b$, where $k, b : \boldsymbol{M} \to \mathrm{T}^*\boldsymbol{M}$, $k\cdot k = k\cdot b = 0$, and $b \equiv \pm *A$; this is strictly related to the photon generalized frame, see Sec. IV E. While a field of this type obeying the Maxwell equations may not exist in arbitrary spacetime, describing a "photon field" in this way makes sense. This idea can be extended to any gauge field W as follows. First, remember that we see all fields as sections of two-fibered bundles $\boldsymbol{Z} \rightarrowtail \boldsymbol{P} \rightarrowtail \boldsymbol{M}$, so that fields explicitly carry the information of their momentum k; then we consider the tensor field $R[W] : \boldsymbol{M} \to \wedge^2\mathrm{T}^*\boldsymbol{M}\otimes\boldsymbol{E}$ (of "curvature type") defined to be

$$R[W] := \mathrm{i}\,k\wedge W - \tfrac{1}{2}\,q\,W\,\bar{\wedge}\,W,$$

where $\bar{\wedge}$ denotes h-contraction in the fibers and antisymmetrization of spacetime indices:

$$(k\wedge W)_{ab}{}^{\alpha\cdot\alpha} = k_{[a}W_{b]}{}^{\alpha\cdot\alpha}, \quad (W\,\bar{\wedge}\,W)_{ab}{}^{\alpha\cdot\alpha} = h_{\beta\cdot\beta}\,W_{[a}{}^{\alpha\cdot\beta}\,W_{b]}{}^{\beta\cdot\alpha}.$$

Next we consider the density

$$\mathcal{L}_W = -\frac{1}{2}\,\langle g^\#\otimes g^\#\otimes R[\bar{W}]\otimes R[W]\rangle\,\eta : \boldsymbol{P} \to \wedge^4\mathrm{T}^*\boldsymbol{M},$$

where the angle bracket indicates that all possible contractions are performed. Of course, this corresponds to the Lagrangian density $\mathcal{L}_X$ of the isospin connection when we make the replacement $\partial_a X_b{}^{\alpha}{}_{\beta} \to \mathrm{i}\,q\;k_a\,W_b{}^{\alpha}{}_{\beta}$. From this we can extract the interactions among the gauge fields (excluding ghosts). Here we are not interested in the "kinetic terms," that is the terms containing two fields and two momenta: free-particles are already described by the appropriate states (by the way, we note that in this context the kinetic terms can be subtracted through a natural operation). We obtain a density $\mathcal{L}'_W \equiv (\ell^{(0)}_W + \ell^{(1)}_W)\,\mathrm{d}^4\mathsf{x}$, with

$$\ell^{(1)}_W = \ell_{AWW} + \ell_{WWZ}, \quad \ell^{(0)}_W = \ell_{AAWW} + \ell_{AWWZ} + \ell_{WWWW} + \ell_{WWZZ},$$

$$\tfrac{1}{|\Theta|}\,\ell_{AWW} = \mathrm{i}\,q\,\sin\theta\,(g^{ad}\,g^{bc} - g^{ac}\,g^{bd})\big(k^{[A]}_a A_b\,W^-_c\,W^+_d + k^+_a\,W^+_b\,A_c\,W^-_d - k^-_a\,W^-_b\,A_c\,W^+_d\big),$$

$$\tfrac{1}{|\Theta|}\,\ell_{WWZ} = \mathrm{i}\,q\,\cos\theta\,(g^{ad}\,g^{bc} - g^{ac}\,g^{bd})\big(k^{[Z]}_a Z_b\,W^-_c\,W^+_d - k^+_a\,W^+_b\,W^-_c\,Z_d + K^-_a\,W^-_b\,W^+_c\,Z_d\big),$$

$$\tfrac{1}{|\Theta|}\,\ell_{AAWW} = \tfrac{1}{2}\,q^2\,\sin^2\theta\,(g^{ad}\,g^{bc} + g^{ac}\,g^{bd} - 2\,g^{ab}\,g^{cd})\,A_a\,A_b\,W^-_c\,W^+_d,$$

$$\tfrac{1}{|\Theta|}\,\ell_{AWWZ} = q^2\,\sin\theta\,\cos\theta\,(2\,g^{ad}\,g^{bc} - g^{ac}\,g^{bd} - g^{ab}\,g^{cd})\,A_a\,W^-_b\,W^+_c\,Z_d,$$

$$\tfrac{1}{|\Theta|}\,\ell_{WWWW} = -\tfrac{1}{2}\,q^2\,(g^{ad}\,g^{bc} - g^{ac}\,g^{bd})\,W^-_a\,W^-_b\,W^+_c\,W^+_d,$$

$$\tfrac{1}{|\Theta|}\,\ell_{WWZZ} = \tfrac{1}{2}\,q^2\,\cos^2\theta\,(g^{ad}\,g^{bc} + g^{ac}\,g^{bd} - 2\,g^{ab}\,g^{cd})\,W^-_a\,W^+_b\,Z_c\,Z_d,$$

where we used the abbreviation $|\Theta| \equiv \det\Theta$ and indicated by $k^{[A]}$, $k^{[Z]}$, and $k^\pm$ the momenta of the corresponding particles. Hence the terms in $\ell^{(1)}$ correspond to one-point interactions of three gauge particles; these depend from the particles' momenta. The terms in $\ell^{(0)}$ correspond to one-point interactions of four gauge particles, which are independent of momenta.

By exchanging index names and rearranging terms we rewrite three-particle interactions as

$$\tfrac{1}{|\Theta|}\,\ell_{AWW} = \mathrm{i}\,q\,\sin\theta\left(g^{ab}\,(k^{+}-k^{[A]})^{c} + g^{bc}\,(k^{-}-k^{+})^{a} + g^{ac}\,(k^{[A]}-k^{-})^{b}\right)A_a\,W_b^{+}\,W_c^{-},$$

$$\tfrac{1}{|\Theta|}\,\ell_{WWZ} = \mathrm{i}\,q\,\cos\theta\left(g^{ab}\,(k^{[Z]}-k^{+})^{c} + g^{bc}\,(k^{+}-k^{-})^{a} + g^{ac}\,(k^{-}-k^{[Z]})^{b}\right)Z_a\,W_b^{+}\,W_c^{-}.$$

To conclude, let us see how to assemble the above "classical" interaction with its quantum counterpart $\underline{\Lambda}$, as indicated in Sec. IV F. We keep the discussion somewhat qualitative, as a thoroughly formal one would require (because of the number of fields and combinations involved) a notational apparatus to heavy for an article of this kind (a more complete notation for the simpler situation of QED was introduced in a previous paper[5]).

Consider, for example, the term indicated as ℓ_{AWW}. Recall that we consider fields as sections $\boldsymbol{M}\to\boldsymbol{Z}$ where $\boldsymbol{Z}\rightarrowtail\boldsymbol{P}\rightarrowtail\boldsymbol{M}$ is some two-fibered bundle. (Incidentally, in the case of the gauge fields we are dealing with, this two-fibered bundle is actually of the "semi-trivial" type: for the field A, for example, it is $\boldsymbol{P}_{0}\times\boldsymbol{E}_A$; but note that this simplification does not hold for the electron bundles $\boldsymbol{W}^{\pm}$.) Now ℓ_{AWW} can be seen (using the notation of Sec. IV F) as a section

$$\boldsymbol{P}_{\times} := \boldsymbol{P}_{0}\underset{\boldsymbol{M}}{\times}\boldsymbol{P}_{m'}\underset{\boldsymbol{M}}{\times}\boldsymbol{P}_{m'}\to\boldsymbol{E}_A^{*}\otimes\boldsymbol{E}_{+}^{*}\otimes\boldsymbol{E}_{-}^{*},$$

since it is a multilinear contraction of the fields depending on their momenta (here m' denotes the mass of the $W^{\pm}$). Then it could be written as $\tilde{\ell}_{(A+-)}\,\overset{*}{\mathsf{e}}_A\otimes\overset{*}{\mathsf{e}}_{+}\otimes\overset{*}{\mathsf{e}}_{-}$, where we denote as $\big(\overset{*}{\mathsf{e}}_A,\overset{*}{\mathsf{e}}_{+},\overset{*}{\mathsf{e}}_{-},\overset{*}{\mathsf{e}}_Z\big)$ the dual frame of $\big(\mathsf{e}_A,\mathsf{e}_{+},\mathsf{e}_{-},\mathsf{e}_Z\big)$ and $\tilde{\ell}_{(A+-)}$ depends on the momenta. Using the finite-dimensional geometric structure we obtain eight different index types of this tensor, each one suitable for describing certain absorptions and creations. For the absorption of three particles we obtain the semi-density

$$\mathrm{e}^{\mathrm{i}\,(k_0^{[A]}+k_0^{+}+k_0^{-})\,\mathrm{t}}\,\tilde{\ell}_{(A+-)}(k_0^{[A]},k_0^{+},k^{-})\,\frac{\delta(k_\perp^{[A]}+k_\perp^{+}+k_\perp^{-})}{\sqrt{8\,k_0^{[A]}\,k_0^{+}\,k_0^{-}}}$$
$$\cdot\big(\sqrt{\mathrm{d}^3k^{[A]}}\otimes\overset{*}{\mathsf{e}}_A\big)\otimes\big(\sqrt{\mathrm{d}^3k^{+}}\otimes\overset{*}{\mathsf{e}}_{+}\big)\otimes\big(\sqrt{\mathrm{d}^3k^{-}}\otimes\overset{*}{\mathsf{e}}_{-}\big).$$

Similarly, we can write $\ell_{AWWZ}=\tilde{\ell}_{(A+-Z)}\,\overset{*}{\mathsf{e}}_A\otimes\overset{*}{\mathsf{e}}_{+}\otimes\overset{*}{\mathsf{e}}_{-}\otimes\overset{*}{\mathsf{e}}_Z$ (where $\tilde{\ell}_{(A+-Z)}$ is now independent of the momenta), and obtain the semi-density

$$\mathrm{e}^{\mathrm{i}\,(k_0^{[A]}+k_0^{+}+k_0^{-}+k_0^{[Z]})\,\mathrm{t}}\,\tilde{\ell}_{(A+-Z)}\,\frac{\delta(k_\perp^{[A]}+k_\perp^{+}+k_\perp^{-}+k_\perp^{[Z]})}{\sqrt{16\,k_0^{[A]}\,k_0^{+}\,k_0^{-}\,k_0^{[Z]}}}$$
$$\cdot\big(\sqrt{\mathrm{d}^3k^{[A]}}\otimes\overset{*}{\mathsf{e}}_A\big)\otimes\big(\sqrt{\mathrm{d}^3k^{+}}\otimes\overset{*}{\mathsf{e}}_{+}\big)\otimes\big(\sqrt{\mathrm{d}^3k^{-}}\otimes\overset{*}{\mathsf{e}}_{-}\big)\otimes\big(\sqrt{\mathrm{d}^3k^{[Z]}}\otimes\overset{*}{\mathsf{e}}_Z\big).$$

The above semi-densities are to be contracted with free states, which read $\mathsf{X}_{k^{[Z]}}\otimes\mathsf{e}_Z$ and the like (Sec. IV D). Eventually, the complete operator $\mathfrak{H}$ is a combination of many more terms. The formalism used by physicists for matrix element calculations, with its tricks and shortcuts, turns out to be handy, but a more explicit and tentatively precise notation may help to grasp underlying concepts.

[1] D. Canarutto, "Possibly degenerate tetrad gravity and Maxwell-Dirac fields," J. Math. Phys. **39**(9), 4814–4823 (1998).
[2] D. Canarutto, "Two-spinors, field theories and geometric optics in curved spacetime," Acta Appl. Math. **62**(2), 187–224 (2000).
[3] D. Canarutto, "Minimal geometric data' approach to Dirac algebra, spinor groups and field theories," Int. J. Geom. Methods Mod. Phys. **4**(6), 1005–1040 (2007); e-print arXiv:math-ph/0703003.
[4] D. Canarutto, "Tetrad gravity, electroweak geometry and conformal symmetry," Int. J. Geom. Methods Mod. Phys. **8**(4), 797–819 (2011); e-print arXiv:1009.2255v1 [math-ph].
[5] D. Canarutto, "Quantum bundles and quantum interactions," Int. J. Geom. Met. Mod. Phys. **2**(5), 895–917 (2005); e-print arXiv:math-ph/0506058v2.
[6] D. Canarutto, "Connections in distributional bundles," Rend. Sem. Mat. Univ. Padova **111**, 71–97 (2004).

[7] N. N. Bogolubov, A. A. Logunov, and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory* (Benjamin, Reading, 1975).
[8] M. Veltman, *Diagrammatica* (Cambridge University Press, 1994).
[9] D. Canarutto, A. Jadczyk, and M. Modugno, "Quantum mechanics of a spin particle in a curved spacetime with absolute time," Rep. Math. Phys. **36**, 95–140 (1995).
[10] J. Janyška and M. Modugno, "Covariant Schrödinger operator," J. Phys. A **35**, 8407–8434 (2002).
[11] J. Janyška and M. Modugno, "Hermitian vector fields and special phase functions," Int. J. Geom. Methods Mod. Phys. **3**(4), 1–36 (2006); e-print arXiv:math-ph/0507070v1.
[12] M. Modugno, D. Saller, and J. Tolksdorf, "Classification of infinitesimal symmetries in covariant classical mechanics," J. Math. Phys. **47**, 062903 (2006).
[13] D. Saller and R. Vitolo, "Symmetries in covariant classical mechanics," J. Math. Phys. **41**(10), 6824–6842 (2000); e-print arXiv:math-ph/0003027v2.
[14] R. Vitolo, "Quantum structures in Galilei general relativity," Ann. Inst. Henri Poincaré **70**(3), 239–257 (1999).
[15] R. Vitolo, "Quantum structures in Einstein general relativity," Lett. Math. Phys. **51**, 119–133 (2000).
[16] J. Janyška, M. Modugno, and R. Vitolo, "An algebraic approach to physical scales," Acta Appl. Math. **110**(3), 1249–1276 (2010); e-print arXiv:0710.1313v1 [math.AC].
[17] D. Canarutto, "Fermi transport of spinors and free QED states in curved spacetime," Int. J. Geom. Methods Mod. Phys. **6**(5), 805–824 (2009); e-print arXiv:0812.0651v1 [math-ph].
[18] D. Canarutto and A. Jadczyk, "Fundamental geometric structures for the Dirac equation in general relativity," Acta Appl. Math. **50**(1), 59–92 (1998).
[19] L. Schwartz, *Théorie Des Distributions* (Hermann, Paris, 1966).
[20] D. Canarutto, "Generalized densities and distributional adjoints of natural operators," Rend. Semin. Mat. Torino **59**(4), 27–36 (2001).
[21] A. Frölicher, *Smooth Structures*, Lecture Notes in Mathematics Vol. 962 (Springer-Verlag, 1982), pp. 69–81.
[22] A. Frölicher and A. Kriegl, *Linear Spaces and Differentiation Theory* (Wiley, 1988).
[23] A. Kriegl and P. Michor, *The Convenient Setting of Global Analysis* (American Mathematical Society, 1997).
[24] A. Cabras and I. Kolář, "Connections on some functional bundles," Czechoslovak Math. J. **45**(3), 529–548 (1995).
[25] I. Kolář and M. Modugno, "The Frölicher-Nijenhuis bracket on some functional spaces," Ann. Polon. Math. **68**, 97–106 (1998).
[26] Y. Choquet-Bruhat and C. DeWitt-Morette, *Analysis, Manifolds and Physics* (North-Holland, Amsterdam, 1982).
[27] P. Nurowski, "Optical geometries and related structures," J. Geom. Phys. **18**, 335–348 (1996).
[28] D. Canarutto, "Quantum connections and quantum fields," Rend. Istit. Mat. Univ. Trieste **36**, 1–21 (2004).