A simple notion of orientability for Fredholm maps of index zero between Banach manifolds and degree theory

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Abstract

Dans cette note on donne une définition du degré topologique pour une classe d'applications (nommées orientables) de Fredholm d'indice zéro entre des variétés de Banach réelles.

On introduit d'abord une notion algébrique d'orientation pour tout opérateur linéaire de Fredholm d'indice zéro entre deux espaces vectoriels réels. Cette notion, qui est définie sans aucune structure topologique, permet de donner une définition d'orientabilité pour des applications (non-linéaires) de Fredholm d'indice zéro définies entre des variétés de Banach.

Le degré que nous présentons vérifie les plus importantes propriétés normalement connues dans la théorie classique, et, en particulier, il est invariant par rapport à des homotopies de classe C^1.

1 Introduction and preliminaries

In this paper we define an integer valued topological degree for a class of Fredholm maps of index zero between real Banach manifolds (i.e. C^1 maps between manifolds such that at any point of the domain the derivative is Fredholm of index zero). The material of the article is a revised and improved version of part of the PhD dissertation of one of the authors (see [3]).

Our intention is to extend and simplify the Elworthy-Tromba approach to degree theory avoiding the concept of Fredholm structure and any related notion of orientation on the source and target manifolds (see [6] and [7]). To this aim we introduce a simple concept of orientation for Fredholm maps of index zero between Banach manifolds. This notion does not coincide with the one introduced by Fitzpatrick, Pejsachowicz and Rabier (see [8] and references therein), it is stable (in the sense that any map "sufficiently close" to an orientable map is orientable), and not based on the Leray-Schauder degree theory (as in [8]). Moreover, in the finite dimensional case, it turns out to be equivalent to the concept of orientation for maps between not necessarily orientable manifolds introduced, with completely different methods, by Dold in [5]. In particular, when $f : M \to N$ is a map acting between finite dimensional orientable manifolds of the same dimension, an orientation of f (in our sense) can be regarded as a pair of orientations of M and N, up to an inversion of both of them (thus, if M is connected and $N = M$, the map f has a canonical orientation).

Our notion is merely based on an elementary, purely algebraic, concept that we introduce here: the orientation of an algebraic Fredholm linear operator of index zero $L : E \to F$ acting between real vector spaces (no additional structure is needed). It turns out that any such an operator has exactly two orientations and, when the vector spaces E and F are actually Banach and the operator L is bounded, any of the two possible orientations of L induces, by a sort of continuity, an orientation on any operator L' sufficiently close to L (in the operator norm). Thus, roughly speaking, an oriented map from an open subset Ω of E into F is a nonlinear Fredholm map of index zero $f : \Omega \to F$ together with a function α which assigns, in a continuous way, an orientation $\alpha(x)$ of the Fréchet derivative $Df(x)$ of f at any

$x \in \Omega$. This notion of oriented map between (real) Banach spaces is easily extended to the context of (real) Banach manifolds.

We will show that any isomorphism L between Banach spaces (or, more generally, vector spaces) admits a natural orientation $\nu(L)$, and this will be crucial in our definition of degree.

As for the case of Brouwer degree, the first step is to introduce our notion of degree in a special case. Namely, given a proper, oriented map $f : M \to N$ between Banach manifolds, and a regular value $y \in N$, the degree of f at y is the integer

$$\deg(f, M, y) := \sum_{x \in f^{-1}(y)} \operatorname{sgn} Df(x),$$

where $\operatorname{sgn} Df(x)$ is 1 if the orientation $\alpha(x)$ of the derivative $Df(x) : T_xM \to T_{f(x)}N$ coincides with the natural orientation $\nu(Df(x))$, and -1 otherwise. In this way, when M and N are finite dimensional oriented manifolds of the same dimension, $f : M \to N$ is a C^1 proper map, and $y \in N$ is a regular value, we get exactly the Brouwer degree $\deg_B(f, M, y)$ of f at y.

The second step is a kind of finite dimensional reduction result (as in the Leray-Schauder degree theory). Roughly speaking we prove that given a proper, oriented map $f : M \to N$ between Banach manifolds, and a finite dimensional oriented submanifold N_1 of N which is transverse to f, then:
a) the manifold $M_1 = f^{-1}(N_1)$, which is of the same dimension as N_1, inherits an orientation from the orientations of f and N_1;
b) any regular value $y \in N_1$ for the restriction of f to the pair of manifolds M_1 (as domain) and N_1 (as codomain) is a regular value for $f : M \to N$ and the Brouwer degree $\deg_B(f, M_1, y)$ coincides with the degree defined above.

Finally, as a consequence of the previous step, we prove that if $f : M \to N$ is a proper, oriented map and $y \in N$ is any value, then there exists a neighborhood V of y such that for any pair of regular values for f, y_1, $y_2 \in V$, one has

$$\deg(f, M, y_1) = \deg(f, M, y_2)\,.$$

It is clear now how we define the degree for any proper, oriented map $f : M \to N$, at any $y \in N$: let R denote the (open dense) subset of N of the regular values of f, then

$$\deg(f, M, y) := \lim_{z \to y,\, z \in R} \deg(f, M, z)\,.$$

Actually our degree will be extended to any triple (f, M, y), where $f : M \to N$ is an oriented map and $y \in N$ is such that $f^{-1}(y)$ is compact. This extension is possible since, we recall, Fredholm maps between Banach manifolds are locally proper.

We conclude our work by proving some fundamental properties of our degree, such as Normalization, Additivity and Homotopy Invariance for C^1 Fredholm homotopies of index 1 (and not merely C^2 as in [6], [7] and [8]).

Regarding a comparison between the content of this paper and the Elworthy-Tromba theory, we observe that a bounded linear automorphism of an infinite dimensional real Banach space which is not a compact perturbation of the identity I (such as $2I$) is not admissible for their degree, unless one considers two different Fredholm structures (and orientations) on the same space (one as domain and the other as codomain). This peculiarity is still present in the extension of the Elworthy-Tromba degree to the case of Fredholm maps with non-negative index due to Borisovich, Zvyagin and Sapronov [4]. In fact, in their extension, the notion of orientation remains unchanged.

However, Fitzpatrick, Pejsachowicz and Rabier in [8], by means of the Leray-Schauder theory, define a concept of orientability for any Fredholm map of index zero between real Banach manifolds which avoids the additional (and often unnatural) notion of Fredholm structure on the manifolds (in this way, any such a map is either orientable or not orientable). Using this concept, they successfully define an integer valued degree for the class of C^2 oriented maps which satisfies fundamental properties such as Normalization, Additivity and Homotopy Invariance, and coincides with our degree when both are defined. This is not surprising since, as proved in [2], there is only one integer valued degree in Banach spaces satisfying these three properties. We believe the same is true in Banach manifolds if we add two more properties: Reduction and Topological Invariance (see section 3).

We observe however that the concept of orientability in [8] does not coincide with our notion. For example a constant map from a non-orientable finite dimensional real manifold into a manifold of the same dimension is orientable according to [8] and not orientable with our definition. With this example one could show that their notion of orientability is unstable (i.e. small perturbations of orientable maps may happen to be non-orientable).

We close this section with some notation.

Given a Banach manifold M and a point $x \in M$, T_xM will denote the tangent space of M at x.

Given a map f between two Banach manifolds M and N, and a point $x \in M$, $Df(x) : T_xM \to T_{f(x)}N$ denotes (when defined) the Fréchet derivative of f at x.

If M_1, M_2 and N are Banach manifolds, $f : M_1 \times M_2 \to N$ is a map, and $(x, y) \in M_1 \times M_2$, $D_1f(x, y) : T_xM_1 \to T_{f(x,y)}N$ stands (when defined) for the partial derivative of f at (x, y) with respect to the first variable; that is, the Fréchet derivative at x of the partial map $f(\cdot, y)$. An analogous notation is given for the derivative in the second variable.

The use of the symbol "$\circ$" to denote the composition of maps is reserved only to the cases when its omission could cause some confusion. In many cases we prefer the multiplicative notation.

2 Orientable maps

In this section we introduce a completely algebraic notion of orientation for Fredholm linear operators of index zero between real vector spaces. We will show that, in the context of Banach spaces, an oriented bounded operator induces, by "continuity", an orientation to any sufficiently close operator. This "continuous transport of orientation" allows us to define a concept of oriented C^1 Fredholm map of index zero between open subsets of Banach spaces (or, more generally, between Banach manifolds).

Let E and F be two real vector spaces. We recall that a linear operator $L : E \to F$ is said to be (*algebraic*) *Fredholm* if both $\mathrm{Ker}L$ and $\mathrm{coKer}L$ have finite dimension. The *index* of a Fredholm operator L is the integer

$$\mathrm{ind}L = \dim \mathrm{Ker}L - \dim \mathrm{coKer}L.$$

Given a Fredholm operator of index zero, $L : E \to F$, we say that a linear operator $A : E \to F$ is a *corrector* of L provided that its range is finite dimensional and $L + A$ is an isomorphism. The set of correctors of L will be denoted by $\mathcal{C}(L)$.

It is easy to see that $\mathcal{C}(L)$ is nonempty. Indeed, if L is an isomorphism the trivial operator is a corrector of L; otherwise, take as a corrector any linear operator $A : E \to F$ which is injective on $\mathrm{Ker}L$ and such that $\mathrm{Ker}A \oplus \mathrm{Ker}L = E$ and $\mathrm{Range}A \oplus \mathrm{Range}L = F$.

We introduce the following equivalence relation in $\mathcal{C}(L)$. Given A, $B \in \mathcal{C}(L)$, consider the composition $(L + B)^{-1}(L + A)$, which is clearly an automorphism of E. It is not difficult to check that the operator

$$K = I - (L + B)^{-1}(L + A),$$

where I is the identity of E, has finite dimensional range. Let E_0 be any finite dimensional subspace of E containing $\mathrm{Range}K$. Since $I - K$ is an isomorphism and $(I - K)(E_0) \subset E_0$, the restriction $(I - K)|_{E_0} : E_0 \to E_0$ is an isomorphism as well. We say that A is *equivalent* to B (or, more precisely, A is $L-$*equivalent* to B), written $A \sim B$, if the determinant of $(I - K)|_{E_0}$, which is well defined, is positive (we use the convention that this determinant is 1 when E_0 is trivial, which may happen only if $A = B$).

To see that this definition is well posed we will show that, given another finite dimensional space E_1 containing $\mathrm{Range}K$, one has

$$\det(I-K)|_{E_1} = \det(I-K)|_{E_0}. \tag{2.1}$$

Since the intersection of E_0 and E_1 is again a finite dimensional space containing Range$K$, it is sufficient to prove that (2.1) holds in the case when E_1 contains E_0. Assume therefore this condition is verified and let E_0' be a complement of E_0 in E_1. That is, let

$$E_1 = E_0' \oplus E_0.$$

With this decomposition $(I-K)|_{E_1}$ is represented by the matrix

$$\begin{pmatrix} I_{E_0'} & 0 \\ -K_{21} & (I-K)|_{E_0} \end{pmatrix}, \tag{2.2}$$

where $I_{E_0'}$ is the identity of E_0' and K_{21} is the projection onto E_0 of the restriction of K to E_0'. The equality (2.1) follows immediately from (2.2).

We claim that "$\sim$" is actually an equivalence relation (with just two equivalence classes). Reflexivity and symmetry are easy to verify. To check transitivity, consider $A, B, C \in \mathcal{C}(L)$ such that $A \sim B$, $B \sim C$ and define the following operators:

$$\begin{aligned} K_{A,B} = &\quad I-(L+B)^{-1}(L+A), \\ K_{B,C} = &\quad I-(L+C)^{-1}(L+B), \\ K_{A,C} = &\quad I-(L+C)^{-1}(L+A). \end{aligned}$$

Given a finite dimensional subspace E_0 of E containing the images of $K_{A,B}$, $K_{B,C}$ and $K_{A,C}$, one has

$$\begin{aligned} &\det((L+C)^{-1}(L+A))|_{E_0} = \\ &\det((L+C)^{-1}(L+B)(L+B)^{-1}(L+A))|_{E_0} = \\ &\det((L+C)^{-1}(L+B))|_{E_0} \cdot \det((L+B)^{-1}(L+A))|_{E_0}, \end{aligned}$$

and, consequently, $A \sim C$.

The set $\mathcal{C}(L)$ is easily seen to be composed of two equivalence classes. We can therefore introduce the following definition of orientation for an algebraic Fredholm operator of index zero.

Definition 2.1 An *orientation* of an algebraic Fredholm operator of index zero $L : E \to F$ is one of the two equivalence classes of $\mathcal{C}(L)$. The operator L is said to be *oriented* if an orientation is actually chosen.

We point out that in the particular case when $L : E \to F$ is a bounded Fredholm operator of index zero between real Banach spaces, a partition in two equivalence classes of the set of compact correctors of L was introduced for the first time (as far as we know) by Pejsachowicz and Vignoli in [12] (see also [11] and reference therein for further applications of this idea). Namely, if A and B are compact (linear) correctors of L, the map $(L+B)^{-1}(L+A)$ is of the form $I-K$, with K a compact operator. Thus the Leray-Schauder degree of $I-K$ is well defined (since $(I-K)^{-1}(0)$ is compact) and equals either 1 or -1 (by a well known result of Leray-Schauder). Now, the operator A is said to be in the same class of B if the degree of $I-K$ is 1 and in a different class otherwise. Clearly, as a consequence of the definition of Leray-Schauder degree, this equivalence relation coincides with our notion in the case when one considers only bounded correctors with finite dimensional image. Apart for the sake of simplicity, in introducing our concept of orientation, the reason why we do not use the equivalence relation in [12] is due to the fact that we want to base our degree just upon the Brouwer theory.

A prelude to the idea of partitioning the set of correctors of an algebraic Fredholm operator of index zero $L : E \to F$ can be found in the pioneering paper of Mawhin [10]. Here is a brief description of this idea. Fix a projector $P : E \to E$ onto Ker$L$ and a subspace F_1 of F such that $F_1 \oplus \text{Range}L = F$. To any isomorphism $J : \text{Ker}L \to F_1$ one can associate the corrector JP of L (this of course does not exhaust

$\mathcal{C}(L)$). Two such correctors, J_1P and J_2P, are equivalent if $\det(J_2^{-1}J_1) > 0$. One can check that, except in the case when L is an isomorphism (which is crucial to us), this equivalence relation produces two equivalence classes, each of them contained in one class of $\mathcal{C}(L)$ (and not both in the same one).

According to Definition 2.1, an oriented operator L is a pair (L, ω), where $L : E \to F$ is a Fredholm operator of index zero and ω is one of the two equivalence classes of $\mathcal{C}(L)$. However, to simplify the notation, we shall not use different symbols to distinguish between oriented and nonoriented operators (unless it is necessary).

Given an oriented operator $L : E \to F$, the elements of its orientation will be called the *positive correctors* of L and denoted by $\mathcal{C}_+(L)$. The complement of $\mathcal{C}_+(L)$ in $\mathcal{C}(L)$ are the negative correctors of L (denoted by $\mathcal{C}_-(L)$).

The composition L_2L_1 of two oriented operators can be naturally oriented by taking as a positive corrector the operator $L_2A_1 + A_2A_1 + A_2L_1$, where A_1 and A_2 are positive correctors of L_1 and L_2, respectively. From now on, unless otherwise stated, the composition of two (or more) oriented operators will be considered as an *oriented composition*.

We point out that any isomorphism L admits a special orientation, namely the equivalence class containing the trivial operator 0. We shall refer to this equivalence class as the *natural orientation* $\nu(L)$ of L. However, if an isomorphism L happens to be already oriented, it is convenient to define its sign as follows: $\mathrm{sgn}L = 1$ if the trivial operator 0 is a positive corrector of L (i.e. if the orientation of L coincides with $\nu(L)$), and $\mathrm{sgn}L = -1$ otherwise.

It is interesting to observe, and not difficult to show, that, in the case when E and F are finite dimensional (necessarily of the same dimension), an orientation of a linear operator $L : E \to F$ determines uniquely an orientation of the product space $E \times F$ (and vice versa). In fact, if L is an oriented operator from E into F, to determine an orientation of $E \times F$ take any of the two orientations of E, take any positive corrector A of L, and consider the orientation of F induced by the chosen one in E through the isomorphism $L + A$ (that is, in such a way that $L + A$ becomes orientation preserving). Thus $E \times F$ turns out to be oriented by considering the product of the two orientations of E and F. It is simple to check that this orientation of the product space does not depend on the chosen one of E and the positive corrector of L. Conversely, given an orientation of $E \times F$, every linear operator $L : E \to F$ can be oriented by choosing as positive correctors of L those linear operators $A : E \to F$ which make $L + A$ an orientation preserving isomorphism (this makes sense since an orientation of $E \times F$ can be regarded as a pair of orientations of E and F, up to an inversion of both of them).

An important particular case is when L is an endomorphism of a finite dimensional space E. In this situation the product $E \times E$ turns out to be *canonically oriented* (as a square of any orientation of E). Thus every $L : E \to E$ turns out to be *canonically oriented* as well (with orientation denoted by $\chi(L)$). It is easy to check that, with this canonical orientation, if $L : E \to E$ is an isomorphism, the sign of L (defined above) is just the sign of its determinant. In other words, if L is an automorphism of a finite dimensional space, $\det(L)$ is positive if and only if the canonical and the natural orientations coincide.

We conclude this algebraic preliminaries by pointing out a property which may be regarded as a sort of reduction of the orientation of an operator to the orientation of its restriction to a convenient pair of subspaces (of the domain and codomain, respectively). This will be useful in the next section where we will show that our degree is just the Brouwer degree of a suitable restriction to finite dimensional manifolds.

Let E and F be two real vector spaces and let $L : E \to F$ be an algebraic Fredholm operator of index zero. Let F_1 be a subspace of F which is transverse to L (that is, $F_1 + \mathrm{Range}L = F$). Observe that in this case the restriction L_1 of L to the pair of spaces $E_1 = L^{-1}(F_1)$ (as domain) and F_1 (as codomain) is again a Fredholm operator of index zero. We claim that an orientation of L gives an orientation of L_1, and vice versa. To see this, let E_0 be a complement of E_1 in E and split E and F as follows: $E = E_0 \oplus E_1$, $F = L(E_0) \oplus F_1$. Thus L can be represented by a matrix

$$\begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix},$$

where L_0 – the restriction of L to the spaces E_0 and $L(E_0)$ – is an isomorphism. Now our claim is a straightforward consequence of the fact that any linear operator $A : E \to F$, represented by

$$\begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix},$$

is a corrector of L if and only if A_1 is a corrector of L_1.

According to the above argument, it is convenient to introduce the following definition.

Definition 2.2 Let $L : E \to F$ be a Fredholm operator of index zero between real vector spaces, let F_1 be a subspace of F which is transverse to L, and denote by L_1 the restriction of L to the pair of spaces $L^{-1}(F_1)$ and F_1. Two orientations, one of L and one of L_1, are said to be *correlated* (or *one induced by the other*) if there exist a projector $P : E \to E$ onto E_1 and a positive corrector A_1 of L_1 such that the operator $A = JA_1P$ is a positive corrector of L, where $J : F_1 \hookrightarrow F$ is the inclusion.

The concept of orientation of an algebraic Fredholm operator of index zero $L : E \to F$ does not require any topological structure on E and F, which are supposed to be just real vector spaces. However, in the context of Banach spaces, this completely algebraic attribute of L has a sort of "local influence" to its "neighbors". More precisely, assume that $L : E \to F$ is a Fredholm operator of index zero between real Banach spaces; that is, L is algebraic Fredholm of index zero and, in addition, bounded. Given an orientation of L, choose a positive bounded corrector A of L (whose existence is ensured by the Hahn-Banach theorem) and observe that A is still a corrector of any bounded operator L' in a convenient neighborhood of L in the Banach space of bounded linear operators from E into F. Therefore, any L' with the property that $L' + A$ is an isomorphism can be oriented by choosing A as a positive corrector. Roughly speaking, this means that L induces, by a sort of stability, an orientation to any sufficiently close bounded operator.

From now on, unless otherwise specified, E and F will denote real Banach spaces, L a Fredholm operator from E into F and $L(E, F)$ the Banach space of bounded linear operators from E into F. If $E = F$ we will write $L(E)$ instead of $L(E, E)$.

For the sake of simplicity, in the context of Banach spaces, the set of continuous correctors of L will be still denoted by $\mathcal{C}(L)$, as in the algebraic case, instead of $\mathcal{C}(L) \cap L(E, F)$. Therefore, from now on, by a corrector of L we shall actually mean a continuous corrector. It is clear that an orientation of L in the previous sense can be regarded as an equivalence class of continuous correctors of L.

We recall that the set $\Phi(E, F)$ consisting of the Fredholm operators from E into F is open in $L(E, F)$, and the integer valued map $\mathrm{ind} : \Phi(E, F) \to \mathbf{Z}$ is continuous. Consequently, given $n \in \mathbf{Z}$, the set $\Phi_n(E, F)$ of Fredholm operators of index n is an open subset of $L(E, F)$.

We introduce now a concept of orientation for a continuous map into $\Phi_0(E, F)$.

Definition 2.3 Let Λ be a topological space and $h : \Lambda \to \Phi_0(E, F)$ a continuous map. An *orientation* of h is a continuous choice of an orientation $\alpha(\lambda)$ of $h(\lambda)$ for each $\lambda \in \Lambda$; where "continuous" means that for any $\lambda \in \Lambda$ there exists $A_\lambda \in \alpha(\lambda)$ which is a positive corrector of $h(\lambda')$ for any λ' in a neighborhood of λ. A map is *orientable* if it admits an orientation and *oriented* when an orientation has been chosen. In particular, a subset $\mathcal{A}$ of $\Phi_0(E, F)$ is said to be *orientable* (or *oriented*) if so is the inclusion $i : \mathcal{A} \hookrightarrow \Phi_0(E, F)$.

Given $L \in \Phi_0(E, F)$, one can prove that if $A \in \mathcal{C}(L)$ is $L-$equivalent to B, then A is $L'-$equivalent to B for any L' sufficiently close to L. This implies that the above notion of "continuous choice of an orientation" is equivalent to the following: "for every $\lambda \in \Lambda$ and every positive corrector A of $h(\lambda)$ there exists a neighborhood U of λ such that $A \in \alpha(\lambda'), \forall \lambda' \in U$".

Clearly any orientable map $h : \Lambda \to \Phi_0(E, F)$ admits at least two orientations. In fact, if h is oriented by α, reverting this orientation at any $\lambda \in \Lambda$, one gets what we call the *opposite orientation* α_- of h.

Observe also that two orientations of h coincide in an open subset of Λ, and for the same reason the set in which two orientations of h are opposite one to the other is open. Therefore, if Λ is connected, two orientations of h are either equal or one is opposite to the other. Thus, in this case, h, if orientable, admits exactly two orientations.

The notion of continuity in the above definition could become the usual one by introducing the following topological space (actually, a real Banach manifold). Let $\widehat{\Phi}_0(E,F)$ denote the set of pairs (L,ω) with $L \in \Phi_0(E,F)$ and ω one of the two orientations of L. Given an open subset U of $\Phi_0(E,F)$ and a bounded linear operator with finite dimensional range $A : E \to F$, consider the set

$$W_{(U,A)} = \{(L,\omega) \in \widehat{\Phi}_0(E,F) : L \in U,\ A \in \omega\}.$$

It is easy to check that the family of sets obtained in this way constitute a base for a topology on $\widehat{\Phi}_0(E,F)$, and the natural projection $p : (L,\omega) \mapsto L$ is a double covering of $\Phi_0(E,F)$. In this way an orientation of a map $h : \Lambda \to \Phi_0(E,F)$ could be regarded as a lifting $\widehat{h} = (h,\alpha) : \Lambda \to \widehat{\Phi}_0(E,F)$ of h, and known results from covering space theory could be used for deducing properties of orientable maps (not quite needed in this article). However, since the situation when E and F are replaced by real Banach manifolds is a little bit more complicated and requires the concept of space of jets (or substitute notions), which would bring us too far away from the purpose of this paper, this argument will be treated in a forthcoming work regarding the orientation, in which we will show, among other results, that if E is a separable infinite dimensional Hilbert space, then $\Phi_0(E)$ is not orientable.

It is clear that the above definition can be used to give a notion of orientation for nonlinear Fredholm maps of index zero between open subsets of Banach spaces. In fact, let $f : \Omega \to F$ be a map from an open subset Ω of a Banach space E into a Banach space F. Assume that f is Fredholm of index zero; that is f is C^1 and the Fréchet derivative $Df(x)$ of f at x belongs to $\Phi_0(E,F)$ for any $x \in \Omega$. An *orientation* of f is just an orientation of the continuous map $Df : x \mapsto Df(x) \in \Phi_0(E,F)$, and f is *orientable* (resp. *oriented*) if so is Df according to Definition 2.3.

We extend now this notion to the context of Banach manifolds. Recall first that a map $f : M \to N$ between Banach manifolds is Fredholm of index n if it is C^1 and its derivative, $Df(x) : T_xM \to T_{f(x)}N$, is Fredholm of index n for any $x \in M$.

Definition 2.4 Consider two Banach manifolds M and N, and let $f : M \to N$ be Fredholm of index zero. An *orientation* α of f is a continuous choice of an orientation $\alpha(x)$ of $Df(x)$ for any $x \in M$; where continuous means that, given a selection of positive correctors $\{A_x \in \alpha(x)\}_{x\in M}$, and two local charts $\varphi : U \to E$ and $\psi : V \to F$ of M and N respectively, with $f(U) \subset V$, the family of linear operators

$$\left\{D\psi(f(\varphi^{-1}(z))) \circ A_{\varphi^{-1}(z)} \circ D\varphi^{-1}(z)\right\}_{z\in\varphi(U)}$$

defines an orientation of the composite map $\psi f \varphi^{-1} : \varphi(U) \to F$.

We observe that when M and N are finite dimensional orientable connected manifolds (of the same dimension), an orientation of $f : M \to N$ can be regarded as a pair of orientations, one of M and one of N, up to an inversion of both of them. The orientation of the composition gf of two oriented maps f and g can be defined as in the case of oriented linear operators. With this induced orientation gf will be called the *oriented composition* of f and g. From now on, unless otherwise stated, the composition of two (or more) oriented maps will be regarded as an oriented composition.

Remark 2.5 A local diffeomorphism $f : M \to N$ between two Banach manifolds can be oriented with orientation ν defined by $0 \in \nu(Df(x)), \forall x \in M$. We will refer to this orientation as the *natural orientation* of the local diffeomorphism f.

Let $f : M \to N$ be an oriented map between Banach manifolds and let N_1 be a submanifold of N which is transverse to f; that is $T_{f(x)}N_1 + \mathrm{Range}Df(x) = T_{f(x)}N$, $\forall x \in f^{-1}(N_1)$. It is known that in this case

$M_1 = f^{-1}(N_1)$ is a submanifold of M and the restriction f_1 of f to M_1 (as domain) and N_1 (as codomain) is again a Fredholm map of index zero. Moreover, for any $x \in M_1$, one has $T_x M_1 = Df(x)^{-1}(T_{f(x)}N_1)$ (see for example [1] and [9] for general results about transversality). Therefore, according to Definition 2.2, given any $x \in M_1$, the orientation of $Df(x) : T_x M \to T_{f(x)}N$ induces an orientation on its restriction $Df_1(x) : T_x M_1 \to T_{f(x)}N_1$, which is just the derivative of the restriction $f_1 : M_1 \to N_1$ of f. As a consequence of the "continuity" assumption in the definition of orientation of a Fredholm map of index zero, such a collection of orientations of $Df_1(x)$, $x \in M_1$, is actually an orientation of $f_1 : M_1 \to N_1$ that, from now on, we shall call the *orientation on f_1 induced by f*. Observe also that, given $x \in M_1$, $Df(x) : T_x M \to T_{f(x)}N$ is an isomorphism if and only if so is $Df_1(x) : T_x M_1 \to T_{f(x)}N_1$ and, with the induced orientation, one has $\mathrm{sgn}Df(x) = \mathrm{sgn}Df_1(x)$; and this will imply one of the fundamental properties of the degree (Reduction property).

The above Definition 2.4 can be slightly modified in order to obtain a notion of orientation for continuous homotopies of Fredholm maps of index zero between Banach manifolds M and N. For simplicity, consider first the case when M is an open subset Ω of a Banach space E, and N is a Banach space F. We say that a continuous map $H : \Omega \times [0,1] \to F$ is an *oriented homotopy* if it is continuously differentiable with respect to the first variable and, for any $(x,t) \in \Omega \times [0,1]$, the partial derivative $D_1 H(x,t) : E \to F$ is an oriented (Fredholm) operator (of index zero) with orientation $\alpha(x,t)$ which depends continuously on (x,t); in the sense that α is an orientation of the map $D_1 H : \Omega \times [0,1] \to \Phi_0(E,F)$ according to Definition 2.3.

Assume now that M and N are two Banach manifolds and $H : M \times [0,1] \to N$ is a continuous map. As above, H is an *oriented homotopy* if it is continuously differentiable with respect to the first variable and, for any $(x,t) \in M \times [0,1]$, the partial derivative $D_1 H(x,t) : T_x M \to T_{H(x,t)}N$ is an oriented operator with orientation $\alpha(x,t)$ which is continuous as a function of (x,t). Where, in this case, continuous means that, given any two local charts $\varphi : U \to E$ and $\psi : V \to F$ of M and N respectively, the map

$$(z,t) \mapsto D\psi(H(\varphi^{-1}(z),t)) \circ D_1 H(\varphi^{-1}(z),t) \circ D\varphi^{-1}(z) \in \Phi_0(E,F),$$

which is defined in an open (possibly empty) subset of $E \times [0,1]$, turns out to be oriented (according to Definition 2.3) by

$$(z,t) \mapsto \left\{ D\psi(H(\varphi^{-1}(z),t)) \circ A \circ D\varphi^{-1}(z) : A \in \alpha(\varphi^{-1}(z),t) \right\}.$$

Clearly, given an oriented homotopy $H : M \times [0,1] \to N$, any partial map $H_t := H(\cdot,t)$ is an oriented map from M into N, according to Definition 2.4. One could actually show that, given a homotopy $H : M \times [0,1] \to N$ of Fredholm maps of index zero, if both H and $D_1 H$ are continuous and H_0 is orientable, then all the partial maps H_t are orientable as well, and an orientation of H_0 induces a unique orientation on any H_t which makes H an oriented homotopy (this result is not needed here and will appear elsewhere).

3 Degree for oriented maps

Let $f : M \to N$ be an oriented map between Banach (boundaryless) manifolds. Given an element $y \in N$, we call the triple (f,M,y) *admissible* (or we say that f is *y-admissible* in M) if $f^{-1}(y)$ is compact. Given f as above, the pair (f,y) contains the same information as the triple (f,M,y) (the domain M and the codomain N of f being implicit in the definition of f). However, the redundant notation (f,M,y) is convenient in order to consider the restriction of f to an open subset U of M (which is still a Banach manifold). In this case we say that the triple (f,U,y) is admissible if so is $(f|_U,U,y)$, that is if the set $f^{-1}(y) \cap U$ is compact.

A triple (f,M,y) is called *strongly admissible* provided that M is an open subset of a Banach manifold R, f admits a continuous extension to the closure $\overline{M}$ of M (again denoted by f), this extension is proper, and $y \notin f(\partial M)$. Clearly any strongly admissible triple is also admissible. Moreover, if (f,M,y) is strongly admissible and U is an open subset of M such that $U \cap f^{-1}(y)$ is compact, then (f, U, y) is strongly admissible as well. Finally, we recall that Fredholm maps are locally proper; thus, if (f, M, y) is admissible, the compactness of $f^{-1}(y)$ implies the existence of an open neighborhood U of $f^{-1}(y)$ which makes (f, U, y) a strongly admissible triple; and this fact will be crucial in this section.

Our aim here is to define a map, called *degree*, which to every admissible triple (f, M, y) assigns an integer, $\deg(f, M, y)$, in such a way that the following five properties hold:

i) (*Normalization*) If $f : M \to N$ is a naturally oriented diffeomorphism and $y \in N$, then

$$\deg(f, M, y) = 1.$$

ii) (*Additivity*) Given an admissible triple (f, M, y) and two open subsets U_1, U_2 of M, if $U_1 \cap U_2 = \emptyset$ and $f^{-1}(y) \subset U_1 \cup U_2$, then (f, U_1, y) and (f, U_2, y) are admissible and

$$\deg(f, M, y) = \deg(f, U_1, y) + \deg(f, U_2, y).$$

iii) (*Topological Invariance*) If (f, M, y) is admissible, $\varphi : R \to M$ is a naturally oriented diffeomorphism from a Banach manifold R onto M and $\psi : N \to Z$ is a naturally oriented diffeomorphism from the codomain N of f onto a Banach manifold Z, then

$$\deg(f, M, y) = \deg(\psi f \varphi, R, \psi(y)),$$

where $\psi f \varphi$ is the oriented composition.

iv) (*Reduction*) Let $f : M \to N$ be an oriented map and let N_1 be a submanifold of N which is transverse to f. Denote by f_1 the restriction of f to the pair of manifolds $M_1 = f^{-1}(N_1)$ and N_1 with the orientation induced by f. Then

$$\deg(f, M, y) = \deg(f_1, M_1, y),$$

provided that $f^{-1}(y)$ is compact.

v) (*Homotopy Invariance*) Let $H : M \times [0, 1] \to N$ be an oriented homotopy and let $y : [0, 1] \to N$ be a continuous path. If the set

$$\{(x, t) \in M \times [0, 1] : H(x, t) = y(t)\}$$

is compact, then $\deg(H_t, M, y(t))$ is well defined and does not depend on $t \in [0, 1]$.

In the sequel we shall refer to i)−v) as the *fundamental properties* of degree.

We define first our notion of degree in the special case when (f, M, y) is a *regular triple*; that is when (f, M, y) is admissible and y is a regular value for f in M. This implies that $f^{-1}(y)$ is a compact discrete set and, consequently, finite. In this case our definition is similar to the classical one in the finite dimensional case. Namely

$$\deg(f, M, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} Df(x), \tag{3.1}$$

where, we recall, $\operatorname{sgn} Df(x) = 1$ if the trivial operator is a positive corrector of the oriented isomorphism $Df(x) : T_x M \to T_y N$, and $\operatorname{sgn} Df(x) = -1$ otherwise.

It is evident that the first four fundamental properties of the degree hold true for the class of regular triples, and after Definition 3.3 below it will be clear that they are still valid in the general case.

A straightforward consequence of the Additivity is the following property that we shall need (for the special case of regular triples) in the proof of Lemma 3.2 below.

vi) (*Excision*) If (f, M, y) is admissible and U is an open neighborhood of $f^{-1}(y)$, then

$$\deg(f, M, y) = \deg(f, U, y).$$

In order to define the degree in the general case we will prove that, given any admissible triple (f, M, y), if U_1 and U_2 are sufficiently small open neighborhoods of $f^{-1}(y)$, and $y_1, y_2 \in N$ are two regular values for f, sufficiently close to y, then

$$\deg(f, U_1, y_1) = \deg(f, U_2, y_2).$$

Let us show first that the degree of a regular triple (f, M, y) can be viewed as the Brouwer degree of the restriction of f to a convenient pair of finite dimensional oriented manifolds.

Consider an admissible triple (f, M, y) (for the moment we do not assume y to be a regular value of f) and let $\psi : V \to F$ be a local chart of N at y such that $\psi(y) = 0$. Given $x \in f^{-1}(y)$, let F_x be a finite dimensional subspace of F which is transverse to ψf at x. This implies the existence of an open neighborhood U_x of x in which ψf is transverse to F_x. Since $f^{-1}(y)$ is compact, one can find a finite dimensional subspace F_0 of F and an open subset U of $f^{-1}(V)$ containing $f^{-1}(y)$ in which ψf is transverse to F_0. Consequently, $N_0 = \psi^{-1}(F_0)$ and $M_0 = (\psi f)^{-1}(F_0) \cap U$ are differentiable manifolds of the same dimension as F_0, f is transverse to N_0 in U, and the restriction $f_0 : M_0 \to N_0$ of f is an oriented map (with orientation induced by f). Since N_0 is orientable (being diffeomorphic to an open subset of a finite dimensional vector space) and f_0 is orientable, M_0 is orientable as well. Therefore, the orientation of f_0 induces a pair of orientations of M_0 and N_0, up to an inversion of both of them (which does not effect the Brouwer degree of f_0 at y). When a pair of these orientations are chosen, we say that the two manifolds M_0 and N_0 are *oriented according to* f.

Before stating Lemma 3.1 below, we point out that if a Fredholm map $f : M \to N$ is transverse to a submanifold N_0 of N, then an element $y \in N_0$ is a regular value for f if and only if it is a regular value for the restriction $f_0 : f^{-1}(N_0) \to N_0$ of f.

Lemma 3.1 *Let (f, M, y) be a regular triple and let N_0 be a finite dimensional orientable submanifold of the codomain N of f, containing y and transverse to f. Then $M_0 = f^{-1}(N_0)$ is an orientable manifold of the same dimension as N_0. Moreover, orienting M_0 and N_0 according to f, the Brouwer degree* $\deg_B(f_0, M_0, y)$ *of f_0 at y coincides with* $\deg(f, M, y)$.

Proof. Observe that, given $x \in M_0$, the tangent space $T_{f(x)}N_0$ of N_0 at $f(x)$ is transverse to $Df(x)$, T_xM_0 coincides with $Df(x)^{-1}(T_{f(x)}N_0)$, and these two spaces can be oriented according to the orientation of $Df(x)$. It is easy to check that, with this pair of orientations, if $x \in f^{-1}(y) = f_0^{-1}(y)$, then both $Df(x)$ and $Df_0(x)$ are (oriented) isomorphisms and $\operatorname{sgn} Df_0(x)$ coincides with $\operatorname{sgn} Df(x)$. The assertion now follows immediately from (3.1) and the definition of Brouwer degree. □

As a consequence of Lemma 3.1 we get the following result which is crucial in our definition of degree.

Lemma 3.2 *Let (f, M, y) be a strongly admissible triple. Given two neighborhoods U_1 and U_2 of $f^{-1}(y)$, there exists a neighborhood V of y such that for any pair of regular values $y_1, y_2 \in V$ one has*

$$\deg(f, U_1, y_1) = \deg(f, U_2, y_2).$$

Proof. Since (f, M, y) is strongly admissible, M is an open subset of a Banach manifold R, and f is actually defined, continuous and proper on the closure $\overline{M}$ of M. Let U_1 and U_2 be two open neighborhoods of $f^{-1}(y)$ and put $U = U_1 \cap U_2$. Since proper maps are closed, there exists a neighborhood V of y with $V \cap f(\overline{M} \setminus U) = \emptyset$. Without loss of generality we may assume that V is the domain of a chart $\psi : V \to F$ whose image, $\psi(V)$, is a ball in F; so that, given two regular values $y_1, y_2 \in V$, $\psi(V)$ contains the line segment S joining $\psi(y_1)$ and $\psi(y_2)$. With an argument similar to the one used just before Lemma 3.1 one can show the existence of a finite dimensional subspace F_0 of F containing $\psi(y_1)$ and $\psi(y_2)$ and transverse to ψf in a convenient neighborhood $W \subset U$ of the compact set $(\psi f)^{-1}(S)$. Thus, both $N_0 = \psi^{-1}(F_0)$ and $M_0 = f^{-1}(N_0) \cap W$ are finite dimensional manifolds of the same dimension as F_0, and they turn out to be oriented according to f (up to an inversion of both orientations). Denote by f_0 the restriction of f to M_0 (as domain) and N_0 (as codomain). From Lemma 3.1 we obtain

$$\deg_B(f_0, M_0, y_1) = \deg(f, W, y_1),$$

$$\deg_B(f_0, M_0, y_2) = \deg(f, W, y_2).$$

On the other hand, since $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are contained in W, by the Excision property for regular triples, we get $\deg(f, U_1, y_1) = \deg(f, W, y_1)$ and $\deg(f, U_2, y_2) = \deg(f, W, y_2)$. Therefore, it remains to show that $\deg_B(f_0, M_0, y_1) = \deg_B(f_0, M_0, y_2)$.

Consider now the path $y(\cdot) : [0, 1] \to N_0$ given by $t \mapsto \psi^{-1}(ty_1 + (1 - t)y_2)$ and observe that the set

$$\{x \in M_0 : f_0(x) = y(t) \text{ for some } t \in [0, 1]\}$$

coincides with $(\psi f)^{-1}(S)$, which is compact. Therefore, from the homotopy invariance of the Brouwer degree we get

$$\deg_B(f_0, M_0, y_1) = \deg_B(f_0, M_0, y_2),$$

and the result is proved. □

Lemma 3.2 justifies the following definition of degree for general admissible triples.

Definition 3.3 Let (f, M, y) be admissible and let U be any open neighborhood of $f^{-1}(y)$ such that $\overline{U} \subset M$ and f is proper on $\overline{U}$. Put

$$\deg(f, M, y) := \deg(f, U, z),$$

where z is any regular value for f in U, sufficiently close to y.

To justify the above definition we point out that the existence of regular values for $f|_U$ which are sufficiently close to y can be directly deduced from Sard's Lemma. In fact, as previously observed, one can reduce the problem of finding regular values of a Fredholm map to its restriction to a convenient pair of finite dimensional manifolds.

Given an oriented map $f : M \to N$, the degree of f at y, $\deg(f, M, y)$, which is defined whenever $f^{-1}(y)$ is compact, does not necessarily depend continuously on $y \in N$, as the following trivial example shows.

Consider the exponential map $\exp : \mathbf{R} \to \mathbf{R}$ with the canonical orientation (which makes sense, since the derivative of any real function is a linear endomorphism of $\mathbf{R}$). Observe that the triple $(\exp, \mathbf{R}, y)$ is admissible for any $y \in \mathbf{R}$ and

$$\deg(\exp, \mathbf{R}, y) = \begin{cases} 0 & \text{if } y \leq 0, \\ 1 & \text{if } y > 0. \end{cases}$$

Thus the map $y \mapsto \deg(\exp, \mathbf{R}, y)$ is discontinuous at $y = 0$.

There are some ways to avoid discontinuities of the map $y \mapsto \deg(f, M, y)$. One is to restrict the attention to the class of strongly admissible triples, and in this case the continuity is a direct consequence of the definition. Another way, more general than the previous one, is the following. Let $f : M \to N$ be an oriented map and consider the open subset of N consisting of those elements $y \in N$, called *proper values* of f in M, which admit an open neighborhood V with the property that $f^{-1}(K)$ is compact whenever $K \subset V$ is compact (this means that the restriction $f : f^{-1}(V) \to V$ is proper). We say that $y \in N$ is a *boundary value* for f in M, and we write $y \in \Delta(f, M)$, if y is not a proper value. One can check that the closed set $\Delta(f, M)$ coincides with $f(\partial M)$ in the case when f is continuous and proper on the closure $\overline{M}$ of an open set M of a Banach manifold Z. Therefore, in the general situation, the limit set $\Delta(f, M)$ is a valid substitute for $f(\partial M)$. Incidentally, observe that $\Delta(\exp, \mathbf{R}) = \{0\}$.

It is easy to see that, given $f : M \to N$ oriented, the map $y \mapsto \deg(f, M, y)$ is well defined and continuous in the open subset $N \setminus \Delta(f, M)$ of N. Thus, $\deg(f, M, y)$ depends only on the connected component of $N \setminus \Delta(f, M)$ containing y.

Theorem 3.4 *The degree satisfies the above five fundamental properties.*

Proof. The first four properties are an easy consequence of the analogous ones for regular triples. Let us prove the Homotopy Invariance. Consider an oriented homotopy $H : M \times [0,1] \to N$ and let $y : [0,1] \to N$ be a continuous path in N. Assume that the set

$$C = \{x \in M : H(x,t) = y(t) \text{ for some } t \in [0,1]\}.$$

is compact. Since H is locally proper, there exists an open neighborhood U of C in M such that H is proper on $\overline{U} \times [0,1]$. Consequently, $H_t = H(\cdot, t)$ is proper on $\overline{U}$ for all $t \in [0,1]$, and, by the definition of degree,

$$\deg(H_t, M, y(t)) = \deg(H_t, U, y(t)), \ \forall t \in [0,1].$$

We need to prove that the function $\sigma(t) = \deg(H_t, U, y(t))$ is locally constant. Let τ be any point in $[0,1]$. Since H is proper on $\overline{U} \times [0,1]$ and $y(\tau) \notin H_\tau(\partial U)$, one can find an open connected neighborhood V of $y(\tau)$ and a compact neighborhood J of τ (in $[0,1]$) such that $y(t) \in V$ for $t \in J$ and $H(\partial U \times J) \cap V = \emptyset$. Thus, if z is any element of V, one has $\sigma(t) = \deg(H_t, U, z)$ for all $t \in J$. To compute this degree we may therefore assume that z is a regular value for H_τ in U, so that $H_\tau^{-1}(z)$ is a finite set $\{x_1, x_2, \dots, x_n\}$ and the partial derivatives $D_1 H(x_i, \tau)$, $i = 1, 2, \dots, n$, are all nonsingular. Consequently, given any x_i in $H_\tau^{-1}(z)$, the Implicit Function Theorem ensures that $H^{-1}(z)$, in a neighborhood $W_i \times J_i$ of (x_i, τ), is the graph of a continuous curve $\gamma_i : J_i \to M$. Since H is proper in $\overline{U} \times J$ (recall that J is compact) and $z \notin H(\partial U \times J)$, the set $H^{-1}(z) \cap (U \times J)$ is compact. This implies the existence of a neighborhood J_0 of τ such that for $t \in J_0$ one has

$$H_t^{-1}(z) = \{\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)\}.$$

Moreover, by the continuity of $D_1 H$, we may assume that z is a regular value for any H_t, $t \in J_0$. Finally, since H is an oriented homotopy, the continuity assumption in the definition of orientation implies that, for any i, $\text{sgn} D_1 H(\gamma_i(t), t)$ does not depend on $t \in J_0$, and from the definition of degree of a regular triple we get that $\sigma(t)$ is constant in J_0. □

We close pointing out that in the finite dimensional context the notion of orientation and the concept of degree can be extended to the continuous case. This is mainly due to the following facts regarding maps acting between finite dimensional (not necessarily orientable) manifolds: 1) any continuous map can be arbitrarily approximated by C^1 maps; 2) if two C^1 maps are sufficiently close, then they are C^1 homotopic; 3) given a C^1 homotopy $H : M \times [0,1] \to N$, an orientation of H_0 (when it makes sense) induces uniquely an orientation on H_1. An exhaustive analysis of these concepts regarding the finite dimensional case will appear in a forthcoming paper. In particular we will show that in the finite dimensional case our notion of degree coincides with the one introduced (with different methods) by A. Dold (see [5], exercise 6, p. 271).

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