ON THE CONCEPT OF ORIENTABILITY FOR FREDHOLM MAPS BETWEEN REAL BANACH MANIFOLDS

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1. Introduction

In [1] we introduced a new degree theory for a class of nonlinear Fredholm maps of index zero between open subsets of (real) Banach spaces (or, more generally, Banach manifolds) called oriented maps. This degree extends the theory given by Elworthy-Tromba in [3] and [4], it is developed starting from the Brouwer degree for maps between finite dimensional manifolds, and it is primarily based on a purely algebraic concept of orientation for Fredholm linear operators of index zero between (real) vector spaces. According to this algebraic concept, any Fredholm operator of index zero has exactly two orientations, no matter whether or not it is an isomorphism. This differs from the definition of Fitzpatrick-Pejsachowicz-Rabier (see [9]), where only the invertible operators have two orientations, and differs from the notion due to Mawhin in [14], where only the noninvertible Fredholm operators of index zero have two orientations.

What was crucial for the construction of our degree is that the concept of orientation of a Fredholm linear operator of index zero is "stable" when embedded in the framework of (real) Banach spaces. In fact, loosely speaking, any bounded oriented operator acting between Banach spaces induces, by a sort of continuity, an orientation on any sufficiently close operator (as well as on any compact linear perturbation of such an operator). Thus, if $f\colon \Omega \to F$ is a nonlinear Fredholm operator of index zero from an open subset of a Banach space E into a Banach space F, this kind of stability allowed us to define an orientation of f as a "continuous" assignment of an orientation of the Fréchet derivative of f at any $x \in \Omega$. After this preliminary definition, the notion of oriented map was extended to the nonflat context (i.e. to the case of Fredholm maps of index zero acting between real Banach manifolds).

In [1] some interesting properties of oriented (and orientable) maps were stated without proof, since they were not essential in the construction of our degree. The purpose of this paper is to give a more detailed analysis of the concept of orientation, including the proofs of some statements appeared in [1]. To this aim the concept of oriented map will be reformulated in terms of covering space theory.

Among other results we prove a homotopy property of the orientability (Theorem 4.3) which, roughly speaking, asserts that the orientation of a nonlinear map can be continuously carried along a homotopy of Fredholm maps of index zero. Thus, in particular, when two nonlinear Fredholm maps of index zero are homotopic (in a sense to be made more precise), either they are both orientable or both nonorientable.

As we shall see, a simple example of nonorientable map consists of a constant map from a nonorientable finite dimensional (real) manifold into a manifold with the same dimension. Since (C^1) maps between orientable finite dimensional manifolds (of the same dimension) are orientable, in the flat, finite dimensional case, a nonorientable map cannot exist (the open sets of $\mathbb{R}^m$ being orientable). In the infinite dimensional (flat) context the situation is different, as we shall see in Section 4, where we will provide an example of a nonorientable Fredholm map of index zero acting between open sets of Banach spaces.

The last part of the paper is dedicated to a comparison with related notions of orientations due to Elworthy-Tromba (see [3] and [4]) and Fitzpatrick-Pejsachowicz-Rabier (see [8] and [9]).

2. Orientation for linear Fredholm operators in vector spaces

This section is devoted to a brief review of the concept of orientation for linear Fredholm operators of index zero between real vector spaces recently introduced in [1].

Let E be a vector space and $T\colon E \to E$ a linear map of the type $T = I - K$, where I denotes the identity of E and K has finite dimensional range. Take any finite dimensional subspace E_0 of E containing $\operatorname{Range} K$ and observe that T maps E_0 into itself. This implies that the determinant, $\det T|_{E_0}$, of the restriction $T|_{E_0}\colon E_0 \to E_0$ is well defined. It is not difficult to show that this determinant does not depend on the choice of the finite dimensional space E_0 containing $\operatorname{Range} K$. Thus, it makes sense to denote by $\det T$ this common value, and this will be done hereafter.

We recall that a linear operator between vector spaces, $L\colon E \to F$, is called (*algebraic*) *Fredholm* if both $\operatorname{Ker} L$ and $\operatorname{coKer} L$ have finite dimension. In this case its *index* is the integer

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{coKer} L.$$

In particular, when $L\colon \mathbb{R}^m \to \mathbb{R}^n$, one easily gets $\operatorname{ind} L = m - n$.

If $L\colon E \to F$ is Fredholm and $A\colon E \to F$ is any linear operator with finite dimensional range, we say that A is a *corrector* of L provided that $L + A$ is an isomorphism. Observe that this may happen only if $\operatorname{ind} L = 0$, since, as well known, $L + A$ is Fredholm of the same index as L. Assume therefore $\operatorname{ind} L = 0$ and notice that, in this case, the set of correctors of L, indicated by $\mathcal{C}(L)$, is nonempty. In fact, any (possibly trivial) linear operator $A\colon E \to F$ such that $\operatorname{Ker} A \oplus \operatorname{Ker} L = E$ and $\operatorname{Range} A \oplus \operatorname{Range} L = F$ is a corrector of L.

We introduce in $\mathcal{C}(L)$ the following equivalence relation. Given $A, B \in \mathcal{C}(L)$, consider the automorphism $T = (L + B)^{-1}(L + A)$ of E. We have

$$T = (L + B)^{-1}(L + B + A - B) = I - (L + B)^{-1}(B - A).$$

Thus $T = I - K$, where $K = (L + B)^{-1}(B - A)$ has finite dimensional range. This implies that $\det T$ is well defined and, in this case, non-vanishing since T is invertible. We say that A is *equivalent* to B or, more precisely, A is L-equivalent to B, if $\det (L + B)^{-1}(L + A) > 0$. This is actually an equivalence relation on $\mathcal{C}(L)$, with just two equivalence classes (see [1]). We can therefore introduce the following definition.

Definition 2.1. An *orientation* of a Fredholm operator of index zero L is one of the two equivalence classes of $\mathcal{C}(L)$. We say that L is *oriented* when an orientation is chosen.

We point out that in the particular case when $L\colon E \to F$ is a bounded Fredholm operator of index zero between real Banach spaces, a partition in two equivalence classes of the set of compact correctors of L was introduced for the first time (as far as we know) by Pejsachowicz and Vignoli in [16]. Namely, if A and B are compact (linear) correctors of L, the map $(L+B)^{-1}(L+A)$ is of the form $I-K$, with K a compact operator. Thus the Leray-Schauder degree of $I-K$ is well defined (since $(I-K)^{-1}(0)$ is compact) and equals either 1 or -1 (by a well known result of Leray-Schauder). Now, the operator A is said to be in the same class of B if the degree of $I-K$ is 1. Clearly, as a consequence of the definition of Leray-Schauder degree, this equivalence relation coincides with our notion in the case when one considers only bounded correctors with finite dimensional image. Apart for the sake of simplicity, the reason why in our concept of orientation we do not use the equivalence relation in [16] is due to the fact that we want our degree to be based just upon the Brouwer theory.

A prelude to the idea of partitioning the set of correctors of an algebraic Fredholm operator of index zero $L\colon E \to F$ can be found in the pioneering paper of Mawhin [14]. Here is a brief description of this idea. Fix a projector $P\colon E \to E$ onto $\operatorname{Ker} L$ and a subspace F_1 of F such that $F_1 \oplus \operatorname{Range} L = F$. To any isomorphism $J\colon \operatorname{Ker} L \to F_1$ one can associate the corrector JP of L (this of course does not exhaust $\mathcal{C}(L)$). Two such correctors, J_1P and J_2P, are equivalent if $\det J_2^{-1}J_1 > 0$. One can check that, except in the case when L is an isomorphism (which is crucial to us), this equivalence relation produces two equivalence classes, each of them contained in one class of $\mathcal{C}(L)$ (and not both in the same one).

According to Definition 2.1, an oriented operator L is a pair (L,ω), where ω is one of the two equivalence classes of $\mathcal{C}(L)$. However, to simplify the notation, we shall not use different symbols to distinguish between oriented and nonoriented operators (unless it is necessary).

Given an oriented operator $L\colon E \to F$, we shall often denote its orientation by $\mathcal{C}_+(L)$, and the elements of this equivalence class will be called the *positive correctors* of L (the elements in the opposite class, $\mathcal{C}_-(L)$, are the *negative correctors*).

A "natural" corrector of an isomorphism L is the trivial operator 0. This corrector defines an equivalence class of $\mathcal{C}(L)$, called the *natural orientation* of L. However, if an isomorphism L has already an orientation (not necessarily the natural one), we define its *sign* as follows: $\operatorname{sign} L = 1$ if the trivial operator 0 is a positive corrector of L (i.e. if L is naturally oriented) and $\operatorname{sign} L = -1$ otherwise. As we shall see, in the case when L is an automorphism of a finite dimensional space, this definition coincides with the sign of the determinant.

In the particular case when the spaces E and F are finite dimensional (of the same dimension), an orientation of a linear operator $L\colon E \to F$ determines uniquely an orientation of the product space $E \times F$ (and vice versa). To see this, suppose first that L is an oriented operator. To determine an orientation of $E \times F$, take any of the two orientations of E and consider a positive corrector A of L. Then orient F in such a way that $L + A$ becomes orientation preserving. Thus $E \times F$ turns out to be oriented by considering the product of the two orientations of E

and F (notice that such an orientation of $E \times F$ does not depend on the chosen orientation of E and the positive corrector of L, but only on the orientation of L). Conversely, if $E \times F$ is oriented, every linear operator $L\colon E \to F$ can be oriented by choosing as positive correctors of L those operators A such that $L + A$ is orientation preserving (this makes sense, since an orientation of $E \times F$ can be regarded as a pair of orientations of E and F, up to an inversion of both of them).

Clearly, if E is a finite dimensional vector space, the product $E \times E$ turns out to be *canonically oriented* by considering the square of any orientation of E. Consequently, any endomorphism of E inherits an orientation that will be called *canonical*. One can check that when an endomorphis L of E is invertible, the natural and the canonical orientations of L coincide if and only if $\det L > 0$. Thus, choosing for L the canonical orientation, one has $\operatorname{sign} L = \operatorname{sign}(\det L)$.

If the spaces E and F are infinite dimensional, and $L\colon E \to F$ is Fredholm of index zero, the above one-to-one correspondence between the orientations of L and of $E \times F$ cannot be stated ($E \times F$ being infinite dimensional). However, let us show that an orientation of L can be regarded as an orientation of the restriction of L to any pair of subspaces E_1 and F_1 of E and F, respectively, with F_1 transverse to L and $E_1 = L^{-1}(F_1)$. In particular, when these two subspaces are finite dimensional, according to the previous argument, an orientation of L can be viewed as an orientation of $E_1 \times F_1$. This is a crucial property of the orientation, very useful in the construction of our degree (see [1]). Consider therefore any subspace F_1 of F, which is transverse to L (i.e. $\operatorname{Range} L + F_1 = F$), and observe that the restriction L_1 of L to the pair of spaces $E_1 = L^{-1}(F_1)$ (as domain) and F_1 (as codomain) is still Fredholm of index zero (thus when F_1 is finite dimensional, E_1 has the same dimension as F_1). Split E and F as follows: $E = E_0 \oplus E_1$, $F = \mathrm{L}(E_0) \oplus F_1$, where E_0 is any direct complement of E_1 in E. In this splitting, L can be represented by means of a matrix

$$L = \begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix},$$

where $L_0\colon E_0 \to \mathrm{L}(E_0)$ is an isomorphism. Thus, a linear operator $A\colon E \to F$, represented by

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix},$$

is a corrector of L if and only if A_1 is a corrector of L_1. It is easy to check that two correctors of L_1 are equivalent if and only if so are the corresponding correctors of L. This establishes a one-to-one correspondence between the two orientations of L_1 and the two orientations of L.

According to the above argument, it is convenient to introduce the following definition.

Definition 2.2. Let $L\colon E \to F$ be a Fredholm operator of index zero between real vector spaces, let F_1 be a subspace of F which is transverse to L, and denote by L_1 the restriction of L to the pair of spaces $L^{-1}(F_1)$ and F_1. Two orientations, one of L and one of L_1, are said to be *correlated* (or *one induced by the other*) if there exists a projector $P\colon E \to E$ onto E_1 and a positive corrector A_1 of L_1 such that the operator $A = JA_1P$ is a positive corrector of L, where $J\colon F_1 \hookrightarrow F$ is the inclusion.

We close this section by pointing out that the concept of orientation for a Fredholm operator of index zero can be extended to any Fredholm operator. In fact, if $L\colon E \to F$ is Fredholm of index $n > 0$, define $\widetilde{L}\colon E \to F \times \mathbb{R}^n$ by $\widetilde{L}x = (Lx, 0)$, which is Fredholm of index 0. An *orientation* of L is just an orientation of the associated operator $\widetilde{L}$. The case of negative index can be treated in a similar way.

3. Stability of the orientation in Banach spaces

As we have seen, the notion of orientation of an algebraic Fredholm operator of index zero $L\colon E \to F$ does not require any topological structure on E and F, which are supposed to be just vector spaces. We will show how, in the context of Banach spaces, an orientation of a continuous Fredholm operator of index zero induces, by a sort of stability, an orientation to any sufficiently close bounded operator. This allows us to define a concept of orientation for continuous maps from a topological space into the set of bounded Fredholm operators of index zero between Banach spaces.

Unless otherwise stated, hereafter E and F will denote two real Banach spaces. We shall indicate, respectively, by $\mathrm{L}(E,F)$ and $\mathrm{Iso}(E,F)$ the Banach space of bounded linear operators from E into F and the open subset of $\mathrm{L}(E,F)$ of the isomorphisms. The special cases $\mathrm{L}(E,E)$ and $\mathrm{Iso}(E,E)$ will be denoted, respectively, $\mathrm{L}(E)$ and $\mathrm{GL}(E)$ (the general linear group of E). Furthermore, $\mathrm{F}(E,F)$ (or $\mathrm{F}(E)$ when $E = F$) will stand for the subspace of $\mathrm{L}(E,F)$ of the operators with finite dimensional range.

From now on, any linear operator between Banach spaces that we shall consider (such as Fredholm operators or correctors) will be assumed to be bounded (even if not explicitly mentioned). For the sake of simplicity, the set of continuous correctors of a Fredholm operator of index zero $L\colon E \to F$ will be still denoted by $\mathcal{C}(L)$, as in the algebraic case, instead of $\mathcal{C}(L) \cap \mathrm{L}(E,F)$. It is clear that an orientation of L can be regarded as an equivalence class of continuous correctors of L.

We recall that the set $\Phi(E,F)$ of Fredholm operators from E into F is open in $\mathrm{L}(E,F)$, and the integer valued map $\mathrm{ind}\colon \Phi(E,F) \to \mathbb{Z}$ is continuous. Consequently, given $n \in \mathbb{Z}$, the set $\Phi_n(E,F)$ of the Fredholm operators of index n (written $\Phi_n(E)$ when $E = F$) is an open subset of $\mathrm{L}(E,F)$.

The following result, which is crucial for us, represents a sort of stability (in the context of Banach spaces) of the equivalence relation introduced in the previous section.

Lemma 3.1. *Let $A, B \in \mathrm{F}(E,F)$ be two L-equivalent correctors of an operator $L \in \Phi_0(E,F)$. Then there exist two neighborhoods U_A and U_B of A and B in $\mathrm{F}(E,F)$ and a neighborhood V_L of L in $\Phi_0(E,F)$ such that A' and B' are L'-equivalent for any $A' \in U_A$, $B' \in U_B$, $L' \in V_L$.*

Proof. Recall that the operator $K = I - (L+B)^{-1}(L+A)$ has finite dimensional range, and the assertion that A and B are L-equivalent means $\det(I-K) > 0$. Therefore, it is enough to show that $\det(I-K') > 0$ for $K' \in \mathrm{F}(E)$ sufficiently close to K. To prove this take a ball $D \subset \mathrm{F}(E)$ with center K such that $I - K'$ is still an automorphism for all $K' \in D$. Choose any $K' \in D$ and consider a finite dimensional subspace E_0 of E containing both $\mathrm{Range}\, K$ and $\mathrm{Range}\, K'$. This implies that E_0 contains the range of any operator $S_t = (1-t)K + tK'$ in the line segment joining K and K'. Thus, by the choice of D, $\det(I - S_t)|_{E_0}$ must be positive for all $t \in [0,1]$, and this implies $\det(I - K') > 0$. □

We recall now the concept of orientation for a subset of $\Phi_0(E,F)$ or, more generally, for a continuous map into $\Phi_0(E,F)$, introduced in [1].

Definition 3.2. Let Λ be a topological space and $h\colon \Lambda \to \Phi_0(E,F)$ a continuous map. An *orientation* α of h is a continuous choice of an orientation $\alpha(\lambda)$ of $h(\lambda)$ for each $\lambda \in \Lambda$; where "continuous" means that for any $\lambda \in \Lambda$ there exists $A_\lambda \in \alpha(\lambda)$ which is a positive corrector of $h(\lambda')$ for any λ' in a neighborhood of λ. A map is *orientable* if it admits an orientation and *oriented* when an orientation has been chosen. In particular, a subset $\mathcal{A}$ of $\Phi_0(E,F)$ is said to be *orientable* (or *oriented*) if so is the inclusion $i\colon \mathcal{A} \hookrightarrow \Phi_0(E,F)$.

Clearly, any restriction of an orientable map is orientable. More generally, if $g_1\colon \Lambda \to \Lambda_1$ is a continuous map between topological spaces and $g_2\colon \Lambda_1 \to \Phi_0(E,F)$ is orientable, then the composition $h = g_2 \circ g_1$ is orientable. Thus, in particular, a map $h\colon \Lambda \to \Phi_0(E,F)$ is orientable whenever its image is contained in an orientable subset of $\Phi_0(E,F)$.

Moreover, since $\mathrm{Iso}(E,F)$ is open in $\mathrm{L}(E,F)$, any given corrector of $L \in \Phi_0(E,F)$ is still a corrector of every L' in a suitable neighborhood of L. Consequently, $\Phi_0(E,F)$ is locally orientable, and so is any continuous map $h\colon \Lambda \to \Phi_0(E,F)$, in the sense that any $\lambda \in \Lambda$ admits a neighborhood U with the property that the restriction $h|_U$ is orientable.

Notice also that $\mathrm{Iso}(E,F)$ is (globally) orientable, and can be oriented with the *natural orientation* (i.e. by choosing the trivial operator $0 \in \mathrm{F}(E,F)$ as a positive corrector of any $L \in \mathrm{Iso}(E,F)$).

Remark 3.3. It is convenient to observe that an orientation of a continuous map $h\colon \Lambda \to \Phi_0(E,F)$ can be given by assigning a family $\{(U_i, A_i)\colon i \in \mathcal{I}\}$, called an *oriented atlas* of h, satisfying the following properties:

- $\{U_i : i \in \mathcal{I}\}$ is an open covering of Λ;
- given $i \in \mathcal{I}$, A_i is a corrector of any $h(\lambda)$, $\forall \lambda \in U_i$;
- if $\lambda \in U_i \cap U_j$, then A_i is $h(\lambda)$-equivalent to A_j.

As a straightforward consequence of Lemma 3.1 we get the following result which asserts that the above definition of orientation can be reformulated in an equivalent way, which turns out to be useful in the proof of some statements.

Proposition 3.4. *Let $h\colon \Lambda \to \Phi_0(E,F)$ be a continuous map. A choice of an orientation $\alpha(\lambda)$ of $h(\lambda)$ for each $\lambda \in \Lambda$ is an orientation for h if and only if the following condition holds:*
given any $\lambda \in \Lambda$ and any positive corrector A of $h(\lambda)$, there exists a neighborhood U of λ such that $A \in \alpha(\lambda')$, $\forall \lambda' \in U$.

The following two results are useful consequences of the above proposition.

Proposition 3.5. *Let $\mathcal{A}$ be an oriented subset of $\Phi_0(E,F)$. Then the map $L \mapsto \operatorname{sign} L$, which is defined on the open subset $\mathcal{A} \cap \mathrm{Iso}(E,F)$ of $\mathcal{A}$, is continuous.*

Proof. If $\mathcal{A} \cap \mathrm{Iso}(E,F)$ is empty, there is nothing to prove. Otherwise, let $L \in \mathcal{A} \cap \mathrm{Iso}(E,F)$ and assume, without loss of generality, that the trivial operator 0 is a positive corrector of L (which means $\operatorname{sign} L = 1$). Proposition 3.4 ensures that 0 remains a positive corrector for the operators of $\mathcal{A}$ in a neighborhood of L, and in such a neighborhood the sign map is constantly 1. □

Proposition 3.6. *An orientable map* $h\colon \Lambda \to \Phi_0(E,F)$ *admits at least two orientations. If, in particular,* Λ *is connected, then* h *admits exactly two orientations.*

Proof. Assume $h\colon \Lambda \to \Phi_0(E,F)$ is orientable and let α be one of its orientations. Taking at any $\lambda \in \Lambda$ the orientation opposite to $\alpha(\lambda)$, one gets an orientation $\alpha_- \neq \alpha$. Now, observe that, as a consequence of Proposition 3.4, the subset of Λ in which two orientations coincide is open, and for the same reason is open also the set in which two orientations are opposite one to the other. Therefore, if Λ is connected, two orientations of h are either equal or one is opposite to the other. □

The following result will be useful in the sequel.

Lemma 3.7. *Let* $L \in \Phi_0(E,F)$ *be a singular operator. Then any oriented neighborhood of* L *contains isomorphisms of opposite signs.*

Proof. Since any operator belonging to $\Phi_0(E,F)\backslash\mathrm{Iso}(E,F)$ admits arbitrarily close operators with one-dimensional kernel, we may assume $\dim \operatorname{Ker} L = \dim \operatorname{coKer} L = 1$. Now, split E and F as follows: $E = E_1 \oplus \operatorname{Ker} L$, $F = \operatorname{Range} L \oplus F_2$, where E_1 and F_2 are closed complements of $\operatorname{Ker} L$ and $\operatorname{Range} L$ respectively. In this decomposition L is represented by means of a matrix

$$L = \begin{pmatrix} L_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

where $L_{11}\colon E_1 \to \operatorname{Range} L$ is an isomorphism. Let $A_{22}\colon \operatorname{Ker} L \to F_2$ be an isomorphism and consider the following corrector of L:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix}.$$

Without loss of generality we may orient L (and, consequently, a neighborhood of L) by choosing A as a positive corrector. To prove the assertion it is enough to check that $\operatorname{sign}(L+tA) = \operatorname{sign} t$ for $t \neq 0$ sufficiently small (observe that $L+tA$ is an isomorphism for all $t \neq 0$). A direct computation shows that the trivial operator and A are $(L+tA)$-equivalent if and only if $1+1/t > 0$. Thus, as far as A is a positive corrector of $L+tA$, and $t \neq 0$, $\operatorname{sign}(L+tA) = \operatorname{sign}(1+1/t)$. Now, Proposition 3.4 ensures that A is a positive corrector of $L+tA$ for all t sufficiently small; and the assertion is proved. □

Obviously, if L belongs to the set $\Phi(E)$ of the Fredholm operators from E into itself and k is a positive integer, the operator $L_k \in \mathrm{L}(E \times \mathbb{R}^k)$ represented by the matrix

$$L_k = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}$$

is again Fredholm of the same index as L. Thus, we have a map $J_k\colon \Phi_0(E) \to \Phi_0(E \times \mathbb{R}^k)$, called the *natural embedding* of $\Phi_0(E)$ into $\Phi_0(E \times \mathbb{R}^k)$, which assigns to every L the operator L_k defined above. We observe that the *natural image* $J_k(\Phi_0(E))$ of $\Phi_0(E)$ is contained in the subset

$$Z = \Phi_0(E \times \mathbb{R}^k)\backslash\mathrm{GL}(E \times \mathbb{R}^k)$$

of the singular operators of $\Phi_0(E \times \mathbb{R}^k)$.

Proposition 3.8. *A subset* $\mathcal{A}$ *of* $\Phi_0(E)$ *is orientable if and only if so is its natural image* $J_k(\mathcal{A}) \subset \Phi_0(E \times \mathbb{R}^k)$*. More generally, the orientations of a continuous map* $h\colon \Lambda \to \Phi_0(E)$ *are in a one-to-one correspondence with the orientations of the composite map* $J_k \circ h$*.*

Proof. Observe first that if A is a corrector of $L \in \mathcal{A}$, then the *associated operator*

$$\bar{A} = \begin{pmatrix} A & 0 \\ 0 & I_k \end{pmatrix},$$

where I_k stands for the identity of $\mathbb{R}^k$, is a corrector of $J_k(L)$. One can easily check that $A, B \in \mathcal{C}(L)$ are L-equivalent if and only if the associated operators $\bar{A}$ and $\bar{B}$ are $J_k(L)$-equivalent. This shows that the orientations of $\mathcal{A}$ are in a one-to-one correspondence with the orientations of its natural image $J_k(\mathcal{A})$. The case of a map $h\colon \lambda \to \Phi_0(E)$ can be treated in the same way. □

The notion of continuity in the definition of oriented map can be regarded as a true continuity by introducing the following topological space (which is actually a real Banach manifold). Let $\widehat{\Phi}_0(E,F)$ denote the set of pairs (L,ω) with $L \in \Phi_0(E,F)$ and ω one of the two equivalence classes of $\mathcal{C}(L)$. Given an open subset W of $\Phi_0(E,F)$ and an element $A \in \mathrm{F}(E,F)$, consider the set

$$O_{(W,A)} = \big\{(L,\omega) \in \widehat{\Phi}_0(E,F) : L \in W,\ A \in \omega\big\}.$$

The collection of sets obtained in this way is clearly a basis for a topology on $\widehat{\Phi}_0(E,F)$, and the natural projection $p\colon (L,\omega) \mapsto L$ is a double covering of $\Phi_0(E,F)$. Observe also that the family of the restrictions of p to the open subsets of $\widehat{\Phi}_0(E,F)$ in which p is injective is an atlas for a Banach manifold structure modeled on $\mathrm{L}(E,F)$.

It is easy to check that the following is an alternative definition of orientation, and has the advantage that many properties of the orientable maps can be directly deduced from well known results in covering space theory.

Definition 3.9. Let $h\colon \Lambda \to \Phi_0(E,F)$ be a continuous map defined on a topological space Λ. An *orientation* of h is a lifting $\widehat{h}$ of h (i.e. a continuous map $\widehat{h}\colon \Lambda \to \widehat{\Phi}_0(E,F)$ such that $p \circ \widehat{h} = h$). The map h is called *orientable* when it admits a lifting, and *oriented* when one of its liftings has been chosen. In particular, a subset $\mathcal{A}$ of $\Phi_0(E,F)$ is *orientable* (*oriented*) when so is the associated inclusion.

According to this definition, an orientation of h is a continuous map $\widehat{h}\colon \Lambda \to \widehat{\Phi}_0(E,F)$ of the form $\widehat{h}\colon \lambda \mapsto (h(\lambda),\alpha(\lambda))$. Thus $\widehat{h}$ is completely determined by its second component α. For this reason, when it is convenient, we shall merely call α an orientation of h, which is in the spirit of Definition 3.2.

The following proposition is a characterization of the orientability for a connected subset of $\Phi_0(E,F)$.

Proposition 3.10. *Let* $\mathcal{A}$ *be a connected subset of* $\Phi_0(E,F)$*. Then* $\mathcal{A}$ *is orientable if and only if* $\widehat{\mathcal{A}} = p^{-1}(\mathcal{A})$ *is composed of two connected components. Moreover, when* $\mathcal{A}$ *is orientable, the restriction of* p *to each one of these components is a homeomorphism onto* $\mathcal{A}$*.*

Proof. First assume that $\mathcal{A}$ is orientable and denote by α and α_- the two orientations of $\mathcal{A}$. It follows that $(\mathcal{A},\alpha)$ and $(\mathcal{A},\alpha_-)$ are nonempty open subsets of $\widehat{\mathcal{A}}$ (here, given an orientation β of $\mathcal{A}$, $(\mathcal{A},\beta)$ denotes the set $\{(L,\beta(L)) : L \in \mathcal{A}\}$). Observe that these two sets are connected (being homeomorphic to $\mathcal{A}$), with empty intersection and such that $(\mathcal{A},\alpha)\cup(\mathcal{A},\alpha') = \widehat{\mathcal{A}}$. Thus $\widehat{\mathcal{A}}$ has exactly two connected components.

Conversely, suppose that $\widehat{\mathcal{A}}$ has two connected components, say $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$. Since the projection $p\colon \widehat{\mathcal{A}} \to \mathcal{A}$ is a covering map, it is also an open map. Moreover, because $\widehat{\mathcal{A}}$ has a finite number of components (just two, in this case), p is a closed map. Thus the two connected sets $p(\widehat{\mathcal{A}}_1)$ and $p(\widehat{\mathcal{A}}_2)$ are open and closed in A and, consequently, $\mathcal{A} = p(\widehat{\mathcal{A}}_1) = p(\widehat{\mathcal{A}}_2)$. Since $p|_{\widehat{\mathcal{A}}_1}\colon \widehat{\mathcal{A}}_1 \to \mathcal{A}$ and $p|_{\widehat{\mathcal{A}}_2}\colon \widehat{\mathcal{A}}_2 \to \mathcal{A}$ are both onto, they are also injective, and then, homeomorphisms. Therefore, as an orientation of $\mathcal{A}$ one can define the map α which assigns to every $L \in \mathcal{A}$ the orientation $\alpha(L)$ of L in such a way that $(L,\alpha(L)) \in \widehat{\mathcal{A}}_1$. Equivalently, according to Definition 3.9, $(p|_{\widehat{\mathcal{A}}_1})^{-1}$ is an orientation of $\mathcal{A}$. □

Let us now recall some basic results in covering space theory that will be useful in the sequel (see, for example, [10]). Theorem 3.11 below implies, in particular, that any continuous map $h\colon \Lambda \to \Phi_0(E,F)$ is orientable, provided that Λ is simply connected and locally path connected.

Theorem 3.11. *Let $\pi\colon Z \to X$ be a covering space and let $h\colon \Lambda \to X$ be a continuous map defined on a connected space Λ. Then, given two liftings of h, either they coincide or they have disjoint images. Moreover, if Λ is simply connected and locally path connected, then, given $\lambda_0 \in \Lambda$ and $z_0 \in \pi^{-1}(h(\lambda_0))$, h admits a lifting $\widehat{h}$ such that $\widehat{h}(\lambda_0) = z_0$.*

Theorem 3.12. *Let $\pi\colon Z \to X$ be a covering space and consider a continuous map $h\colon \Lambda \to X$ which admits a lifting $\widehat{h}$. Given a homotopy $H\colon \Lambda \times [0,1] \to X$ with $H(\cdot,0) = h$, there exists a unique lifting $\widehat{H}$ of H such that $\widehat{H}(\cdot,0) = \widehat{h}$.*

From Theorem 3.11 we deduce the following characterization of the orientability. Observe first that, according to this theorem, given a covering space $\pi\colon Z \to X$, given a path $\sigma\colon [0,1] \to X$ and given a point $z_0 \in \pi^{-1}(\sigma(0))$, there exists a unique lifting $\widehat{\sigma}$ such that $\widehat{\sigma}(0) = z_0$. Moreover, if $\widehat{\sigma}'$ is a different lifting of σ, then $\widehat{\sigma}([0,1])$ and $\widehat{\sigma}'([0,1])$ are disjoint. In particular, when $\pi\colon Z \to X$ is a 2-fold covering space, any closed path in X admits two liftings which are either both closed or both not closed.

Proposition 3.13. *Let $h\colon \Lambda \to \Phi_0(E,F)$ be a continuous map defined on a locally path connected topological space Λ. Then h is orientable if and only if, given any closed path $\gamma\colon [0,1] \to \Lambda$, the two liftings of $h\circ\gamma$ are closed. In particular, an open subset $\mathcal{A}$ of $\Phi_0(E,F)$ is orientable if and only if the two liftings of any closed path in $\mathcal{A}$ are closed.*

Proof. Since any connected component of Λ is also locally path connected, without loss of generality we may suppose that Λ is connected.

Assume first that h is orientable and let $\widehat{h}_1$ and $\widehat{h}_2$ denote the two orientations of h. Given a closed path $\gamma\colon [0,1] \to \Lambda$, the two liftings of $h\circ\gamma$ are $\widehat{h}_1\circ\gamma$ and $\widehat{h}_2\circ\gamma$, which are clearly closed.

Conversely, assume that for any closed path $\gamma\colon [0,1] \to \Lambda$, the two liftings of $h \circ \gamma$ are closed. Choose $\lambda_0 \in \Lambda$ and an orientation ω_0 of $h(\lambda_0)$. Let us show that there exists a unique orientation $\widehat{h}$ of h such that $\widehat{h}(\lambda_0) = (h(\lambda_0), \omega_0)$. Given $\lambda \in \Lambda$, consider any path $\sigma\colon [0,1] \to \Lambda$ such that $\sigma(0) = \lambda_0$ and $\sigma(1) = \lambda$ (recall that a connected, locally path connected space is path connected). Define $\widehat{h}(\lambda) := \widehat{h \circ \sigma}(1)$, where $\widehat{h \circ \sigma}$ is the unique lifting of $h \circ \sigma$ such that $\widehat{h \circ \sigma}(0) = (h(\lambda_0), \omega_0)$. As a consequence of the assumption, $\widehat{h}(\lambda)$ does not depend on the path joining λ_0 with λ. Thus $\widehat{h}\colon \Lambda \to \Phi_0(E,F)$ is well defined. The continuity of $\widehat{h}$ is an easy consequence of the fact that Λ is locally path connected. □

The following straightforward consequence of Theorem 3.12 states a crucial property of our notion of orientation. In particular, it implies that, given a homotopy $H\colon \Lambda \times [0,1] \to \Phi_0(E,F)$, the partial maps $H_s = H(\cdot, s)$ are either all orientable or all nonorientable.

Theorem 3.14. *Let $H\colon \Lambda \times [0,1] \to \Phi_0(E,F)$ be a homotopy. Then H is orientable if and only if so is the partial map $H_0 = H(\cdot,0)$. In particular, if H_0 is oriented with orientation $\widehat{H}_0$, there exists a unique orientation $\widehat{H}$ of H such that $\widehat{H}(\cdot,0) = \widehat{H}_0$.*

We conclude this section with a result that should clarify our notion of orientation (Theorem 3.15 below). As we have already seen, $\Phi_0(E,F)$ is locally orientable. However, if E and F have the same finite dimension, then $\Phi_0(E,F)$ coincides with $\mathrm{L}(E,F)$, and thus, being simply connected, it is actually globally orientable. If otherwise E and F are infinite dimensional, $\Phi_0(E,F)$ has a more complicated topological structure, and, as we shall see below, it may happen to be nonorientable.

An interesting result of Kuiper (see [12]) asserts that the linear group $\mathrm{GL}(E)$ of an infinite dimensional separable Hilbert space is contractible. It is also known that $\mathrm{GL}(l^p)$, $1 \le p < \infty$, and $\mathrm{GL}(c_0)$ are contractible as well. There are, however, examples of infinite dimensional Banach spaces whose linear group is disconnected (see [5], [15] and references therein).

The following result shows, in particular, that when $\mathrm{GL}(E)$ is connected, then $\Phi_0(E)$ is not orientable. We do not know if $\Phi_0(E)$ turns out to be nonorientable with the weaker assumption that E is infinite dimensional.

Theorem 3.15. *Assume $\mathrm{Iso}(E,F)$ is nonempty and connected. Then there exists a nonorientable map $\gamma\colon S^1 \to \Phi_0(E,F)$ defined on the unit circle of $\mathbb{R}^2$. In particular $\Phi_0(E,F)$ is nonorientable and, consequently, it is connected and not simply connected.*

Proof. Let S^1_+ and S^1_- denote, respectively, the two arcs of S^1 with nonnegative and nonpositive second coordinate. By Lemma 3.7 there exists an oriented open connected subset U of $\Phi_0(E,F)$ containing two points in $\mathrm{Iso}(E,F)$, say L_- and L_+, such that $\operatorname{sign} L_- = -1$ and $\operatorname{sign} L_+ = 1$. Let $\gamma_+\colon S^1_+ \to U$ be a path such that $\gamma_+(-1,0) = L_-$ and $\gamma_+(1,0) = L_+$. Since $\mathrm{Iso}(E,F)$ is an open connected subset of $\mathrm{L}(E,F)$, it is also path connected. Therefore there exists a path $\gamma_-\colon S^1_- \to \mathrm{Iso}(E,F)$ such that $\gamma_-(-1,0) = L_-$ and $\gamma_-(1,0) = L_+$. Define $\gamma\colon S^1 \to \Phi_0(E,F)$ by

$$\gamma(x,y) = \begin{cases} \gamma_+(x,y) & \text{if } y \ge 0 \\ \gamma_-(x,y) & \text{if } y \le 0 \end{cases}$$

and assume, by contradiction, it is orientable. This implies that also the image $\gamma(S^1)$ of γ is orientable, with just two possible orientations. Orient, for example, $\gamma(S^1)$ with the unique orientation compatible with the oriented subset U of $\Phi_0(E,F)$. Thus, being $\gamma(S^1_+) \subset U$, we get $\operatorname{sign} L_- = -1$ and $\operatorname{sign} L_+ = 1$. On the other hand, since the image of γ_- is contained in $\mathrm{Iso}(E,F)$, from Proposition 3.5 it follows $\operatorname{sign} L_- = \operatorname{sign} L_+$, which is a contradiction.

Clearly, since γ is not orientable, any subset of $\Phi_0(E,F)$ containing $\gamma([0,1])$ is not orientable as well. Thus, $\Phi_0(E,F)$ is not simply connected, since otherwise it would be orientable.

Finally, $\Phi_0(E,F)$ is connected, as contained in the closure of the connected set $\mathrm{Iso}(E,F)$. □

4. Orientation for Fredholm maps between Banach manifolds and degree

This section is devoted to a notion of orientation for Fredholm maps of index zero between Banach manifolds based on the concept of orientation for continuous maps into $\Phi_0(E,F)$. This concept was introduced in [1] in order to define a topological degree for oriented Fredholm maps of index zero between Banach manifolds. We give here just the general idea of this notion of degree, since our interest is mainly dedicated to the properties of the orientation.

From now on M and N will denote two differentiable manifolds modeled on two real Banach spaces E and F respectively. We recall that a map $f\colon M \to N$ is Fredholm of index n if it is C^1 and its derivative $Df(x)\colon T_xM \to T_{f(x)}N$ is a linear Fredholm operator of index n for all $x \in M$ (here, given a point $x \in M$, T_xM denotes the tangent space of M at x).

Consider first the special case when f is a Fredholm map of index zero from an open subset Ω of E into F. An *orientation* of f is, by definition, just an orientation of the continuous map $Df\colon x \mapsto Df(x)$, and f is *orientable* (resp. *oriented*) is so is Df according to Definition 3.9.

Let now $f\colon M \to N$ be a Fredholm map of index zero between two manifolds. For any $x \in M$ one can choose an orientation ω_x of the derivative $Df(x) \in \Phi_0(T_xM, T_{f(x)}N)$. However, in order to define an orientation of f, we need a notion of continuity for the map $x \mapsto \omega_x$, and this cannot be immediately stated as in the flat case. For this purpose we make the following construction.

Consider the set

$$J(M,N) = \big\{(x,y,L) : (x,y) \in M \times N,\ L \in \mathrm{L}(T_xM, T_yN)\big\},$$

and denote by $\pi\colon (x,y,L) \mapsto (x,y)$ the natural projection of $J(M,N)$ onto $M \times N$. The set $J(M,N)$ has a natural topology defined as follows. Given two charts $\phi\colon U \to E$ and $\psi\colon V \to F$ of M and N respectively, and given an open subset W of $\mathrm{L}(E,F)$, consider the (possibly empty) set

$$Q_{(\phi,\psi,W)} = \big\{(x,y,L) \in \pi^{-1}(U \times V) : D\psi(y) \circ L \circ D\phi^{-1}(\phi(x)) \in W\big\}. \tag{4.1}$$

Clearly, the collection of the sets obtained in this way is a basis for a topology on $J(M,N)$, and π is continuous. Moreover, given two charts (U,ϕ) and (V,ψ) as above, the map

$$\Gamma_{(\phi,\psi)}\colon \pi^{-1}(U \times V) \to E \times F \times \mathrm{L}(E,F)$$

defined by $(x, y, L) \mapsto (\phi(x), \psi(y), D\psi(y) \circ L \circ D\phi^{-1}(\phi(x)))$ is a homeomorphism onto the open subset $\phi(U) \times \psi(V) \times \mathrm{L}(E, F)$ of the Banach space $E \times F \times \mathrm{L}(E, F)$. Actually – but it is not important for our purposes – the family of maps $\Gamma_{(\phi,\psi)}$, with ϕ and ψ local charts of M and N, is a vector bundle atlas for the natural projection π. The reader who is familiar with the notion of space of jets has probably noticed that $J(M, N)$ is just the bundle of first order jets from M into N, usually denoted $J^1(M, N)$ (see e.g. [11]).

Consider now the subset $\Phi_0 J(M, N)$ of $J(M, N)$ defined as

$$\Phi_0 J(M, N) = \big\{(x, y, L) \in J(M, N) : L \in \Phi_0(T_x M, T_y N)\big\},$$

which is clearly open in $J(M, N)$ and, consequently, inherits the structure of a Banach manifold. As a topological space, $\Phi_0 J(M, N)$ is essential in the definition of oriented Fredholm map of index zero $f\colon M \to N$. In fact, analogously to the flat case, where an orientation of $f\colon \Omega \to F$ is just an orientation of the derivative $Df\colon \Omega \to \Phi_0(E, F)$, in the general case we associate to $f\colon M \to N$ the continuous map $jf\colon M \to \Phi_0 J(M, N)$ given by $jf(x) = (x, f(x), Df(x))$ and we define a concept of orientation for such a map (or, more generally, for continuous maps into $\Phi_0 J(M, N)$). The task will be accomplished by introducing the following 2-fold covering space of $\Phi_0 J(M, N)$, which plays the same role as $\widehat{\Phi}_0(E, F)$ in the flat case:

$$\widehat{\Phi}_0 J(M, N) = \big\{(x, y, L, \omega) : (x, y, L) \in \Phi_0 J(M, N),\ \omega \text{ an orientation of } L\big\}.$$

The topology of $\widehat{\Phi}_0 J(M, N)$ is defined as follows. Let $\phi\colon U \to E$ and $\psi\colon V \to F$ be two charts of M and N respectively, and let W be an open subset of $\mathrm{L}(E, F)$. Consider the associated open subset $Q_{(\phi,\psi,W)}$ of $J(M, N)$ defined in formula (4.1) and let A be a given operator in $\mathrm{F}(E, F)$. Define the (possibly empty) set

$$\begin{aligned} O_{(\phi,\psi,W,A)} \quad = \quad & \big\{(x, y, L, \omega) \in \widehat{\Phi}_0 J(M, N) : (x, y, L) \in Q_{(\phi,\psi,W)}, \\ & D\psi^{-1}(\psi(y)) \circ A \circ D\phi(x) \in \omega\big\}. \end{aligned}$$

The family of sets obtained in this way is clearly a basis for a topology on $\widehat{\Phi}_0 J(M, N)$, and the natural projection $p\colon (x, y, L, \omega) \mapsto (x, y, L)$ is a double covering of $\Phi_0 J(M, N)$. Thus $\widehat{\Phi}_0 J(M, N)$ inherits (from $\Phi_0 J(M, N)$) the structure of Banach manifold (on $E \times F \times \mathrm{L}(E, F)$).

We can now introduce our notion of orientation for a Fredholm map of index zero $f\colon M \to N$. Consider first a continuous map $h\colon \Lambda \to \Phi_0 J(M, N)$, where Λ is a topological space. We say that h is *orientable* when it admits a lifting $\widehat{h}\colon \Lambda \to \widehat{\Phi}_0 J(M, N)$, and a chosen lifting is an *orientation* of h, which, in this case, is called *oriented*.

Definition 4.1. A map $f\colon M \to N$ between two real Banach manifolds is said to be *orientable* if it is Fredholm of index zero and $jf\colon M \to \Phi_0 J(M, N)$ admits a lifting $\widehat{jf}\colon M \to \widehat{\Phi}_0 J(M, N)$. A lifting of jf is an *orientation* of f, and f is *oriented* when it is orientable and one of its orientations has been chosen.

Let us see now some properties of this notion of orientation.

First of all observe that, as in the flat case, an orientation of $f\colon M \to N$ can be regarded as a continuous map α which assigns to any $x \in M$ an orientation $\alpha(x)$

of the derivative $Df(x)\colon T_xM \to T_{f(x)}N$, where "continuous" means that the map $\widehat{jf}\colon M \to \widehat{\Phi}_0 J(M,N)$ given by $x \mapsto (x, f(x), Df(x), \alpha(x))$ is continuous.

Clearly, if a map $f\colon M \to N$ is orientable, it admits at least two orientations, and exactly two when M is connected.

Notice also that any local diffeomorphism $f\colon M \to N$ is orientable. In fact, f can be *naturally oriented* by choosing, for any $x \in M$, the natural orientation of the isomorphism $Df(x)$.

The simplest example of a nonorientable Fredholm map (of index zero) consists of a constant function from a finite dimensional nonorientable manifold M into a manifold N of the same dimension as M. The following less trivial example provides a nonorientable Fredholm map in the flat case, i.e. acting between open sets of Banach spaces.

Example 4.2. Let E be a Banach space with $\mathrm{GL}(E)$ connected, S^1 be the unit circle in $\mathbb{R}^2$, and $\gamma\colon S^1 \to \Phi_0(E)$ be a nonorientable C^1 path. Consider the open subset $\Omega = E \times (\mathbb{R}^2 \setminus \{0\})$ of $E \times \mathbb{R}^2$ and define the map $f\colon \Omega \to E \times \mathbb{R}^2$ by $f(x,y) = (\gamma(y/\|y\|)x, y)$, which is clearly of class C^1. Let us show that f is actually Fredholm of index zero. The Fréchet derivative of f at a point $(x_0, y_0) \in \Omega$ is given by

$$Df(x_0,y_0)(u,v) = (\gamma(y_0/\|y_0\|)u + K(x_0,y_0)v, v)$$

where $K(x_0,y_0) = D_2 f_1(x_0,y_0)$ is the second partial derivative of the first component $f_1\colon \Omega \to E$ of f (i.e. $f_1(x,y) = \gamma(y/\|y\|)x$). Since the operator $L = \gamma(y_0/\|y_0\|)$ is Fredholm of index zero, so is $J_2(L)\colon E \times \mathbb{R}^2 \to E \times \mathbb{R}^2$, where J_2 is the natural embedding of $\Phi_0(E)$ into $\Phi_0(E \times \mathbb{R}^2)$, i.e. $J_2(L)(u,v) = (Lu, 0)$. Consequently, since the difference $Df(x_0,y_0) - J_2(L)$ has finite dimensional range, $Df(x_0,z_0)$ is Fredholm of the same index as $J_2(L)$. Thus, as claimed, the nonlinear map f is Fredholm of index zero.

Let us prove that f is not orientable. By the definition of orientability for Fredholm maps of index zero, we need to show that it is not orientable the derivative $Df\colon \Omega \to \Phi_0(E \times \mathbb{R}^2)$ of f. Now the map $h\colon \Omega \to \Phi_0(E)$ given by $h(x,y) = \gamma(y/\|y\|)$ is nonorientable, since so is its restriction $\gamma\colon S^1 \to \Phi_0(E)$. (Here the unit circle S^1 of $\mathbb{R}^2$ is regarded as a subset of $\{0\} \times \mathbb{R}^2 \subset E \times \mathbb{R}^2$.) Thus, Proposition 3.8 ensures that also the composition $J_2 \circ h\colon \Omega \to \Phi_0(E \times \mathbb{R}^2)$ is nonorientable. Finally, the non-orientability of Df is a straightforward consequence of Theorem 3.14, since Df and $J_2 \circ h$ are homotopic (as maps into $\Phi_0(E \times \mathbb{R}^2)$) via the map

$$H(x,y,s) = Df(x,y) + s((J_2 \circ h)(x,y) - Df(x,y)), \quad s \in [0,1].$$

To see that the homotopy H is actually a map into $\Phi_0(E \times \mathbb{R}^2)$, observe that, given $(x,y) \in \Omega$, the difference $(J_2 \circ h)(x,y) - Df(x,y)$ is a compact operator.

Consider a homotopy of Fredholm maps of index zero from M into N, i.e. a continuous map $H\colon M \times [0,1] \to N$ which is continuously differentiable with respect to the first variable and such that, for any $(x,s) \in M \times [0,1]$, the partial derivative $D_1 H(x,s)$ is a Fredholm operator of index zero from T_xM into $T_{H(x,s)}N$. We say that H is *orientable* if so is the continuous map $j_1 H\colon M \times [0,1] \to \Phi_0 J(M,N)$ defined by $j_1 H(x,s) = (x, H(x,s), D_1 H(x,s))$, and *oriented* when an orientation of H has been assigned. Clearly, an oriented $H\colon M \times [0,1] \to N$ induces, by restriction, an orientation to any partial map H_s.

The following direct consequence of Theorem 3.12 implies, in particular, that the partial maps of a Fredholm homotopy are either all orientable or all nonorientable.

Theorem 4.3. *Let $H\colon M \times [0,1] \to N$ be a homotopy of Fredholm maps of index zero and assume that H_0 is orientable. Then H is orientable and an orientation of H_0 is the restriction of a unique orientation of H.*

In the remaining part of this section we give a brief idea of our notion of degree (for a complete discussion see [1]).

Definition 4.4. Let M and N be two Banach manifolds and $f\colon M \to N$ be an oriented map. Given an open subset U of M and an element $y \in N$, we say that the triple (f, U, y) is *admissible* (or, equivalently, f is y-admissible in U) if $f^{-1}(y) \cap U$ is compact.

Our topological degree is an integer valued function defined in the class of admissible triples and satisfying the following main properties:

i) (*Normalization*) If $f\colon M \to N$ is a naturally oriented diffeomorphism and $y \in N$, then

$$\deg(f, M, y) = 1.$$

ii) (*Additivity*) If (f, M, y) is an admissible triple and U_1, U_2 are two open disjoint subsets of M such that $f^{-1}(y) \subset U_1 \cup U_2$, then

$$\deg(f, M, y) = \deg(f, U_1, y) + \deg(f, U_2, y).$$

iii) (*Homotopy invariance*) Let $H\colon M \times [0,1] \to N$ be an oriented homotopy of Fredholm maps of index zero. Then, given any path $y\colon [0,1] \to N$ such that the set $\{(x,t) \in M \times [0,1] : H(x,t) = y(t)\}$ is compact, $\deg(H_t, M, y(t))$ does not depend on t.

The degree of an admissible triple (f, U, y) is preliminary defined when y is a regular value for f in U. In this case

$$\deg(f, U, y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} Df(x).$$

This restrictive assumption on y is then removed by means of the following lemma of [1].

Lemma 4.5. *Let (f, U, y) be admissible and let W_1 and W_2 be two open neighborhoods of $f^{-1}(y)$ such that $\overline{W}_1 \cup \overline{W}_2 \subset U$ and f is proper in $\overline{W}_1 \cup \overline{W}_2$. Then there exists a neighborhood V of y such that for any pair of regular values $y_1, y_2 \in V$ one has*

$$\deg(f, W_1, y_1) = \deg(f, W_2, y_2).$$

Lemma 4.5 justifies the following definition of degree for general admissible triples (recall first that Fredholm maps are locally proper).

Definition 4.6. Let (f, U, y) be admissible and let W be any open neighborhood of $f^{-1}(y)$ such that $\overline{W} \subset U$ and f is proper on $\overline{W}$. The degree of (f, U, y) is given by

$$\deg(f, U, y) := \deg(f, W, z),$$

where z is any regular value for f in W, sufficiently close to y.

As pointed out in [1], no infinite dimensional version of the Sard Theorem is needed in the above definition, since the existence of a sequence of regular values for $f|_W$ which converges to y is a consequence of the Implicit Function Theorem and the classical Sard-Brown Lemma.

Observe that, in particular, our degree is defined for any oriented map between compact (not necessarily orientable) manifolds. In a forthcoming paper we will show that, in the finite dimensional context and for C^1 maps, our degree coincides with the extension of the Brouwer degree given by Dold in [2]. Moreover, our notion of oriented map between finite dimensional manifolds will be easily interpreted in the continuous case in order to coincide with the concept of continuous oriented map given by Dold (see [2], exercise 6, p. 271).

5. Comparison with other notions of orientability

In this section we compare our concept of orientation with two strictly related notions: the first one due to Elworthy-Tromba and the other one to Fitzpatrick-Pejsachowicz-Rabier. As pointed out in the Introduction, the first $\mathbb{Z}$-valued degree theory in the context of Banach manifolds is due to Elworthy and Tromba (see [3] and [4]). Their construction is based on an extension to the infinite dimensional case of the usual notion of orientation for finite dimensional manifolds. We give here a brief summary of their ideas and results.

Let M be a differentiable manifold modeled on a real Banach space E. A *Fredholm structure* on M is an atlas $\mathcal{A}$ which satisfies the following property and is maximal with respect to this property:

for any (U,ϕ), $(V,\psi) \in \mathcal{A}$, and for each $x \in U \cap V$, the derivative $D(\psi \circ \phi^{-1})(\phi(x))$ of $\psi \circ \phi^{-1}$ at $\phi(x)$ is of the form $I - K_x$, where $K_x \colon E \to E$ is a compact linear operator.

A *Fredholm manifold* is a Banach manifold with a Fredholm structure, and the charts of the structure are the *Fredholm charts* of the manifold. A Fredholm manifold $(M,\mathcal{A})$ is *orientable* (or, equivalently, $\mathcal{A}$ is *orientable*) if there exists a subatlas $\mathcal{U}$ of $\mathcal{A}$ such that, given any two charts (U_1,ϕ_1) and (U_2,ϕ_2) in $\mathcal{U}$, one has

$$\deg_{LS}(D(\phi_1 \circ \phi_2^{-1})(\phi_2(x))) = 1 \tag{5.1}$$

for all $x \in U_1 \cap U_2$, where $\deg_{LS}(D(\phi_1 \circ \phi_2^{-1})(\phi_2(x)))$ denotes the Leray-Schauder degree of $D(\phi_1 \circ \phi_2^{-1})(\phi_2(x))$. A subatlas of $\mathcal{A}$ which is maximal with respect to condition (5.1) is called an *orientation* of M, and M is said to be an *oriented manifold* when one of its orientations is assigned. In this case, the charts of the selected orientation are said to be the *oriented charts* of M.

Notice that an infinite dimensional (real) Banach space admits infinitely many Fredholm structures. One of these, the *trivial structure*, is defined as the unique Fredholm structure containing the identity. From the properties of the Leray-Schauder degree it follows immediately that such a structure admits exactly two orientations, and one of these is the unique orientation containing the identity. The fact that in this case one of the two orientations is distinguished is a peculiarity of the Elworthy-Tromba theory (observe that with the classical notion, a finite dimensional Banach space has no distinguished orientations).

Let M and N be two Fredholm manifolds based on the same Banach space E. A Fredholm map of index zero $f \colon M \to N$ is said to be *admissible* if, for every pair of oriented charts (U, ϕ) and (V, ψ) of M and N respectively, and for any $x \in U \cap f^{-1}(V)$, one has

$$D(\psi \circ f \circ \phi^{-1})(\phi(x)) = I - K_x, \tag{5.2}$$

where K_x is a compact endomorphism of E.

In the case of a Fredholm map of index zero $f\colon M \to E$, where M is a real Banach manifold modeled on a Banach space E, there exists a unique Fredholm structure on M in such a way that f becomes admissible as a map into E with its trivial structure.

The topological degree introduced by Elworthy and Tromba is defined for the class of proper C^2 admissible maps between oriented Fredholm manifolds. The following is the first fundamental step in their definition (see the two cited papers [3] and [4] for the complete construction).

Definition 5.1. Let M and N be two oriented manifolds and $f\colon M \to N$ be C^2, proper and admissible. The degree of f with respect to a regular value y, written $\deg_{ET}(f, y)$, is defined as

$$\deg_{ET}(f, y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} Df(x),$$

where $\operatorname{sign} Df(x) = 1$ or -1, depending on whether $\deg_{LS} D(\psi \circ f \circ \phi^{-1})(\phi(x)) = 1$ or -1, for any given pair of oriented charts ϕ and ψ at x and $f(x)$ respectively.

In [3] it is shown that when N is connected $\deg_{ET}(f, y)$ does not depend on the regular value $y \in N$. Notice that the infinite dimensional version of Sard's Theorem due to Smale ([17]) ensures that the regular values of f are almost all of N (in the sense of Baire category).

The following result shows that the class of maps for which the Elworthy-Tromba degree is well defined is strictly contained in the class of orientable maps according to our definition.

Proposition 5.2. *If $f\colon M \to N$ is an admissible map between two orientable Fredholm manifolds, then it is orientable in the sense of Definition 4.1.*

Proof. Assume that M and N are oriented. Given $x \in M$, define the orientation $\alpha(x)$ of $Df(x)$ in such a way that

$$\deg_{LS}(D\psi(f(x)) \circ (Df(x) + A) \circ D\phi^{-1}(\phi(x))) = 1$$

for any $A \in \alpha(x)$ and any pair of oriented charts ϕ and ψ at x and $f(x)$ respectively. From the properties of the Leray-Schauder degree it follows that $\alpha(x)$ is well defined. One can check that the map $\widehat{jf}\colon M \to \widehat{\Phi}_0 J(M, N)$ given by $\widehat{jf}(x) = (x, f(x), Df(x), \alpha(x))$ is continuous, which means it is a lifting of jf. □

Since our notion of orientation is defined for the class of Fredholm maps of index zero between Banach manifolds and not merely for the subclass of admissible maps between Fredholm manifolds, the converse of the above proposition does not make sense, unless M and N are finite dimensional. However, even in the restricted context of finite dimensional compact manifolds, one may find smooth orientable maps for which the Elworthy-Tromba degree is not defined, as shown by the following two classes of maps.

According to our construction, given a finite dimensional manifold M,

- the canonical projection $p\colon \widehat{M} \to M$ of the oriented double covering $\widehat{M}$ of M is orientable, being a local diffeomorphism,
- the identity $I\colon M \to M$ is orientable.

Thus, in the above cases, if M is compact, our degree is well defined, no matter whether or not M is orientable. This is not the case for the Elworthy-Tromba degree, since in the finite dimensional context their notion of orientation coincides with the classical one and their degree is just the Brouwer degree.

A deep analysis of the Elworthy-Tromba theory is not the object of this paper. We only point out that their theory encounters some difficulties. First of all, it is not easy to verify when an infinite dimensional Banach manifold admits a Fredholm structure. Moreover, even when a manifold admits such a structure, this is not necessarily unique, and one could find both orientable and nonorientable structures on the same manifold (as the two authors observe). In addition, the Elworthy-Tromba degree does not verify a general homotopy invariance property. In fact, given two oriented manifolds M and N, and given a C^2 proper Fredholm homotopy of index one, $H\colon M \times [0,1] \to N$, with admissible partial end-maps H_0 and H_1, it does not necessarily follow that each H_t is admissible. However, an absolute value homotopy invariance property is verified. That is, given a regular value $y \in N$ for both H_0 and H_1, one has

$$\deg_{ET}(H_0, y) = \pm \deg_{ET}(H_1, y),$$

and not necessarily

$$\deg_{ET}(H_0, y) = \deg_{ET}(H_1, y),$$

unless each H_t is admissible.

Finally, still regarding the comparison with the Elworthy-Tromba theory, we point out that their construction depends on the Leray-Schauder degree, while our definition is merely based on the Brouwer degree.

The theory of Elworthy and Tromba has been recently and considerably improved by Fitzpatrick, Pejsachowicz and Rabier in [8] and [9], where they introduce a topological degree for C^2 Fredholm maps of index zero between Banach (and not merely Fredholm) manifolds. In their construction, rather than defining a concept of orientation for some class of infinite dimensional differentiable manifolds, they develop a general theory of orientation for C^2 Fredholm maps of index zero which, as they observe, is more general and simpler than the Elworthy-Tromba theory. Before comparing their notion of orientation with our one, we summarize their ideas.

Consider two Banach spaces E and F and a path $\gamma\colon [0,1] \to \Phi_0(E,F)$. There exists a continuous path $k\colon [0,1] \to \mathrm{L}(E,F)$ such that $k(t)$ is compact for every $t \in [0,1]$, and $\gamma(t)+k(t) \in \mathrm{Iso}(E,F)$ (see for instance [13]). Define $g\colon [0,1] \to \mathrm{Iso}(E,F)$ by $g(t) = \gamma(t) + k(t)$. One has $g(t)^{-1} \circ \gamma(t) = I - h(t)$, where $h(t)$ is a compact operator from E into itself. The path $t \mapsto g(t)^{-1}$ is called a *parametrix* of γ. Assume now that $\gamma(0)$ and $\gamma(1)$ are isomorphisms. Given a parametrix $\beta\colon [0,1] \to \mathrm{Iso}(F,E)$ of γ, the number

$$\sigma(\gamma,[0,1]) = \deg_{LS}(\beta(0) \circ \gamma(0)) \deg_{LS}(\beta(1) \circ \gamma(1)),$$

which is either 1 or -1, does not depend on the parametrix β and, consequently, it can be actually associated to γ (as before $\deg_{LS}$ stands for the Leray-Schauder degree). The number $\sigma(\gamma,[0,1])$ is called *parity* of γ.

The parity of a path of Fredholm operators verifies some interesting properties, discussed in [6] and [7]. In the sequel we will need the following two:

Homotopy invariance. Given a homotopy $\Gamma\colon [0,1]\times[0,1] \to \Phi_0(E,F)$ such that $\Gamma(0,s)$ and $\Gamma(1,s)$ are isomorphisms for all $s \in [0,1]$, one has $\sigma(\Gamma(\cdot,0),[0,1]) = \sigma(\Gamma(\cdot,1),[0,1])$.

Multiplicativity under partition. Given a continuous path $\gamma\colon [0,1] \to \Phi_0(E,F)$ such that $\gamma(0), \gamma(1) \in \mathrm{Iso}(E,F)$, and given $t_0 \in [0,1]$ such that $\gamma(t_0) \in \mathrm{Iso}(E,F)$, one has

$$\sigma(\gamma,[0,1]) = \sigma(\gamma,[0,t_0])\,\sigma(\gamma,[t_0,1]).$$

By means of the parity, Fitzpatrick, Pejsachowicz and Rabier introduce the following notion of orientability for a continuous map $h\colon \Lambda \to \Phi_0(E,F)$ defined on a topological space Λ.

A point $\lambda \in \Lambda$ is said to be *regular* for h if $h(\lambda) \in \mathrm{Iso}(E,F)$. The set of regular points for h is denoted R_h. The map h is said to be *orientable* if there exists a map $\varepsilon\colon R_h \to \{-1,1\}$, called *orientation*, such that, given any continuous path $\gamma\colon [0,1] \to \Lambda$ with $\gamma(0), \gamma(1) \in R_h$, it follows

$$\sigma(h\circ\gamma,[0,1]) = \varepsilon(\gamma(0))\,\varepsilon(\gamma(1)).$$

If h has no regular points, then it is clearly orientable with the unique orientation given by $\varepsilon\colon \emptyset \to \{-1,1\}$. A subset $\mathcal{A}$ of $\Phi_0(E,F)$ is orientable if so is the inclusion $i\colon \mathcal{A} \hookrightarrow \Phi_0(E,F)$.

By Multiplicativity under partition of the parity, one can prove that h is orientable if and only if, given a continuous path $\gamma\colon [0,1] \to \Lambda$ such that $\gamma(0) = \gamma(1) \in R_h$, one has $\sigma(h\circ\gamma,[0,1]) = 1$.

Remark 5.3. If Λ is simply connected, then (because of the Homotopy invariance of the parity) any continuous map $h\colon \Lambda \to \Phi_0(E,F)$ is orientable.

The following proposition states that the orientation can be transported along a homotopy (provided it admits at least a regular point).

Proposition 5.4. *Let $H\colon \Lambda\times[0,1] \to \Phi_0(E,F)$ be a continuous homotopy and assume that, for some $t \in [0,1]$, $H_t = H(\cdot,t)\colon \Lambda \to \Phi_0(E,F)$ admits regular points and is orientable. Then H is orientable (in particular every H_s is orientable).*

The successive step in the construction of Fitzpatrick, Pejsachowicz and Rabier concerns a notion of orientability for nonlinear Fredholm maps of index zero between Banach spaces.

Definition 5.5. Given two Banach spaces E and F, and an open subset Ω of E, let $f\colon \Omega \to F$ be a Fredholm map of index zero. Then f is said to be *orientable* if so is the derivative $Df\colon \Omega \to \Phi_0(E,F)$. An *orientation* of f is an orientation of Df, and f is *oriented* when an orientation is chosen.

The final step in their construction is the extension of the notion of orientability and degree for Fredholm maps between Banach manifolds. To make this paper not too long we omit this construction, which can be found in [9]. For the same reason we limit our comparison to the case of maps between Banach spaces.

The following is the first step in the construction of the Fitzpatrick-Pejsachowicz-Rabier degree (FPR-degree for short). Consider an open subset U of E and a C^2

Fredholm map $f : U \to F$ which is proper on closed bounded subsets of U. Let f be oriented with orientation ε. Let Ω be an open bounded subset of U, with $\overline{\Omega} \subset U$. If $y \notin f(\partial\Omega)$ and it is a regular value for $f|_\Omega$, then the degree of f in Ω with respect to y is defined as

$$\deg_{FPR}(f, \Omega, y) = \sum_{x \in (f|_\Omega)^{-1}(y)} \varepsilon(x).$$

After this preliminary definition, the assumption that y is a regular value is removed by means of the infinite dimensional version of Sard's Theorem (see [9] for the complete construction and the properties of the FPR-degree).

Since terms such as "orientable map" and "orientation" are used in both our sense and the Fitzpatrick-Pejsachowicz-Rabier theory, to avoid confusion, from now on we will add the prefix FPR to any term whose meaning is in the sense of Fitzpatrick, Pejsachowicz and Rabier. Thus, for example, a map is FPR-orientable when it is orientable in their theory. Clearly, when no prefix is used, the meaning is according to us.

Our notion of orientation has close links with the FPR-theory. One of these concerns the parity, as the following proposition shows.

Proposition 5.6. *Let $\gamma\colon [0,1] \to \Phi_0(E,F)$ be continuous and such that $\gamma(0)$ and $\gamma(1)$ are isomorphisms. Given any orientation of γ, one has*

$$\sigma(\gamma, [0,1]) = \operatorname{sign}\gamma(0)\operatorname{sign}\gamma(1).$$

Consequently, when $\gamma(0) = \gamma(1)$, then $\sigma(\gamma, [0,1]) = 1$ if and only if the two liftings of γ are closed paths in $\widehat{\Phi}_0(E,F)$.

Proof. Let $\beta\colon [0,1] \to \mathrm{L}(E,F)$ be a continuous path of correctors of γ (whose existence is ensured by Lemma 34.4 in [13]). Without loss of generality assume that $\beta(0)$ is a positive corrector of $\gamma(0)$. From Lemma 3.1 it follows that $\beta(t)$ is a positive corrector of $\gamma(t)$ for all $t \in [0,1]$. Now, $\operatorname{sign}\gamma(0) = 1$ if and only if only if $\beta(0)$ is $\gamma(0)$-equivalent to the trivial operator 0 and (by the definition of the Leray-Schauder degree) if and only if $\deg_{LS}((\gamma(0) + \beta(0))^{-1}\gamma(0)) = 1$. In other words $\deg_{LS}((\gamma(0)+\beta(0))^{-1}\gamma(0)) = \operatorname{sign}\gamma(0)$. Analogously, $\deg_{LS}((\gamma(1)+\beta(1))^{-1}\gamma(1)) = \operatorname{sign}\gamma(1)$, and the statement follows easily. □

By the above proposition it follows that an orientable map $h\colon \Lambda \to \Phi_0(E,F)$ is also FPR-orientable. In fact, an orientation of h induces an FPR-orientation by the formula $\varepsilon(\lambda) = \operatorname{sign} h(\lambda)$, $\lambda \in R_h$ (here R_h may be empty). The converse is not true since, given for example a Banach space E with $\Phi_0(E)$ nonorientable, as a consequence of Proposition 3.8 the natural image $J_1(\Phi_0(E)) \subset \Phi_0(E \times \mathbb{R})$ is still nonorientable, but clearly FPR-orientable (as totally composed of singular operators). However, the following result holds.

Proposition 5.7. *Let Λ be a connected, locally path connected topological space and let $h\colon \Lambda \to \Phi_0(E,F)$ be a continuous map with $R_h \neq \emptyset$. Then h is orientable if and only if it is FPR-orientable.*

Proof. As pointed out above, if h has an orientation, one can define an FPR-orientation $\varepsilon\colon R_h \to \{-1,1\}$ of h by $\varepsilon(\lambda) = \operatorname{sign} h(\lambda)$. Proposition 5.6 ensures that ε is actually an FPR-orientation.

Assume h is not orientable. By Proposition 3.13 there is a closed path $\gamma_1 : [0,1] \to \Lambda$ such that the two liftings of $h \circ \gamma_1$ (which, we recall, are paths in $\widehat{\Phi}_0(E,F)$) are not closed. By assumption, there exists a point $\lambda_0 \in \Lambda$ such that $h(\lambda_0) \in \mathrm{Iso}(E,F)$. Since Λ is connected and locally path connected, it is also path connected. Therefore there exists a path $\gamma_0\colon [0,1] \to \Lambda$ joining λ_0 with $\gamma_1(0) = \gamma_1(1)$. Define the closed path $\gamma\colon [0,1] \to \Lambda$ by

$$\gamma(t) = \begin{cases} \gamma_0(3t) & \text{if } t \in [0,1/3] \\ \gamma_1(3t-1) & \text{if } t \in [1/3,2/3] \\ \gamma_0(3-3t) & \text{if } t \in [2/3,1]. \end{cases}$$

Since $(h \circ \gamma)(0) = (h \circ \gamma)(1) \in \mathrm{Iso}(E,F)$, it makes sense to consider the parity $\sigma(h \circ \gamma, [0,1])$. Let us show that this parity is -1, which implies that h is not FPR-orientable. By Proposition 5.6 it is enough to show that one of the two liftings of $h \circ \gamma$ is not closed (in this case also the other one is not closed). Let β_0^+ and β_0^- be the two liftings of $h \circ \gamma_0$, and consider the unique lifting β_1 of $h \circ \gamma_1$ such that $\beta_1(0) = \beta_0^+(1)$. Define $\beta\colon [0,1] \to \widehat{\Phi}_0(E,F)$ by

$$\beta(t) = \begin{cases} \beta_0^+(3t) & \text{if } t \in [0,1/3] \\ \beta_1(3t-1) & \text{if } t \in [1/3,2/3] \\ \beta_0^-(3-3t) & \text{if } t \in [2/3,1]. \end{cases}$$

The continuity of β is ensured by the fact that β_1 is not closed and $\beta_0^+(1) \neq \beta_0^-(1)$. Thus β is a lifting of γ. Since, by construction, β is not closed, Proposition 5.6 implies $\sigma(h \circ \gamma, [0,1]) = -1$, and this shows that h is not FPR-orientable. □

Simply connected subsets of $\Phi_0(E,F)$ are FPR-orientable. However, the following example shows that without the assumption of local path connectedness, a simply connected subset of $\Phi_0(E,F)$ may be nonorientable.

Example 5.8. Let E be a Banach space with $\mathrm{GL}(E)$ connected and consider a closed path $\gamma\colon [0,1] \to \Phi_0(E)$ with nonorientable image (whose existence is ensured by Theorem 3.15). Assume also that γ is simple (so that its image is homeomorphic to S^1). The two arcs $C_0 = \gamma([0,1/2])$ and $C_1 = \gamma([1/2,1])$ are orientable, being connected and locally path connected. Notice that these arcs have exactly two common points: $\gamma(0)$ and $\gamma(1/2)$. Now, let us fix an orientation ω of $\gamma(0)$ and denote by α_0 and α_1 the orientations induced by ω on C_0 and C_1 respectively. Observe that, $\gamma([0,1])$ being not orientable, the two orientations $\alpha_0(1/2)$ and $\alpha_1(1/2)$ of $\gamma(1/2)$ must be opposite one to the other. Let U be an open connected orientable neighborhood of $\gamma(1/2)$ in $\Phi_0(E)$ with the property that $C_0' = C_0 \cup U$ and $C_1' = C_1 \cup U$ turn out to be orientable. Let $\delta > 0$ be such that $\gamma(t)$ belongs to U for all $t \in [1/2-\delta, 1/2+\delta]$, and define the function $g\colon [0,1] \to \Phi_0(E)$ by

$$g(t) = \begin{cases} \gamma(t) & \text{if } t \in [0,1/2-\delta] \cup [1/2+\delta,1] \\ (1+\varepsilon \sin \frac{\delta\pi}{t-1/2})\gamma(t) & \text{if } t \in [1/2-\delta,1/2) \cup (1/2,1/2+\delta] \\ \gamma(1/2) & \text{if } t = 1/2, \end{cases}$$

where ε is such that $g(t) \in U$ for all $t \in [1/2-\delta, 1/2+\delta]$. Notice that g is discontinuous at $t = 1/2$ and its image, $g([0,1])$, is simply connected, hence FPR-orientable. Let us show that $g([0,1])$ is not orientable. Observe that $g([0,1/2])$ and $g([1/2,1])$ are contained in C_0 and C_1 respectively. Thus, they are orientable and, being connected, they can be oriented with orientations α_0' and α_1' induced by the orientation ω of $\gamma(0)$. Since U is orientable and connected, it admits exactly two orientations. Thus, $\alpha_0'(1/2) = \alpha_0(1/2)$ and $\alpha_1'(1/2) = \alpha_1(1/2)$, and this implies that the two orientations $\alpha_0'(1/2)$ and $\alpha_1'(1/2)$ of $\gamma(1/2)$ are opposite one to the other. Consequently $g([0,1])$ is not orientable.

As we have seen above, any orientable map $h\colon \Lambda \to \Phi_0(E,F)$ is FPR-orientable as well. This fact suggests that the notion of orientability defined by Fitzpatrick, Pejsachowicz and Rabier is more general than our one. However, our concept is simpler and has a sort of stability property which is not valid for the FPR-orientability. First of all observe that the Fitzpatrick-Pejsachowicz-Rabier notion of orientability (as well as the Elworthy-Tromba theory) is based on the Leray-Schauder degree, while our concept is strictly related to the purely algebraic notion of orientation of a Fredholm linear operator of index zero. Moreover, our simple definition, in some sense, contains the concept of FPR-orientation (which, consequently, could be freed from the Leray-Schauder dependence). In fact, as previously pointed out, given $h\colon \Lambda \to \Phi_0(E,F)$, the assignment of an orientation $\alpha(\lambda)$ of $h(\lambda)$ for each $\lambda \in \Lambda$ induces a sign function on the regular subset R_h of Λ. Consequently, when the assignment is continuous (namely, when it is actually an orientation of h), because of Proposition 5.6, this sign function is actually an FPR-orientation.

By slightly modifying the proof of Proposition 5.7 one can easily show that when Λ is locally path connected there is a one-to-one correspondence between the orientations and the FPR-orientations of h, provided that any connected component of Λ contains a regular point. Of course if Λ is connected and totally composed of singular points, this correspondence breaks down, since h admits exactly one FPR-orientation, but either two orientations (if orientable) or no one (if nonorientable). This shows that our concept of orientation goes insight the structure of the singular maps, and this is conceived in such a way to make the orientability a stable property under small perturbations, as shown in Theorem 3.14. Of course we could change our definition (gaining in generality and loosing in properties) by saying that any continuous mapping $h\colon \Lambda \to \Phi_0(E,F)$ is orientable except those which do not admit a lifting $\widehat{h}$ and have at least a regular point (but we think this is not convenient).

We emphasize that the FPR-orientability does not verify a general property of continuous transport along a homotopy (as Theorem 3.14 for the orientability). To see this, consider a Banach space E with $\mathrm{GL}(E)$ connected and let $\gamma\colon [0,1] \to \Phi_0(E)$ be a closed path with $\gamma(0) \in \mathrm{GL}(E)$ and parity $\sigma(\gamma,[0,1]) = -1$. Thus γ, if regarded as a map defined on the circle S^1, is not FPR-orientable (and neither orientable because of Propositions 3.13 and 5.6). Given the product space $E \times \mathbb{R}$, consider the homotopy $\Gamma\colon S^1 \times [0,1] \to \Phi_0(E \times \mathbb{R})$ with block decomposition

$$\Gamma(\lambda, s) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & s \end{pmatrix},$$

and observe that the partial map Γ_0 is FPR-orientable (since any $\Gamma(\lambda,0)$ is a singular operator) while Γ_1 is not FPR-orientable (since, as one can check, $\sigma(\Gamma_1,[0,1]) = -1$). This is, of course, in accord with our theory, since all the maps Γ_s, $s \in [0,1]$, are nonorientable. Notice also that the FPR-nonorientable maps Γ_s, $0 < s \leq 1$, are arbitrarily close to Γ_0, and this is a sort of instability in the FPR-theory of orientation.

So far we have compared the relationship between the notions of orientation according to us and to Fitzpatrick, Pejsachowicz and Rabier in the case of maps from a topological space Λ into $\Phi_0(E,F)$. This automatically gives a comparison for Fredholm maps of index zero acting between open sets of Banach spaces. Indeed, in both the two notions, a map $f\colon \Omega \to F$ is orientable if so is the Fréchet derivative $Df\colon \Omega \to \Phi_0(E,F)$. Here, as usual in this paper, E and F are real Banach spaces, Ω is an open subset of E and f is Fredholm of index zero. Recalling that Ω (as an open subset of a Banach space) is locally path connected, the comparison in this case can be easily carried out. For a complete analysis for maps acting between real Banach manifolds we should have reported here the FPR-notion of orientation in such a case. However, taking into account that Banach manifolds are locally path connected, the interested reader can check that in this context the situation is similar to the special local case of maps between open sets of Banach spaces. Namely, for a map $f\colon M \to N$ between real Banach manifolds which is Fredholm of index zero, there is a one-to-one correspondence between the orientations and the FPR-orientations, provided that any connected component of M contains a regular point. If M is connected and f has no regular points, then f is FPR-orientable with only one FPR-orientation, no matter whether or not it is orientable.

The simplest example of a map $f\colon M \to N$ which is FPR-orientable but not orientable is given by taking M finite dimensional and nonorientable, N with the same dimension as M, and f constant. Clearly, in this case, small perturbations of f may produce maps with regular points which are FPR-nonorientable. An example in the infinite dimensional context of an FPR-orientable Fredholm map f that is not orientable can be given by repeating the construction in Example 4.2 starting from a nonorientable curve $\gamma\colon S^1 \to \Phi_0(E)$ with image in the set of singular operators (this is possible by Proposition 3.8).

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