Orientation and degree for Fredholm maps of index zero between Banach spaces

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Abstract

We define a notion of topological degree for a class of maps (called orientable), defined between real Banach spaces, which are Fredholm of index zero. We introduce first a notion of orientation for any linear Fredholm operator of index zero between two real vector spaces. This notion (which does not require any topological structure) allows to define a concept of orientability for nonlinear Fredholm maps between real Banach spaces.

The degree which we present verifies the most important properties, usually taken into account in other degree theories, and it is invariant with respect to continuous homotopies of Fredholm maps.

1 Introduction

We define an integer valued topological degree for a class of Fredholm maps of index zero between real Banach spaces (i.e. C^1 maps such that at any point of the domain the derivative is Fredholm of index zero). This is a revised and simplified version of a recent study in collaboration with Prof. M. Furi (see [2] and [3]).

The starting point is a definition of orientation for Fredholm linear operators of index zero between real vector spaces. Such a notion is completely algebraic (no topological framework is needed). Any operator admits exactly two orientations. Furthermore, in the domain of real Banach spaces, given a (bounded) Fredholm operator of index zero L, an orientation of L induces by a sort of continuity an orientation on any operator L' sufficiently close to L.

This "local stability" of the orientation allows us to define a notion of orientation for (nonlinear) Fredholm maps of index zero between Banach spaces. More precisely, given two real Banach spaces E and F and an open subset Ω of E, a Fredholm map of index zero, $f : \Omega \to F$, is called orientable if one can assign in a "continuous way" (we will precise the sense of such a continuity) an orientation to the derivative of f at any point of its domain. The degree is consequently defined for the class of oriented maps. Such a degree verifies the most classical and important properties usually taken into account in degree theory.

Our construction extends and simplifies the theories given by Elworthy and Tromba and, later, by Fitzpatrick, Pejsachowicz, and Rabier. The first two authors introduced the notion of *Fredholm structure* in order to extend to infinite dimensional manifolds the classical concept of orientation of finite dimensional manifolds (on which is based the Brouwer degree) (see [4] and [5]). By a different approach, Fitzpatrick, Pejsachowicz and Rabier define a notion of orientation for Fredholm maps of index zero between real Banach spaces and then a degree for the class of oriented maps (see [7] and references therein).

The concept of orientation introduced by Fitzpatrick, Pejsachowicz and Rabier has interesting similarities and differences with our one and the reader can find a comparison (also with the work of Elworthy and Tromba) in [2] and [3].

We conclude this paper with an application to bifurcation theory. Precisely we present a global bifurcation theorem in the domain of Fredholm maps which are oriented according with our construction. More precise results in bifurcation theory and a comparison with analogous results obtained by Fitzpatrick, Pejsachowicz and Rabier will appear in a forthcoming paper in collaboration with M. Furi.

2 Orientable maps

We introduce here a completely algebraic notion of orientation for Fredholm linear operators of index zero between real vector spaces. Therefore, we will show that, in the context of Banach spaces, an oriented bounded operator induces, by a sort of continuity, an orientation to any sufficiently close operator. This "continuous transport of orientation" allows us to define a concept of oriented C^1 Fredholm map of index zero between open subsets of Banach spaces (and, more generally, between Banach manifolds).

Let E be a vector space and consider a linear map $T : E \to E$ of the form $T = I - K$, where I denotes the identity of E and K has finite dimensional range. Given any finite dimensional subspace E_0 of E containing $\operatorname{Range} K$, T maps E_0 into itself. Then, consider the restriction $T|_{E_0} : E_0 \to E_0$. It is not difficult to prove that the determinant, $\det T|_{E_0}$, is well defined and does not depend on the choice of the finite dimensional space E_0 containing $\operatorname{Range} K$. Thus, it makes sense to denote by $\det T$ this common value.

We recall that a linear operator between vector spaces, $L : E \to F$, is called (*algebraic*) *Fredholm* if $\operatorname{Ker} L$ and $\operatorname{coKer} L$ have finite dimension. Its *index* is the integer

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{coKer} L.$$

Notice that, when $L : \mathbf{R}^m \to \mathbf{R}^n$, one easily gets $\operatorname{ind} L = m - n$.

If $L : E \to F$ is Fredholm and $A : E \to F$ is any linear operator with finite dimensional range, we say that A is a *corrector* of L provided that $L + A$ is an isomorphism. This can be verified only if $\operatorname{ind} L = 0$, since, as well known, $L + A$ is Fredholm of the same index as L. Assume therefore $\operatorname{ind} L = 0$ and notice that, in this case, the set of correctors of L, indicated by $\mathcal{C}(L)$, is nonempty. In fact, any (possibly trivial) linear operator $A : E \to F$ such that $\operatorname{Ker} A \oplus \operatorname{Ker} L = E$ and $\operatorname{Range} A \oplus \operatorname{Range} L = F$ is a corrector of L.

We introduce in $\mathcal{C}(L)$ the following equivalence relation. Given $A, B \in \mathcal{C}(L)$, consider the automorphism $T = (L + B)^{-1}(L + A)$ of E. We have

$$T = (L + B)^{-1}(L + B + A - B) = I - (L + B)^{-1}(A - B).$$

Clearly $(L+B)^{-1}(A-B)$ has finite dimensional range. This implies that $\det T$ is well defined and, in this case, not zero since T is invertible. We say that A is *equivalent* to B or, more precisely, A is *L-equivalent* to B, if $\det (L + B)^{-1}(L + A) > 0$. This is actually an equivalence relations on $\mathcal{C}(L)$, with just two equivalence classes (see [2]). We can therefore introduce the following definition.

Definition 2.1 An *orientation* of a Fredholm operator of index zero L is one of the two equivalence classes of $\mathcal{C}(L)$. We say that L is *oriented* when an orientation is chosen.

The idea of dividing in two classes the set of correctors of a Fredholm operator is already present in Mawhin (see [10] and [11]) and Pejsachowicz-Vignoli ([13]). In [3] the reader can find a comparison between these approach and our one.

According to Definition 2.1, an oriented operator L is a pair (L, ω), where $L : E \to F$ is a Fredholm operator of index zero and ω is one of the two equivalence classes of $\mathcal{C}(L)$. However, to simplify the notation, we shall not use different symbols to distinguish between oriented and nonoriented operators (unless it is necessary).

Given an oriented operator $L : E \to F$, the elements of its orientation will be called the *positive correctors* of L.

Any isomorphism L admits a special orientation, namely the equivalence class containing the trivial operator 0. We shall refer to this orientation as the *natural orientation* $\nu(L)$ of L. Moreover, if an isomorphism L is actually oriented, we define its sign as follows: $\operatorname{sgn} L = 1$ if the trivial operator 0 is a positive corrector of L (i.e. if the orientation of L coincides with $\nu(L)$), and $\operatorname{sgn} L = -1$ otherwise.

Let us now see a property which may be regarded as a sort of reduction of the orientation of an operator to the orientation of its restriction to a convenient pair of subspaces of the domain and codomain. This will be useful in the next section where we will show that our degree is exactly the Brouwer degree of a suitable restriction to finite dimensional manifolds.

Let E and F be two real vector spaces and let $L : E \to F$ be an algebraic Fredholm operator of index zero. Let F_1 be a subspace of F which is transverse to L (that is, $F_1 + \operatorname{Range} L = F$). Observe that in this case the restriction L_1 of L to the pair of spaces $E_1 = L^{-1}(F_1)$ (as domain) and F_1 (as codomain) is again a Fredholm operator of index zero. We can prove that an orientation of L gives an orientation of L_1, and vice versa. To see this, let E_0 be a complement of E_1 in E and split E and F as follows: $E = E_0 \oplus E_1$, $F = L(E_0) \oplus F_1$. Thus L can be represented by a matrix

$$\begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix},$$

where L_0 is an isomorphism. Then our claim follows immediately from the fact that any linear operator $A : E \to F$, represented by

$$\begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix},$$

is a corrector of L if and only if A_1 is a corrector of L_1. According to the above argument, it is convenient to introduce the following definition.

Definition 2.2 Let $L : E \to F$ be a Fredholm operator of index zero between real vector spaces, let F_1 be a subspace of F which is transverse to L, and denote by L_1 the restriction of L to the pair of spaces $L^{-1}(F_1)$ and F_1. Two orientations, one of L and one of L_1, are said to be *correlated* (or *one induced by the other*) if there exist a projector $P : E \to E$ onto E_1 and a positive corrector A_1 of L_1 such that the operator $A = JA_1P$ is a positive corrector of L, where $J : F_1 \hookrightarrow F$ is the inclusion.

The above concept of orientation of an algebraic Fredholm operator of index zero $L : E \to F$ does not require any topological structure on E and F, which are supposed to be just real vector spaces. However, in the context of Banach spaces, an orientation of L can induce an orientation on the operators sufficiently close to L. Precisely, assume that $L : E \to F$ is a bounded Fredholm operator of index zero between real Banach spaces. Given an orientation of L and a positive bounded corrector A of L (whose existence is ensured by the Hahn-Banach theorem), A is still a corrector of any operator L' in a convenient neighborhood U of L in the Banach space of bounded linear operators from E into F. Therefore, if L' belongs to U, then it can be oriented by choosing A as a positive corrector.

From now on, unless otherwise specified, E and F will denote real Banach spaces, $L(E, F)$ the Banach space of bounded linear operators from E into F and $Iso(E, F)$ the open subset of isomorphisms.

For the sake of simplicity, in the context of Banach spaces, the set of continuous correctors of a Fredholm operator of index zero L will be still denoted by $\mathcal{C}(L)$, as in the algebraic case, instead of $\mathcal{C}(L) \cap L(E, F)$. Therefore, from now on, by a corrector of L we shall actually mean a continuous corrector. It is clear that an orientation of L can be regarded as an equivalence class of continuous correctors of L.

We recall that the set $\Phi(E, F)$ of the Fredholm operators from E into F is open in $L(E, F)$, and the integer valued map $\operatorname{ind} : \Phi(E, F) \to \mathbf{Z}$ is continuous. Consequently, given $n \in \mathbf{Z}$, the set $\Phi_n(E, F)$ of Fredholm operators of index n is an open subset of $L(E, F)$.

We introduce now a concept of orientation for a continuous map into $\Phi_0(E,F)$.

Definition 2.3 Let Λ be a topological space and $h : \Lambda \to \Phi_0(E,F)$ a continuous map. An *orientation* of h is a continuous choice of an orientation $\alpha(\lambda)$ of $h(\lambda)$ for each $\lambda \in \Lambda$; where "continuous" means that for any $\lambda \in \Lambda$ there exists $A_\lambda \in \alpha(\lambda)$ which is a positive corrector of $h(\lambda')$ for any λ' in a neighborhood of λ. A map is *orientable* if it admits an orientation and *oriented* when an orientation has been chosen.

Remark 2.4 We can give a notion of orientation for subsets of $\Phi_0(E,F)$. Precisely, a subset $\mathcal{A}$ of $\Phi_0(E,F)$ is said to be *orientable* if so is the inclusion $i : \mathcal{A} \hookrightarrow \Phi_0(E,F)$.

Any orientable map $h : \Lambda \to \Phi_0(E,F)$ admits at least two orientations. In fact, if h is oriented by α, reverting this orientation at all $\lambda \in \Lambda$, one gets what we call the *opposite orientation* α_- of h. Observe also that two orientations of h coincide in an open subset of Λ, and for the same reason the set in which two orientations of h are opposite one to the other is open. Therefore, if Λ is connected, two orientations of h are either equal or one is opposite to the other. Thus, in this case, an orientable map h admits exactly two orientations.

Remark 2.5 An orientation of a continuous map $h : \Lambda \to \Phi_0(E,F)$ can be given by assigning a family $\{(U_i, A_i) : i \in \mathcal{I}\}$, called an *oriented atlas* of h, satisfying the following properties:

- $\{U_i : i \in \mathcal{I}\}$ is an open covering of Λ;
- given $i \in \mathcal{I}$, A_i is a corrector of any $h(\lambda)$, $\forall \lambda \in U_i$;
- if $\lambda \in U_i \cap U_j$, then A_i is $h(\lambda)$-equivalent to A_j.

Let us now define a notion of orientation for nonlinear Fredholm maps of index zero between open subsets of Banach spaces.

Definition 2.6 Let Ω be an open subset of E and $f : \Omega \to F$ be Fredholm of index zero. An *orientation* of f is an orientation of $Df : x \mapsto Df(x) \in \Phi_0(E,F)$, and f is *orientable* (resp. *oriented*) if so is Df according to Definition 2.3.

Remark 2.7 A local diffeomorphism $f : \Omega \to F$ between two open subsets of E and F, can be oriented with the orientation ν defined by $0 \in \nu(Df(x)), \forall x \in \Omega$. We will refer to this orientation as the *natural orientation* of the local diffeomorphism f.

We have previously seen that the orientation of an algebraic Fredholm operator of index zero can be regarded as the orientation of the operator to a suitable pair of subspaces. Analogously, let $f : \Omega \to F$ be oriented and let F_1 be a subspace of F which is transverse to f; that is $F_1 + \operatorname{Range} Df(x) = F$, $\forall x \in f^{-1}(F_1)$. It is known that, in this case, $M_1 = f^{-1}(F_1)$ is a submanifold of Ω and the restriction f_1 of f to M_1 (as domain) and F_1 (as codomain) is again a Fredholm map of index zero. Moreover, for any $x \in M_1$, the tangent space to M_1 at x, T_xM_1 coincides with $Df(x)^{-1}(F_1)$ (see for example [1] and [8] for general results about transversality). Therefore, according to Definition 2.2, given any $x \in M_1$, the orientation of $Df(x) \in \Phi_0(E,F)$ induces an orientation on its restriction $Df_1(x) \in \Phi_0(T_xM_1, F_1)$, which is the derivative of f_1 at x. Such a collection of orientations of $Df_1(x)$, $x \in M_1$, is actually an orientation of $f_1 : M_1 \to F_1$ that, from now on, we shall call the *orientation on f_1 induced by f*. Observe also that, given $x \in M_1$, $Df(x) : E \to F$ is an isomorphism if and only if so is $Df_1(x) : T_xM_1 \to F_1$ and, with the induced orientation, one has $\operatorname{sgn} Df(x) = \operatorname{sgn} Df_1(x)$; and this will imply one of the fundamental properties of the degree (Reduction property).

Definition 2.6 can be slightly modified in order to obtain a notion of orientation for continuous homotopies of Fredholm maps. Let Ω be open in E. We say that a continuous map $H : \Omega \times [0,1] \to F$ is a *continuous family of Fredholm maps of index zero* if it is differentiable with respect to the first variable, the partial derivative $D_1H(x,t) : E \to F$ is a Fredholm operator of index zero for any $(x,t) \in \Omega \times [0,1]$, and it depends continuously on (x,t).

We will say that a continuous family of Fredholm maps of index zero H is *orientable* if so is the map $D_1H : (x,t) \mapsto D_1H(x,t)$ and its orientation is an *orientation* of D_1H.

Clearly, given an oriented continuous family of Fredholm maps of index zero, $H : \Omega \times [0,1] \to F$, any partial map $H_t := H(\cdot,t)$ is an oriented map from Ω into F, according to Definition 2.6. One could actually show that, given a continuous family of Fredholm maps of index zero $H : \Omega \times [0,1] \to F$, if H_0 is orientable, then all the partial maps H_t are orientable as well, and an orientation of H_0 induces a unique orientation on any H_t which makes H oriented (see [3]).

The properties of this concept of orientation are treated in depth in [2] and [3]. Here we limit ourselves to some remarks. If E and F are two finite dimensional (of the same dimension) Banach spaces, $\Phi_0(E,F)$ coincides with $L(E,F)$. In this case, one can prove that any continuous map with values in $L(E,F)$ is orientable (moreover, recalling Remark (2.4), $L(E,F)$ is orientable).

If E and F are infinite dimensional, it is proved that continuous maps defined on simply connected and locally path connected topological spaces into $\Phi_0(E,F)$ are orientable. However, because of the topological structure of $\Phi_0(E,F)$, we can find nonorientable maps with values in $\Phi_0(E,F)$. This is based on the fact that there exist Banach spaces E whose linear group $GL(E)$ is connected. For example, an interesting result of Kuiper (see [9]) asserts that the linear group $GL(E)$ of an infinite dimensional separable Hilbert E space is contractible. It is also known that $GL(l^p)$, $1 \le p < \infty$, and $GL(c_0)$ are contractible as well. There are, however, examples of infinite dimensional Banach spaces whose linear group is disconnected (see [6], [12] and references therein). When $GL(E)$ is connected, then it is possible to define nonorientable maps with values into $\Phi_0(E)$.

Theorem 2.8 *Assume $Iso(E,F)$ is nonempty and connected. Then there exists a nonorientable map $\gamma : S^1 \to \Phi_0(E,F)$ defined on the unit circle of $\mathbf{R}^2$.*

Proof. We give here a sketch of the proof (see [3] for more details). Let S^1_+ and S^1_- denote, respectively, the two arcs of S^1 with nonnegative and nonpositive second coordinate. One can prove that there exists an oriented open connected subset U of $\Phi_0(E,F)$ containing two points in $Iso(E,F)$, say L_- and L_+, such that $\operatorname{sign} L_- = -1$ and $\operatorname{sign} L_+ = 1$. Let $\gamma_+ : S^1_+ \to U$ be a path such that $\gamma_+(-1,0) = L_-$ and $\gamma_+(1,0) = L_+$. Since $Iso(E,F)$ is an open connected subset of $L(E,F)$, it is also path connected. Therefore there exists a path $\gamma_- : S^1_- \to Iso(E,F)$ such that $\gamma_-(-1,0) = L_-$ and $\gamma_-(1,0) = L_+$. Define $\gamma : S^1 \to \Phi_0(E,F)$ by

$$\gamma(x,y) = \begin{cases} \gamma_+(x,y) & \text{if } y \ge 0 \\ \gamma_-(x,y) & \text{if } y \le 0 \end{cases}$$

and assume, by contradiction, it is orientable. This implies that also the image $\gamma(S^1)$ of γ is orientable, with just two possible orientations. Orient, for example, $\gamma(S^1)$ with the unique orientation compatible with the oriented subset U of $\Phi_0(E,F)$. Thus, being $\gamma(S^1_+) \subset U$, we get $\operatorname{sign} L_- = -1$ and $\operatorname{sign} L_+ = 1$. On the other hand, since the image of γ_- is contained in $Iso(E,F)$, it follows $\operatorname{sign} L_- = \operatorname{sign} L_+$, which is a contradiction. □

Remark 2.9 The above result can be used to prove that $\Phi_0(E,F)$ is not simply connected, since it can be verified that simply connected and locally path connected subsets of $\Phi_0(E,F)$ are orientable (see [3]).

By means of this example of nonorientable map it is possible to define an example of nonorientable Fredholm map of index zero from an open subset of a Banach space into another Banach space. To make this paper not too long, we omit this construction (which can be found in [3]).

3 Degree for oriented maps

As previously, E and F stands for two real Banach spaces. Let Ω be open in E and $f : \Omega \to F$ be oriented. Given an element $y \in F$, we call the triple (f,Ω,y) *admissible* if $f^{-1}(y)$ is compact. A triple (f,Ω,y) is called *strongly admissible* provided that f admits a proper continuous extension to the closure

$\overline{\Omega}$ of Ω (again denoted by f), and $y \notin f(\partial\Omega)$. Clearly any strongly admissible triple is also admissible. Moreover, if (f, Ω, y) is strongly admissible and U is an open subset of Ω such that $U \cap f^{-1}(y)$ is compact, then (f, Ω, y) is strongly admissible as well.

Our aim here is to define a map, called *degree*, which to every admissible triple (f, Ω, y) assigns an integer, $\deg(f, \Omega, y)$, in such a way that the following five properties hold:

i) (*Normalization*) If $f : \Omega \to F$ is a naturally oriented diffeomorphism and $y \in f(\Omega)$, then

$$\deg(f, \Omega, y) = 1.$$

ii) (*Additivity*) Given an admissible triple (f, Ω, y) and two open subsets U_1, U_2 of Ω, if $U_1 \cap U_2 = \emptyset$ and $f^{-1}(y) \subset U_1 \cup U_2$, then (f, U_1, y) and (f, U_2, y) are admissible and

$$\deg(f, \Omega, y) = \deg(f, U_1, y) + \deg(f, U_2, y).$$

iii) (*Topological Invariance*) If (f, Ω, y) is admissible, $\varphi : U \to \Omega$ is a naturally oriented diffeomorphism from an open subset of a Banach space onto M and $\psi : f(\Omega) \to V$ is a naturally oriented diffeomorphism from $f(\Omega)$ onto an open subset of a Banach space, then

$$\deg(f, M, y) = \deg(\psi f \varphi, U, \psi(y)),$$

where $\psi f \varphi$ is oriented with orientation induced the orientations of ψ, f and φ.

iv) (*Reduction*) Let $f : \Omega \to F$ be oriented and let F_1 be a subspace of F which is transverse to f. Denote by f_1 the restriction of f to the manifold $M_1 = f^{-1}(F_1)$ with the orientation induced by f. Then

$$\deg(f, \Omega, y) = \deg(f_1, M_1, y),$$

provided that $f^{-1}(y)$ is compact.

v) (*Homotopy Invariance*) Let $H : \Omega \times [0, 1] \to F$ be an oriented continuous family of Fredholm maps of index zero, and let $y : [0, 1] \to F$ be a continuous path. If the set

$$\{(x, t) \in \Omega \times [0, 1] : H(x, t) = y(t)\}$$

is compact, then $\deg(H_t, \Omega, y(t))$ is well defined and does not depend on $t \in [0, 1]$.

In the sequel we shall refer to i)$-$v) as the *fundamental properties* of degree.

We define first our notion of degree in the special case when (f, Ω, y) is a *regular triple*; that is when (f, Ω, y) is admissible and y is a regular value for f in Ω. This implies that $f^{-1}(y)$ is a compact discrete set and, consequently, finite. In this case our definition is similar to the classical one in the finite dimensional case. Namely

$$\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} Df(x), \tag{3.1}$$

where, we recall, $\operatorname{sgn} Df(x) = 1$ if the trivial operator is a positive corrector of the oriented isomorphism $Df(x)$, and $\operatorname{sgn} Df(x) = -1$ otherwise.

It is easy to prove that the first four fundamental properties of the degree hold for the class of regular triples, and we will prove below that they are still valid in the general case.

A straightforward consequence of the Additivity is the following property that we shall need (for the special case of regular triples) in the proof of Lemma 3.2 below.

vi) (*Excision*) If (f, Ω, y) is admissible and U is an open neighborhood of $f^{-1}(y)$, then

$$\deg(f, \Omega, y) = \deg(f, U, y).$$

In order to define the degree in the general case we will prove that, given any admissible triple (f, Ω, y), if U_1 and U_2 are sufficiently small open neighborhoods of $f^{-1}(y)$, and $y_1, y_2 \in F$ are two regular values for f, sufficiently close to y, then

$$\deg(f, U_1, y_1) = \deg(f, U_2, y_2).$$

Let us show first that the degree of a regular triple (f, Ω, y) can be viewed as the Brouwer degree of the restriction of f to a convenient pair of finite dimensional oriented manifolds.

Consider an admissible triple (f, Ω, y) (for the moment we do not assume y to be a regular value of f). Since $f^{-1}(y)$ is compact, there exists a finite dimensional subspace F_0 of F and an open subset U of $f^{-1}(y)$ in which f is transverse to F_0. Consequently, $M_0 = f^{-1}(F_0) \cap U$ is a differentiable manifold of the same dimension as F_0, f is transverse to F_0 in U, and the restriction $f_0 : M_0 \to F_0$ of f is an oriented map (with orientation induced by f). Since F_0 is a finite dimensional vector space and f_0 is orientable, M_0 is orientable as well. Therefore, the orientation of f_0 induces a pair of orientations of M_0 and F_0, up to an inversion of both of them (which does not effect the Brouwer degree of f_0 at y). When a pair of these orientations are chosen, we say that M_0 and F_0 are *oriented according to* f.

Before stating the following lemma, we point out that if a Fredholm map $f : \Omega \to F$ is transverse to a subspace F_0 of F, then an element $y \in F_0$ is a regular value for f if and only if it is a regular value for the restriction $f_0 : f^{-1}(F_0) \to F_0$ of f.

Lemma 3.1 *Let (f, Ω, y) be a regular triple and let F_0 be a finite dimensional suspace F, containing y and transverse to f. Then $M_0 = f^{-1}(F_0)$ is an orientable manifold of the same dimension as F_0. Moreover, orienting M_0 and F_0 according to f, the Brouwer degree* $\deg_B(f_0, M_0, y)$ *of f_0 at y coincides with* $\deg(f, \Omega, y)$.

The proof is not difficult and can be found in [2]. As a consequence of this lemma we get the following result which is crucial in our definition of degree.

Lemma 3.2 *Let (f, Ω, y) be a strongly admissible triple. Given two neighborhoods U_1 and U_2 of $f^{-1}(y)$, there exists a neighborhood V of y such that for any pair of regular values $y_1, y_2 \in V$ one has*

$$\deg(f, U_1, y_1) = \deg(f, U_2, y_2).$$

Proof. Since (f, Ω, y) is strongly admissible, f is actually well defined and proper on the closure $\overline{\Omega}$ of Ω. Let U_1 and U_2 be two open neighborhoods of $f^{-1}(y)$ and put $U = U_1 \cap U_2$. Since proper maps are closed, there exists a neighborhood V of y with $V \cap f(\overline{\Omega} \setminus U) = \emptyset$. Without loss of generality we may assume that V is convex. With an argument similar to that used just before Lemma 3.1 one can show the existence of a finite dimensional subspace F_0 of F containing y_1 and y_2 and transverse to f in a convenient neighborhood $W \subset U$ of the compact set $f^{-1}(S)$, where S is the segment joining y_1 and y_2. Thus, $M_0 = f^{-1}(F_0) \cap W$ is a finite dimensional manifold of the same dimension as F_0, and they turn out to be oriented according to f (up to an inversion of both orientations). Denote by f_0 the restriction of f to M_0 (as domain) and F_0 (as codomain). From Lemma 3.1 we obtain

$$\deg_B(f_0, M_0, y_1) = \deg(f, W, y_1),$$

$$\deg_B(f_0, M_0, y_2) = \deg(f, W, y_2).$$

On the other hand, since $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are contained in W, by the Excision property for regular triples, we get $\deg(f, U_1, y_1) = \deg(f, W, y_1)$ and $\deg(f, U_2, y_2) = \deg(f, W, y_2)$. Therefore, it remains to show that $\deg_B(f_0, M_0, y_1) = \deg_B(f_0, M_0, y_2)$.

Consider now the path $y(\cdot) : [0, 1] \to F_0$ given by $t \mapsto ty_1 + (1 - t)y_2$. Clearly, the set

$$\{x \in M_0 : f_0(x) = y(t) \text{ for some } t \in [0, 1]\}$$

coincides with $f^{-1}(S)$, which is compact. Therefore, from the homotopy invariance of the Brouwer degree we get

$$\deg_B(f_0, M_0, y_1) = \deg_B(f_0, M_0, y_2),$$

and the result is proved. □

Lemma 3.2 justifies the following definition of degree for general admissible triples.

Definition 3.3 Let (f, Ω, y) be admissible and let U be an open neighborhood of $f^{-1}(y)$ such that $\overline{U} \subset \Omega$ and f is proper on $\overline{U}$. Put

$$\deg(f, M, y) := \deg(f, U, z),$$

where z is any regular value for f in U, sufficiently close to y.

To justify the above definition we point out that the existence of regular values for $f|_U$ which are sufficiently close to y can be directly deduced from Sard's Lemma. In fact, as previously observed, one can reduce the problem of finding regular values of a Fredholm map to its restriction to a convenient pair of finite dimensional manifolds.

Theorem 3.4 *The degree satisfies the above five fundamental properties.*

Proof. The first four properties are an easy consequence of the analogous ones for regular triples. Let us prove the Homotopy Invariance. Consider an oriented continuous family of Fredholm maps of index zero $H : \Omega \times [0, 1] \to F$ and let $y : [0, 1] \to F$ be a continuous path in F. Assume that the set

$$C = \{x \in \Omega : H(x, t) = y(t) \text{ for some } t \in [0, 1]\}.$$

is compact. Since H is locally proper, there exists an open neighborhood U of C in Ω such that H is proper on $\overline{U} \times [0, 1]$. Consequently, $H_t = H(\cdot, t)$ is proper on $\overline{U}$ for all $t \in [0, 1]$, and, by the definition of degree,

$$\deg(H_t, \Omega, y(t)) = \deg(H_t, U, y(t)), \ \forall t \in [0, 1].$$

We need to prove that the function $\sigma(t) = \deg(H_t, U, y(t))$ is locally constant. Let τ be any point in $[0, 1]$. Since H is proper on $\overline{U} \times [0, 1]$ and $y(\tau) \notin H_\tau(\partial U)$, one can find an open connected neighborhood V of $y(\tau)$ and a compact neighborhood J of τ (in $[0, 1]$) such that $y(t) \in V$ for $t \in J$ and $H(\partial U \times J) \cap V = \emptyset$. Thus, if z is any element of V, one has $\sigma(t) = \deg(H_t, U, z)$ for all $t \in J$. To compute this degree we may therefore assume that z is a regular value for H_τ in U, so that $H_\tau^{-1}(z)$ is a finite set $\{x_1, x_2, \ldots, x_n\}$ and the partial derivatives $D_1H(x_i, \tau)$, $i = 1, 2, \ldots, n$, are all nonsingular. Consequently, given any x_i in $H_\tau^{-1}(z)$, the Implicit Function Theorem ensures that $H^{-1}(z)$, in a neighborhood $W_i \times J_i$ of (x_i, τ), is the graph of a continuous curve $\gamma_i : J_i \to \Omega$. Since H is proper in $\overline{U} \times J$ (recall that J is compact) and $z \notin H(\partial U \times J)$, the set $H^{-1}(z) \cap (U \times J)$ is compact. This implies the existence of a neighborhood J_0 of τ such that for $t \in J_0$ one has

$$H_t^{-1}(z) = \{\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)\}.$$

Moreover, by the continuity of D_1H, we may assume that z is a regular value for any H_t, $t \in J_0$. Finally, since H is oriented, the continuity assumption in the definition of orientation implies that, for any i,

$\operatorname{sgn} D_1 H(\gamma_i(t), t)$ does not depend on $t \in J_0$, and from the definition of degree of a regular triple we get that $\sigma(t)$ is constant in J_0. □

The notions of orientation and topological degree we have defined for nonlinear Fredholm maps of index zero between Banach spaces can be extended to Fredholm maps of index zero between Banach manifolds. This is treated in [2] and [3].

The degree theory we have introduced can be applied to obtain some bifurcation results. In a forthcoming paper, written in collaboration with M. Furi, this topic will be treated in depth. Here we are going to state the following global bifurcation result.

Let Ω be an open subset of E and $H : \Omega \times \mathbf{R} \to F$ be a continuous family of Fredholm maps of index zero. Assume that $H(0, \lambda) = 0$ for all $\lambda \in \mathbf{R}$. Let us recall that a real number λ_0 is called a *bifurcation point* if any neighborhood of $(0, \lambda_0)$ contains pairs (x, λ) such that $H(x, \lambda) = 0$ and $x \neq 0$. Those solutions of $H(x, \lambda) = 0$ belonging to $\{0\} \times \mathbf{R}$ are usually called *trivial* whereas the other ones are said to be *nontrivial*.

Theorem 3.5 *Let $H : \Omega \times \mathbf{R} \to F$ be a continuous family of Fredholm maps of index zero with $H(0, \lambda) = 0$ for all $\lambda \in \mathbf{R}$. Let also λ_1, $\lambda_2 \in \mathbf{R}$ be such that $\operatorname{sign} D_1 H(0, \lambda_1) \operatorname{sign} D_1 H(0, \lambda_2) = -1$. Then there exists a connected set of nontrivial solutions of $H(x, \lambda) = 0$ whose closure in $\Omega \times \mathbf{R}$ has nonempty intersection with $\{0\} \times [\lambda_1, \lambda_2]$ and either is not compact or contains a point $(0, \overline{\lambda})$ with $\overline{\lambda} \notin [\lambda_1, \lambda_2]$.*

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