Geometry in the Large

July 15, 2016

1 Problem Session

Problem 1. *What can we say about curvature-homogeneous Riemannian 3-manifolds? What about when assuming completeness or compactness? Can they be classified?*

Curvature-homogeneous means that the eigenvalues of the Riemannian curvature are constant. The higher dimensional cases are likely easier.

Problem 2. *Find a smooth (compact) manifold with 2 metrics g_0, g_1 such that Ricci $(g_0) = 0$ and Ricci $(g_1) = \lambda g_1$ for some $\lambda \neq 0$. (So it has both a Ricci-flat metric and an Einstein (but not Ricci-flat) metric.*

It is not even known if S^4 has a Ricci-flat metric. There do exist manifolds with a Ricci-flat metric and also a positive scalar curvature metric.

Problem 3. *Does any homogeneous space diffeomorphic to $\mathbb{R}^n$ ever admit a closed geodesic?*

This is unknown even for 3-manifolds. The stating example to consider is the universal cover of $SL(2, \mathbb{R})$.

Problem 4. *Does the space of closed $Sp\,(2)Sp\,(1)$-structures on an 8-manifold satisfy the h-principle? What about the same question for half-flat $SU(3)$ structures on 6-manifolds?*

$SU(3)$ structures are equivalent to specifying ω, γ where ω is a 2-form which point-wise looks like $dq^1 \wedge d\bar{q}^1 + dq^2 \wedge d\bar{q}^2 + dq^3 \wedge d\bar{q}^3$ and γ is a 3-form that point-wise looks like $dz^1 \wedge dz^2 \wedge dz^3$. Half-flat means that $d(\omega^2) = 0$ and $(\text{Im }\gamma) = 0$

An $\text{Sp}\,(2)\text{Sp}\,(1)$ structure is equivalent to specifying a 4-form Φ such that Φ point-wise looks like $\frac{1}{6}(\omega_I^2 + \omega_J^2 + \omega_K)^2$ where I, J, K are the standard quaternion structure on $\mathbb{H}^2$.

Problem 5. *Classify compact 3-manifolds M with metrics such that the Riemannian curvature tensor has a non-trivial kernel everywhere. This means that there is some non-zero $X \in T_pM$ for each $p \in M$ with $R(X, \cdot) = 0$.*

It is known that outside flat points it locally splits with a Euclidean factor. If the set of flat points is small enough, then it is a graph manifold.

Problem 6. *Classify the Riemannian submersions $\pi : \mathbb{E}^n \to M^{n-r}$ for $r > 0$. Are they all homogeneous?*

Earlier work by Gromov and Walschap was not conclusive for $r > 3$.

Problem 7. *Is the moduli space of all Zoll metrics on S^2 connected?*

A metric on S^2 is called Zoll if every geodesic is simple and closed. It is known (Zoll, Guillemin) that the moduli space is infinite dimensional even if you assume rotational symmetry.

Problem 8. *Which conformal classes of metrics on S^6 do not admit a compatible complex structure?*

Since compatibility with complex structures places point-wise conditions on the Weyl tensor, it's reasonable to investigate the conformal stuctures. LeBrun showed that (S^6, g_{round}), and all conformal structures sufficiently close to g_{round}, do not admit such a structure. Another question is to classify what "sufficiently close" means.

Problem 9. *Is there an algebraic proof that homogeneous and Ricci-flat implies flat?*

This appears to be a hard problem. Known proofs are non-algebraic, see for example in Besse "Einstein Manifolds" which uses the splitting principle.

Problem 10. *Given two closed hyperbolic 3-manifolds M_1, M_2 and $\epsilon > 0$, do there exist finite covers $\tilde{M}_1, \tilde{M}_2$ with subsets $S_i \subseteq \tilde{M}_i$ so that S_1 is $(1+\epsilon)$-Lipshitz isomorphic to S_2 and $vol(\tilde{M}_i \setminus S_i) < \epsilon vol(\tilde{M}_i)$.*

See the paper by Kahn-Markovič about the Ehrenpreis conjecture.