PERIODIC SOLUTIONS FOR NONLINEAR EQUATIONS WITH MEAN CURVATURE-LIKE OPERATORS

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Abstract. We give an existence result for a periodic boundary value problem involving mean curvature-like operators in the scalar case. Following [3], we use an approach based on the Leray-Schauder degree.

1. Introduction

In [3] (see also [4]) Manásevich and Mawhin prove an existence result for the periodic boundary value problem

$$(\phi(u'))' = f(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{1.1}$$

where $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism satisfying particular monotonicity conditions including for instance p-Laplacian-like operators. They use a topological approach: the properties of ϕ and f allow to apply the Leray-Schauder degree to prove that (1.1) admits a solution (see [3, Theorem 3.1]).

In this paper, proceeding in the general spirit of Manásevich-Mawhin's ideas, we obtain an existence result (Theorem 3.1 below) for a different problem. Precisely, we study the nonlinear scalar equation with periodic boundary conditions

$$(\phi(u'))' = f(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{1.2}$$

where $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is still a Carathéodory function, but $\phi : \mathbb{R} \to \mathbb{R}$ is, in our case, an increasing homeomorphism between $\mathbb{R}$ and the open interval $(-1,1)$, with $\phi(0) = 0$.

The interest in the above class of nonlinear operators $u \mapsto (\phi(u'))'$ is mainly due to the fact that they include the scalar version of the mean curvature operator

$$u \mapsto \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

The paper is organized as follows. In the next section we consider problem (1.2) in the particular case when f is independent of u and u'. The study of this simplified problem is the first step in the direction of tackling problem (1.2) by the Leray-Schauder degree, as done in Section 3. That section is, in particular, devoted to the main theorem of this work, that is, an existence result for (1.2). In the last section we present an application of the main theorem to a particular system.

We refer to [1] or [2] for the definition and the main properties of the Leray-Schauder degree.

In what follows I will denote the closed interval $[0,T]$, with T fixed. In addition, we will put $\mathcal{C} = C(I,\mathbb{R})$, $\mathcal{C}^1 = C^1(I,\mathbb{R})$, $\mathcal{C}_{T,0} = \{u \in \mathcal{C} : u(0) = u(T) = 0\}$, $\mathcal{C}^1_T = \{u \in \mathcal{C}^1 : u(0) = u(T),\ u'(0) = u'(T)\}$, $L^1 = L^1(I,\mathbb{R})$, and, finally, $L^1_m = \{h \in L^1 : \int_0^T h(t)dt = 0\}$.

Remark 1.1. We point out that by a solution of (1.2) we mean a C^1 real function u on $[0,T]$, satisfying the boundary conditions, such that $\phi(u')$ is absolutely continuous and verifies (1.2) a.e. on $[0,T]$.

2. A simplified problem

Consider the following periodic boundary value problem

$$(\phi(u'))' = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{2.1}$$

where h is in L^1_m and ϕ is an increasing homeomorphism between $\mathbb{R}$ and $(-1,1)$, with $\phi(0) = 0$. If a C^1 function $u : I \to \mathbb{R}$ solves the equation $(\phi(u'))' = h(t)$, of course there exists a real a such that

$$\phi(u'(t)) = a + H(h)(t), \tag{2.2}$$

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where H is the continuous linear integral operator, that is,

$$H(h)(t) = \int_0^t h(s)ds.$$

Remark 2.1. Notice that the condition $u'(0) = u'(T)$ implies that $\int_0^T h(t)dt = 0$ and this justifies the assumption that $h \in L^1_m$.

By the inversion of ϕ in (2.2), we have

$$u'(t) = \phi^{-1}(a + H(h)(t)), \tag{2.3}$$

and thus the image of $H(h)$, which is a closed interval containing 0, has measure smaller than 2.

Call $\widetilde{D}$ the set of functions h in L^1_m such that there exists a real a with

$$a + H(h)(t) \in (-1, 1), \quad \forall t \in I.$$

The set $\widetilde{D}$ is unbounded in L^1_m. Indeed, take for simplicity $T = 1$ and consider the sequence of functions $\{h_n\}$ in L^1_m, defined by

$$h_n(t) = \begin{cases} n & t \in [k/n, (2k+1)/(2n)) \\ -n & t \in [(2k+1)/(2n), (k+1)/n) \cup \{1\}, \quad k = 0, \dots, n-1. \end{cases} \tag{2.4}$$

It is straightforward to see that $\{h_n\} \subseteq \widetilde{D}$ and that, in particular, for each n, $H(h_n)$ is a nonnegative function with norm 1/2. On the other hand, $\|h_n\|_{L^1} = n$ and this shows that $\widetilde{D}$ is not bounded in L^1_m.

Moreover $\widetilde{D}$ is open in L^1_m. To see this, let $h \in \widetilde{D}$ be given and consider any ε in L^1_m. For any t one has

$$\int_0^t h(s)ds - \|\varepsilon\|_{L^1} \le \int_0^t h(s)ds + \int_0^t \varepsilon(s)ds \le \int_0^t h(s)ds + \|\varepsilon\|_{L^1}.$$

If $[c, d]$ is the image of $H(h)$, then $\widetilde{D}$ contains the open ball in L^1_m of center h and radius $[2 - (d - c)]/2$.

Coming back to problem (2.1), if u is a solution, we have

$$u(t) = u(0) + \int_0^t \phi^{-1}(a + H(h)(s))ds.$$

The boundary condition $u(0) = u(T)$ implies that

$$\int_0^T \phi^{-1}(a + H(h)(t))dt = 0. \tag{2.5}$$

Therefore problem (2.1) admits a solution in $\mathcal{C}^1_T$ if and only if h belongs to the subset D of $\widetilde{D}$ defined as the set of functions $h \in \widetilde{D}$ such that there exists $a \in \mathbb{R}$ verifying the (2.5). The following proposition lists some properties of D.

Proposition 2.2. *The following conditions hold:*
1. *the set D is open and unbounded in L^1_m;*
2. *D contains the open ball in L^1_m centered at 0 with radius 1;*
3. *for any $h \in D$ the real a such that*

$$\int_0^T \phi^{-1}(a + H(h)(t))dt = 0$$

is unique and then defines a map $\alpha : D \to \mathbb{R}$ which is bounded and continuous.

Proof. (1) The unboundedness of D can be proved in the same way as done for $\widetilde{D}$. In the simple case $T = 1$, take the sequence $\{h_n\}$ defined by formula (2.4). For any n the function

$$G_n(a) = \int_0^1 \phi^{-1}(a + H(h_n)(t))dt$$

is well defined in $(-1, 1/2)$. Of course, $G_n(a) > 0$ if $a \ge 0$ and $G_n(a) < 0$ if $a \le -1/2$. As G_n is continuous, it admits a zero in its domain. Therefore $\{h_n\} \subseteq D$.

To prove that D is open in L^1_m, define the set

$$C = \left\{ l \in \mathcal{C}_{T,0} : \exists\, a \in \mathbb{R} \text{ with } -1 < a + l(t) < 1, \forall t \in I, \text{ and } \int_0^T \phi^{-1}(a + l(t))dt = 0 \right\}.$$

The set C is open in $\mathcal{C}_{T,0}$. Indeed, fix $\bar{l} \in C$ and let $a_{\bar{l}}$ be such that

$$\int_0^T \phi^{-1}(a_{\bar{l}} + \bar{l}(t))dt = 0.$$

Denote by (a_1, a_2) the open interval, containing $a_{\bar{l}}$, such that $\phi^{-1}(a + \bar{l}(t))$ is well defined for every $a \in (a_1, a_2)$ and every $t \in I$. Since ϕ^{-1} is strictly increasing, we can take $a_1 < \bar{a}_1 < a_{\bar{l}} < \bar{a}_2 < a_2$ such that

$$\int_0^T \phi^{-1}(\bar{a}_1 + \bar{l}(t))dt < 0 \quad \text{and} \quad \int_0^T \phi^{-1}(\bar{a}_2 + \bar{l}(t))dt > 0.$$

Then, consider a neighborhood U of $\bar{l}$ in $\mathcal{C}_{T,0}$ such that for each $m \in U$ one has

$$\int_0^T \phi^{-1}(\bar{a}_1 + m(t))dt < 0 \quad \text{and} \quad \int_0^T \phi^{-1}(\bar{a}_2 + m(t))dt > 0.$$

The existence of U is a consequence of the fact that, for $i = 1, 2$, the map

$$l \mapsto \int_0^T \phi^{-1}(\widehat{a}_i + l(t))dt$$

is well defined and continuous in a neighborhood of $\bar{l}$. It follows that $U \subseteq C$ which turns out to be open in $\mathcal{C}_{T,0}$. Now, as $D = H^{-1}(C)$, where H is here the integral operator restricted to L^1_m and valued in $\mathcal{C}_{T,0}$, we have that D is open in L^1_m.

(2) The set C, defined above, contains the open ball B in $\mathcal{C}_{T,0}$ of center zero and radius $1/2$. Indeed let $l \in B$ be given. If $\int_0^T \phi^{-1}(l(t))dt = 0$, clearly l is in C. Otherwise, without loss of generality, suppose that the above integral is positive. As $\sup_t l(t) < 1/2$ and $\inf_t l(t) > -1/2$, one can find a real a, close enough to $-1/2$, such that $-1 < a + l(t) < 0$ for each $t \in I$, and thus

$$\int_0^T \phi^{-1}(a + l(t))dt < 0.$$

Hence, there exist $\widehat{a} \in (a, 0)$ such that

$$\int_0^T \phi^{-1}(\widehat{a} + l(t))dt = 0$$

and this proves that $l \in C$.

Now, let $h \in L^1_m$ with $||h||_{L^1} < 1$. Define

$$h_+(t) = \begin{cases} h(t) & \text{if } h(t) \geq 0 \\ 0 & \text{if } h(t) < 0 \end{cases} \quad \text{and} \quad h_-(t) = \begin{cases} 0 & \text{if } h(t) \geq 0 \\ -h(t) & \text{if } h(t) < 0. \end{cases}$$

As $\int_0^T h(t)dt = 0$, one has that $||h_+|| = ||h_-|| < 1/2$. It follows that $|H(h)(t)| < 1/2$ for any $t \in I$ and hence $H(h) \in C$. Thus $h \in D$ and the claim follows.

(3) Since ϕ^{-1} is strictly increasing, for any $h \in D$ the real a such that

$$\int_0^T \phi^{-1}(a + H(h)(t))dt = 0$$

is unique and then defines a map $\alpha : D \to \mathbb{R}$. The boundedness of α is a consequence of the fact that, for each $h \in D$, $H(h)(0) = 0$ and thus $|H(h)(t)| < 2$, for any $t \in I$.

To see the continuity of α we proceed as follows. For any function $l \in C$ the real a such that

$$\int_0^T \phi^{-1}(a + l(t))dt = 0$$

is unique. Therefore it is well defined the map $\widetilde{\alpha} : C \to \mathbb{R}$, such that, for each $l \in C$,

$$\int_0^T \phi^{-1}(\widetilde{\alpha}(l) + l(t))dt = 0.$$

Let us prove the continuity of $\widetilde{\alpha}$. Let $\{l_n\}$ be a sequence in C, converging to $l \in C$. Since $\widetilde{\alpha}$ is bounded, any subsequence of $\widetilde{\alpha}(l_n)$ admits a convergent subsequent, say $\widetilde{\alpha}(l_{n_j}) \to \bar{a}$ as $j \to \infty$. Let us show first that $\phi^{-1}(\bar{a} + l(t))$ is well defined. To see this, call (a_1, a_2) the domain of the map G_l, defined as

$$G_l(a) = \int_0^T \phi^{-1}(a + l(t))dt.$$

Then, consider $\overline{a}_1$ and $\overline{a}_2$ in (a_1, a_2) such that

$$G_l(\overline{a}_1) < 0 \quad \text{and} \quad G_l(\overline{a}_2) > 0.$$

Let U be a neighborhood of l in C such that, for any m in U,

$$G_m(\overline{a}_1) < 0 \quad \text{and} \quad G_m(\overline{a}_2) > 0.$$

This implies that, for j sufficiently large, $\alpha(l_{n_j}) \in [\overline{a}_1, \overline{a}_2]$ and hence $\overline{a} \in [\overline{a}_1, \overline{a}_2]$.

Now, from the continuity of the map $x \mapsto \int_0^T \phi^{-1}(x(t))dt$ and since

$$\int_0^T \phi^{-1}(\alpha(l_{n_j}) + l_{n_j}(t))dt = 0,$$

it follows that

$$\int_0^T \phi^{-1}(\overline{a} + l(t))dt = 0$$

and this proves the continuity of $\widetilde{\alpha}$. Finally, α is continuous being the composition $\widetilde{\alpha} = H \circ \alpha$. □

For any $h \in D$, problem (2.1) has infinite solutions which differ by a constant and can be written as

$$u(t) = u(0) + H\left(\phi^{-1}[\alpha(h) + H(h)]\right)(t), \tag{2.6}$$

where, by an abuse of notation, in the above formula ϕ^{-1} is the operator which associates to any map g the map $t \mapsto \phi^{-1}(g(t))$.

Define $P : \mathcal{C}^1_T \to \mathcal{C}^1_T$ as $Pu = u(0)$. Observe that $\mathcal{C}^1_T$ admits the splitting

$$\mathcal{C}^1_T = E_1 \oplus E_2, \tag{2.7}$$

where E_1 contains the maps $\widetilde{u}$ such that $\widetilde{u}(0) = 0$ and E_2 is the one-dimensional subspace of constant maps. It is immediate to see that P is the continuous projection onto E_2 by the above decomposition.

In addition consider $Q : L^1 \to L^1$, defined as $Qh = \frac{1}{T}\int_0^T h(t)dt$. One can split L^1 as

$$L^1 = L^1_m \oplus F_2, \tag{2.8}$$

where F_2 is the one-dimensional subspace of constant maps. The operator Q turns easily out to be the continuous projection on F_2 with the above splitting of L^1.

Then, consider the subset $\widehat{D}$ of L^1, $\widehat{D} = D + F_2$, and the nonlinear operator $K : \widehat{D} \to \mathcal{C}^1_T$, defined as

$$K(h)(t) = H\left(\phi^{-1}[\alpha((I-Q)h) + H((I-Q)h)]\right)(t). \tag{2.9}$$

If a C^1 function u is a solution of (2.1), for a given $h \in D$, of course u solves the equation

$$u = Pu + Qh + K(h). \tag{2.10}$$

Conversely, if $u \in \mathcal{C}^1_T$ is a solution of (2.10), for a given $h \in \widehat{D}$, it follows that h actually belongs to D and u solves (2.1).

Proposition 2.3. *The map K is continuous and sends equi-integrable sets of $\widehat{D}$ into relatively compact sets in $\mathcal{C}^1_T$.*

Proof. The continuity of K is a straightforward consequence of the fact that this map is a composition of continuous maps.

Consider an equi-integrable set S of L^1, contained in $\widehat{D}$, and let $g \in L^1$ be such that

$$|h(t)| \le g(t) \quad \text{a.e. in } I.$$

Let us show that $\overline{K(S)}$ is compact. To see this consider first a sequence $\{k_n\}$ of $K(S)$ and let $\{h_n\} \subseteq S$ be such that $K(h_n) = k_n$. For any $t_1, t_2 \in I$ we have

$$\begin{aligned} |H(I-Q)(h_n)(t_1) - H(I-Q)(h_n)(t_2)| &\le \left|\int_{t_2}^{t_1} h_n(s)ds\right| + |Qh_n|\,|t_1 - t_2| \\ &\le \left|\int_{t_2}^{t_1} g(s)ds\right| + \frac{t_1 - t_2}{T}\int_0^T g(s)ds. \end{aligned}$$

Therefore the sequence $\{H(I-Q)(h_n)\}$ is bounded and equicontinuous and then, by Ascoli-Arzelà Theorem, it admits a convergent subsequence in $\mathcal{C}$, say $\{H(I-Q)(h_{n_j})\}$. Up to a subsequence, $\{\alpha((I-Q)(h_{n_j})) + H((I-Q)(h_{n_j})\}$ converges in $\mathcal{C}$. In addition we have that

$$(K(h_{n_j}))'(t) = \phi^{-1}\left[\{\alpha((I-Q)(h_{n_j})) + H((I-Q)(h_{n_j})\}\right](t)$$

and, by the continuity of ϕ^{-1}, $(K(h_{n_j}))'$ is convergent in $\mathcal{C}$. Therefore $\{k_{n_j}\} = \{K(h_{n_j})\}$ converges in $\mathcal{C}^1_T$.

Now consider a sequence $\{k_n\}$ belonging to $\overline{K(S)}$ (that is, not necessarily to $K(S)$). Let $\{l_n\} \subseteq K(S)$ be such that $||l_n - k_n|| \to 0$ as $n \to \infty$. Let in addition $\{l_{n_j}\}$ be a subsequence of $\{l_n\}$ that converges to l. Therefore, $l \in \overline{K(S)}$ and $\{k_{n_j}\} \to l$, and this completes the proof. □

3. Main result

In this section we present the main result of the paper, that is, an existence theorem for the periodic boundary value problem

$$(\phi(u'))' = f(t, u, u'), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{3.1}$$

where ϕ is as in the above section and $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is,

i) for almost every $t \in I$, $f(t, \cdot, \cdot)$ is continuous;
ii) for any $(x, y) \in \mathbb{R}^2$, $f(\cdot, x, y)$ is measurable;
iii) for any $\rho > 0$ there exists $g \in L^1$ such that, for almost every $t \in I$ and every $(x, y) \in \mathbb{R}^2$, with $|x| \leq \rho$ and $|y| \leq \rho$, we have
$$|f(t, x, y)| \leq g(t).$$

Theorem 3.1. *Let Ω be a bounded open subset of $\mathcal{C}^1_T$ such that the following conditions hold:*

(1) *for any $u \in \overline{\Omega}$ the map $t \mapsto f(t, u(t), u'(t))$ belongs to $\widehat{D}$;*
(2) *for each $\lambda \in (0, 1)$ the problem*
$$(\phi(u'))' = \lambda f(t, u, u'), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{3.2}$$
has no solution on $\partial\Omega$;
(3) *the equation*
$$F(a) := \int_0^T f(t, a, 0)dt = 0 \tag{3.3}$$
has no solution on $\partial\Omega_2$, where $\Omega_2 := \Omega \cap E_2$ and E_2 is the subspace of $\mathcal{C}^1_T$ in the splitting (2.7);
(4) *the Brouwer degree*
$$\deg_B(F, \Omega_2, 0)$$
is well defined and nonzero.

Then problem (3.1) has a solution in $\overline{\Omega}$.

Proof. Let N_f denote the Nemytski operator associated to f, that is,

$$N_f : \mathcal{C}^1_T \to L^1, \quad N_f(u)(t) = f(t, u(t), u'(t)).$$

Consider the problem

$$(\phi(u'))' = \lambda N_f(u) + (1 - \lambda)QN_f(u), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{3.4}$$

For $\lambda \in (0, 1]$, if u is a solution of (3.2), then, as seen in the previous section, condition $u'(0) = u'(T)$ implies $QN_f(u) = 0$ and hence u solves problem (3.4) as well. Conversely, if u is a solution of problem (3.4), then $QN_f(u) = 0$ since it is easy to see that

$$Q[\lambda N_f(u) + (1 - \lambda)QN_f(u)] = QN_f(u),$$

and thus u solves problem (3.2) (λ still belongs to $(0, 1]$).

Let us now consider problem (3.4). It can be written in the equivalent form

$$u = \mathcal{K}(u, \lambda), \tag{3.5}$$

where

$$\begin{aligned}\mathcal{K}(u, \lambda) &= Pu + QN_f(u) + (K \circ [\lambda N_f + (1 - \lambda)QN_f])(u) \\ &= Pu + QN_f(u) + (K \circ [\lambda(I - Q)N_f])(u)\end{aligned}$$

is well defined in $\overline{\Omega} \times [0, 1]$. Suppose that (3.4) has no solution on $\partial\Omega$ for $\lambda = 1$, since, otherwise, the theorem is proved. Take $\lambda = 0$. Problem (3.4) becomes

$$(\phi(u'))' = \frac{1}{T}\int_0^T f(t, u(t), u'(t))dt, \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{3.6}$$

It follows that $\int_0^T f(t,u(t),u'(t))dt = 0$ and this implies that u is a constant function, say $u(t) = c$. Therefore, we have

$$\int_0^T f(t,c,0)dt = 0.$$

By assumption (2) $c \notin \partial\Omega_2$. Hence the equation

$$u - \mathcal{K}(u,\lambda) = 0$$

has no solution on $\partial\Omega \times [0,1]$. In addition, as f is Carathéodory, the nonlinear map $\mathcal{N} : \mathcal{C}_T^1 \times [0,1] \to L^1$, defined by

$$\mathcal{N}(u,\lambda) = \lambda N_f(u) + (1-\lambda)QN_f(u), \tag{3.7}$$

is continuous and takes bounded sets into equi-integrable sets. This implies that, recalling Proposition 2.3, $\mathcal{K}$ is completely continuous. We can apply the homotopy invariance property of the Leray-Schauder degree to the map $(u,\lambda) \mapsto u - \mathcal{K}(u,\lambda)$, obtaining

$$\deg_{LS}(I - \mathcal{K}(\cdot,0),\Omega,0) = \deg_{LS}(I - \mathcal{K}(\cdot,1),\Omega,0). \tag{3.8}$$

We can now say that problem (3.1) has a solution in $\overline{\Omega}$ if we prove that $\deg_{LS}(I - \mathcal{K}(\cdot,0),\Omega,0) \neq 0$. To see this we apply a finite-dimensional reduction property of the Leray-Schauder degree. Observe first that $K(0) = 0$, then

$$\mathcal{K}(u,0) = Pu + QN_f(u).$$

To compare the Leray-Schauder degree of the triple $(I-\mathcal{K}(\cdot,0),\Omega,0)$ with the Brouwer degree of $(F,\Omega_2,0)$, consider the splitting (2.7) of $\mathcal{C}_T^1$. The operator $I - \mathcal{K}(\cdot,0)$ can be represented in block-matrix form as

$$I - \mathcal{K}(\cdot,0) = \begin{pmatrix} I_{E_1} & -\mathcal{K}_{12} \\ 0 & -F \end{pmatrix}.$$

By the properties of the Leray-Schauder degree we have that

$$\deg_{LS}(I - \mathcal{K}(\cdot,0),\Omega,0) = -\deg_B(F,\Omega_2,0)$$

and this completes the proof. □

4. An application

In this section we show an application of Theorem 3.1. Consider the problem

$$\frac{u''}{(1+(u')^2)^{3/2}} = (t^2 - t + 1/2)(u^3 + (u')^4) + \eta, \quad u(0) = u(1), \quad u'(0) = u'(1), \tag{4.1}$$

where η is a real constant.

Remark 4.1. Recalling Remark 1.1, if $u \in \mathcal{C}_T^1$ solves the equation, $\phi(u')$ is absolutely continuous (ϕ being defined as $\phi(t) = t/\sqrt{1+t^2}$). It is immediate to verify that u' is absolutely continuous as well. Now, observe that a solution u of (4.1) is such that u'' coincides a.e. with a continuous function, that is, can be continuously extended to $[0,1]$. This implies that u' is actually C^1 and then any solution of the problem is actually a C^2 function.

Observe first that be the open ball B of $\mathcal{C}_T^1$ with center zero and radius 1 is admissible for problem (4.1) in the sense that, for any $u \in B$, the map

$$t \mapsto (t^2 - t + 1/2)(u^3(t) + (u'(t))^4) + \eta$$

belongs to the set $\widehat{D}$, introduced in the above section. To see this, it is sufficient to show that, for any $u \in B$, the map $t \mapsto (t^2 - t + 1/2)(u^3(t) + (u'(t))^4)$ belongs to D. This follows easily from the inequality

$$\sup_t |(t^2 - t + 1/2)(u^3(t) + (u'(t))^4)| < 1,$$

and the fact that D contains the open ball in L_m^1 centered at 0 with radius 1 (Proposition 2.2).

Call Ω the open ball in $\mathcal{C}_T^1$ with center zero and radius 3/4. Our purpose is to show that, by applying Theorem 3.1, problem (4.1) admits a solution in $\overline{\Omega}$ if $|\eta|$ is sufficiently small. We start by showing that (4.1) has no solution on $\partial\Omega$ for η in a suitable neighborhood of zero. Let $u \in \partial\Omega$ be given, that is,

$$||u||_1 = ||u||_0 + ||u'||_0 = 3/4.$$

We will consider different cases.

(i) Suppose $||u||_0 > 1/2$. This implies that $|u(t)| > 1/4$ for each t, since, otherwise, by the mean value theorem, we have $|u'(t')| > 1/4$ for some t', and hence $||u||_0 + ||u'||_0 > 3/4$. Notice that in this case u does not change sign.

Assume first that u is positive. If $\eta > -1/256$, then the right hand side of the equation in (4.1) is positive. If the left hand side is positive, then u' is stricly increasing, but this is not possible because $u'(0) = u'(T)$. If otherwise u is negative, the right hand side of the equation is negative if $\eta < 3/1024$. Analogously to the above case, the left hand side cannot be negative for each t.

(ii) Suppose $1/3 < ||u||_0 \leq 1/2$. This implies that $1/4 \leq ||u'||_0 < 5/12$ and thus $|u'(t)| \geq 1/4$, for some t. In addition, $u(0) = u(1)$ implies that $u'(t') = 0$ for some t', and thus, by the mean value theorem, $|u''(t'')| \geq 1/4$ for some t''. Let us show that

$$\left(1 + (u'(t))^2\right)^{3/2} \left|(t^2 - t + 1/2)(u^3(t) + (u'(t))^4) + \eta\right| < 1/4, \tag{4.2}$$

for each $t \in [0, 1]$ and any $\eta \in (-1/256, 3/1024)$. First we have that

$$\left(1 + (u'(t))^2\right)^{3/2} < \left(\frac{13}{12}\right)^3 < 1.28, \quad \forall t \in [0, 1],$$

then

$$\left|(t^2 - t + 1/2)(u^3(t) + (u'(t))^4) + \eta\right| \leq \frac{\left|u^3(t)\right|}{2} + \frac{(u(t)')^4}{2} + |\eta| < \frac{1}{16} + \frac{1}{2}\left(\frac{5}{12}\right)^4 + \frac{1}{256} < 0.09,$$

for each $t \in [0, 1]$ and $\eta \in (-1/256, 3/1024)$. The left hand side of (4.2) turns out to be smaller than 0.12 and hence (4.2) holds.

(iii) Finally, the case when $||u||_0 \leq 1/3$ is analogous to the previous one.

Summarizing this argument, problem (4.1) has no solution on $\partial\Omega$, with $\eta \in (-1/256, 3/1024)$. Let us apply Theorem 3.1 to show that our problem has a solution in $\overline{\Omega}$. To this purpose, observe that the problem

$$\frac{u''}{(1 + (u')^2)^{3/2}} = \lambda\left[(2t^2 - 2t + 1)(u^3 + (u')^4) + \eta\right], \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{4.3}$$

has no solution for any $\lambda \in (0, 1]$, and this can be easily seen by the same argument used in the case when $\lambda = 1$. Recalling points (3) and (4) in the statement of Theorem 3.1, the equation

$$F_\eta(a) = \int_0^1 \left[(t^2 - t + 1/2)a^3 + \eta\right] dt = \frac{1}{3}a^3 + \eta = 0$$

has no solution on $\partial\Omega_2 = (-3/4, 3/4)$ for any given $\eta \in (-1/256, 3/1024)$. It is immediate to see that

$$\deg_B(F_\eta, \Omega_2, 0) = 1.$$

Thus we can apply Theorem 3.1 to conclude that (4.1) admits a solution in Ω for $\eta \in (-1/256, 3/1024)$. It is also immediate that any solution is nontrivial if $\eta \neq 0$.

References

[1] K. Deimling, *Nonlinear Functional Analysis*, Springer Verlag, Berlin, 1985.

[2] N. Lloyd, *Degree Theory*, Cambridge University press, 1978.

[3] R. Manásevich and J. Mawhin, *Periodic Solutions for Nonlinear Systems with p-Laplacian-Like Operators*, J. Differential Equations **145** (1998), 367–393.

[4] J. Mawhin, *Periodic solutions of systems with p-Laplacian-like operators*, Nonlinear analysis and its application to differential equations (Lisbon 1998), 37–63, Prog. Nonlinear Differential Equations Appl., 43, Birkhäuser, Boston, 2001.

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