Homogeneity for Riemannian Quotient Manifolds

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Background

- (M, ds^2): connected simply connected Riemannian homogeneous space
- $M = G/K$ where $G = \mathcal{I}(M, ds^2)$ is the isometry group
- $M \to \Gamma\backslash M$ Riemannian covering
- In other words Γ is a discrete subgroup of G and if $\gamma \in \Gamma$ has a fixed point on M then $\gamma = 1$.
- Problem: when is $\Gamma\backslash M$ homogeneous?
- First step: If $\Gamma\backslash M$ is homogeneous then every $\gamma \in \Gamma$ is an isometry of constant displacement.
- Example: if $\Gamma\backslash\mathbb{R}^n$ is homogeneous then Γ consists of pure translations so $\Gamma\backslash\mathbb{R}^n$ is the product of a torus and an euclidean space.

More Background

- A less trivial example:
- (M, ds^2) has every sectional curvature $\leqq 0$
- $(M, ds^2) = (M_1, ds_1^2) \times (M_2, ds_2^2)$ (de Rham) where
  - (M_1, ds_1^2) is the flat factor in the de Rham decomposition
  - (M_2, ds_2^2) is the product of the irreducible factors
- $M \to \Gamma\backslash M$ universal Riemannian covering
- Then the following are equivalent.
  - $\Gamma\backslash M$ is homogeneous
  - Every $\gamma \in \Gamma$ is an isometry of constant displacement
  - Every $\gamma \in \Gamma$ is an isometry of bounded displacement
  - Every $\gamma \in \Gamma$ is just a pure translation along the Euclidean factor (M_1, ds_1^2) of (M, ds^2)

Yet More Background

- A nontrivial example:
- $S^{n-1} \subset \mathbb{R}^n$ usual round sphere of dimension $n-1$ in $\mathbb{R}^n$
- Γ finite group of fixed point free isometries of S^{n-1}, in other words $\Gamma \backslash S^{n-1}$ is a spherical space form
- Suppose that $\Gamma \backslash S^{n-1}$ is Riemannian homogeneous
- Let L denote the normalizer of Γ in $\mathcal{I}(S^{n-1}) = O(n)$
- Then L^0 centralizes Γ and is transitive on S^{n-1}
- Schur's Lemma: L^0 is contained in the multiplicative group of a real division algebra $\mathbb{A} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. So
- (1) If $\mathbb{A} = \mathbb{R}$: $\Gamma \subset \{\pm 1\}$
  (2) If $\mathbb{A} = \mathbb{C}$: Γ is cyclic of order > 2
  (3) If $\mathbb{A} = \mathbb{H}$: Γ is binary dihedral, binary tetrahedral, binary octahedral or binary icosahedral

Constant Curvature

- $M \to \Gamma\backslash M$ universal Riemannian covering
- Theorem. Suppose that M is complete and has constant sectional curvature K. Then $\Gamma\backslash M$ is Riemannian homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement
- For $K < 0$: the less–trivial example says that $\Gamma\backslash M$ is Riemannian homogeneous if and only if $\Gamma = \{1\}$
- For $K = 0$: this is covered by the trivial example
- For $K > 0$: this involves some nontrivial finite group theory based on (i) $\gamma \neq \pm I$ has constant displacement if and only if it has eigenvalues $\{\lambda, \overline{\lambda}; \ldots ; \lambda, \overline{\lambda}\}$ and (ii) an induction involving binary polyhedral and $SL(2; \mathbb{Z}_p)$ groups

Riemannian Symmetric

- $M \to \Gamma\backslash M$ universal Riemannian covering
- Theorem. Suppose that M is a Riemannian symmetric space. Then $\Gamma\backslash M$ is Riemannian homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement
- First reduction: to case where M is irreducible
- Second reduction: to case where M is compact irreducible
- Compact irreducible case 1: M is a group manifold
- Compact irreducible case 2: $M = G/K$ with G compact simple classical
- Compact irreducible case 3: $M = G/K$ with G compact simple exceptional

Finsler Symmetric

- $M \to \Gamma\backslash M$ universal Finsler covering
- Theorem. Suppose that (M, F) is a Finsler symmetric space. Then $\Gamma\backslash M$ is Finsler homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement
- (M, F) is Berwald and $(M, F) = (M_0, F_0) \times (M_1, F_1) \times (M_2, F_2)$ with (M_0, F_0) Minkowski, (M_1, F_1) compact type, (M_2, F_2) noncompact type
- First reduction: constant displacement isometries decompose so reduced to cases $(M, F) = (M_i, F_i)$
- Second reduction: take care of Minkowski and noncompact type cases
- Third reduction: reduce to irreducible Riemannian cases

Dichotomy – Unbounded Cases

- Putting Euclidean factors aside now there are two disparate cases
- Unbounded: here the evidence is that isometries of bounded displacement are ordinary translations along the Euclidean factor
  - Riemannian manifolds of sectional curvature $\leqq 0$
  - Riemannian manifolds without focal points
  - Riemannian manifolds homogeneous under a semisimple group with no compact factor
  - Riemannian manifolds homogeneous under an exponential solvable Lie group of isometries

Dichotomy – Bounded Cases

- Bounded: here much of the progress on the conjecture has been case by case verification
  - Riemannian or Finsler symmetric spaces
  - Compact homogeneous with a certain Weyl group condition, e.g. Stieffel manifolds
  - Twistor bundles over Grassmann manifolds, hermitian or quaternionic symmetric spaces, nearly-Kähler (3–symmetric) spaces, the 5–symmetric E_8/A_4A_4 , ...
- Example: $M = G/K_1$ fibered over $N = G/K_1K_2$.
  - M and N carry normal Riemannian metrics from G
  - Γ: finite subgroup of $Z_G K_2$
  - Then Γ acts on M: by $(z, k_2)(gK_1) = zgk_2^{-1}K_1$
  - This is isometric and centralizes the (transitive) isometric action of G on M so $\Gamma\backslash M$ is homogeneous

Idea of Proof: Sectional Curvature $\leqq 0$

- M is a complete simply connected manifold with every sectional curvature $\leqq 0$
- γ is an isometry of M of bounded displacement
- $t \mapsto \sigma(t)$ geodesic $\Rightarrow d_\gamma(t) = dist(\sigma(t), \gamma(\sigma(t)))$ bounded
- Geodesic segments $\overline{\sigma(t), \gamma(\sigma(t))}$ fill out a flat totally geodesic strip in M
- So γ is ordinary translation along the euclidean factor of the de Rham decomposition of M
- Theorem. Suppose that M is homogeneous and $M \to \Gamma \backslash M$ is a Riemannian covering. Then $\Gamma \backslash M$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement. In that case Γ is a discrete group of ordinary translations along the euclidean factor of M.

Idea of Proof: Group Structure

- If G is noncompact simple and α is a bounded automorphism then $\alpha = 1$. Essentially the same argument as for sectional curvature $\leqq 0$: distinct 1–parameter subgroups of hyperbolic type must diverge apart,
- If $M = G/K$, G semisimple with no compact factor, and γ is a bounded isometry then $\gamma = 1$. This uses $\mathcal{I}(M)^0 = \{xK \mapsto gxu^{-1}K \mid g \in G, u$ isometry $, u$ normalizes K
- If $M = G/K$, G exponential solvable, and γ is a bounded isometry then $\alpha = 1$. This uses some basic unipotent group theory, and includes the case of nilpotent G.

Idea of Proof: Riemannian Symmetric

- M: complete simply conn. Riemannian symmetric space
- γ is an isometry of constant displacement
- $\gamma = \gamma_0 \times \gamma_1 \times \cdots \times \gamma_r$ along the de Rham decomposition $M = M_0 \times M_1 \times \cdots \times M_r$, each γ_i constant displ. on M_i
- So can assume that M is compact and irreducible
- $\Gamma \subset \mathcal{I}(M)$, every $\gamma \in \Gamma$ const. displ, $M \to \Gamma \backslash M$ covering
- If $M = (L \times L)/(diag\, L)$ group manifold then Γ is $\mathcal{I}(M)$-conjugate to a subgroup of $L \times \{1\}$.
- If $M = G/K$ with G simple: run through the classification
- Theorem. Let $M \to \Gamma \backslash M$ be a Riemannian covering. Then $\Gamma \backslash M$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement.

Idea of Proof: Twistor Bundles

- Using Borel – de Siebenthal theory one obtains a list of all bundles $\widetilde{M} = G/K_1 \to G/K = M$ where
  - G compact simple, $K = K_1 K_2$ maximal rank,
  - $K \simeq K_1 \times K_2$ with $\dim \mathfrak{k}_1 \neq 0 \neq \dim \mathfrak{k}_2$
  - M and $\widetilde{M}$ are normal homogeneous spaces of G
  - $\widetilde{M} \to M$ is a Riemannian submersion
- $\mathcal{I}(\widetilde{M})^0 = G \times r(K_2) = \{xK_1 \mapsto gxk_2^{-1}K_1 \mid g \in G, k_2 \in K_2\}$
- $\mathcal{I}(\widetilde{M}) = \bigcup_{\alpha \in \mathrm{Out}(G,K_1)} \mathcal{I}(\widetilde{M})^0 \alpha$ where $\mathrm{Out}(\bullet) = \mathrm{Aut}(\bullet)/\mathrm{Int}(\bullet)$ outer automorphisms, and $\mathrm{Out}(G,K_1) = \{\alpha \in \mathrm{Aut}(G) \mid \alpha(K_1) = K_1\,,\, \alpha|_{K_1} \in \mathrm{Out}(K_1)\}$
- Theorem. Let $\widetilde{M} \to \Gamma\backslash\widetilde{M}$ be a Riemannian covering. Equivalent: (1) $\gamma \in \Gamma$ is of constant displacement (2) $\Gamma \subset Z_G \times r(K_2)$, (3) $\Gamma\backslash\widetilde{M}$ is homogeneous.

Thank you for your attention